On the Lie algebroid of a derived self-intersection

Damien Calaque*, Andrei Căldăraru†, Junwu Tu‡

Abstract. Let $i : X \hookrightarrow Y$ be a closed embedding of smooth algebraic varieties. Denote by N the normal bundle of X in Y. We describe the construction of two Lie-type structures on the shifted bundle $N[-1]$ which encode the information of the formal neighborhood of X inside Y. We also present applications of classical Lie theoretic constructions (universal enveloping algebra, Chevalley-Eilenberg complex) to the understanding of the geometry of embeddings.

Contents

1. Introduction 1

2. The dg-Lie algebroid of a closed embedding 6

3. Monads 13

4. Obstructions 16

A. Deformation theory of sheaves of algebras 21

1. Introduction

The aim of this paper is to study the derived self-intersection of a closed subspace X of an ambient space Y, when X and Y are smooth algebraic varieties. The word "derived" appears because self-intersections are badly-behaved, in the sense that they are not transverse and thus their actual dimension does not coincide with the expected one. If X and Y were differentiable manifolds the standard approach would be to consider a small perturbation X_ϵ of X and to take $X \cap X_\epsilon$ as the self-intersection. However, this approach has several drawbacks:

- it does not preserve the set-theoretical intersection;

*Department of Mathematics, ETH Zürich, 8092 Zürich, Switzerland, *e-mail:* damien.calaque@math.ethz.ch. On leave from Institut Camille Jordan, CNRS & Université Lyon 1, France.

†Department of Mathematics, University of Wisconsin–Madison, 480 Lincoln Drive, Madison WI 53706-1388, USA, *e-mail:* andreic@math.wisc.edu.

‡Mathematics Department, University of Oregon, Eugene OR 97403, USA, *e-mail:* junwut@uoregon.edu.

- it does lead to some category theoretic problems (there is no functorial choice for X_ϵ);
- we can't do this with algebraic varieties.

It has long been known that a possible replacement for geometric perturbation is homological perturbation. More precisely, if $i : X \hookrightarrow Y$ is a closed embedding we will consider a resolution $\mathcal{R}$ of $\mathcal{O}_X$ by a differential graded (dg) $i^{-1}\mathcal{O}_Y$-algebra, and equip X with the sheaf of dg-rings $\mathcal{O}_X \otimes^{\mathbb{L}}_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X := \mathcal{R} \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X$. The pair $(X, \mathcal{O}_X \otimes^{\mathbb{L}}_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X)$ is a dg-scheme that we will denote by $X \times^h_Y X$; it is called the derived self-intersection of X inside Y (or more generally the homotopy fiber product of X with itself over Y for a general morphism $X \to Y$). Different choices of resolution give rise to weakly equivalent results (in the derived category of dg-schemes), and there exist functorial choices for such resolutions.

In the remainder of this introduction we shall make constant use of an important analogy between derived algebraic geometry (where objects of study are dg schemes), and homotopy theory (where one studies topological spaces and homotopy classes of maps between them). In both settings the notion of homotopy fiber product makes sense, and the two constructions have similar properties[1]. The advantage of studying the topological setting is that in this case the role of resolutions is taken by fibrant replacements, and the homotopy fiber product becomes a familiar space of paths. Since we can understand the derived self-intersection in more concrete terms, we gain insights into what results can be expected for the algebro-geometric problem we are studying. This motivates the precise results which we state and prove in this paper (and which ultimately do not rely on the analogy with topological spaces).

1.1. The diagonal embedding (after M. Kapranov)

Let us consider the example of the diagonal embedding $\Delta : X \hookrightarrow X \times X$. In the topological setting, one way to compute the homotopy fiber product is to factor $\Delta : X \to X \times X$ into an acyclic cofibration followed by a fibration: $X \overset{\sim}{\rightarrowtail} \tilde{X} \twoheadrightarrow X \times X$. This is achieved by taking

$$\tilde{X} := \{\gamma : [0,1] \to X \times X \,|\, \gamma(0) \in \Delta(X)\}$$

with $\gamma \mapsto \gamma(1)$ as the fibration map. Therefore the derived self-intersection of the diagonal is

$$\tilde{X} \times_{X \times X} X = \{\gamma : [0,1] \to X \times X \,|\, \gamma(0), \gamma(1) \in \Delta(X)\}.$$

This space is immediately seen to be weakly equivalent to the loop space of X via the map sending a path γ in $X \times X$ with both ends in the diagonal to the loop $\tilde{\gamma}$ in X defined by

$$\tilde{\gamma}(t) = \begin{cases} \pi_1\big(\gamma(2t)\big) & \text{if } 0 \leq t \leq 1/2 \\ \pi_2\big(\gamma(2-2t)\big) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Therefore $X \times^h_{X \times X} X$ fibers over X, and the fibers have the structure of a (homotopy) group.

[1]This can be made precise using model categories and the advanced technology of homotopical and derived algebraic geometry, after Toën-Vezzosi [19, 20] and Lurie [15]. Despite the fact that this is certainly the appropriate framework to work with, we will nevertheless stay within the realm of dg-schemes (after [6]), which are sufficient for our purposes.

Returning to the world of derived algebraic geometry, the homotopy fiber product $X \times^h_{X\times X} X$ can be shown to have a similar structure of derived group scheme over X^2. We are interested in understanding what the associated Lie algebra of this group is.

What is the Lie algebra of $X \times^h_{X\times X} X$?

The answer to this question is contained in a beautiful paper of Kapranov [13]: informally speaking this is the Atiyah extension $S^2(T_X) \to T_X[1]$, viewed as a Lie bracket $\wedge^2(T_X[-1]) \to T_X[-1]$ on the object $T_X[-1]$ of $\mathbf{D}(X)$. The existence of this bracket is not so mysterious. Associated to the embedding Δ is the dg Lie algebra $\mathfrak{g}_X := \mathbb{T}_{X/X\times X}$, the derived relative tangent space. Its Lie structure arises as in the case of any tangent space, from commutation of derivations. Since $T_X[-1]$ is the cohomology of $\mathfrak{g}_X$, it is naturally endowed with a (strict) Lie algebra structure, which Kapranov further identifies with the Atiyah bracket. Moreover, since this Lie structure is obtained by passing to the homology of a dg Lie algebra, it is natural to expect that homotopy transfer should induce higher (Massey) brackets on the cohomology $T_X[-1]$. These higher brackets are also constructed by other means in Kapranov's paper.

What is the universal enveloping algebra of this Lie algebra ?

The answer to this question was given by Markarian [16]: $\mathbf{U}(T_X[-1])$ is the Hochschild cochain complex $\mathcal{HH}_X := (\pi_1)_*\mathbb{R}\mathcal{H}om_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$ of X. The PBW theorem then gives an isomorphism in $\mathbf{D}(X)$

$$S\big(T_X[-1]\big) \xrightarrow{\sim} \mathcal{HH}_X,$$

which is nothing but the HKR theorem [11, 21].

Observe moreover that any object E in $\mathbf{D}(X)$ is naturally a representation of the Lie algebra $T_X[-1]$, via its own Atiyah class, and that morphisms in $\mathbf{D}(X)$ are all $T_X[-1]$-linear. The $T_X[-1]$-invariant space of an object E is then the space of morphisms from $\mathcal{O}_X$, the trivial representation, to E: these are (derived) global sections. Assuming that the Duflo isomorphism[3] is valid in dg or triangulated categories (which is unknown), we would have an isomorphism of algebras

$$H^*\Big(X, S\big(T_X[-1]\big)\Big) \xrightarrow{\sim} HH^*_X := \mathrm{Ext}^*_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)\,.$$

This result has been proved using a different approach in [5, 7].

What is the Chevalley-Eilenberg algebra of $\mathfrak{g}_X$?

The answer to this question is again in Kapranov's paper: $C^*(\mathfrak{g}_X)$ is quasi-isomorphic to the structure ring $\mathcal{O}_{X^{(\infty)}_{X\times X}}$ of the formal neighborhood $X^{(\infty)}_{X\times X}$ of the diagonal in $X \times X$.

[2] It is an X-scheme because $X \times X$ itself is an X-scheme through one of the projections $X \times X \to X$. Moreover, the choice of one or the other is "homotopically irrelevant" in our context. Notice nevertheless that it would be a bit more subtle if we were looking at the 2-groupoid structure on $X \times^h_{X\times X} X$.

[3] Which states that for a finite dimensional Lie algebra $\mathfrak{g}$, $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$ are isomorphic as algebras.

1.2. The general case of a closed embedding (this paper)

For a general closed embedding $i : X \hookrightarrow Y$ of smooth schemes we start again by taking cues from topology. If i were an arbitrary embedding of topological spaces, we could use as a fibrant replacement the path space

$$\tilde{X} := \{\gamma : [0,1] \to Y \,|\, \gamma(0) \in i(X)\}$$

with $\gamma \mapsto \gamma(1)$ as the fibration morphism. Therefore a topological model for the derived self-intersection of X into Y would be

$$\tilde{X} \times_Y X = \{\gamma : [0,1] \to Y \,|\, \gamma(0), \gamma(1) \in i(X)\},$$

which this time does not have the structure of a group scheme over X. Nevertheless $X \times^h_Y X$ is the space of arrows of a homotopy groupoid having X as space of objects.

The same structure is carried over to the algebro-geometric setting: for a closed embedding of smooth schemes, $X \times^h_Y X$ has the structure of a derived groupoid Y-scheme having X as the space of objects. One can again pose the problem of determining the Lie algebroid associated to this groupoid.

What is the Lie algebroid of $X \times^h_Y X$?

The informal answer to this question is that the Lie algebroid of the groupoid scheme $X \times^h_Y X$ is the shifted normal bundle $N[-1]$ of X in Y, with anchor map the Kodaira-Spencer class $N[-1] \to T_X$. As before, this structure arises as the shadow on cohomology of a richer dg Lie algebroid structure on the relative derived tangent space $\mathbb{T}_{X/Y}$. From this point of view the anchor map is the map induced on cohomology from the base change morphism $\mathbb{T}_{X/Y} \to \mathbb{T}_{X/\mathsf{pt}}$. A more precise statement of the above discussion is given below.

Proposition 1.1. *Let $X \xrightarrow{j} \widetilde{X} \xrightarrow{\pi} Y$ be a smooth resolution of the inclusion morphism $i : X \hookrightarrow Y$. Then the pair $(\mathcal{O}_{\widetilde{X}}, T_{\widetilde{X}/Y})$ is a dg-Lie algebroid over $\mathcal{O}_Y$ and it is the dg-Lie algebroid of the groupoid in dg-Y-schemes $(\widetilde{X}, \widetilde{X} \times_Y \widetilde{X})$.*

In order to pass to more familiar objects, note that $j^* : \mathbf{D}(\widetilde{X}) \to \mathbf{D}(X)$ is a triangulated equivalence, and under this equivalence the object $T_{\widetilde{X}/Y}$ corresponds to the object $\mathbb{T}_{X/Y} := j^* T_{\widetilde{X}/Y} \cong N[-1]$ in $\mathbf{D}(X)$. In particular, we get from the above that $(i_*\mathcal{O}_X, i_*N[-1])$ is a Lie algebroid in $\mathbf{D}(Y)$.

What is the universal enveloping algebra of this Lie algebroid ?

The enveloping algebra of a Lie algebroid with base X is naturally an $\mathcal{O}_{X\times X}$-module set-theoretically supported on the diagonal. This is consistent with the the answer we give to the above question: the universal enveloping algebra of the Lie algebroid of $X \times^h_Y X$ is the kernel K of $i^*i_! = (i^*i_*)^\vee$, where the algebra structure $K \circ K \to K$ is given by the natural transformation $i^*i_!i^*i_! \implies i^*i_!$. More precisely (see §2.5 and Section 3) we have:

Theorem 1.2. *Let $X \xrightarrow{j} \widetilde{X} \xrightarrow{\pi} Y$ be a smooth resolution of the inclusion morphism $i : X \hookrightarrow Y$. Then we have a natural quasi-isomorphism of dg-algebras $\mathbf{U}(T_{\widetilde{X}/Y}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}})$. In other words, $\mathbf{U}(T_{\widetilde{X}/Y})$ represents the monad $i^*i_!$.*

We also prove that the above identification is also "compatible with coproducts". This can be seen *via* the following dual statement:

Theorem 1.3. *Let* $X \xrightarrow{j} \widetilde{X} \xrightarrow{\pi} Y$ *be a smooth resolution of the inclusion morphism* $i : X \hookrightarrow Y$*. Then we have a natural quasi-isomorphism of dg-algebras* $J(T_{\widetilde{X}/Y}) \longrightarrow \mathcal{O}_{\widetilde{X}\times_Y \widetilde{X}}$*. In other words,* $J(T_{\widetilde{X}/Y})$ *represents the monad* i^*i_**.*

What is the Chevalley-Eilenberg algebra of this Lie algebroid ?

We answer that this is quasi-isomorphic to the structure ring $\mathcal{O}_{X_Y^{(\infty)}}$ of the formal neighborhood $X_Y^{(\infty)}$ of the embedding $i : X \hookrightarrow Y$, generalizing Kapranov's result in the case diagonal embeddings. See Theorem 2.4.

The above result uses the full structure of dg Lie algebroid on $T_{\widetilde{X}/Y}$. One would be interested in obtaining an ∞-type structure on its cohomology $N[-1]$, by constructing higher operations on this object. This would be a complete generalization of the construction that Kapranov gave in the diagonal embedding case.

