

# NON-FINE MODULI SPACES OF SHEAVES ON K3 SURFACES

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ABSTRACT. In general, if  $M$  is a moduli space of stable sheaves on  $X$ , there is a unique element  $\alpha$  in the Brauer group of  $M$  such that a  $\pi_M^* \alpha^{-1}$ -twisted universal sheaf exists on  $X \times M$ . In this paper we study the situation when  $X$  and  $M$  are K3 surfaces, and we identify  $\alpha$  in terms of Mukai's map between the cohomology of  $X$  and of  $M$  (defined by means of a quasi-universal sheaf). We prove that the derived category of sheaves on  $X$  and the derived category of  $\alpha$ -twisted sheaves on  $M$  are equivalent.

This suggests a conjecture which describes, in terms of Hodge isometries of lattices, when derived categories of twisted sheaves on two K3 surfaces are equivalent. If proven true, this conjecture would generalize a theorem of Orlov and a recent result of Donagi-Pantev.

## 1. INTRODUCTION

1.1. Ever since Mukai's seminal paper [13], moduli spaces of sheaves on K3 surfaces have attracted constant interest, largely because of the wealth of geometric properties they possess. Such moduli spaces are always even-dimensional, so the first case with non-trivial geometry is when  $M$  is a 2-dimensional moduli space of sheaves on a K3 surface  $X$ . In this paper we'll focus on this situation, whose study was initiated by Mukai, and later continued in a slightly different direction by Orlov [15].

The guiding principle when studying such moduli spaces can be loosely stated by saying that the geometry of  $M$  is inherited from  $X$ . In our case, for example,  $M$  is again a K3; however,  $M$  could be quite different from  $X$ , and this is why this situation is so interesting. The question that arises is what relations can be found between geometric objects on  $X$  and on  $M$ , and most of the time we will be interested in relating cohomology groups (integral or rational, plus Hodge structure) of  $X$  and of  $M$ , as well as certain derived categories of sheaves on  $X$  and  $M$ .

1.2. Mukai's original paper was concerned with studying the cohomology of  $M$ , and we'll start by presenting a sketch of his ideas. This can be done most easily in the case when  $M$  is fine, in other words when a universal sheaf exists on  $X \times M$ .

The *Mukai lattice* of  $X$  is defined as the group

$$\tilde{H}(X, \mathbf{Z}) = H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z}),$$

endowed with the Mukai product (3.3), which is a slight modification of the usual product in cohomology. Using the Chern classes of a universal sheaf one defines (3.7) a class in  $H^*(X \times M, \mathbf{Z})$ , which induces a correspondence

$$\varphi : \tilde{H}(X, \mathbf{Z}) \rightarrow \tilde{H}(M, \mathbf{Z}).$$

This correspondence is in fact an isomorphism, which respects the Mukai product and the natural Hodge structure on the Mukai lattice induced from the Hodge structure of  $X$  (3.4). Thus the integral Mukai lattices of  $X$  and of  $M$  are Hodge isometric, and from this Mukai is able to give a complete description of  $M$ , by means of the Torelli theorem.

1.3. Although it can be presented entirely in terms of cohomology, the above construction relies heavily on the fact that there is an underlying equivalence of derived categories between  $X$  and  $M$ . More precisely, the integral functor defined using a universal sheaf is a Fourier-Mukai transform, i.e., an equivalence  $\mathbf{D}_{\text{coh}}^b(M) \cong \mathbf{D}_{\text{coh}}^b(X)$ . It is an interesting question to ask for what other K3's  $M$  do we have such equivalences, and Orlov's work has provided an answer to this question. His main result can be stated as follows:

**Theorem (Orlov [15]).** *Let  $X$  and  $M$  be K3 surfaces. Then the following are equivalent:*

1.  $M$  is a fine, compact, 2-dimensional moduli space of stable sheaves on  $X$ ;
2. there is a Hodge isometry  $T_X \cong T_M$  between the transcendental lattices of  $X$  and of  $M$ ;
3. the derived categories  $\mathbf{D}_{\text{coh}}^b(X)$  and  $\mathbf{D}_{\text{coh}}^b(M)$  are equivalent.

The implication (1)  $\Rightarrow$  (2) is an immediate consequence of Mukai's results: since the map  $\varphi$  is defined by means of an algebraic class, it takes algebraic classes on  $X$  to algebraic classes on  $M$ . Being a Hodge isometry, it follows that the transcendental lattice of  $X$  will be mapped Hodge isometrically by  $\varphi$  onto the transcendental lattice of  $M$ .

1.4. Orlov's theorem should be viewed as a derived version of the Torelli theorem: given a Mukai lattice with a Hodge structure, singling out the  $H^2$  lattice will determine the K3 surface (by Torelli), and therefore the category of coherent sheaves; singling out a smaller lattice as the transcendental lattice of a K3 will no longer determine the surface itself, but it will determine the derived category of sheaves on the K3.

1.5. The issue with the above results is the fact that the condition that  $M$  be fine is quite restrictive (at the moment, only one explicit class of examples of this type is known). There are many situations in which relaxing this condition would be relevant – for example, the first example studied by Mukai ([14, 2.2]) is not fine. (In this case one calculates the moduli space of spinor bundles on a  $(2, 2, 2)$  complete intersection in  $\mathbf{P}^5$  and finds it to be a double cover of  $\mathbf{P}^2$  branched over a sextic.) On the level of cohomology, Mukai is able to avoid the condition of fineness, defining the map  $\varphi$  by means of a *quasi-universal sheaf* (3.6), as a replacement for the universal sheaf in the original construction. (One needs to move to rational cohomology, instead of the integral one used before.) While a universal sheaf may not exist in general, a quasi-universal sheaf always exists.

