Math 521 **Midterm 1** **Fall 2012**

NAME:

(1) (10 points) For each of the following, give an example if one exists. If there is none, state that there is no example. You do not need to give any justification.

(a) A non-empty bounded subset of $\mathbb{Q}$ with no infimum in $\mathbb{Q}$.
(b) A subset of $\mathbb{R}$ containing $\mathbb{N}$ in which $\{1\}$ is open but $\{2\}$ is not.
(c) An unbounded convergent sequence in $\mathbb{R}$.
(d) An infinite metric space where every subset is open.
(e) A continuous function f from $\mathbb{R}$ to $\mathbb{R}$ and a pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ so that $(x_n - y_n)_{n\in\mathbb{N}}$ converges to 0 but $(f(x_n) - f(y_n))_{n\in\mathbb{N}}$ does not converge to 0.

Solution:

(a) $\{x \mid x^2 < 2\}$
(b) $\mathbb{N} \cup (2 - \frac{1}{2}, 2 + \frac{1}{2})$
(c) There is no example.
(d) $\mathbb{N}$ as a subset of $\mathbb{R}$ or $\mathbb{R}$ with the discrete metric.
(e) $f(x) = x^2$, $x_n = n$, and $y_n = n + \frac{1}{n}$.

(2) (10 points) Recall that a sequence (p_n) is Cauchy if for every $\epsilon > 0$ there is an integer N such that for all $k, n \geq N$, $d(p_k, p_n) < \epsilon$. We say a metric space is *complete* if every Cauchy sequence converges. Recall that we proved that $\mathbb{R}$ is complete.

a) Show that if M and N are complete, then so is $M \times N$ with the metric $d_{\max}$.

b) Use part a to show that if M and N are complete, then so is $M \times N$ with the metric d_E or d_{sum}.

c) Conclude that $\mathbb{R}^m$ is complete for any m.

Solution:

a) We want to show that $M \times N$ is complete. Suppose $(p_n, q_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then for each $\epsilon > 0$, there is an N so that $n, k \geq N \to d_{\max}((p_n, q_n), (p_k, q_k)) < \epsilon$. But then $n, k \geq N \to d_M(p_n, p_k) < \epsilon$ and $d_N(q_n, q_k) < \epsilon$. Thus $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are both Cauchy. Thus they both converge. But we know that if $(p_n)_{n\in\mathbb{N}}$ converges and $(q_n)_{n\in\mathbb{N}}$ converges, then $(p_n, q_n)_{n\in\mathbb{N}}$ converges. Thus we have shown that an arbitrary Cauchy sequence in $M \times N$ converges. Thus $M \times N$ is complete.

b) We know that a sequence in $M \times N$ converges with respect to the metric $d_{\max}$ if and only if it converges with respect to the metric d_E if and only if it converges with respect to the metric d_{sum}. Thus if every Cauchy sequence converges with respect to the metric $d_{\max}$, then every Cauchy sequence converges with respect the the other two metrics as well.

c) Induct: Base case: $m = 1$. We know $\mathbb{R}$ is complete.

$m = k + 1$: By the inductive hypothesis, $\mathbb{R}^k$ is complete. Also, $\mathbb{R}$ is complete. So, using part a, $\mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R}$ is complete.

(3) (10 points) Show that a subset S of a metric space M is closed in M if and only if $\partial S \subseteq S$; show that S is open in M if and only if $S \cap \partial S = \emptyset$.

Solutions: There were several ways to do this problem.

Claim 1. If S is closed in M, then $\partial S \subseteq S$.

Proof. Since S is closed, $\text{cl}(S) = S$. But then $\partial S = \text{cl}(S) \setminus \text{int}(S) \subseteq \text{cl}(S) = S$. □

Claim 2. If $\partial S \subseteq S$, then S is closed.

Proof. $\partial S = \text{cl}(S) \setminus \text{int}(S) \subseteq S$. Thus $(\text{cl}(S) \setminus \text{int}(S)) \cup \text{int}(S) \subseteq S \cup \text{int}(S)$. So, $\text{cl}(S) \subseteq (\text{cl}(S) \setminus \text{int}(S)) \cup \text{int}(S) \subseteq S \cup \text{int}(S) \subseteq S$. The last $\subseteq$ is because $\text{int}(S) \subseteq S$. □

Claim 3. S is open if and only if $S \cap \partial S = \emptyset$.

Proof. S is open if and only if S^C is closed if and only if $\partial(S^C) \subseteq S^C$ if and only if $\partial(S^C) \cap S = \emptyset$.

But $\partial S = \partial S^C$, (this was a homework assignment), so S is open if and only if $\partial(S^C) \cap S = \partial(S^C) \cap S = \emptyset$.

□

(4) (10 points) For which intervals $[a, b]$ in $\mathbb{R}$ is the intersection $[a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$ a clopen subset of the metric space $\mathbb{R} \setminus \mathbb{Q}$.

The answer: $[a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$ is a clopen subset of $\mathbb{R} \setminus \mathbb{Q}$ if and only if both a and b are rational.

Claim 4. If either a or b is irrational, then $[a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$ is not open in $\mathbb{R} \setminus \mathbb{Q}$.

Proof. Without loss of generality, we assume a is irrational. Then $a \in [a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$. But for any $r > 0$, $B_r(a)$ contains irrational numbers less than a, thus $B_r(a) \cap (\mathbb{R} \setminus \mathbb{Q}) \not\subseteq [a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$. Thus $[a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$ is not open in $\mathbb{R} \setminus \mathbb{Q}$. □

Claim 5. If a and b are rational, then $[a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$ is a clopen subset of $\mathbb{R} \setminus \mathbb{Q}$.

Proof. By the inheritance principle, $[a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$ is a clopen subset of $\mathbb{R} \setminus \mathbb{Q}$, since it is the intersection of $[a, b]$ with $\mathbb{R} \setminus \mathbb{Q}$. Similarly, it is open, as it the intersection of (a, b) with $\mathbb{R} \setminus \mathbb{Q}$. □