To achieve this goal we would need to employ a homotopy transfer procedure to get a minimal L_∞-algebroid structure on $N[-1]$ (see Definition 4.2). We are not actually able to do homotopy transfer for complexes of sheaves (as opposed to complexes of vector spaces), but we are able to prove the following result (see Proposition 4.4 in Section 4):

Proposition 1.4. *There exists a minimal* L_∞*-algebroid structure on* $N[-1]$ *whose Chevalley-Eilenberg algebra is quasi-isomorphic to* $\mathcal{O}_{X_Y^{(\infty)}}$*.*

As a consequence of this construction we obtain "higher anchor maps" which can be interpreted as successive obstructions to splitting the embedding of X into its l-th infinitesimal neighborhood $X_Y^{(l)}$. (These obstructions already appeared in problems in complex geometry, see [1].) We also obtain "higher brackets" which are interpreted as successive obstructions (which also appeared in [1]) to linearizing $X_Y^{(l)}$.

1.3. Plan of the paper

In Section 2 we describe the dg-Lie algebroid associated with a closed embedding. We first recall quickly some known facts about the (co)tangent complex and about Lie algebroids, with emphasis on relative derivations. The Lie algebroid structure on the relative tangent complex of a morphism $f : X \to Y$ then becomes obvious. We finally prove that, in the case of a closed embedding $i : X \hookrightarrow Y$, the Chevalley-Eilenberg algebra of this Lie algebroid is quasi-isomorphic to the structure sheaf of $X_Y^{(\infty)}$, its jet algebra represents i^*i_* and its universal envelopping algebra represents $i^*i_!$.

In the third section we show that the above identifications are compatible with natural Hopf-like structures on these objects.

Section 4 is devoted to the construction of an L_∞-algebroid structure on the shifted normal bundle $N[-1]$, which is such that its Chevalley-Eilenberg algebra is also quasi-isomorphic to the structure sheaf of $X_Y^{(\infty)}$. We prove that the structure maps of this L_∞-algebroid structure provide obstructions to splitting the inclusion $X \to X_Y^{(k)}$ and linearizing $X_Y^{(k)}$. This allows us to recover similar results of [1].

We conclude the paper with an appendix about cosimplicial methods and a correspondence betwen Maurer-Cartan elements and non-abelian 1-cocycles.

1.4. Acknowledgements

Shilin Yu [22] independently obtained similar results for complex manifolds (closer in spirit to Kapranov's original formulation). We warmly thank him for the numerous enlightening discussions we had before and during the writing of the present paper. The second author was inspired by numerous conversations with Dima Arinkin and Tony Pantev during this project. The work of the first author is supported by the Swiss National Science Foundation (grant number 200021_137778). The second and third authors were supported by the National Science Foundation under Grants No. DMS-0901224 and DMS-1200721.

1.5. Notation

We work over an algebraically closed field $\mathbf{k}$ of characteristic zero. Unless otherwise specified all (dg-)algebras, (dg-)schemes, varieties, etc. are over $\mathbf{k}$.

We often denote by pt the terminal dg-scheme. Namely, $\mathsf{pt} := \mathsf{Spec}(\mathbf{k})$.

Whenever we have a dg-algebra A we denote by $A^\sharp$ the underlying graded algebra and by d_A its differential. Then $A = (A^\sharp, d_A)$. Similarly, given a dg-scheme X we denote by $X_\sharp$ the dg-scheme having the same underlying scheme and structure sheaf $\mathcal{O}_{X_\sharp} := \mathcal{O}_X^\sharp$.

For a graded module E and an integer k, we denote by $E[k]$ the graded module whose l-th graded piece is the $k+l$-th graded piece of E.

We freely extend the above notation to sheaves.

By a Lie algebroid we mean a sheaf of Lie-Rinehart algebras (L-R algebras in [18], to which we refer for more details).

2. The dg-Lie algebroid of a closed embedding

In this section we construct a differential graded Lie algebroid associated to a closed embedding $i : X \hookrightarrow Y$ of smooth algebraic varieties. We show that the Chevalley-Eilenberg algebra of this differential graded Lie algebroid is quasi-isomorphic to the formal neighborhood algebra of X in Y; while the dual of its universal enveloping algebra is quasi-isomorphic to the formal neighborhood of X inside the derived self-intersection $X \times_Y^h X$. We shall work with Ciocan-Fontanine and Kapranov's dg-schemes [6, Section 2].

2.1. Short review of the relative (co)tangent complex

Let $i : X \hookrightarrow Y$ be a closed embedding of smooth algebraic varieties with normal bundle N. Let us also assume that Y is quasi-projective to ensure existence of resolutions by locally free sheaves on Y.

By the constructions of [6, Theorem 2.7.6] the map i can be factored as

$$X \xrightarrow{j} \widetilde{X} \xrightarrow{\pi} Y,$$

where j is a quasi-isomorphic closed embedding and π is smooth. Moreover since we are assuming Y is quasi-projective and smooth, the ordinary scheme underlying the dg-scheme $\widetilde{X}$ can in fact be taken to be just Y. Its structure sheaf is of the form

$$\mathcal{O}_{\widetilde{X}} := \big(\mathbf{S}_{\mathcal{O}_Y}(E[1]), Q\big)$$

for some non-positively graded locally free sheaf E on Y, and Q is a degree one $\mathcal{O}_Y$-linear derivation on $\mathbf{S}_{\mathcal{O}_Y}(E[1])$ that squares to zero. The appearance of a shift on E is to mimic the case when X is a complete intersection in Y. By construction there is a quasi-isomorphism of dg-$\mathcal{O}_Y$-algebras

$$\mathcal{O}_{\widetilde{X}} \longrightarrow j_*\mathcal{O}_X .$$

The relative cotangent complex $\mathbb{L}_{X/Y}$ is by definition

$$\mathbb{L}_{X/Y} := j^*L_{\widetilde{X}/Y}, \quad \text{where} \quad L_{\widetilde{X}/Y} := \big(\Omega^1_{\mathcal{O}^\sharp_{\widetilde{X}}/\mathcal{O}_Y}, L_Q\big) .$$

The differential L_Q is given by the Lie derivation action on the graded space of 1-forms $\Omega^1_{\mathcal{O}^\sharp_{\widetilde{X}}/\mathcal{O}_Y}$. Since Q is an odd derivation that squares to zero, its Lie derivative L_Q also squares to zero because $L_Q^2 = \frac{1}{2}L_{[Q,Q]} = 0$. It is well-defined up to quasi-isomorphism (see [6, Proposition 2.7.7] for a precise statement). Moreover, the restriction morphism $L_{\widetilde{X}/Y} \longrightarrow j_*j^*L_{\widetilde{X}/Y} = j_*\mathbb{L}_{X/Y}$ is a quasi-isomorphism. In particular this implies that the dg-sheaf $L_{\widetilde{X}/Y}$ is isomorphic to $j_*N^\vee[1]$ as an object in $\mathbf{D}(\widetilde{X})$. Namely, we have two distinguished triangles

$$i^*\Omega^1_{Y/pt} \longrightarrow \Omega^1_{X/pt} \longrightarrow \mathbb{L}_{X/Y} \xrightarrow{+1} \qquad \text{and} \qquad N^\vee \longrightarrow i^*\Omega^1_{Y/pt} \longrightarrow \Omega^1_{X/pt} \xrightarrow{+1} .$$

Dually we can define the tangent complex $\mathbb{T}_{X/Y}$ as $j^*T_{\widetilde{X}/Y}$, where $T_{\widetilde{X}/Y} := \mathsf{Der}_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}})$ is endowed with the differential $[Q, -]$. The fact that this differential squares to zero follows from the graded version of Jacobi identity for vector fields. The dg-sheaf $T_{\widetilde{X}/Y}$ is isomorphic to $i_*N[-1]$ in $\mathbf{D}(Y)$[4].

2.2. Recollection on (dg-)Lie algebroids

There is a natural Lie bracket on $T_{\widetilde{X}/Y}$, being the space of $\mathcal{O}_Y$-linear derivations of $\mathcal{O}_{\widetilde{X}}$. More precisely, the pair $(\mathcal{O}_{\widetilde{X}}, T_{\widetilde{X}/Y})$ is a dg-Lie algebroid. In the next three sections we recall the definitions of three dg-algebras naturally associated with a dg-Lie algebroid: the Chevalley-Eilenberg algebra, the universal enveloping algebra, and the jet algebra.

[4] We are making here a slight abuse of language. Strictly speaking, $T_{\widetilde{X}/Y}$ is quasi-isomorphic to $j_*N[-1]$ as an $\mathcal{O}_{\widetilde{X}}$-module and thus $\pi_*T_{\widetilde{X}/Y}$ is quasi-isomorphic to $i_*N[-1]$ as an $\mathcal{O}_Y$-module. But for an $\mathcal{O}_{\widetilde{X}}$-module F, π_*F is simply the same sheaf (on Y) viewed as an $\mathcal{O}_Y$-module in the obvious way. We will therefore allow ourselves to omit π_* from the notation when the context is clear enough.

2.2.1. Chevalley-Eilenberg algebra of a Lie algebroid

Let $(\mathcal{A}, \mathfrak{g})$ be a dg-Lie algebroid with anchor map ρ and Lie bracket μ. We first consider the dg-commutative algebra $\widehat{\mathbf{S}}_{\mathcal{A}}(\mathfrak{g}^\vee[-1])$, which is the adic-completion of the symmetric algebra with respect to the kernel of the augmentation $\mathbf{S}_{\mathcal{A}}(\mathfrak{g}^\vee[-1]) \to \mathcal{A}$. We then replace the differential d by $D := d + d_{\mathsf{CE}}$, where d_{CE} is defined on generators $a \in \mathcal{A}$ and $\xi \in \mathfrak{g}^\vee[-1]$ as follows[5]:

- $d_{\mathsf{CE}}(a) := s \circ \rho^\vee(da)$, and
- the operator $d_{\mathsf{CE}}(\xi) \in \mathbf{S}^2_{\mathcal{A}} \mathfrak{g}^\vee[-1]$ acts on a pair of $(v, w) \in \mathfrak{g}^2$ by formula

$$d_{\mathsf{CE}}(\xi)(v, w) := \rho(v)(\xi(w)) - \rho(w)(\xi(v)) - \xi(\mu(v, w)).$$

Then the Chevalley-Eilenberg algebra $\mathsf{C}^*(\mathfrak{g})$ is the dg-commutative algebra $\big(\widehat{\mathbf{S}}_{\mathcal{A}}(\mathfrak{g}^\vee[-1])^\sharp, D\big)$. We also denote by $\mathsf{C}^{(k)}(\mathfrak{g})$ its quotient by the $k+1$-th power of the augmentation kernel.

Example 2.1 (Relative De Rham complex)**.** Let $\mathcal{B} \to \mathcal{A}$ be a morphism of (sheaves of) dg-commutative algebras, and set $\mathfrak{g} := \mathsf{Der}_{\mathcal{B}}(\mathcal{A})$. We equip $(\mathcal{A}, \mathfrak{g})$ with a dg-Lie algebroid structure: $\mathfrak{g}$ has $[d_{\mathcal{A}}, -]$ as differential, the graded commutator as Lie bracket, and the inclusion $\mathsf{Der}_{\mathcal{B}}(\mathcal{A}) \hookrightarrow \mathsf{Der}_{\mathbf{k}}(\mathcal{A})$ as anchor map. The Chevalley-Eilenberg algebra is then the completion of the relative De Rham algebra $\Omega^*_{\mathcal{A}/\mathcal{B}}$. As a graded algebra $\Omega^*_{\mathcal{A}/\mathcal{B}}$ is $\mathbf{S}_{\mathcal{A}^\sharp}(\Omega^1_{\mathcal{A}^\sharp/\mathcal{B}^\sharp}[-1])$, and the two commuting differentials are $d = L_{d_B}$ and $d_{\mathsf{CE}} = d_{\mathsf{DR}}$.

2.2.2. Universal enveloping algebra of a Lie algebroid

We borrow the notation from the previous subsection. First observe that $\mathcal{A}$ being acted on by $\mathfrak{g}$ one can consider the semi-direct product dg-Lie algebra $\mathcal{A} \rtimes \mathfrak{g}$ and take its (ordinary) universal enveloping algebra $\mathsf{U}(\mathcal{A} \rtimes \mathfrak{g})$, which is $\mathbf{k}$-augmented. We define the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of $(\mathcal{A}, \mathfrak{g})$ as the quotient of the augmentation ideal $\mathsf{U}^+(\mathcal{A} \rtimes \mathfrak{g})$ by the following relations: for any $a \in \mathcal{A}$ and $x \in \mathcal{A} \oplus \mathfrak{g}$, $ax = a \cdot x$, where $\cdot$ denotes the $\mathcal{A}$-module structure on $\mathcal{A} \oplus \mathfrak{g}$. It is a (non-central) dg-$\mathcal{A}$-algebra. As such it is naturally endowed with a compatible $\mathcal{A}$-bimodule structure.

Moreover, $\mathbf{U}(\mathfrak{g})$ is also endowed with the structure of a cocommutative coring in the category of *left* $\mathcal{A}$-modules: we have a coproduct $\Delta : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbf{U}(\mathfrak{g})$, where we only consider the *left* $\mathcal{A}$-module structure on $\mathbf{U}(\mathfrak{g})$ when taking the tensor product. Notice that $\mathbf{U}(\mathfrak{g})$ also acts on $\mathcal{A}$. All these algebraic structures and their compatibilities can be summarized in the following way: $\mathbf{U}(\mathfrak{g})$ is a (dg-)bialgebroid (see [3] and references therein for a survey of the equivalent definitions).

Example 2.2 (Relative differential operators)**.** Let $\mathcal{B} \to \mathcal{A}$ and $\mathfrak{g}$ be as in Example 2.1. In this case $\mathbf{U}(\mathfrak{g})$ coincides with $\mathit{Diff}_{\mathcal{B}}(\mathcal{A})$, the sheaf of $\mathcal{B}$-linear differential operators on $\mathcal{A}$ endowed with the differential $[d_{\mathcal{B}}, -]$. We easily see that the $\mathcal{A}$-bimodule structure factors through an action of $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$. Finally, the coproduct is given by $\Delta(P)(a, a') = P(aa')$.