1.6. Since Mukai's main interest was to obtain a correspondence between the cohomology of  $M$  and the cohomology of  $X$ , constructing  $\varphi$  by means of a quasi-universal sheaf is enough. What is lost in this approach is the equivalence of derived categories – one can not hope to get a Fourier-Mukai transform by using a quasi-universal sheaf. In this paper we propose a different approach, which makes apparent an underlying equivalence of derived categories even in the non-fine case. Using this idea, we are able to generalize to the case of non-fine moduli problems the implications (1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (3) of Orlov's result, and to suggest a conjectural generalization of the equivalence (2)  $\Leftrightarrow$  (3). The key idea is that instead of replacing the universal sheaf by a quasi-universal one, we can replace it by a *twisted* universal sheaf. This yields an equivalence

$$\mathbf{D}_{\text{coh}}^b(M, \alpha) \cong \mathbf{D}_{\text{coh}}^b(X),$$

where  $\alpha$  is the twisting (an element of the Brauer group of  $M$ , determined by the original moduli problem data), and  $\mathbf{D}_{\text{coh}}^b(M, \alpha)$  is the derived category of  $\alpha$ -twisted sheaves on  $M$ . (A brief review of the Brauer group and twisted sheaves is included in Section 2. For further details, the reader is referred to [3] or [4].)

1.7. We can rephrase this idea in the language of modules over an Azumaya algebra (which we'll avoid in the sequel, preferring the more intuitive language of twisted sheaves). In [13], Mukai constructs a module over an Azumaya algebra  $\mathcal{A}$  over  $\mathcal{O}_X$  (i.e., a twisted sheaf), but then he forgets the extra structure as an  $\mathcal{A}$ -module, and only uses the  $\mathcal{O}_X$ -module structure. We make use of the extra structure available to get the equivalence of derived categories.

1.8. Using twisted sheaves (or modules over an Azumaya algebra) is particularly relevant in view of current developments of mirror symmetry. Recall that one of the fundamental ingredients of Kontsevich's Homological Mirror Symmetry [10] is the derived category of sheaves on a Calabi-Yau manifold. Recently there have been suggestions ([3, 6.8], [9]) that in order to get a mathematical description of the full physical picture, one needs to study not only derived categories of sheaves, but also derived categories of twisted sheaves. In the physical context, the equivalence  $\mathbf{D}_{\text{coh}}^b(M, \alpha) \cong \mathbf{D}_{\text{coh}}^b(X)$  suggests that turning on discrete torsion  $\alpha$  on  $M$  yields the same physical theory as having no discrete torsion on  $X$ .

1.9. Reverting back to questions about moduli spaces of sheaves, it is a general fact [3, 3.3.2 and 3.3.4] that although a universal sheaf may not exist, there is always a twisting  $\alpha \in \text{Br}(M)$  such that a  $\pi_M^* \alpha^{-1}$ -twisted universal sheaf exists on  $X \times M$ , when  $M$  is a moduli space of stable sheaves on  $X$ . (Here  $\pi_M$  is the projection from  $X \times M$  on the second factor.) An  $\alpha$  with this property is unique, and it is called the obstruction to the existence of a universal sheaf on  $X \times M$  (because an untwisted universal sheaf exists only if  $\alpha = 0$ ). Once a twisted universal sheaf is found, the proof that it induces an equivalence of derived categories is almost the same as in the untwisted case. What this paper actually brings new is the calculation of the

obstruction  $\alpha$  in terms of the map  $\varphi$  introduced earlier by Mukai. (This map has a natural interpretation in the context of twisted sheaves, see Sections 2 and 3.)

1.10. Before we can state our results, we need to list a few facts about the situation we study –  $X$  is a K3 surface,  $v \in \tilde{H}(X, \mathbf{Z})$  is an isotropic Mukai vector, and  $M$  is the moduli space of stable sheaves on  $X$  whose Mukai vector is  $v$  (3.5). The vector  $v$  and the polarization of  $X$  are such that  $M$  is non-empty and compact, so it is a K3 surface (Theorem 3.1). We do not assume that  $M$  is fine, so Mukai’s map we consider is defined on the rational cohomology,  $\varphi : \tilde{H}(X, \mathbf{Q}) \rightarrow \tilde{H}(M, \mathbf{Q})$ .

1. The Brauer group  $\text{Br}(M)$  of any smooth surface  $M$  is isomorphic to the cohomological Brauer group  $\text{Br}'(M)$ , defined as

$$\text{Br}'(M) = H_{\text{ét}}^2(M, \mathcal{O}_M^*).$$

For a K3 surface there is a natural identification (2.4)

$$\text{Br}'(M) \cong T_M^\vee \otimes \mathbf{Q}/\mathbf{Z} \cong \text{Hom}_{\mathbf{Z}}(T_M, \mathbf{Q}/\mathbf{Z}).$$

2. The map  $\varphi$  restricts to an injection  $T_X \hookrightarrow T_M$ , which fits into an exact sequence

$$0 \rightarrow T_X \xrightarrow{\varphi} T_M \rightarrow \mathbf{Z}/n\mathbf{Z} \rightarrow 0,$$

where  $n$  is an integer determined by the initial moduli problem ([13, 6.4], Theorem 3.4). Although  $\varphi$  depends on the choice of quasi-universal sheaf, its restriction to  $T_X$  is independent of this choice.

3. Applying  $\text{Hom}_{\mathbf{Z}}(\cdot, \mathbf{Q}/\mathbf{Z})$  to the above exact sequence yields

$$0 \rightarrow \mathbf{Z}/n\mathbf{Z} \rightarrow \text{Hom}_{\mathbf{Z}}(T_M, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\varphi^\vee} \text{Hom}_{\mathbf{Z}}(T_X, \mathbf{Q}/\mathbf{Z}) \rightarrow 0,$$

or, in view of (1),

$$0 \rightarrow \mathbf{Z}/n\mathbf{Z} \rightarrow \text{Br}(M) \xrightarrow{\varphi^\vee} \text{Br}(X) \rightarrow 0.$$

Elements of  $\text{Ker } \varphi^\vee$  are those  $\alpha \in \text{Hom}_{\mathbf{Z}}(T_M, \mathbf{Q}/\mathbf{Z})$  that satisfy  $\varphi(T_X) \subseteq \text{Ker } \alpha$ .