[5] We omit (de)suspension maps from the second formula.

2.2.3. The jet algebra of a Lie algebroid

We still borrow the notation from the above subsection. We define the jet $\mathcal{A}$-bimodule $\mathbf{J}(\mathfrak{g})$ as the left $\mathcal{A}$-dual of $\mathbf{U}(\mathfrak{g})$: $\mathbf{J}(\mathfrak{g}) := \underline{\mathsf{Hom}}_{\mathcal{A}\text{-mod}}\big(\mathbf{U}(\mathfrak{g}),\mathcal{A}\big)$. Therefore $\mathbf{J}(\mathfrak{g})$ becomes a commutative ring in the category of *left* $\mathcal{A}$-modules. The $\mathcal{A}$-bimodule structure on $\mathbf{J}(\mathfrak{g})$ can then be described by two dg-algebra morphisms $\mathcal{A} \longrightarrow \mathbf{J}(\mathfrak{g})$: $a \longmapsto \big(P \mapsto aP(1)\big)$ and $a \longmapsto \big(P \mapsto P(a)\big)$.

Observe that $\mathbf{U}(\mathfrak{g})$ is endowed with an increasing filtration obtained by assiging degree 0, resp. degree 1, to elements of $\mathcal{A}$, resp. $\mathfrak{g}$. All the algebraic structures we have seen on $\mathbf{U}(\mathfrak{g})$ are compatible with the filtration; in particular $\mathbf{U}(\mathfrak{g})^{\leq k}$ is a sub-$\mathcal{A}$-coring in $\mathbf{U}(\mathfrak{g})$. Therefore the quotient $\mathbf{J}^{(k)}(\mathfrak{g}) := \underline{\mathsf{Hom}}_{\mathcal{A}\text{-mod}}\big(\mathbf{U}(\mathfrak{g})^{\leq k},\mathcal{A}\big)$ of $\mathbf{J}(\mathfrak{g})$ inherits from it an $\mathcal{A}$-algebra structure.

Example 2.3 (Relative jets)**.** Let $\mathcal{B} \to \mathcal{A}$ and $\mathfrak{g}$ be as in Example 2.1, and denote by $\mathcal{J}$ the (dg-)ideal of the multiplication map $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A}$. We have a morphism of dg-algebras

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathbf{J}(\mathfrak{g})\,,\ a \otimes a' \longmapsto \big(P \mapsto aP(a')\big)\,.$$

This induces morphisms $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}/\mathcal{J}^{k+1} \longrightarrow \mathbf{J}^{(k)}(\mathfrak{g})$, which happen to be isomorphisms whenever the underlying morphism of graded algebra $\mathcal{B}^\sharp \to \mathcal{A}^\sharp$ is smooth. This gives an isomorphism between $\mathcal{A}\hat{\otimes}_{\mathcal{B}}\mathcal{A} := \varprojlim \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}/\mathcal{J}^{k+1}$ and $\mathbf{J}(\mathfrak{g})$. This is essentially (a dg-version of) Grothendieck's description of differential operators via formal neighborhood of the diagonal map [8, (16.8.4)].

2.3. The Chevalley-Eilenberg algebra is the formal neighborhood of X in Y

Let $\mathcal{I} := \mathsf{ker}(\mathcal{O}_Y \to i_*\mathcal{O}_X)$ be the ideal sheaf of X into Y, $\mathcal{O}_X^{(k)} := \mathcal{O}_Y/\mathcal{I}^{k+1}$, and $\mathcal{O}_X^{(\infty)} := \varprojlim \mathcal{O}_X^{(k)}$. We also write $\Omega^*_{\tilde{X}/Y} := \Omega^*_{\mathcal{O}_{\tilde{X}}/\mathcal{O}_Y}$, and $\Omega^{(k)}_{\tilde{X}/Y}$ for its quotient by the $(k+1)$-th power of the augmentation ideal. Recall from Example 2.1 that we have $C^*(T_{\tilde{X}/Y}) = \varprojlim \Omega^{(k)}_{\tilde{X}/Y}$.

Theorem 2.4. *There are quasi-isomorphisms of sheaves of dg-algebras on* Y

$$\varphi^{(k)} : \Omega^{(k)}_{\tilde{X}/Y} \longrightarrow \mathcal{O}_X^{(k)}\,.$$

Moreover these maps are compatible with the inverse systems on both sides, and taking inverse limits induces a quasi-isomorphism of algebras

$$\varphi : C^*(T_{\tilde{X}/Y}) \longrightarrow \mathcal{O}_X^{(\infty)}\,.$$

Proof. We define an $\mathcal{O}^\sharp_{\tilde{X}}$-linear map $\varphi : \Omega^1_{\tilde{X}_\sharp/Y}[-1] \longrightarrow \mathcal{O}_Y$ by the composition

$$\Omega^1_{\tilde{X}_\sharp/Y}[-1] \stackrel{-\iota_Q}{\longrightarrow} \mathcal{O}^\sharp_{\tilde{X}} = \mathbf{S}_{\mathcal{O}_Y}\big(E[1]\big) \stackrel{\epsilon}{\twoheadrightarrow} \mathcal{O}_Y\,,$$

where the last map is just the canonical $\mathcal{O}_Y$-augmentation.

Lemma 2.5. *The map* φ *constructed above has image inside the ideal sheaf* $\mathcal{I}$.

Proof. Locally on Y the space of one-forms is generated by elements of the form $a d_{DR}(e)$ for local sections $a \in \mathcal{O}_{\widetilde{X}}$ and $e \in E$. Then we have

$$\varphi\big(a d_{DR}(e)\big) = \epsilon\big(-\iota_Q\big(a d_{DR}(e)\big)\big) = -\epsilon\big(aQ(e)\big) = -\epsilon(a)\epsilon\big(Q(e)\big)\,.$$

The composed map $\mathcal{O}_{\widetilde{X}} \xrightarrow{\epsilon} \mathcal{O}_Y \to i_*\mathcal{O}_X$ being a quasi-isomorphism, we have in particular that the image of $Q(e)$ through it is zero. Therefore $\epsilon\big(Q(e)\big) \in \mathcal{J}$. ☐

By the universal property of symmetric algebras, the map φ induces a morphism of $\mathcal{O}^{\sharp}_{\widetilde{X}}$-algebras

$$\Omega^*_{\widetilde{X}_\sharp/Y} = \mathbf{S}_{\mathcal{O}^\sharp_{\widetilde{X}}}\Big(\Omega^1_{\widetilde{X}_\sharp/Y}[-1]\Big) \longrightarrow \mathcal{O}_Y\,.$$

We shall still denote this map by φ.

Lemma 2.6. *φ is a morphism of $\mathcal{O}_{\widetilde{X}}$-algebras. In other words, it is a cochain map.*

Proof. Locally on Y, a k-form α can be written as $f d_{DR}(e_1)\cdots d_{DR}(e_k)$ for local sections $f \in \mathcal{O}_{\widetilde{X}}$ and $e_i \in E$, $1 \le i \le k$. We want to show that $\varphi\big(D(\alpha)\big) = 0$. This is a direct computation, we have

$$\begin{aligned}
\varphi\big(D(\alpha)\big) &= \varphi\Big(L_Q\big(f d_{DR}(e_1)\cdots d_{DR}(e_k)\big) + d_{DR}\big(f d_{DR}(e_1)\cdots d_{DR}(e_i)\big)\Big)\\
&= \varphi\Big(Q(f) d_{DR}(e_1)\cdots d_{DR}(e_k) + \sum_{1\le i\le k}(-1)^{|f|+|e_1|+\cdots+|e_{i-1}|+(i-1)} f d_{DR}(e_1)\cdots L_Q\big(d_{DR}(e_i)\big)\cdots d_{DR}(e_k)\\
&\quad + d_{DR}(f) d_{DR}(e_1)\cdots d_{DR}(e_k)\Big)\\
&= \epsilon\Big((-1)^k Q(f)Q(e_1)\cdots Q(e_k) + (-1)^{k+|f|+|e_1|+\cdots+|e_{i-1}|+(i-1)}\sum_{1\le i\le k} fQ(e_1)\cdots\big(\iota_Q L_Q d_{DR}(e_i)\big)\cdots Q(e_k)\\
&\quad + (-1)^{k+1}Q(f)Q(e_1)\cdots Q(e_k)\Big).
\end{aligned}$$

The first and the last terms in the above sum cancel each other. For the middle term observe that $\iota_Q L_Q d_{DR} = \iota_Q d_{DR}\iota_Q d_{DR} = QQ = 0$. ☐

By Lemma 2.5 the map φ sends the augmentation ideal (which is generated by $\Omega^1_{\widetilde{X}/Y}[-1]$) into the ideal $\mathcal{J}$. Hence φ induces an inverse system of $\mathcal{O}_{\widetilde{X}}$-algebra morphisms

$$\varphi^{(k)} : \Omega^{(k)}_{\widetilde{X}/Y} \longrightarrow \mathcal{O}^{(k)}_X\,.$$

We have to show that the maps $\varphi^{(k)}$ defined as above are quasi-isomorphisms. We proceed by induction. For the case $k = 0$ this reduces to the fact that $\mathcal{O}_{\widetilde{X}}$ is a resolution of $i_*\mathcal{O}_X = \mathcal{O}_Y/\mathcal{J}$. Assuming $\varphi^{(k)}$ is a quasi-isomorphism we would like to show it is also the case for $\varphi^{(k+1)}$. For this we consider the following morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{S}_{\mathcal{O}_Y}(E[1]) \otimes_{\mathcal{O}_Y} \mathbf{S}^{k+1}_{\mathcal{O}_Y}(E) & \longrightarrow & \Omega^{(k+1)}_{\widetilde{X}/Y} & \longrightarrow & \Omega^{(k)}_{\widetilde{X}/Y} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle\varphi^{(k+1)}} & & \downarrow{\scriptstyle\varphi^{(k)}} & & \\
0 & \longrightarrow & i_*\Big(\mathbf{S}^{k+1}_{\mathcal{O}_Y}(N^\vee)\Big) & \longrightarrow & \mathcal{O}^{(k+1)}_X & \longrightarrow & \mathcal{O}^{(k)}_X & \longrightarrow & 0.
\end{array}$$

The leftmost vertical map is a quasi-isomorphism by the projection formula, and $\varphi^{(k)}$ is so by the induction hypothesis. Thus $\varphi^{(k+1)}$ is also a quasi-isomorphism. The Theorem is proved. ☐

Remark 2.7. This result means that the dg-Lie algebroid structure on $T_{\widetilde{X}/Y}$ encodes all the formal neighborhood of X into Y. Moreover our proof of the theorem works for a general closed embedding of quasi-projective dg-schemes. For a general morphism $X \to Y$ of quasi-projective dg-schemes this gives a *definition* of the "derived" formal neighborhood of X in Y: take the Chevalley-Eilenberg algebra of $T_{\widetilde{X}/Y}$ for a smooth factorization of $X \to Y$ through $\widetilde{X}$.

Example 2.8 (The case of a global complete intersection)**.** As an example we now write down an explicit free resolution for the formal neighborhood algebra of an embedding $i : X \hookrightarrow Y$ in the case when X is a global complete intersection in Y. It turns out that our construction recovers the classical Eagon-Northcott resolution of formal neighborhood of X in Y (see [4]).

To fix the notation we assume that X is defined as the zero locus of a section s of a vector bundle $E^\vee$ on Y. Then the sheaf $i_*\mathcal{O}_X$ has a locally free resolution of the form

$$(\mathbf{S}_{\mathcal{O}_Y}(E[1]), \lrcorner s) = \cdots \xrightarrow{\lrcorner s} \wedge^2 E \xrightarrow{\lrcorner s} E \xrightarrow{\lrcorner s} \mathcal{O}_Y \longrightarrow 0 \cdots .$$

Note that this is a resolution of $i_*\mathcal{O}_X$ as an $\mathcal{O}_Y$-algebra. Thus we take $\widetilde{X}$ to be the dg-scheme defined by $(\mathbf{S}_{\mathcal{O}_Y}(E[1]), \lrcorner s)$ on Y. Let us write down the Chevalley-Eilenberg complex of $T_{\widetilde{X}/Y}$. This is by definition the completed De Rham complex of $(\mathbf{S}_{\mathcal{O}_Y}(E[1]), \lrcorner s)$ which is given by $\mathbf{S}_{\mathcal{O}_Y}(E[1]) \otimes_{\mathcal{O}_Y} \widehat{\mathbf{S}}_{\mathcal{O}_Y}(E)$ as a quasi-coherent sheaf on Y, with differential acting on the component $\mathbf{S}^i_{\mathcal{O}_Y}(E[1]) \otimes_{\mathcal{O}_Y} \mathbf{S}^j_{\mathcal{O}_Y}(E)$ by the sum of the following two operators:

$$L_{\lrcorner s}\big((e_1 \cdots e_i) \otimes (f_1 \cdots f_j)\big) = \sum_{l=1}^{i} (-1)^{l-1} \Big(\big(e_1 \cdots (\langle s, e_l \rangle) \cdots e_i\big) \otimes (f_1 \cdots f_j)\Big);$$

$$d_{DR}\big((e_1 \cdots e_i) \otimes (f_1 \cdots f_j)\big) = \sum_{l=1}^{i} (-1)^{l-1} \big((e_1 \cdots \widehat{e_l} \cdots e_i) \otimes (e_l f_1 \cdots f_j)\big).$$

2.4. The jet algebra represents i^*i_*

Our goal in this subsection is to understand the jet algebra $\mathbf{J}(T_{\widetilde{X}/Y})$ of the dg-Lie algebroid $T_{\widetilde{X}/Y}$. Our main result identifies $\mathbf{J}(T_{\widetilde{X}/Y})$ with the formal neighborhood of $\widetilde{X}$ inside $\widetilde{X} \times_Y \widetilde{X}$.