- 1.11. As a matter of notation, for  $w \in \tilde{H}(M, \mathbf{Q})$  the functional

$$(w \cdot \cdot)|_{T_M} \bmod \mathbf{Z} \in \text{Hom}_{\mathbf{Z}}(T_M, \mathbf{Q}/\mathbf{Z}) = \text{Br}(M)$$

will be denoted by  $[w]$ . Note that since it is restricted to  $T_M$ ,  $[w]$  only depends on the  $H^2(M, \mathbf{Q})$ -component of  $w$ . The condition in (3) above can be written as

$$[w] \in \text{Ker } \varphi^\vee \text{ for } w \in \tilde{H}(M, \mathbf{Q}) \Leftrightarrow (w \cdot \varphi(t)) \in \mathbf{Z} \text{ for all } t \in T_X.$$

- 1.12. Now we can state our main results:

**Theorem 1.1.** *Let  $X$  be a polarized K3 surface, let  $v$  be a primitive isotropic Mukai vector, and let  $M$  be the moduli space of stable sheaves whose Mukai vector is  $v$ . Assume that  $M$  is compact and non-empty, and let  $\varphi : T_X \rightarrow T_M$  be the restriction of Mukai’s map (defined by means of a quasi-universal or twisted universal sheaf).*

*If  $u \in \tilde{H}(X, \mathbf{Z})$  is such that  $(u \cdot v) = 1$  then  $[\varphi(u)] \in \text{Br}(M)$  is the obstruction to the existence of a universal sheaf on  $X \times M$ .*

**Theorem 1.2.** *Under the assumptions of the previous theorem, let  $\varphi^\vee$  be the dual of  $\varphi$ , tensored with  $\mathbf{Q}/\mathbf{Z}$ . Then we have:*

1. *the kernel of  $\varphi^\vee : \mathrm{Br}(M) \rightarrow \mathrm{Br}(X)$  is a cyclic group of order  $n$ , generated by the obstruction  $\alpha$  to the existence of a universal sheaf on  $X \times M$ ;*
2. *the map  $\varphi$  restricts to a Hodge isometry  $T_X \cong \mathrm{Ker} \alpha$ ;*
3. *any  $\pi_M^* \alpha^{-1}$ -twisted universal sheaf on  $X \times M$  induces an equivalence of derived categories*

$$\mathbf{D}_{\mathrm{coh}}^b(M, \alpha) \cong \mathbf{D}_{\mathrm{coh}}^b(X).$$

1.13. Theorem 1.2 provides the desired generalization of the implications (1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (3) in Orlov's theorem. For the equivalence (2)  $\Leftrightarrow$  (3) we can only conjecture what the result should be:

**Conjecture 1.3.** *Let  $X$  and  $Y$  be K3 surfaces, and let  $\alpha$  and  $\beta$  be elements in  $\mathrm{Br}(X)$  and  $\mathrm{Br}(Y)$ , respectively. Using (1.10.1),  $\alpha$  can be identified with a group homomorphism  $T_X \rightarrow \mathbf{Q}/\mathbf{Z}$ , and similarly for  $\beta$ . Then the following are equivalent:*

1. *the derived categories  $\mathbf{D}_{\mathrm{coh}}^b(X, \alpha)$  and  $\mathbf{D}_{\mathrm{coh}}^b(Y, \beta)$  are equivalent;*
2. *the lattices  $\mathrm{Ker} \alpha \subseteq T_X$  and  $\mathrm{Ker} \beta \subseteq T_Y$  are Hodge isometric, where  $\mathrm{Ker} \alpha$  and  $\mathrm{Ker} \beta$  inherit the Hodge structure from the overlying lattices  $T_X$  and  $T_Y$ , respectively.*

1.14. This generalizes the original Orlov result, for  $\alpha = \beta = 0$ . If  $Y$  is a moduli space of stable sheaves on  $X$ ,  $\beta \in \mathrm{Br}(Y)$  is the obstruction to the existence of a universal sheaf on  $X \times Y$ , and  $\alpha = 0$ , the conjecture is a restatement of Theorem 1.2:  $\varphi$  induces a Hodge isometry between  $T_X = \mathrm{Ker} \alpha$  and  $\mathrm{Ker} \beta$ .

1.15. Conjecture 1.3 also generalizes the following result of Donagi-Pantev, which is relevant in the study of moduli spaces of twisted Higgs bundles:

**Theorem (Donagi-Pantev [6]).** *Let  $J/S$  be an elliptic K3 surface with a section, and let  $\alpha, \beta \in \mathrm{Br}(J)$ . Identifying  $\mathrm{Br}(J)$  with  $\mathrm{III}_S(J)$ , the Ogg-Shafarevich group of  $J$ , yields elliptic K3 surfaces  $J_\alpha$  and  $J_\beta$  (in general without a section) which correspond to  $\alpha$  and  $\beta$ , respectively. Viewing  $J$  as a moduli space of stable sheaves on  $J_\alpha$  and  $J_\beta$  gives surjections  $\mathrm{Br}(J) \rightarrow \mathrm{Br}(J_\alpha)$  and  $\mathrm{Br}(J) \rightarrow \mathrm{Br}(J_\beta)$ ; let  $\bar{\beta}$  be the image of  $\beta$  in  $\mathrm{Br}(J_\alpha)$  and  $\bar{\alpha}$  the image of  $\alpha$  in  $\mathrm{Br}(J_\beta)$ . Then there exists an equivalence of derived categories of twisted sheaves  $\mathbf{D}_{\mathrm{coh}}^b(J_\alpha, \bar{\beta}) \cong \mathbf{D}_{\mathrm{coh}}^b(J_\beta, \bar{\alpha})$ .*

1.16. A few words of explanation are in order here. Ogg-Shafarevich theory associates to an elliptic fibration  $X/S$  without a section an element  $\alpha_X$  of the Tate-Shafarevich group  $\mathrm{III}_S(J)$  of the relative Jacobian  $J/S$  of  $X/S$ , and this association gives rise to a bijection between the set of all elliptic fibrations whose relative Jacobian is  $J/S$  and the group  $\mathrm{III}_S(J)$ . When  $J$  is a surface,  $\mathrm{III}_S(J)$  is naturally isomorphic to  $\mathrm{Br}(J)$ .

Given an elliptic fibration  $X/S$  without a section, one obtains  $J/S$  as the relative moduli space of rank 1, degree 0 semistable sheaves on the fibers of  $X/S$ , and thus one

gets an obstruction to the existence of a universal sheaf on  $X \times J$  which is an element  $\alpha$  in  $\text{Br}(J)$ . In [5] it was shown that  $\alpha$  coincides with the element  $\alpha_X \in \text{III}_S(J) = \text{Br}(J)$  that classifies  $X/S$ .