Let us begin with a description of the derived self-intersection $\widetilde{X} \times_Y \widetilde{X}$: by the definition of fiber product of dg-schemes its underlying scheme is $\widetilde{X}^0 \times_Y \widetilde{X}^0 = Y$ (recall that $\widetilde{X}^0 = Y$) and its structure sheaf is $\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{\widetilde{X}}$. The diagonal map $\Delta_{\widetilde{X}/Y} : \widetilde{X} \to \widetilde{X} \times_Y \widetilde{X}$ is defined by the identity map on the underlying scheme Y and the multiplication map $\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}$ on the structure sheaves. We denote its kernel by $\mathcal{J}$, and we define $\mathcal{O}_\Delta^{(k)} := \mathcal{O}_{\widetilde{X} \times_Y \widetilde{X}}/\mathcal{J}^{k+1}$ $(k \geq 0)$ and $\mathcal{O}_\Delta^{(\infty)} := \varprojlim \mathcal{O}_\Delta^{(k)}$. According to Example 2.3 we have the following description of $\mathbf{J}(T_{\widetilde{X}/Y})$.

Proposition 2.9. *There is an isomorphism $\mathcal{O}_\Delta^{(\infty)} \cong \mathbf{J}(T_{\widetilde{X}/Y})$ of $\mathcal{O}_{\widetilde{X} \times_Y \widetilde{X}}$-algebras.*

This result in fact holds for any morphism of dg-schemes $X \to Y$. The specificity of a closed embedding $i : X \hookrightarrow Y$ of varieties is that the formal neighborhood of the diagonal in $\widetilde{X} \times_Y \widetilde{X}$ is quasi-equivalent to $\widetilde{X} \times_Y \widetilde{X}$ itself, as we prove now.

Theorem 2.10. *For a closed embedding $i : X \hookrightarrow Y$ of smooth schemes assume that Y is quasi-projective. Then the formal neighborhood algebra $\mathcal{O}_\Delta^{(\infty)}$ of the embedding $\widetilde{X} \hookrightarrow \widetilde{X} \times_Y \widetilde{X}$ is quasi-isomorphic to the structure sheaf $\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}}$.*

In paticular, combined with Proposition 2.9, we conclude that $\mathbf{J}(T_{\widetilde{X}/Y})$ is the structure sheaf of $\widetilde{X}\times_Y\widetilde{X}$.

Proof. There is a natural map of sheaves of algebras $\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}} \to \mathcal{O}_\Delta^{(\infty)}$ induced from the natural quotient maps $\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}} \to \mathcal{O}_\Delta^{(k)}$. Moreover as the derived tensor product the cohomologies of $\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}}$ are given by $\underline{\mathrm{Tor}}^i_Y(\mathcal{O}_X, \mathcal{O}_X)$ where the cohomological grading is compatible with the filtration degree on $\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}}$. Since $\underline{\mathrm{Tor}}^i_Y(\mathcal{O}_X, \mathcal{O}_X)$ vanishes for large i for a smooth space Y and the dg-ideal $\mathcal{J}$ of the diagonal map $\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}$ is concentrated on strictly negative degrees, the natural quotient maps $\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}} \to \mathcal{O}_\Delta^{(k)}$ are quasi-isomorphisms for large k. The theorem follows after taking inverse limit. (Note that taking inverse limit does not in general commute with taking cohomology. However in our case this is true since the cohomologies of the inverse system stabilize.) ☐

Proof of Theorem 1.3. We should prove that, for a closed embedding $i : X \hookrightarrow Y$, $\mathbf{J}(T_{\widetilde{X}/Y})$ represents the functor i^*i_*. This has to be interpreted correctly. Namely we have factored the map i as $\pi \circ j$ for a quasi-equivalence $j : X \to \widetilde{X}$ and a smooth map $\pi : \widetilde{X} \to Y$. Since the map π is smooth, the sheaf $h_*\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}}$ where h is the embedding $\widetilde{X} \times_Y \widetilde{X} \hookrightarrow \widetilde{X} \times \widetilde{X}$, gives the kernel for $\pi^*\pi_* : \mathbf{D}(\widetilde{X}) \to \mathbf{D}(\widetilde{X})$. Thanks to Proposition 2.9 and Theorem 2.10 we have a quasi-isomorphism

$$\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}} \cong \mathbf{J}(T_{\widetilde{X}/Y})\,.$$

Thus we conclude that $h_*\mathbf{J}(T_{\widetilde{X}/Y}) \in \mathbf{D}(\widetilde{X} \times \widetilde{X})$ is the kernel representing the functor $\pi^*\pi_*$. Finally we observe that the functor $\pi^*\pi_*$ corresponds to i^*i_* via the equivalence $\mathbf{D}(X) \cong \mathbf{D}(\widetilde{X})$. Hence the sheaf $(j \times j)^*h_*\mathbf{J}(T_{\widetilde{X}/Y})$ on $X \times X$ represents i^*i_*.

As for the monad structure on $\pi^*\pi_*$, it **is** the algebra structure on $\mathcal{O}_{\widetilde{X}\times_Y\widetilde{X}}$. ☐

Remark 2.11. It is interesting to observe that there is an iterative process arising from an embedding of algebraic varieties $i : X \hookrightarrow Y$. Namely we have associated to the embedding a natural dg-Lie algebroid T_i (here we changed the notation for $T_{\widetilde{X}/Y}$ to emphasize the morphism i). But the morphism i defines another embedding of dg-schemes $\Delta_i : \widetilde{X} \to \widetilde{X} \times_Y \widetilde{X}$. The dg-Lie algebroid associated to this latter embedding Δ_i will be denoted by T_{Δ_i}. There is an interesting relationship between T_i and T_{Δ_i} in view of Theorem 2.4 and Proposition 2.9:

$$C^*(T_{\Delta_i}) \xrightarrow{2.4} \mathcal{O}^{(\infty)}_{\widetilde{X}/\widetilde{X}\times_Y\widetilde{X}} \xrightarrow{2.9} \mathbf{J}(T_i).$$

In particular we see that the Lie algebroid T_{Δ_i} encodes less information than that of T_i since in the above isomorphism we do not see the Lie structure on T_i from that of T_{Δ_i}. Moreover observe that

$T_{\Delta_i} \cong T_i[-1]$, hence its Chevalley-Eilenberg algebra is of the form $\widehat{\mathbf{S}}(T_i^\vee)$ endowed with the Chevalley-Eilenberg differential. The isomorphism $C^*(T_{\Delta_i}) \cong \mathbf{J}(T_i)$ then certainly resembles the PBW theorem for T_i. It is thus natural from this isomorphism to conjecture that the PBW property holds for T_i if-and-only-if certain Lie structure vanishes on T_{Δ_i}. This is in fact the main point of the paper [2] where the authors study the HKR property for an embedding $i : X \to Y$. Their result on the derived level asserts that the vanishing of the Lie structure on $N[-2] = T_{\Delta_i}$ is equivalent to the HKR property for $N[-1] = T_i$.

2.5. The universal enveloping algebra represents $i^*i_!$

Let us sketch a proof of Theorem 1.2. By the universal property of $\mathbf{U}(T_{\widetilde{X}/Y})$ we get a morphism $\mathbf{U}(T_{\widetilde{X}/Y}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}})$ of dg-algebras in $\mathcal{O}_{\widetilde{X}}$-bimodules over $\mathcal{O}_Y$. One can prove that, in the case of a closed embedding, it is a quasi-isomorphism (the proof is very similar to the one of Theorem 2.10, with Tor's being replaced by Ext's). In particular we have an algebra isomorphism $\mathbf{U}(i_*N[-1]) \to \mathcal{E}xt_{\mathcal{O}_Y}(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$ in $\mathbf{D}^b_{coh}(Y)$, and thus by seeing $\mathbf{U}(i_*N[-1])$ as an object of $\mathbf{D}^b_{coh}(X \times X)$ set-theoretically supported on the diagonal, we get that it is the kernel representing the functor $i^*i_! : \mathbf{D}^b_{coh}(X) \to \mathbf{D}^b_{coh}(X)$. The monad structure on $i^*i_!$ coming from the product on $\mathcal{E}xt_{\mathcal{O}_Y}(i_*\mathcal{O}_X, i_*\mathcal{O}_X)$ easily identifies with the one coming from the projection formula $i^*i_!i^*i_! \Rightarrow i^*i_!$. □

Let us provide yet another approach. Borrowing the notation from §2.2.2, we have the following

Lemma 2.12. *The functor* $\mathbf{U}(\mathfrak{g})\underset{\mathcal{A}}{\otimes} - : \mathcal{A}\text{-mod} \longrightarrow \mathcal{A}\text{-mod}$ *is left adjoint to* $\mathrm{Hom}_{\mathcal{A}\text{-mod}}\big(\mathbf{U}(\mathfrak{g}), -\big)$*. Here we use that for any left* $\mathcal{A}$*-module* M*, the right* $\mathcal{A}$*-module structure on* $\mathbf{U}(\mathfrak{g})$ *turns* $\mathrm{Hom}_{\mathcal{A}\text{-mod}}\big(\mathbf{U}(\mathfrak{g}), M\big)$ *into a left* $\mathcal{A}$*-module.*

Proof. Let M, N be $\mathcal{A}$-modules. As usual the one-to-one correspondence

$$\varphi \in \mathrm{Hom}_{\mathcal{A}\text{-mod}}\Big(\mathbf{U}(\mathfrak{g}) \underset{\mathcal{A}}{\otimes} M, N\Big) \xleftrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}\text{-mod}}\Big(M, \mathrm{Hom}_{\mathcal{A}\text{-mod}}\big(\mathbf{U}(\mathfrak{g}), N\big)\Big) \ni \psi$$

is given by $\varphi(P \otimes m) = \psi(m)(P)$. □

Then notice that $\mathrm{Hom}_{\mathcal{A}\text{-mod}}\big(\mathbf{U}(\mathfrak{g}), -\big) = \mathbf{J}(\mathfrak{g})\underset{\mathcal{A}}{\widehat{\otimes}} -$, where the $\mathcal{A}$-bimodule structure on $\mathbf{J}(\mathfrak{g})$ is the one described in §2.2.3.

We finally consider the case when $\mathcal{A} = \mathcal{O}_{\widetilde{X}}$ and $\mathfrak{g} = T_{\widetilde{X}/Y}$. It follows from Theorem 1.3 (which we proved in the previous subsection) that $\mathbf{U}(T_{\widetilde{X}/Y})$ represents the kernel for the left adjoint functor to i^*i_*, which is $i^*i_!$. This has to be understood as $(j \times j)^*h_*\mathbf{U}(T_{\widetilde{X}/Y})$, which is nothing but $\mathbf{U}(i_*N[-1])$ viewed as an object of $\mathbf{D}^b_{coh}(X \times X)$, being the kernel representing $i^*i_!$.

3. Monads

In the previous ection we have identified $\mathbf{U}(T_{\widetilde{X}/Y})$ with the kernel of $i^*i_!$. Both are equipped with an associative product (they even carry a Hopf-like structure). Identifying these additional structures is the subject of the present section.

3.1. The Hopf monad associated with the universal enveloping algebra

Let $(\mathcal{A}, \mathfrak{g})$ be a Lie algebroid. We have seen that $\mathbf{U}(\mathfrak{g})$ is a bialgebroid. It actually has a very specific feature: source and target maps $\mathcal{A} \to \mathbf{U}(\mathfrak{g})$ are the same. Therefore, the forgetful functor $U : \mathbf{U}(\mathfrak{g})\text{-mod} \longrightarrow \mathcal{A}\text{-mod}$ is *strong* monoidal (recall that $\mathbf{U}(\mathfrak{g})$ being a bialgebroid its category of left modules is monoidal, see e.g. [3] and references therein)[6].

Observe that U has a left adjoint: $F : \mathbf{U}(\mathfrak{g}) \underset{\mathcal{A}}{\otimes} - : \mathcal{A}\text{-mod} \longrightarrow \mathbf{U}(\mathfrak{g})\text{-mod}$. Moreover, U being strong monoidal, then its left adjoint F is *colax* monoidal and hence the monad $T := UF$ is a *Hopf monad* in the sense of [17]: it is a monad in the 2-category OpMon having monoidal categories as objects, colax monoidal functors as 1-morphisms and natural transformations of those as 2-morphisms.

3.1.1. The dual Hopf comonad associated with the jet algebra

Notice that the strong monoidal functor U also has a right adjoint $G := \underline{\mathsf{Hom}}_{\mathcal{A}\text{-mod}}\big(\mathbf{U}(\mathfrak{g}), -\big)$, which is *lax* monoidal. Going along the same lines as above one sees that $S := UG$ (which is right adjoint to T) is a *Hopf comonad*, meaning that it is a comonad in the 2-category Mon having monoidal categories as objects, lax monoidal functors as 1-morphisms and natural transformations of those as 2-morphisms.