Now we can see that Conjecture 1.3 would imply the Donagi-Pantev result: we have  $T_{J_\alpha} = \text{Ker } \alpha$  and, viewing elements of the Brauer group as group homomorphisms from the transcendental lattice to  $\mathbf{Q}/\mathbf{Z}$ ,  $\bar{\beta}$  is just the restriction of  $\beta$  to  $\text{Ker } \alpha$ . Therefore

$$\text{Ker}_{T_{J_\alpha}} \bar{\beta} = \text{Ker}_{T_J} \alpha \cap \text{Ker}_{T_J} \beta,$$

and similarly,

$$\text{Ker}_{T_{J_\beta}} \bar{\alpha} = \text{Ker}_{T_J} \alpha \cap \text{Ker}_{T_J} \beta,$$

both these equalities respecting the Hodge structures (being induced on all the lattices from the Hodge structure of  $T_J$ ). By Conjecture 1.3,

$$\mathbf{D}_{\text{coh}}^b(J_\alpha, \bar{\beta}) \cong \mathbf{D}_{\text{coh}}^b(J_\beta, \bar{\alpha}),$$

which is the Donagi-Pantev result.

In full honesty, the actual statement proven in [6] is much stronger: one of  $\alpha$  or  $\beta$  can be *non-algebraic*, i.e. an element of  $H_{\text{an}}^2(J, \mathcal{O}_J^*)$  which is not necessarily torsion. In this situation the corresponding  $J_\alpha$  or  $J_\beta$  is non-algebraic, a situation which we cannot handle. Therefore Conjecture 1.3 should be thought of as a generalization of the result in [6] to the case of algebraic K3 surfaces.

1.17. In the spirit of our earlier comment (1.4), Conjecture 1.3 describes the relationship between sublattices of the Mukai lattice and twisted derived categories. The following is a dictionary of the correspondence between sublattices of a Mukai lattice endowed with a Hodge structure and categories on a K3  $X$  ( $\alpha \in \text{Br}(X)$ ):

Lattice	Category
$H^2(X, \mathbf{Z})$	$\mathfrak{Coh}(X)$
$T_X$	$\mathbf{D}_{\text{coh}}^b(X)$
$\text{Ker}(\alpha)$	$\mathbf{D}_{\text{coh}}^b(X, \alpha)$

1.18. The paper is organized as follows: in Section 2 we present general results about the Brauer group on a K3, twisted sheaves, and derived categories. We follow up in the next section with facts about K3 surfaces, along with Mukai's results on moduli spaces of stable sheaves on them. Section 4 deals with deformations of vector bundles as twisted sheaves, and in Section 5 we prove the main theorem and discuss some possible ways of approaching the proof of Conjecture 1.3.

**Conventions.** All our spaces are complex manifolds over  $\mathbf{C}$ , and the topology used is either the analytic or étale one. When referring to derived categories, we mean the bounded derived category of complexes with coherent cohomology.

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## 2. TWISTED SHEAVES AND DERIVED CATEGORIES

In this section we list some results about the Brauer group of a scheme. Our reference for this topic is [11, Chapter IV]. We also include a sketch of the definition and main properties of twisted sheaves. The reader unfamiliar with the subject is referred to [3, Chapters 1 and 2] or [4]. The topology used, unless otherwise mentioned, is the étale or analytic topology.

2.1. The *cohomological Brauer group* of a scheme  $X$  is the group

$$\mathrm{Br}'(X) = H_{\text{ét}}^2(X, \mathcal{O}_X^*).$$

It naturally occurs in many aspects of algebraic geometry, as a higher generalization of the Picard group. To have an example in mind consider the question of classifying projective bundles over a space  $X$ , up to those bundles that are projectivizations of vector bundles. In the étale topology the sequence of sheaves of groups

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \mathrm{GL}(n) \rightarrow \mathrm{PGL}(n) \rightarrow 0,$$

is exact, hence it yields the exact sequence

$$H^1(X, \mathrm{GL}(n)) \rightarrow H^1(X, \mathrm{PGL}(n)) \xrightarrow{\delta_n} H^2(X, \mathcal{O}_X^*) = \mathrm{Br}'(X).$$

We read this as saying that the obstruction to lifting a projective bundle (given by an element  $p$  of  $H^1(X, \mathrm{PGL}(n))$ ) to a vector bundle (element of  $H^1(X, \mathrm{GL}(n))$ ) is the image of  $p$  in the cohomological Brauer group, under the coboundary map  $\delta_n$ .

2.2. In fact, we are more interested in a subgroup of  $\mathrm{Br}'(X)$ , namely the image of the coboundary maps  $\delta_n$  for all  $n$ . This subgroup is the *Brauer group* of  $X$ , denoted by  $\mathrm{Br}(X)$ . We list below some of its main properties.

The Brauer group is torsion; this follows from the short exact sequence

$$0 \rightarrow \mathbf{Z}/n\mathbf{Z} \rightarrow \mathrm{SL}(n) \rightarrow \mathrm{PGL}(n) \rightarrow 0,$$

by taking the cohomology long exact sequence, and deducing that the image of  $\mathrm{PGL}(n)$  in  $\mathrm{Br}'(X)$  is contained in the image of the map  $H^2(X, \mathbf{Z}/n\mathbf{Z}) \rightarrow H^2(X, \mathcal{O}_X^*)$ , and hence is  $n$ -torsion.

If  $X$  is smooth,  $\mathrm{Br}'(X)$  is torsion as well ([11, II, 1.4]). If  $X$  is a smooth curve,  $\mathrm{Br}(X) = \mathrm{Br}'(X) = 0$ . If  $X$  is a smooth surface,  $\mathrm{Br}(X) = \mathrm{Br}'(X)$  ([11, IV, 2.16]).