Finally recall that $\underline{\mathsf{Hom}}_{\mathcal{A}\text{-mod}}\big(\mathbf{U}(\mathfrak{g}), -\big) \cong \mathbf{J}(\mathfrak{g}) \widehat{\underset{\mathcal{A}}{\otimes}} -$, where the $\mathcal{A}$-bimodule structure on $\mathbf{J}(\mathfrak{g})$ is the one described in §2.2.3. Notice that, on the one hand, the lax monoidal structure on S is given by the coproduct on $\mathbf{U}(\mathfrak{g})$, and thus by the product on $\mathbf{J}(\mathfrak{g})$:

$$\Big(\mathbf{J}(\mathfrak{g}) \widehat{\underset{\mathcal{A}}{\otimes}} - \Big) \otimes_{\mathcal{A}} \Big(\mathbf{J}(\mathfrak{g}) \widehat{\underset{\mathcal{A}}{\otimes}} - \Big) \cong \big(\mathbf{J}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbf{J}(\mathfrak{g})\big) \widehat{\underset{\mathcal{A}\otimes\mathcal{A}}{\otimes}} (- \otimes -) \Longrightarrow \mathbf{J}(\mathfrak{g}) \widehat{\underset{\mathcal{A}}{\otimes}} (- \otimes_{\mathcal{A}} -)\,.$$

On the other hand, the comonad structure on S is given by the product on $\mathbf{U}(\mathfrak{g})$, and hence by the coproduct on $\mathbf{J}(\mathfrak{g})$:

$$\mathbf{J}(\mathfrak{g}) \widehat{\underset{\mathcal{A}}{\otimes}} - \Longrightarrow \Big(\mathbf{J}(\mathfrak{g}) \widehat{\underset{\mathcal{A}}{\otimes}} \mathbf{J}(\mathfrak{g})\Big) \widehat{\underset{\mathcal{A}}{\otimes}} - \cong \mathbf{J}(\mathfrak{g}) \widehat{\underset{\mathcal{A}}{\otimes}} \Big(\mathbf{J}(\mathfrak{g}) \widehat{\underset{\mathcal{A}}{\otimes}} - \Big)\,.$$

3.2. The Hopf (co)monad associated with a closed embedding

Similarly to the above, the left adjoint $\mathsf{i}_!$, resp. the right adjoint i_*, to the strong monoidal functor i^* is colax monoidal, resp. lax monoidal. Hence $\mathsf{i}^*\mathsf{i}_!$ is a Hopf monad and $\mathsf{i}^*\mathsf{i}_*$ is a Hopf comonad.

We already know (see §2.4 and §2.5) that there are isomorphisms of functor $\mathsf{i}^*\mathsf{i}_* \cong UG$ and $\mathsf{i}^*\mathsf{i}_! \cong UF$ whenever $(\mathcal{A}, \mathfrak{g}) = (\mathcal{O}_{\widetilde{\mathsf{X}}}, \mathsf{T}_{\widetilde{\mathsf{X}}/\mathsf{Y}})$. It remains to be shown that the Hopf (co)monad structures coincide.

3.2.1. The Hopf comonad associated with a morphism of algebras

In this § functors are not derived. Let $\mathcal{B} \to \mathcal{A}$ be a morphism of (sheaves of dg-)algebras. The strong monoidal (dg-)functor $\mathcal{A} \otimes_{\mathcal{B}} - : \mathcal{B}\text{-mod} \to \mathcal{A}\text{-mod}$ admits the "forgetful" functor $\mathcal{A}\text{-mod} \to \mathcal{B}\text{-mod}$ as a right adjoint, which is then lax monoidal. Again, this turns $\mathcal{A} \otimes_{\mathcal{B}} - : \mathcal{A}\text{-mod} \to \mathcal{A}\text{-mod}$ into a

[6] The forgetful functor usually goes to $\mathcal{A}$-bimodules, but here its essential image is the monoidal subcategory consisting of those bimodules which have the same underlying left and right module structure. It is isomorphic to the monoidal category of $\mathcal{A}$-modules.

(dg-)Hopf comonad. Notice that the lax monoidal structure $(\mathcal{A} \otimes_{\mathcal{B}} -) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{B}} -) \Rightarrow \mathcal{A} \otimes_{\mathcal{B}} (- \otimes_{\mathcal{A}} -)$ is given by the product of $\mathcal{A}$ while the comonad structure $\mathcal{A} \otimes_{\mathcal{B}} - \Rightarrow \mathcal{A} \otimes_{\mathcal{B}} (\mathcal{A} \otimes_{\mathcal{B}} -)$ is given by the morphism $\mathcal{A} \to \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$, $\mathfrak{a} \mapsto \mathfrak{a} \otimes 1$.

Observe that this Hopf comonad can be seen as the Hopf comonad associated with a Hopf algebroid in a similar way to what happens with the jet algebra. Namely, the algebra $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$ is equipped with two obvious algebra maps $\mathcal{A} \to \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$ and a coproduct

$$\begin{aligned} \Delta : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} &\longrightarrow (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}) \underset{\mathcal{A}}{\otimes} (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}) \cong \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \\ \mathfrak{a} \otimes \mathfrak{a}' &\longmapsto \mathfrak{a} \otimes 1 \otimes \mathfrak{a}' \end{aligned}$$

which satisfy (similarly to $\mathbf{J}(\mathfrak{g})$) the axioms of a cogroupoid object in (sheaves of dg-)commutative algebras. This proves the functor $(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}) \underset{\mathcal{A}}{\otimes} - : \mathcal{A}\text{-}\mathtt{mod} \to \mathcal{A}\text{-}\mathtt{mod}$ with the structure of a Hopf comonad: as in the case of $\mathbf{J}(\mathfrak{g})$, the lax monoidal structure comes from the product of $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$ while the comonad structure comes from its coproduct.

Proposition 3.1. *There is a natural isomorphism* $\mathcal{A} \otimes_{\mathcal{B}} - \tilde{\Rightarrow} (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}) \underset{\mathcal{A}}{\otimes} -$ *of Hopf comonads.*

Proof. Let $\mathcal{M}$ be an $\mathcal{A}$-module. One observes that the natural isomorphism $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{M} \tilde{\to} (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}) \underset{\mathcal{A}}{\otimes} \mathcal{M}$ is given by $\mathfrak{a} \otimes \mathfrak{m} \mapsto \mathfrak{a} \otimes 1 \otimes \mathfrak{m}$ and obviously commutes with the comonad and lax monoidal structures. □

3.2.2. Identifying i^*i_* and UG as Hopf comonads

Proposition 3.2. *Let* $\mathcal{B} \to \mathcal{A}$ *and* $\mathfrak{g}$ *be as in Example 2.1. Then the dg-algebra morphism*

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathbf{J}(\mathfrak{g})$$

introduced in Example 2.3 is a dg-Hopf algebroid morphism.

Proof. Straightforward. □

Let $\mathcal{A} = \mathcal{O}_{\widetilde{X}}$ and $\mathfrak{g} = T_{\widetilde{X}/Y}$. Then it follows from Theorem 2.10 that $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \hat{\otimes}_{\mathcal{B}} \mathcal{A} \cong \mathbf{J}(\mathfrak{g})$ is a quasi-isomorphism. Recall that $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} = \mathcal{O}_{\widetilde{X} \times_Y \widetilde{X}}$ and $\mathcal{A} \hat{\otimes}_{\mathcal{B}} \mathcal{A} = \mathcal{O}_{\Delta}^{(\infty)}$. The next result then follows from Propositions 3.1 and 3.2.

Theorem 3.3. *There is a natural quasi-isomorphism* $\pi^*\pi_* \Rightarrow S$ *of dg-Hopf comonads on* $\mathcal{O}_{\widetilde{X}}\text{-}\mathtt{mod}$.

If one denotes by the same symbols dg-functors and the induces functors on the homotopy category, one then has

Corollary 3.4. *The Hopf comonads* i^*i_* *and* S *on* $\mathbf{D}(X)$ *are natually isomorphic. Therefore their respective left adjoint Hopf monads* $i^*i_!$ *and* T *are isomorphic too.*

4. Obstructions

In this section we present the second construction of a certain Lie-type structure on $N[-1]$ associated to a closed embedding $i : X \hookrightarrow Y$ of smooth algebraic varieties. In spirit this construction might be thought of as obtained from the one of Section 2 by applying homological transfer technique. Indeed we shall end up with a strong homotopy Lie (or L_∞-) algebroid structure on $N[-1]$.

4.1. $X_Y^{(\infty)}$ versus $X_N^{(\infty)}$

Let k be either a positive integer or ∞, and denote by $X_Y^{(k)}$ the k-th infinitesimal neighbourhood of X into Y, where by convention $X_Y^{(\infty)} := \varinjlim X_Y^{(k)}$. We borrow the notation from Subsection 2.3. Notice in particular that $i_*\mathcal{O}_X = \mathcal{O}_Y/\mathcal{J}$ and $i_*N^\vee = \mathcal{J}/\mathcal{J}^2$.

Lemma 4.1. *The canoncial isomorphism* $\mathrm{gr}\big(\mathcal{O}_X^{(1)}\big) \longrightarrow i_*\mathcal{O}_X \oplus i_*N^\vee$ *is multiplicative.*

Let U_{i_0} be an affine open subset of Y, and denote by $\rho_{i_0} : U_{i_0} \hookrightarrow Y$ the embedding morphism. There exists a filtered isomorphism of algebras

$$\Phi_{i_0} : \rho_{i_0}^*\left(\mathcal{O}_X^{(\infty)}\right) \longrightarrow \rho_{i_0}^*\left(\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)\right)$$

such that $\mathrm{gr}^{\leq 1}\big(\Phi_{i_0}\big)$ is the isomophism appearing in Lemma 4.1. Let U_{i_0} and U_{i_1} be two such open subsets, and denote by Φ_{i_0} and Φ_{i_1} the associated isomorphisms. On the intersection $U_{i_0 i_1}$, we get an automorphism

$$\Phi_{i_0 i_1} := \Phi_{i_1} \circ \Phi_{i_0}^{-1} : \rho_{i_0 i_1}^*\left(\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)\right) \longrightarrow \rho_{i_0 i_1}^*\left(\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)\right)$$

where $\rho_{i_0 i_1}$ denotes the embedding morphism $U_{i_0} \cap U_{i_1} \hookrightarrow Y$. It moreover has the property that $\mathrm{gr}^{\leq 1}\big(\Phi_{i_0 i_1}\big)$, which is an automorphism of $\rho_{i_0 i_1}^*\,(i_*\mathcal{O}_X \oplus i_*N^\vee)$, is the identity.

Let $\mathcal{U} := \{U_{i_0}\}_{i_0 \in \mathcal{I}}$ be an open affine covering of Y, so that we have local trivializations Φ_{i_0} $(i_0 \in \mathcal{I})$. Then the collection of isomorphisms $\{\Phi_{i_0 i_1}\}_{(i_0,i_1)\in\mathcal{I}^2}$ forms a non-abelian 1-cocycle of the sheaf of groups $\mathsf{Aut}^+\left(\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)\right)$ consisting of those automorphisms Φ of $\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)$ such that $\mathrm{gr}^{\leq 1}\big(\Phi\big) = \mathrm{id}$.

Let us denote by $\mathsf{Der}^+\left(\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)\right)$ the sheaf of Lie algebras consisting of those derivations θ of $\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)$ satisfying $\mathrm{gr}^{\leq 1}\big(\theta\big) = 0$. The exponential map

$$\exp : \mathsf{Der}^+\left(\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)\right) \longrightarrow \mathsf{Aut}^+\left(\widehat{\mathbf{S}}_{i_*\mathcal{O}_X}(i_*N^\vee)\right)$$

is an isomorphism.

4.2. Homotopy Lie algebroid structure on $N[-1]$

Let X be a topological space, and let $\mathcal{U}$ be an open covering of X. For a sheaf of abelian groups F on X, we denote by $C^*_{TW}(F)$ the Thom-Whitney resolution of F associated to the covering $\mathcal{U}$. Basic properties of the functor C^*_{TW} are recalled in the Appendix A.

Definition 4.2. Let E be a finitely generated locally free sheaf on a smooth algebraic variety X. An L_∞-*algebroid* structure on $\mathsf{E}[-1]$ is a $\mathbf{k}$-linear filtered derivation Q of degree one on the differential graded algebra $\hat{\mathbf{S}}_{C^*_{TW}(\mathcal{O}_X)}(C^*_{TW}(\mathsf{E}^\vee))$ such that $(d_{TW}+Q)^2=0$, where d_{TW} is the original differential.

It is called *minimal* if $\mathrm{gr}^{\leq 1}(Q)=0$.

In other words, an L_∞-algebroid structure is a Maurer-Cartan element Q in $\mathsf{Der}\left(\hat{\mathbf{S}}_{C^*_{TW}(\mathcal{O}_X)}(C^*_{TW}(\mathsf{E}^\vee))\right)$. In the following we shall only consider minimal L_∞-algebroids which are Maurer-Cartan elements in $\mathsf{Der}\left(\hat{\mathbf{S}}_{C^*_{TW}(\mathcal{O}_X)}(C^*_{TW}(\mathsf{E}^\vee))\right)$.

Since Q is a k-linear derivation, it is uniquely determined by its restriction to the subspaces $\mathcal{C}^*_{TW}(\mathcal{O}_X)$ and $\mathcal{C}^*_{TW}(\mathsf{E}^\vee)$. On these subspaces Q is the direct product of the following maps

$$a_k : \mathcal{C}^*_{TW}(\mathcal{O}_X) \to \left(\mathcal{C}^*_{TW}(\mathsf{E}^\vee)\right)^k \quad \text{and}$$
$$l_k : \mathcal{C}^*_{TW}(\mathsf{E}^\vee) \to \left(\mathcal{C}^*_{TW}(\mathsf{E}^\vee)\right)^k.$$

These maps are both of homological degree one. The minimality condition implies that $a_0=0$, $l_0=l_1=0$. Thus they induce an actual Lie algebroid structure on the pair $(\mathcal{O}_X, \mathsf{E}[-1])$ in $\underline{D}(\mathbf{k}_X)$ [7].The equation $(d_{TW}+Q)^2=0$ implies a system of quadratic relations among a_k and l_k's. The first few of them are

$$\begin{aligned}
&[d_{TW}, a_1]=0\\
&[d_{TW}, a_2]+l_2\circ a_1=0\\
&[d_{TW}, l_2]=0\\
&[d_{TW}, l_3]+(l_2\otimes \mathsf{id}+\mathsf{id}\otimes l_2)\circ l_2=0
\end{aligned}$$

The first and the third equations imply that both a_1 and l_2 are cochain morphisms. We also observe that there are more structure maps due to the fact that Q is a derivation. This gives, for each $k\geq 1$, the equation

$$[l_k, e(f)] = e(a_{k-1}(f))$$

where $e(-)$ is the operator given by multiplication by $-$. The following lemma is a direct consequence of the above identities.

Lemma 4.3. *Let* $\mathsf{E}[-1]$ *be a* L_∞ *Lie algebroid over* X *such that* a_1 *and* a_2 *vanish. Then* $\mathsf{E}[-1]$ *is a Lie algebra object in the symmetric monoidal category* $\mathbf{D}(X)$.

Proof. Using the above compatibilities, one sees that the vanishing of a_1 implies l_2 is $\mathcal{O}_X$-linear, and the vanishing of a_2 implies that l_3 is $\mathcal{O}_X$-linear, which gives the Jacobi identity. □

Later we will be able to give a geometric interpretation for the vanishing of a_1 and a_2 in the case of $\mathsf{N}[-1]$ associated to $i: X \hookrightarrow Y$, see Proposition 4.6.

Proposition 4.4. *Let* $i: X \hookrightarrow Y$ *be a closed embedding of smooth algebraic varieties. Then there is a minimal* L_∞-*algebroid structure on* $\mathsf{N}[-1]$ *such that its Chevalley-Eilenberg algebras is quasi-isomorphic to* $\mathcal{O}_X^\infty$.

Proof. We apply Theorem A.6 in the case $\mathcal{A}=\hat{\mathbf{S}}_{\mathcal{O}_X}\mathsf{N}^\vee$, $T^+_{\mathcal{A}}=\mathsf{Der}^+\left(\hat{\mathbf{S}}_{\mathcal{O}_X}\mathsf{N}^\vee\right)$, $\mathcal{B}=\mathcal{O}_X^\infty$, and the 1-cocycle Φ constructed in the previous subsection. □

[7] It coincides with the one induced from $T_{\tilde{X}/Y}$.

4.3. Obstructions

The structure maps of the L_∞ algebroid $N[-1]$ can be used to describe various cohomology obstructions when comparing $X_N^{(k)}$ with $X_Y^{(k)}$. Let us set

$$\mathcal{F} := \left(\hat{\mathbf{S}}_{\mathcal{C}^*_{TW}(\mathcal{O}_X)}\big(\mathcal{C}^*_{TW}(N^\vee)\big), d_{TW} + Q\right)$$

the Chevalley-Eilenberg algebra of $N[-1]$ defined in Proposition 4.4. Observe that since $\mathrm{gr}(\Phi_{i_0 i_1}) = 0$, the operator Q preserves the augmentation ideal $\mathcal{F}^+$ consisting of symmetric tensors of degree at least 1. Thus $\mathcal{F}^+$ is a differential ideal in $\mathcal{F}$. We define a quotient differential graded algebra $\mathcal{F}^{(k)} := \mathcal{F}/(\mathcal{F}^+)^{k+1}$ for each $k \geq 0$. By Theorem A.6 we have a quasi-isomorphism

$$\mathcal{O}^{(k)}_{X/Y} \to \mathcal{F}^{(k)}.$$

4.3.1. Splittings of $X \hookrightarrow X_Y^{(k)}$

Let $s_k : X_Y^{(k)} \to X$ be a splitting of the embedding $X \hookrightarrow X_Y^{(k)}$. We again use $s_k : \mathcal{O}_X \to \mathcal{O}_X^{(k)}$ to denote the corresponding morphism on rings. In this case when forming the cocycle Φ we can further require that Φ_{i_0} to fill the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X^{(k)}|_{U_{i_0}} & \xrightarrow{\Phi_{i_0}^{(k)}} & \mathbf{S}^{\leq k}_{\mathcal{O}_X} N^\vee |_{U_{i_0}} \\ {\scriptstyle s_k}\uparrow & & \uparrow \\ \mathcal{O}_X|_{U_{i_0}} & = & \mathcal{O}_X|_{U_{i_0}} \end{array}$$

where the right vertical arrow is the obvious inclusion map. This restriction implies that the corresponding derivations $T_{i_0 i_1}$ is contained in the subset of $\mathrm{Der}^+\left(\hat{\mathbf{S}}_{\mathcal{O}_X} N^\vee\right)$ consisting of derivations such that the component

$$\mathcal{O}_X \to \mathbf{S}^i N^\vee$$

vanishes for all $0 \leq i \leq k$. Observe that this is a Lie subalgebra due to the minimality condition. We shall call such a 1-cocycle Φ compatible with s_k.

If Φ is a 1-cocycle compatible with s_k, then we have a corresponding homotopy L_∞ algebra structure on $N[-1]$. Recall from Lemma A.5 that the operator Q is the image of a Maurer-Cartan element θ under the morphism

$$\alpha : C^*_{TW}\Big(\mathrm{Der}\left(\hat{\mathbf{S}}_{\mathcal{O}_X} N^\vee\right)\Big) \to \mathrm{Der}\left(C^*_{TW}(\hat{\mathbf{S}}_{\mathcal{O}_X} N^\vee)\right)$$

defined in *loc. cit.*. Hence the structure maps a_k and l_k, being the component maps of Q, also lies in the image of α. If we denote by $\theta_{0,k}$ and $\theta_{1,k}$ the components of θ such that

$$\theta^{0,k} \in C^*_{TW}\Big(\mathrm{Der}\left(\mathcal{O}_X, (N^\vee)^k\right)\Big), \quad \text{and}$$
$$\theta^{1,k} \in C^*_{TW}\Big(\mathrm{Hom}_{k_X}\left(N^\vee, (N^\vee)^k\right)\Big),$$

then we have

$$a_k = \alpha(\theta^{0,k}), \quad \text{and} \quad l_k = \alpha(\theta^{1,k}).$$

Lemma 4.5. *Let* $s_k : \mathcal{O}_X \to \mathcal{O}_X^{(k)}$ *be a splitting, and let* Φ *be a* 1*-cocycle compatible with* s_k*. Let* $a_i = \alpha(\theta^{0,i})$ *be the associated structure maps in the* L_∞ *algebroid structure on* $N[-1]$*. Then we have* $\theta^{0,i} = 0$ *and* $a_i = 0$ *for* $0 \le i \le k$*. Moreover both* a_{k+1} *and* $\theta^{0,k+1}$ *are cocycles.*

Proof. The vanishing of $\theta^{0,i}$ and a_i is clear from the definition. Now let us show that $\theta^{0,k+1}$ is a cocycle. Indeed since θ satisfies the Maurer-Cartan equation, we have

$$[d_{TW}, \theta^{0,k+1}] + \sum_{i+j=k+2} \theta^{1,j} \circ \theta^{0,i} = 0.$$

Due to minimality condition the sum above is over $j \ge 2$, which in particular implies that $i \le k$. Hence the sum $\sum_{i+j=k+2} \theta^{1,j} \circ \theta^{0,i}$ vanishes, and so we get

$$[d_{TW}, \theta^{0,k+1}] = 0.$$

This proves that $\theta^{0,k+1}$ is a cocycle. Since a_{k+1} is the image of $\theta^{0,k+1}$ under the cochain map α, it is also a cocycle. □

Thus the element $\theta^{0,k+1}$ is a degree one cocycle in the complex $\mathcal{C}^*_{TW}\Big(\mathsf{Der}\left(\mathcal{O}_X, (N^\vee)^k\right)\Big)$. Its class $[\theta^{0,k+1}]$ is then a class in $\mathsf{Ext}^1(\mathbf{S}^{k+1}N, T_X)$. Similarly the class $[a_{k+1}]$ is in the first cohomology group $H^1\big(\mathsf{Hom}(\mathcal{C}^*_{TW}(\mathbf{S}^{k+1}_{\mathcal{O}_X}N), T_{\mathcal{C}^*_{TW}(\mathcal{O}_X)})\big)$. It is plausible the two cohomology groups are in fact isomorphic, but the authors do not know a proof of this claim.

Proposition 4.6. *Assume that we are in the same setup as Lemma 4.5, then the following are equivalent:*

(A) *there exists a splitting* $s_{k+1} : \mathcal{O}_X \to \mathcal{O}_X^{(k+1)}$ *lifting* s_k*;*

(B) *the cohomology class* $[\theta^{0,k+1}] \in \mathsf{Ext}^1(\mathbf{S}^{k+1}N, T_X)$ *vanishes;*

(C) *the cohomology class* $[a_{k+1}] \in H^1\big(\mathsf{Hom}(\mathcal{C}^*_{TW}(\mathbf{S}^{k+1}_{\mathcal{O}_X}N), T_{\mathcal{C}^*_{TW}(\mathcal{O}_X)})\big)$ *vanishes.*

Proof. • (A) ⇒ (B): Assuming (A), we can choose another 1-cocycle Φ' to be compatible with s_{k+1}, which in particular implies that it is compatible with s_k. Denote by θ' the corresponding Maurer-Cartan element. Since different choices of Φ give rise to gauge equivalent Maurer-Cartan element by Corollary A.4, there exist a degree zero element η in the differential graded Lie algebra $\mathcal{C}^*_{TW}(\mathfrak{g})$ such that

$$e^\eta * \theta' = \theta$$

where $\mathfrak{g}$ is the Lie subalgebra of $\mathsf{Der}^+(\hat{\mathbf{S}}_{\mathcal{O}_X}N^\vee)$ whose $(0,i)$ components vanishes for $0 \le i \le k$. Writing out the above equation in the $(0,k+1)$-component implies that

$$[d_{TW}, \eta^{0,k+1}] = \theta^{(0,k+1)}.$$

This proves (B).

• (B) ⇒ (C): It suffices to note that on the chain level we have $a_{k+1} = \alpha(\theta^{(0,k+1)})$.

• (C) ⇒ (A): Assume that $a_{k+1} = [d_{TW}, h]$ for some degree zero $h \in \mathsf{Hom}(\mathcal{C}^*_{TW}(\mathbf{S}^{k+1}_{\mathcal{O}_X}N), T_{\mathcal{C}^*_{TW}(\mathcal{O}_X)})$. We define a morphism

$$\tilde{s}_{k+1} : \mathcal{F}^{(0)} \to \mathcal{F}^{(k+1)}$$

by $s_{k+1}(f) := f + h(f)$. Recall that $\mathcal{F}^{(k)}$ is the k-truncation of $\mathcal{F} := \Big(\hat{\mathbf{S}}_{\mathcal{C}^*_{TW}(\mathcal{O}_X)}(\mathcal{C}^*_{TW}(N^\vee)), d_{TW} + Q\Big)$. One checks that $\tilde{s}_{k+1}$ is a morphism of differential graded algebras. Moreover the composition

$$\mathcal{F}^{(0)} \overset{\tilde{s}_{k+1}}{\to} \mathcal{F}^{(k+1)} \twoheadrightarrow \mathcal{F}^{(k)}$$

is simply $f \mapsto f$. Thus the induced map on cohomology gives a map of algebras

$$s_{k+1} : \mathcal{O}_X \to \mathcal{O}_X^{(k+1)}$$

that lifts the splitting map s_k. □

Corollary 4.7. *Let $i : X \to Y$ be an embedding of smooth algebraic varieties. Assume that there exists a splitting s_2 of the natural morphism $X \hookrightarrow X^{(2)}$. Then $N[-1]$ is a Lie algebra object in the symmetric monoidal category $\mathbf{D}(X)$.*

Proof. This is a consequence of Proposition 4.6 and Lemma 4.3. □

Remark 4.8. The Lie structure on $N[-1]$ depends on the choice of s_2.

4.3.2. When is $X_N^{(k)}$ isomorphic to $X_Y^{(k)}$

Let us assume that there is an isomorphism $t_{k-1} : X_N^{(k-1)} \cong X_Y^{(k-1)}$. We would like to understand when t_{k-1} lifts to an isomorphism between $X_N^{(k)}$ and $X_Y^{(k)}$.

First note that the isomorphism t_{k-1} induces a splitting $s_{k-1} : X_Y^{(k-1)} \to X$. Hence we can use Proposition 4.6 to analyze the lifting of s_{k-1} to a splitting s_k, which should necessarily exist if t_{k-1} lifts. Thus in the following we assume that there is a splitting $s_k : X_Y^{(k)} \to X$ compatible with t_{k-1} in the sense that the induced splitting from t_{k-1} agrees with that of s_k.

Given a compatible pair (t_{k-1}, s_k) as above, we can require the isomorphisms Φ_{i_0} to be compatible with (t_{k-1}, s_k) in the sense that it is compatible with s_k, and that there is a commutative diagram.

$$\begin{array}{ccc} \mathcal{O}_X^{(k-1)}|_{U_{i_0}} & \xrightarrow{\Phi_{i_0}^{(k-1)}} & \mathbf{S}^{\leq k-1}_{\mathcal{O}_X} N^\vee|_{U_{i_0}} \\ {\scriptstyle t_{k-1}}\uparrow & & \| \\ \mathbf{S}^{\leq k-1}_{\mathcal{O}_X} N^\vee|_{U_{i_0}} & = & \mathbf{S}^{\leq k-1}_{\mathcal{O}_X} N^\vee|_{U_{i_0}} \end{array}$$

The corresponding 1-cocycle $\Phi_{i_0 i_1}$ is then called compatible with (t_{k-1}, s_k).