2.3. Consider the Kummer sequence

$$0 \rightarrow \mathbf{Z}/n\mathbf{Z} \rightarrow \mathcal{O}_X^* \xrightarrow{\cdot n} \mathcal{O}_X^* \rightarrow 0,$$

which is exact in both the étale and analytic topologies. Taking the associated long exact sequence yields

$$\mathrm{Pic}(X) \xrightarrow{\cdot n} \mathrm{Pic}(X) \xrightarrow{c_1 \bmod n} H^2(X, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \mathrm{Br}'(X) \xrightarrow{\cdot n} \mathrm{Br}'(X),$$

which implies that the  $n$ -torsion part of  $\mathrm{Br}'(X)$ ,  $\mathrm{Br}'(X)_n$ , fits in the exact sequence

$$0 \rightarrow \mathrm{Pic}(X) \otimes \mathbf{Z}/n\mathbf{Z} \rightarrow H^2(X, \mathbf{Z}/n\mathbf{Z}) \rightarrow \mathrm{Br}'(X)_n \rightarrow 0.$$

Taking the direct limit over all  $n$ , we conclude that on any scheme or analytic space  $X$  we have the exact sequence

$$0 \rightarrow \mathrm{Pic}(X) \otimes \mathbf{Q}/\mathbf{Z} \rightarrow H^2(X, \mathbf{Q}/\mathbf{Z}) \rightarrow \mathrm{Br}'(X)_{\mathrm{tors}} \rightarrow 0.$$

If  $X$  is a smooth scheme over  $\mathbf{C}$ , and  $X^{\mathrm{an}}$  is the associated analytic space, then we have

$$\mathrm{Br}'(X) = \mathrm{Br}'_{\mathrm{an}}(X)_{\mathrm{tors}},$$

because  $\mathrm{Pic}(X)$  and  $H^2(X, \mathbf{Q}/\mathbf{Z})$  are the same in the étale and analytic topologies.

2.4. Specializing to the case of a K3 surface  $X$ , we have  $H^2(X, \mathbf{Q}/\mathbf{Z}) \cong H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$  (because  $H^3(X, \mathbf{Z}) = 0$ ), and hence

$$\mathrm{Br}(X) \cong (H^2(X, \mathbf{Z})/\mathrm{NS}(X)) \otimes \mathbf{Q}/\mathbf{Z}.$$

There is a natural isomorphism

$$H^2(X, \mathbf{Z})/\mathrm{NS}(X) \cong T_X^\vee,$$

which maps  $v \in H^2(X, \mathbf{Z})/\mathrm{NS}(X)$  to the functional  $(v, \cdot)$  restricted to  $T_X$ . (To prove that this map is an isomorphism one needs to use the fact that  $T_X$  is a primitive sublattice of the unimodular lattice  $H^2(X, \mathbf{Z})$ .) We conclude that on any K3 surface  $X$  there is a natural isomorphism

$$\mathrm{Br}(X) \cong T_X^\vee \otimes \mathbf{Q}/\mathbf{Z} = \mathrm{Hom}_{\mathbf{Z}}(T_X, \mathbf{Q}/\mathbf{Z}).$$

2.5. We now shift our attention to the topic of twisted sheaves. Let  $X$  be a scheme or analytic space, and let  $\alpha \in \mathrm{Br}'(X) = H^2(X, \mathcal{O}_X^*)$  be represented by a Čech 2-cocycle, given along a fixed open cover  $\{U_i\}_{i \in I}$  by sections

$$\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*).$$

An  $\alpha$ -twisted sheaf  $\mathcal{F}$  (along the fixed cover) consists of a pair

$$(\{\mathcal{F}_i\}_{i \in I}, \{\varphi_{ij}\}_{i, j \in I}),$$

where  $\mathcal{F}_i$  is a sheaf on  $U_i$  for all  $i \in I$  and

$$\varphi_{ij} : \mathcal{F}_j|_{U_i \cap U_j} \rightarrow \mathcal{F}_i|_{U_i \cap U_j}$$

is an isomorphism for all  $i, j \in I$ , subject to the conditions:

1.  $\varphi_{ii} = \mathrm{id}$ ;
2.  $\varphi_{ij} = \varphi_{ji}^{-1}$ ;
3.  $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$ .

The class of  $\alpha$ -twisted sheaves together with the obvious notion of homomorphism is an abelian category, denoted by  $\mathfrak{Mod}(X, \alpha)$ , the category of  $\alpha$ -twisted sheaves. If one requires all the sheaves  $\mathcal{F}_i$  to be coherent, one obtains the category of coherent  $\alpha$ -twisted sheaves, denoted by  $\mathfrak{Coh}(X, \alpha)$ .

This notation is consistent, since one can prove that these categories are independent of the choice of the covering  $\{U_i\}$  ([3, 1.2.3]) or of the particular cocycle  $\{\alpha_{ijk}\}$  ([3, 1.2.8]) (all the resulting categories are equivalent to one another).

2.6. For  $\mathcal{F}$  an  $\alpha$ -twisted sheaf, and  $\mathcal{G}$  an  $\alpha'$ -twisted sheaf, one can define  $\mathcal{F} \otimes \mathcal{G}$  (which is an  $\alpha\alpha'$ -twisted sheaf), as well as  $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$  (which is  $\alpha^{-1}\alpha'$ -twisted), by gluing together the corresponding sheaves. If  $f : Y \rightarrow X$  is any morphism,  $f^*\mathcal{F}$  is an  $f^*\alpha$ -twisted sheaf on  $Y$ . Finally, if  $\mathcal{F} \in \mathfrak{Mod}(Y, f^*\alpha)$ , one can define  $f_*\mathcal{F}$ , which is  $\alpha$ -twisted on  $X$ . It is important to note here that one can not define arbitrary push-forwards of twisted sheaves.

These operations satisfy all the usual relations (adjointness of  $f_*$  and  $f^*$ , relations between  $\underline{\mathrm{Hom}}$  and  $\otimes$ , etc.)

The category  $\mathfrak{Mod}(X, \alpha)$  has enough injectives, and enough  $\mathcal{O}_X$ -flats ([3, 2.1.1, 2.1.2]).

2.7. We are mainly interested in  $\mathbf{D}_{\mathrm{coh}}^b(\mathfrak{Mod}(X, \alpha))$ , the derived category of complexes of  $\alpha$ -twisted sheaves on  $X$  with coherent cohomology. For brevity, we'll denote it by  $\mathbf{D}_{\mathrm{coh}}^b(X, \alpha)$ . Since the category  $\mathfrak{Coh}(X, \alpha)$  does not have locally free sheaves of finite rank if  $\alpha \notin \mathrm{Br}(X)$ , from here on we'll only consider the case  $\alpha \in \mathrm{Br}(X)$ .

The technical details of the inner workings of  $\mathbf{D}_{\mathrm{coh}}^b(X, \alpha)$  can be found in [4] or [3, Chapter 2]. The important facts are that one can define derived functors for all the functors considered in (2.6), and they satisfy the same relations as the untwisted ones (see for example [7, II.5]). One can prove duality for a smooth morphism  $f : X \rightarrow Y$ , which provides a right adjoint

$$f^!(\cdot) = \mathbf{L}f^*(\cdot) \otimes_X \omega_{X/Y}[\dim_X Y]$$

to  $\mathbf{R}f_*(\cdot)$ , as functors between  $\mathbf{D}_{\mathrm{coh}}^b(Y, \alpha)$  and  $\mathbf{D}_{\mathrm{coh}}^b(X, f^*\alpha)$ .

2.8. If  $X$  and  $Y$  are smooth schemes or analytic spaces,  $\alpha \in \mathrm{Br}(Y)$ , and  $\mathcal{E} \in \mathbf{D}_{\mathrm{coh}}^b(X \times Y, \pi_Y^*\alpha^{-1})$  (where  $\pi_X$  and  $\pi_Y$  are the projections from  $X \times Y$  to  $X$  and  $Y$ , respectively), we define the integral functor

$$\Phi_{Y \rightarrow X}^{\mathcal{E}} : \mathbf{D}_{\mathrm{coh}}^b(Y, \alpha) \rightarrow \mathbf{D}_{\mathrm{coh}}^b(X),$$

given by

$$\Phi_{Y \rightarrow X}^{\mathcal{E}}(\cdot) = \pi_{X,*}(\pi_Y^*(\cdot) \otimes^{\mathbf{L}} \mathcal{E}).$$

The following criterion for determining when  $\Phi_{Y \rightarrow X}^{\mathcal{E}}$  is an equivalence (whose proof can be found in [4] or [3, 3.2.1]) is entirely similar to the corresponding ones for untwisted derived categories due to Mukai [12], Bondal-Orlov [1] and Bridgeland [2].

**Theorem 2.1.** *The functor  $F = \Phi_{Y \rightarrow X}^{\mathcal{E}}$  is fully faithful if and only if for each point  $y \in Y$ ,*

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{coh}}^b(X)}(F\mathcal{O}_y, F\mathcal{O}_y) = \mathbf{C},$$

*and for each pair of points  $y_1, y_2 \in Y$ , and each integer  $i$ ,*

$$\mathrm{Ext}_{\mathbf{D}_{\mathrm{coh}}^b(X)}^i(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0,$$

*unless  $y_1 = y_2$  and  $0 \leq i \leq \dim Y$ . (Here  $\mathcal{O}_y$  is the skyscraper sheaf  $\mathbf{C}$  on  $y$ , which is naturally an  $\alpha$ -sheaf.)*

*Assuming the above conditions satisfied, then  $F$  is an equivalence of categories if and only if for every point  $y \in Y$ ,*

$$F\mathcal{O}_y \otimes^{\mathbf{L}} \omega_X \cong F\mathcal{O}_y.$$

2.9. Since we want to relate derived categories to cohomology, we'd like to define the notion of Chern character for twisted sheaves on a space  $X$ . We do this as follows: we fix, once and for all, a locally free  $\alpha^{-1}$ -twisted sheaf  $\mathcal{E}$ , and define the Chern character of an  $\alpha$ -twisted sheaf  $\mathcal{F}$  to be

$$\mathrm{ch}_{\mathcal{E}}(\mathcal{F}) = \frac{1}{\mathrm{rk}(\mathcal{E})} \mathrm{ch}(\mathcal{F} \otimes \mathcal{E}),$$

where the right hand side of the equality is computed as the Chern character of the usual (untwisted) sheaf  $\mathcal{F} \otimes \mathcal{E}$ . Note that if we define Chern classes in this way, they will live in the rational cohomology of  $X$ , not in the integral cohomology as the usual ones. The factor  $1/\mathrm{rk}(\mathcal{E})$  is introduced so that the Chern character of a point is the expected one.

This definition is dependent on the choice of  $\mathcal{E}$ , and therefore it is important to find out how our Chern character changes when using different  $\mathcal{E}$ 's. Given any two locally free  $\alpha$ -twisted sheaves  $\mathcal{E}$  and  $\mathcal{E}'$ , there exist locally free, untwisted sheaves  $\mathcal{G}, \mathcal{G}'$  such that  $\mathcal{E} \otimes \mathcal{G} \cong \mathcal{E}' \otimes \mathcal{G}'$  (for example, one can take  $\mathcal{G} = \mathcal{E}^{\vee} \otimes \mathcal{E}'$  and  $\mathcal{G}' = \mathcal{E}^{\vee} \otimes \mathcal{E}$ ), so it is enough to assume that  $\mathcal{E}' = \mathcal{E} \otimes \mathcal{G}'$  for a locally free, untwisted  $\mathcal{G}'$ . Then an easy calculation shows that

$$\mathrm{ch}_{\mathcal{E}'}(\mathcal{F}) = \mathrm{ch}_{\mathcal{E}}(\mathcal{F}) \cdot \frac{\mathrm{ch}(\mathcal{G}')}{\mathrm{rk}(\mathcal{G}')}.$$

(see also [13, Proof of Theorem 1.4 and 1.5, p. 385]). We'll see (3.9) that this implies that when we define maps on transcendental parts of the cohomology of  $X$  using the above definition of the Chern character for twisted sheaves, the choice of  $\mathcal{E}$  is irrelevant.

### 3. MUKAI'S RESULTS

In order to fix the notation, we present in this section a few results about K3 surfaces and moduli spaces of stable sheaves on them. Most of these results are either classical, or are due to Mukai [13], although we present them using the language of twisted sheaves.

3.1. Mukai's main idea in [13] is to use a universal sheaf (possibly twisted, or a quasi-universal one) to define a map between sheaves on  $X$  and sheaves on  $M$  (more precisely, between the derived categories of  $X$  and  $M$ ). Taking a modified version of the Chern character yields a map between the algebraic parts of the cohomology of  $X$  and of  $M$ , which can be extended to the full cohomology. Since the map on derived categories was an equivalence, the map on cohomology is an isomorphism, even after being extended to the total cohomology groups of  $X$  and  $M$ . Furthermore, since an equivalence of categories preserves the relative Euler characteristic of two complexes of sheaves (the alternating sum of the dimensions of the Ext groups), this gives a bilinear form that is preserved by the map on cohomology. We expand these ideas in the following few paragraphs.