Lemma 4.9. *Let (t_{k-1}, s_k) be a compatible pair, and let Φ be a compatible 1-cocycle as described above. Then we have $a_i = 0$ for all $0 \leq i \leq k$, and $l_i = 0$ for all $0 \leq i \leq k-1$. Moreover the morphism l_k is a cocycle.*

Proof. The proof is similar to that of Lemma 4.5. Indeed the vanishing of $\mathfrak{a}_i$ and $\mathfrak{l}_i$ follow from definition while the identity $[d_{\mathsf{TW}}, \mathfrak{l}_k] = 0$ follows from the Maurer-Cartan equation $(d_{\mathsf{TW}} + Q) = 0$ and the vanishing of lower degree $\mathfrak{a}_i$'s and $\mathfrak{l}_i$'s. □

The morphism $\mathfrak{l}_k$ is a degree one cocycle in the complex $\mathsf{Hom}_{C^*_{\mathsf{TW}}(\mathcal{O}_X)}\big(C^*_{\mathsf{TW}}(N^\vee), C^*_{\mathsf{TW}}(\mathbf{S}^k_{\mathcal{O}_X} N^\vee)\big)$. Since the functor TW is monoidal this complex computes $\mathsf{Ext}^*_X(N^k, N)$.

Proposition 4.10. *Let the setup be the same as in Lemma 4.9. Then the following are equivalent:*

(A) *the class* $[\mathfrak{l}_k] \in \mathsf{Ext}^1_X(N^k, N)$ *vanishes;*

(B) *the isomorphism* t_{k-1} *lifts to an isomorphism* $t_k : \mathbf{S}^{\leq k}_{\mathcal{O}_X} N^\vee \to \mathcal{O}^{(k)}_X$.

Proof. • (A) ⇒ (B): Assume that $\mathfrak{l}_k = [d_{\mathsf{TW}}, h]$ for some degree zero morphism h in the morphism complex $\mathsf{Hom}_{C^*_{\mathsf{TW}}(\mathcal{O}_X)}\big(C^*_{\mathsf{TW}}(N^\vee), C^*_{\mathsf{TW}}(\mathbf{S}^k_{\mathcal{O}_X} N^\vee)\big)$. We can then define a morphism of algebras $\tilde{t}_k$: $C^*_{\mathsf{TW}}(\mathbf{S}^k_{\mathcal{O}_X} N^\vee) \to \mathcal{F}^{(k)}$ by formula $\mathsf{id} + h$. Notice that this is a morphism of algebras since h has image in $\mathbf{S}^k$. That it also commutes with differential is a computation using Lemma 4.9 and the identity $[d_{\mathsf{TW}}, h] = \mathfrak{l}_k$. Thus the induced map on cohomology defines a required lifting. Note that it lifts t_{k-1} since $\mathsf{id} + h$ modulo $\mathbf{S}^k$ is just id.

• (B) ⇒ (A): Assuming that there exists such a lifting, then we can choose a different cocycle Φ' such that the corresponding structure maps satisfy $\mathfrak{a}_i = 0$ and $\mathfrak{l}_i = 0$ for all $0 \leq i \leq k$. Same as in the proof of Proposition 4.6, we can use the fact that Φ and Φ' are gauge equivalent to show that $\mathfrak{l}_k$ is exact. □

Remark 4.11. Similar obstruction classes as in Proposition 4.6 and Proposition 4.10 were discussed in [1, Proposition 2.2] and [1, Corollary 3.4 & Corollary 3.6] using complex analytic methods. The existence of such classes can be formulated using the language of gerbes and stacks, see for example [14, Section 4]. Its relationship with an L_∞-algebroid structure seems to be new.

A. Deformation theory of sheaves of algebras

Let k be a base field which is of characteristic zero. All algebras are k-algebras, and derivations of algebras are k-linear. In this section we recollect basic facts on the folklore that deformations of a sheaf of commutative algebras $\mathcal{A}$ is governed by a differential graded Lie algebra resolving the tangent lie algebra $T_{\mathcal{A}}$. To ensure convergence we restrict to a pronilpotent Lie subalgebra $T^+_{\mathcal{A}}$ of $T_{\mathcal{A}}$, and assume that the exponential map

$$\exp : T^+_{\mathcal{A}} \to \mathsf{Aut}(\mathcal{A})$$

is well-defined. We denote the image $\exp(T^+_{\mathcal{A}})$ by $\mathsf{Aut}^+(\mathcal{A})$. This is a sheaf of prounipotent groups.

A.1. Flasque resolutions of sheaves of algebras

Let $\mathcal{U} := \{U_i\}_{i \in \mathcal{I}}$ be an open cover of a topological space X. For an open subset $U_i \subset X$ we denote by ρ_i the inclusion $U_i \hookrightarrow X$. For a multi-index $I = (i_0, \cdots, i_k) \in \mathcal{I}^{k+1}$ we denote by U_I the intersection

$U_{i_0} \cap \cdots U_{i_k}$, and by ρ_I the inclusion $U_I \hookrightarrow X$. Associated to this data is a simplicial space

$$\mathcal{U}_\bullet := \left(\cdots \coprod_{I \in \mathcal{I}^{k+1}} U_I \cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \coprod_{(i_0,i_1)\in\mathcal{I}^2} U_{i_0,i_1} \rightrightarrows \coprod_{i_0\in\mathcal{I}} U_{i_0} \right).$$

Let $\mathcal{A}$ be a sheaf of algebras on a topological space X. We get a cosimplicial sheaf of algebras

$$\mathcal{A}^\bullet_{\mathcal{U}} := \left(\coprod_{i_0\in\mathcal{I}} (\rho_{i_0})_*(\rho_{i_0})^*\mathcal{A} \rightrightarrows \coprod_{(i_0,i_1)\in\mathcal{I}^2} (\rho_{i_0,i_1})_*(\rho_{i_0,i_1})^*\mathcal{A} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\cdots \coprod_{I\in\mathcal{I}^{k+1}} (\rho_I)_*(\rho_I)^*\mathcal{A} \right)$$

on X. Its global sections $\mathcal{A}^\bullet_{\mathcal{U}}(X)$ form a cosimplicial algebra.

The (sheaf version) Čech complex $\mathcal{C}^*(\mathcal{U},\mathcal{A}) := \mathsf{Tot}(\mathcal{A}^\bullet_{\mathcal{U}})$ (or simply $\mathcal{C}^*(\mathcal{A})$) of the cosimplicial sheaf of algebras $\mathcal{A}^\bullet_{\mathcal{U}}$ gives a flasque resolution of $\mathcal{A}$. We denote by $C^*(\mathcal{A})$ its global sections which form a differential graded algebra. However if $\mathcal{A}$ were commutative, the Čech resolution $\mathcal{C}^*(\mathcal{A})$ is not commutative. For purposes of this paper we need to consider commutative (or Lie) resolution of sheaves of commutative (Lie respectively) algebras. Since the Čech resolution does not respect the symmetric monoidal structure, we need to use a better resolution.

A.1.1. Thom-Whitney complex

We now describe a symmetric monoidal functor TW from cosimplicial cochain complexes to cochain complexes. We refer to the Appendix in [9] for a more detailed discussion. Given a cosimplicial cochain complex $V^\bullet = V^0 \rightrightarrows V^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} V^2 \cdots$, we define a cochain complex $\mathsf{TW}(V^\bullet)$ by

$$\mathsf{TW}(V^\bullet) := \mathsf{eq}\left(\prod_n V^n \otimes \Omega^*_{\Delta^n} \underset{\mathrm{id}\otimes\varphi^*}{\overset{\varphi_*\otimes\mathrm{id}}{\rightrightarrows}} \prod_{\varphi\in\Delta([k],[l])} V^k \otimes \Omega^*_{\Delta^l} \right).$$

The cochain complex $\mathsf{TW}(V^\bullet)$ is quasi-isomorphic to $\mathsf{Tot}(V^\bullet)$. In fact there is an homotopy retraction between the two complexes

$$\begin{aligned} I &: \mathsf{Tot}(V^\bullet) \to \mathsf{TW}(V^\bullet) \\ P &: \mathsf{TW}(V^\bullet) \to \mathsf{Tot}(V^\bullet) \\ H &: \mathsf{TW}(V^\bullet) \to \mathsf{TW}(V^\bullet) \end{aligned}$$

where the maps I, P and H are given by explicit formulas.

The functor TW provides a nice (i.e. functorial) way of describing homotopy limits of cosimplicial diagrams within the category of cochain complexes. Being symmetric monoidal it sends commutative, Lie and associative cosimplicial dg-algebras to commutative, Lie and associative dg-algebras, respectively. It also sends cosimplicial dg-modules (over a cosimplicial dg-algebra) to dg-modules, and preserves the tensor product of those (because TW commutes with finite colimits, and thus with push-outs). Observe that TW extends to complete topological cochain complexes, with the completed tensor product $\hat{\otimes}$ as monoidal product.

Given a sheaf of algebras $\mathcal{A}$ and an open cover $\mathcal{U}$, we have a cosimplicial sheaf of algebras $\mathcal{A}_{\mathcal{U}}^{\bullet}$ as described above. Applying the Thom-Whitney functor (in the category of sheaves) we get a quasi-isomorphism of sheaves of differential graded algebras

$$\mathcal{A} \longrightarrow \mathcal{C}^*(\mathcal{U},\mathcal{A}) = \mathsf{Tot}(\mathcal{A}_{\mathcal{U}}^{\bullet}) \xrightarrow{I} \mathsf{TW}(\mathcal{A}_{\mathcal{U}}^{\bullet}).$$

This provides a flasque resolution of $\mathcal{A}$ as soon as $\mathcal{A}$ is flasque on every open subset U_{i_0}. In the following we use the notation $\mathcal{C}^*_{\mathsf{TW}}(\mathcal{U},\mathcal{A})$ (or simply $\mathcal{C}^*_{\mathsf{TW}}(\mathcal{A})$) to denote the Thom-Whitney resolution. Its global sections will be denoted by $C^*_{\mathsf{TW}}(\mathcal{A})$. Due to the monoidal properties of the functor TW, the resolution $\mathcal{C}^*_{\mathsf{TW}}(\mathcal{A})$ is commutative (or Lie) whenever $\mathcal{A}$ is.

A.1.2. The Totalization functor

We will need to descibe another kind of homotopy limit. Namely, there is a functor Tot [8] from cosimplicial simplicial sets to simplicial sets defined, on a given cosimplicial simplicial set $X^{\bullet}$, by

$$\mathsf{Tot}(X^{\bullet}) := \mathsf{eq}\left(\prod_n (X^n)^{\Delta^n} \overset{\varphi_*}{\underset{\varphi^*}{\rightrightarrows}} \prod_{\varphi\in\Delta([k],[l])} (X^k)^{\Delta^l}\right).$$

Notice that $(X^k)^{\Delta^l} = X^k_l$, where we put the simplicial degree in the lower index. The functor Tot provides a nice way of describing homotopy limits of cosimplicial diagrams within the category of simplicial sets.

A.2. Maurer-Cartan elements versus non-abelian 1-cocycles

We shall define two groupoids naturally associated to a cosimplicial pronilpotent Lie algebra $\mathfrak{g}^{\bullet}$. Recall the Maurer-Cartan set $\mathsf{MC}(\mathfrak{g})$ of a pronilpotent differential graded Lie algebra is defined by

$$\mathsf{MC}(\mathfrak{g}) := \{\theta \in \mathfrak{g}_1 \mid d\theta + \frac{1}{2}[\theta,\theta] = 0\}.$$

A degree zero element $a \in \mathfrak{g}_0$ acts on the set $\mathsf{MC}(\mathfrak{g})$ by

$$\theta \mapsto e^a * \theta := e^{\mathsf{ad}(a)}(\theta) - \frac{e^{\mathsf{ad}(a)}-1}{\mathsf{ad}(a)}(da).$$

The Deligne groupoid $\mathsf{Del}(\mathfrak{g})$ of $\mathfrak{g}$ is the action groupoid associated to this action. We define the Deligne groupoid of a cosimplicial pronilpotent Lie algebra $\mathfrak{g}^{\bullet}$ to be $\mathsf{Del}(\mathsf{TW}(\mathfrak{g}^{\bullet}))$.

The second groupoid $Z^1(\exp(\mathfrak{g}^{\bullet}))$ associated to $\mathfrak{g}^{\bullet}$ is defined as follows. We denote by $G^{\bullet} := \exp(\mathfrak{g}^{\bullet})$ the cosimplicial prounipotent group associated to $\mathfrak{g}^{\bullet}$ [9]. The set of objects of $Z^1(\exp(\mathfrak{g}^{\bullet}))$ is the set of nonabelian 1-cocycles, i.e. $T \in \mathfrak{g}^1$ such that $e^{\partial_0(T)}e^{-\partial_1(T)}e^{\partial_2(T)} = \mathsf{id}$. An element $a \in G^0$ acts on the set of nonabelian 1-cocycles by

$$e^T \mapsto e^{-\partial_1(a)}e^T e^{\partial_0(a)}.$$

We define $Z^1(\exp(\mathfrak{g}^{\bullet}))$ to be the groupoid associated to this action.

[8] We used the same notation Tot when forming total complexes, but this should not cause any confusion.

[9] As a set G is $\mathfrak{g}$ and the group structure is given by the Campbell-Hausdorff formula.