3.2. Let  $X$  be a complex K3 surface, in other words, a compact complex manifold of complex dimension 2, simply connected and with  $K_X = 0$ . We have  $H^2(X, \mathbf{Z}) = \mathbf{Z}^{22}$ , and considering this group with the intersection pairing we obtain a lattice which is isomorphic to

$$\mathcal{L}_{\text{K3}} = E_8^{\oplus 2} \oplus U(1)^{\oplus 3}.$$

Inside the  $H^2(X, \mathbf{Z})$  lattice there are two natural sublattices, the Néron-Severi sublattice of  $X$ ,  $\text{NS}(X)$ , (consisting of first Chern classes of holomorphic vector bundles), and its orthogonal complement, the transcendental lattice  $T_X = \text{NS}(X)^\perp$ . Both these lattices are primitive sublattices of  $H^2(X, \mathbf{Z})$ , but may be non-unimodular.

The complex structure of  $X$  is reflected in the Hodge decomposition of  $H^2(X, \mathbf{C})$ ,

$$H^2(X, \mathbf{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

and these groups are 1-, 20-, and 1-dimensional, respectively. This decomposition induces in turn a Hodge structure on  $T_X$  (since  $H^{2,0}(X)$  is orthogonal to any algebraic vector).

Two lattices  $\mathcal{L}, \mathcal{L}'$ , endowed with Hodge structures, will be said to be *Hodge isometric* if there is an isometry between them, preserving the Hodge structure. As an application of this concept, the Torelli theorem can be stated by saying that two K3 surfaces  $X$  and  $Y$  are isomorphic if and only if  $H^2(X, \mathbf{Z})$  and  $H^2(Y, \mathbf{Z})$  are Hodge isometric.

3.3. The *Mukai lattice* of  $X$  is defined to be

$$\tilde{H}(X, \mathbf{Z}) = H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z}),$$

endowed with the product

$$((r, l, s) \cdot (r', l', s')) = \int_X l.l' - r.s' - r'.s,$$

where the dot product on the right hand side is the cup product in  $H^*(X, \mathbf{Z})$ .

3.4. The *Hodge decomposition* on  $\tilde{H}(X, \mathbf{Z})$  is given by

$$\begin{aligned}\tilde{H}^{2,0}(X) &= H^{2,0}(X) \\ \tilde{H}^{1,1}(X) &= H^0(X, \mathbf{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbf{C}) \\ \tilde{H}^{0,2}(X) &= H^{0,2}(X)\end{aligned}$$

Elements in  $\tilde{H}^{1,1}(X)$  will be called *algebraic*.

We'll sometimes also consider  $\tilde{H}(X, \mathbf{Q}) = \tilde{H}(X, \mathbf{Z}) \otimes \mathbf{Q}$ , with intersection product and Hodge decomposition defined in a similar fashion.

3.5. For a coherent sheaf  $\mathcal{F}$  (or, more generally, an element of  $\mathbf{D}_{\text{coh}}^b(X)$ ), define  $v(\mathcal{F}) \in \tilde{H}(X, \mathbf{Z})$  by

$$v(\mathcal{F}) = \text{ch}(\mathcal{F}) \cdot \sqrt{\text{td}(X)} = (\text{rk}(\mathcal{F}), c_1(\mathcal{F}), \text{rk}(\mathcal{F})\omega + \frac{1}{2}c_1(\mathcal{F})^2 - c_2(\mathcal{F})),$$

(where  $\omega \in H^4(X, \mathbf{Z})$  is the fundamental class of  $X$ ). This element is called the *Mukai vector* of  $\mathcal{E}$ . From Grothendieck-Riemann-Roch it follows that

$$\chi(\mathcal{E}, \mathcal{F}) = \sum (-1)^i \dim \text{Ext}_X^i(\mathcal{E}, \mathcal{F}) = -(v(\mathcal{E}) \cdot v(\mathcal{F})),$$

for any  $\mathcal{E}, \mathcal{F} \in \mathbf{D}_{\text{coh}}^b(X)$ .

If  $\mathcal{F}$  is an  $\alpha$ -twisted sheaf on  $M$ , we define its Mukai vector using the same formula as in the untwisted case, but using the definition of the Chern character in (2.9). Since this definition depends upon the choice of a locally free,  $\alpha^{-1}$ -twisted sheaf  $\mathcal{E}$  on  $M$ , we'll denote this type of Mukai vector by  $v_{\mathcal{E}}(\mathcal{F})$ .

**Theorem 3.1** ([13, Theorem 1.4]). *Let  $X$  be a polarized K3 surface, and let  $v \in \tilde{H}(X, \mathbf{Z})$  be a primitive (indivisible) vector that lies in the algebraic part of  $\tilde{H}(X)$ . Assume that  $v$  is isotropic (i.e.  $(v, v) = 0$ ), and that the moduli space of stable sheaves of Mukai vector  $v$ ,  $M(v)$ , is non-empty and compact. Then  $M(v)$  is a K3 surface.*

3.6. Under the assumptions of the previous theorem, let  $M = M(v)$ . There exists a unique element  $\alpha \in \text{Br}(M)$ , such that a  $\pi_M^* \alpha^{-1}$ -twisted universal sheaf  $\mathcal{P}$  exists on  $X \times M$  ([3, 3.3.2 and 3.3.4]). The twisting  $\alpha$  is called the *obstruction to the existence of a universal sheaf* on  $X \times M$ . Let  $\mathcal{E}$  be a fixed  $\alpha$ -twisted locally free sheaf of finite rank on  $M$ , (which exists by [3, 1.3.5]), and let  $\tilde{\mathcal{P}} = \mathcal{P} \otimes \pi_M^* \mathcal{E}$ . It is an untwisted sheaf on  $X \times M$ , and any sheaf obtained in this way will be called a *quasi-universal sheaf*. It has the property that

$$\tilde{\mathcal{P}}|_{X \times [\mathcal{F}]} \cong \mathcal{F}^{\oplus n}$$

for some  $n$  that only depends on  $\tilde{\mathcal{P}}$ . (Here,  $\mathcal{F}$  is a stable sheaf on  $X$  with Mukai vector  $v$ , and  $[\mathcal{F}]$  is the point of  $M$  that corresponds to it.) In fact,  $\tilde{\mathcal{P}}$  satisfies a certain universal property which is very similar to that enjoyed by a universal sheaf; see [13, Appendix 2]. Mukai uses a quasi-universal sheaf to define the correspondence between the cohomology of  $X$  and  $M$ , however, since it seems more natural, we'll avoid using this quasi-universal sheaf and use the twisted universal sheaf  $\mathcal{P}$  instead.