Theorem A.1. *Let $\mathfrak{g}^\bullet$ be a cosimplicial pronilpotent Lie algebra and $\mathsf{G}^\bullet := \exp(\mathfrak{g}^\bullet)$ the corresponding cosimplicial* $\mathbf{k}$*-prounipotent group. Then there is an equivalence*

$$\mathsf{Del}(\mathsf{TW}(\mathfrak{g}^\bullet)) \cong \mathsf{Z}^1(\exp(\mathfrak{g}^\bullet)).$$

Proof. We start with some recollection of homotopy theory. We temporary forget about sheaves and let $\mathfrak{g}$ be a pronilpotent Lie algebra, and $\mathsf{G} := \exp(\mathfrak{g})$. We consider the two following simplicial sets: the nerve $\mathsf{N}(\mathsf{G}) := \left(\cdots \mathsf{G}^2 \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathsf{G} \rightrightarrows *\right)$ of the group G and the simplicial set $\mathsf{MC}_\bullet(\mathfrak{g})$ defined by

$$\mathsf{MC}_\bullet(\mathfrak{g}) := \mathsf{MC}\left(\Omega_{\Delta^\bullet}\hat{\otimes}\mathfrak{g}\right) = \left\{\theta \in \Omega^1_{\Delta^\bullet}\hat{\otimes}\mathfrak{g} \,\middle|\, d_{dR}(\theta) + \frac{1}{2}[\theta,\theta] = 0\right\}.$$

Lemma A.2 (Folklore)**.** $\mathsf{N}(\mathsf{G})$ *is weakly equivalent to* $\mathsf{MC}_\bullet(\mathfrak{g})$*.*

Sketch of proof. We construct an explicit map $e : \mathsf{MC}_\bullet(\mathfrak{g}) \longrightarrow \mathsf{BG}$. Any $\theta \in \mathsf{MC}_n(\mathfrak{g})$ determines a flat connection $\nabla_\theta := d_{dR} + \theta$ on the trivial G-bundle over Δ^n. We then define an element $e(\theta) \in \mathsf{G}^n$ as follows: recalling that the vertices of Δ^n are labelled by $\{0, \dots, n\}$, we define $e(\theta)_i$, $i \in \{1, \dots, n\}$, to be the holonomy of ∇_θ along the segment joining $i-1$ to i. It turns out that e induces a weak equivalence $\mathsf{MC}_\bullet(\mathfrak{g}) \to \mathsf{N}(\mathsf{G})$. □

Now let $\mathfrak{g}^\bullet$ be a cosimplicial pronilpotent Lie algebra, so that $\mathsf{G}^\bullet = \exp(\mathfrak{g}^\bullet)$ is a cosimplicial group.

Lemma A.3. *There is a natural weak equivalence*

$$\mathsf{Tot}\left(\mathsf{MC}_\bullet(\mathfrak{g}^\bullet)\right) \cong \mathsf{MC}_\bullet\left(\mathsf{TW}(\mathfrak{g}^\bullet)\right).$$

Sketch of proof. One knows that this is true for nilpotent Lie algebras after [10, Theorem 4.1]. The pronilpotent case comes from the fact that the functor MC commutes with limits, as well as small limits mutually commute. □

The above lemma, together with the fact that $\pi_{\leq 1}\left(\mathsf{Tot}\big(\mathsf{N}(\mathsf{G}^\bullet)\big)\right) = \mathsf{Z}^1(\mathsf{G}^\bullet)$, give the result. □

A.3. Deformations of sheaves of algebras

Back to the geometric situation, we consider the cosimplicial pronilpotent Lie algebra $\mathfrak{g}^\bullet = \mathsf{T}^{+\bullet}_{\mathcal{A}}(\mathsf{X})$. Its corresponding sosimplicial prounipotent Lie group $\mathsf{G}^\bullet$ which governs the (positive) deformation theory of the sheaf $\mathcal{A}$. Indeed objects of $\mathsf{Z}^1(\mathsf{G}^\bullet)$ are non-abelian 1-cocycles, i.e. a collection $\{\Phi_{i_0i_1}\}$ such that $\Phi_{i_0i_1} \in \mathsf{Aut}^+(\mathcal{A})\,|_{U_{i_0i_1}}$ and on $U_{i_0i_1i_2}$ we have

$$\Phi_{i_2i_0} \circ \Phi_{i_1i_2} \circ \Phi_{i_0i_1} = \mathsf{id}\,.$$

Two such deformations are equivalent if one can be transformed to the other by an element $(\Psi_{i_0})_{i_0\in\mathcal{I}}$ of the non-abelian Čech 0-cochains acting on non-abelian 1-cocycles by

$$(\Phi_{i_0i_1}) \mapsto (\Psi^{-1}_{i_1} \circ \Phi_{i_0i_1} \circ \Psi_{i_0}).$$

Thus Theorem A.1 implies the following corollary.

Corollary A.4. *There is an equivalence of groupoid*

$$\mathsf{Del}\big(\mathsf{C}^*_{\mathsf{TW}}(\mathsf{T}^+_{\mathcal{A}})\big) \cong \mathsf{Z}^1\big(\exp(\mathsf{T}^{+\bullet}_{\mathcal{A}}(\mathsf{X})\big).$$

A.4. Deformation of flasque resolutions

Let $\{\Phi_{i_0i_1} = \exp(T_{i_0i_1})\}$ be a non-abelian 1-cocycle in G^1. Using $\Phi_{i_0i_1}$'s to glue local piece $\mathcal{A}|_{U_{i_0}}$ we get another sheaf of algebras which is locally isomorphic to $\mathcal{A}$, but not globally. We denote this new sheaf of algebras by $\mathcal{B}$. Since the sheaf $\mathcal{C}^*_{TW}(\mathcal{A})$ is a flasque resolution of $\mathcal{A}$, it is natural to ask how to deform this flasque resolution to give a flasque resolution of $\mathcal{B}$.

By Corollary A.4, there is a Maurer-Cartan element θ in the differential graded Lie algebra $C^*_{TW}(T^+_{\mathcal{A}})$ corresponding to the non-abelian 1-cocycle Φ. This element may be considered as an operator Q acting on $C^*_{TW}(\mathcal{A})$ thanks to the following Lemma.

Lemma A.5. *There is a morphism of differential graded Lie algebras*

$$\alpha : C^*_{TW}(T^+_{\mathcal{A}}) \to \mathrm{Der}\left(C^*_{TW}(\mathcal{A}), C^*_{TW}(\mathcal{A})\right).$$

Proof. Using that $\mathrm{Der}(\mathcal{A})$ acts on $\mathcal{A}$ together with the monoidal structure of TW, we get a cochain map $C^*_{TW}(T^+_{\mathcal{A}})\otimes C^*_{TW}(\mathcal{A}) \to C^*_{TW}(\mathcal{A})$, which leads to $\alpha : C^*_{TW}(T^+_{\mathcal{A}}) \to \mathrm{End}\left(C^*_{TW}(\mathcal{A})\right)$. It is a straightforward computation to check that α is a Lie algebra map and that its image lies in derivations. ☐

We denote by $Q := \alpha(\theta)$ the operator associated to θ corresponding to the non-abelian 1-cocycle Φ. Thus Q is a Maurer-Cartan element of the differential graded Lie algebra $\mathrm{Der}(C^*_{TW}(\mathcal{A}))$, which implies that we can deform the differential d on $C^*_{TW}(\mathcal{A})$ to $d + Q$. Similarly, on the level of sheaves, we get a deformation of $\mathcal{C}^*_{TW}(\mathcal{A})$ whose differential will again be denoted by $d + Q$.

Theorem A.6. *There is a quasi-isomorphism*

$$\mathcal{B} \to \left(\mathcal{C}^*_{TW}(\mathcal{A}), d + Q\right)$$

of sheaves of differential graded algebras.

Proof. For each $i \in \mathcal{I}$, we first show that the cohomology of $\left(\mathcal{C}^*_{TW}(\mathcal{A}), d+Q\right)|_{U_i}$ is quasi-isomorphic to $\mathcal{A}_i$. For this we apply Corollary A.4 on U_i using the covering $\{U_i \cap U_{i_0}\}_{i_0\in\mathcal{I}}$. Observe that the 1-cocycle $\Phi|_{U_i}$ is isomorphic to the trivial cocycle by the gauge transformation defined by the degree 0 element $\{\Phi_{ii_0}\}_{i_0\in\mathcal{I}}$. By Corollary A.4 we conclude that the Maurer-Cartan section θ corresponding to Φ, when restricted to U_i, is gauge equivalent to 0 via a unique element $a_i \in \mathcal{C}^0_{TW}(T^+_{\mathcal{A}})(U_i)$ (corresponding to the gauge transformation $\{\Phi_{ii_0}\}_{i_0\in\mathcal{I}}$) such that

$$e^{a_i} * (0) = \theta|_{U_i} .$$

Moreover the degree zero Thom-Whitney cochain a_i is of the form $\left(\prod_{i_0\in\mathcal{I}} T_{ii_0}, \cdots\right)$. This latter assertion follows from the proof of [10, Theorem 4.1]. Via the representation α we get an isomorphism of complexes of sheaves

$$\left(\mathcal{C}^*_{TW}(\mathcal{A}), d\right)|_{U_i} \overset{\alpha(e^{a_i})}{\longrightarrow} \left(\mathcal{C}^*_{TW}(\mathcal{A}), d + Q\right)|_{U_i} .$$

The complex on the left hand side is quasi-isomorphic to $\mathcal{A}_i$ via composition

$$\mathcal{A}_i \longrightarrow \mathcal{C}^*(\mathcal{A})|_{U_i} \overset{I}{\longrightarrow} \left(\mathcal{C}^*_{TW}(\mathcal{A}), d\right)|_{U_i} .$$

Let U_i and U_j be two open subsets in the covering $\mathcal{U}$. we consider the following diagram of maps on the intersection U_{ij}

$$\begin{array}{ccccccc}
\mathcal{A}|_{U_{ij}} & \longrightarrow & \mathcal{C}^*(\mathcal{A})|_{U_{ij}} & \xrightarrow{I} & (\mathcal{C}^*_{TW}(\mathcal{A}), d)|_{U_{ij}} & \xrightarrow{\alpha(e^{a_i})} & (\mathcal{C}^*_{TW}(\mathcal{A}), d+Q)|_{U_{ij}} \\
 & & & & & & \downarrow \mathrm{id} \\
\mathcal{A}|_{U_{ij}} & \longrightarrow & \mathcal{C}^*(\mathcal{A})|_{U_{ij}} & \xleftarrow{P} & (\mathcal{C}^*_{TW}(\mathcal{A}), d)|_{U_{ij}} & \xleftarrow{\alpha(e^{-a_j})} & (\mathcal{C}^*_{TW}(\mathcal{A}), d+Q)|_{U_{ij}} .
\end{array}$$

We can use this diagram to calculate the induced gluing map on the cohomology of the sheaf $(\mathcal{C}^*_{TW}(\mathcal{A}), d+Q)$. Using explicit formulas for I, P, and the fact that a_i is of the form $\left(\prod_{i_0\in\mathcal{J}} T_{ii_0}, \cdots\right)$, we conclude that this gluing map is Φ_{ij}. The proof is complete. □

References

[1] M. Abate, F. Bracci, F. Tovena, Embeddings of submanifolds and normal bundles, Advances in Math. **220** (2009), 620–656.

[2] D. Arinkin, A. Căldăraru, When is the self-intersection of a subvariety a fibration?, preprint `arXiv:1007.1671`.

[3] T. Brzezinski, G. Militaru, Bialgebroids, $\times_A$-bialgebras and duality, J. Algebra **251** (2002), no. 1, 279–294.

[4] A. Buchsbaum, D. Eisenbud, Generic free resolutions and a family of generically perfect ideals, Advances in Math. 18 (1975), no. 3, 245-301

[5] D. Calaque, M. Van den Bergh, Hochschild cohomology and Atiyah classes, Advances in Math. **224** (2010), no. 5, 1839–1889.

[6] I. Ciocan-Fontanine, M. Kapranov, Derived Quot schemes, Ann. Scient. ENS, **34** (2001), no. 3, 403–440.

[7] V. Dolgushev, D. Tamarkin, B. Tsygan, The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal, J. Noncommut. Geom. **1** (2007), no. 1, 1–25.

[8] A. Grothendieck, Éléments de géométrie algébrique IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie. Publ. Math. IHÉS **32** (1967), 361 pp.

[9] D. Fiorenza, M. Manetti, E. Martinengo Semicosimplicial DGLAs in deformation theory, Communications in Algebra Vol. 40, 2012, 2243-2260.

[10] V. Hinich, Descent of Deligne groupoids, IMRN 1997, no. 5, 223–239.

[11] G. Hochschild, B. Kostant, A. Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. **102** (1962), 383–408.

[12] L. Illusie, Complexe cotangent et déformations II, Lecture Notes in Mathematics **283**, Springer-Verlag, Berlin-New York, 1972. vii+304 pp.

[13] M. Kapranov, Rozansky-Witten invariants via Atiyah classes, Compositio Math. **115** (1999), no. 1, 71–113.

[14] M. Kapranov, Noncommutative geometry based on commutator expansions, J. Reine Angew. Math. 505 (1998), 73118.

[15] J. Lurie, Derived Algebraic Geometry V: Structured Spaces, preprint available at `http://www.math.harvard.edu/~lurie/papers/DAG-V.pdf`.

[16] N. Markarian, The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem, preprint `arXiv:math/0610553`.

[17] I. Moerdijk, Monads on tensor categories, J. Pure Appl. Algebra **168** (2002), no. 2-3, 189–208.

[18] G.S. Rinehart, Differential forms on general commutative algebras, Trans. Amer. Math. Soc. **108** (1963), no. 2, 195–222.

[19] B. Toën, G. Vezzosi, From HAG to DAG: derived moduli stacks, in Axiomatic, enriched and motivic homotopy theory, 173–216, NATO Sci. Ser. II Math. Phys. Chem., 131, KluwerAcad. Publ., Dordrecht, 2004.

[20] B. Toën, G. Vezzosi, Homotopical Algebraic Geometry II: Geometric Stacks and Applications, Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224 pp.

[21] A. Yekutieli, The Continuous Hochschild Cochain Complex of a Scheme, Canadian J. Math **54** (2002), no. 6, 1319–1337.

[22] S. Yu, Dolbeault dga of formal neighborhoods and L_∞-algebroids, preprint available at `http://www.math.psu.edu/yu/papers/DolbeaultDGA.pdf`.