3.7. The dual  $\mathcal{Q}$  of  $\mathcal{P}$  is defined by

$$\mathcal{Q} = \mathbf{R}\underline{\mathrm{Hom}}_{X \times M}(\mathcal{P}, \mathcal{O}_{X \times M}),$$

as an element of  $\mathbf{D}_{\mathrm{coh}}^b(X \times M, \pi_M^* \alpha)$ . Using it, we can define the correspondence

$$\varphi_{\mathcal{E}} = \varphi_{X \rightarrow M}^{\mathcal{Q}, \mathcal{E}} : \tilde{H}(X, \mathbf{Q}) \rightarrow \tilde{H}(M, \mathbf{Q}),$$

given by the formula

$$\varphi_{\mathcal{E}}(\cdot) = \pi_{M,*}(\pi_X^*(\cdot) \cdot v_{\pi_M^* \mathcal{E}}(\mathcal{Q})),$$

where

$$v_{\pi_M^* \mathcal{E}}(\mathcal{Q}) = \mathrm{ch}_{\pi_M^* \mathcal{E}}(\mathcal{Q}) \cdot \sqrt{\mathrm{td}(X \times M)},$$

and  $\mathrm{ch}_{\pi_M^* \mathcal{E}}(\mathcal{Q})$  is the Chern character of the twisted sheaf  $\mathcal{Q}$ , defined in (2.9), computed using the  $\alpha^{-1}$ -twisted locally free sheaf  $\mathcal{E}$  on  $M$ .

The reason one uses  $\mathcal{Q}$  is the fact that if  $\Phi_{M \rightarrow X}^{\mathcal{P}}$  is the integral transform associated to  $\mathcal{P}$  (which is an equivalence), then  $\Phi_{X \rightarrow M}^{\mathcal{Q}}$  is its inverse. Also, from Grothendieck-Riemann-Roch it follows that

$$v_{\mathcal{E}}(\Phi_{X \rightarrow M}^{\mathcal{Q}}(\mathcal{F})) = \varphi_{X \rightarrow M}^{\mathcal{Q}, \mathcal{E}}(v(\mathcal{F})).$$

3.8. This is the core of the following result:

**Theorem 3.2** ([13, Theorem 1.5]). *Under the hypotheses of Theorem 3.1, the map  $\varphi$  is a Hodge isometry between  $\tilde{H}(X, \mathbf{Q})$  and  $\tilde{H}(M, \mathbf{Q})$ . It maps  $v \in \tilde{H}(X, \mathbf{Q})$  to the vector  $(0, 0, \omega) \in \tilde{H}(M, \mathbf{Q})$ , and it therefore induces a Hodge isometry*

$$v^\perp / v \cong H^2(M, \mathbf{Q}),$$

*the former computed inside  $\tilde{H}(X, \mathbf{Q})$ . Restricted to  $v^\perp$ , this isometry is independent of the choice of  $\mathcal{E}$ , and is integral, i.e., it takes integral vectors to integral vectors. It therefore induces a Hodge isometry*

$$H^2(M, \mathbf{Z}) \cong v^\perp / v,$$

*the latter now computed in  $\tilde{H}(X, \mathbf{Z})$ .*

*Remark 3.3.* Using the Torelli theorem, this gives a complete description of the moduli space  $M$ .

**Theorem 3.4.** *Let  $n$  be the greatest common divisor of the numbers  $(u.v)$ , where  $u$  runs over all  $\tilde{H}^{1,1}(X) \cap \tilde{H}(X, \mathbf{Z})$  ( $v$  is the Mukai vector referred to in Theorem 3.1). Then the following statements hold:*

1. *There exist  $\alpha$ -twisted locally free sheaves on  $M$  of ranks  $r_1, r_2, \dots, r_k$ , with  $\mathrm{gcd}(r_1, r_2, \dots, r_k) = n$ , and therefore  $\alpha$  is  $n$ -torsion.*
2. *Any map  $\varphi_{\mathcal{E}} = \varphi_{X \rightarrow M}^{\mathcal{Q}, \mathcal{E}}$  maps  $T_X$  into  $T_M$  (viewing  $T_X$  as a sublattice of  $\tilde{H}(X, \mathbf{Z})$  via the inclusion  $\lambda \mapsto (0, \lambda, 0)$ ), and the restriction  $\varphi_{\mathcal{E}}|_{T_X}$  is independent of the choice of  $\mathcal{E}$ .*

3. There exists  $\lambda \in T_X$  such that  $v - \lambda$  is divisible by  $n$  (in  $\tilde{H}(X, \mathbf{Z})$ ); for such a  $\lambda$  we have  $\varphi(\lambda)$  divisible by  $n$  in  $T_M$ .
4.  $\varphi|_{T_X}$  is injective, and its cokernel is a finite, cyclic group of order  $n$ , generated by  $\varphi(\lambda)/n$  for any  $\lambda$  satisfying the condition in 3.

*Proof.* For the first statement, see [13, Remark A.7]. The other statements are [13, 6.4].  $\square$

3.9. The calculation in (2.9) shows that for any  $u \in \tilde{H}(X, \mathbf{Q})$ , changing  $\mathcal{E}$  will only change  $\varphi_{\mathcal{E}}(u)$  by an algebraic amount, and therefore  $[\varphi_{\mathcal{E}}(u)]$  (as defined in (1.11)) is independent of the choice of  $\mathcal{E}$ .

#### 4. TWISTED DEFORMATIONS OF VECTOR BUNDLES

In this section we study what happens when we try to deform a vector bundle from the central fiber of a family, if the first Chern class of the vector bundle fails to deform to neighboring fibers. We show that if such a deformation exists as a twisted sheaf, then there is a simple formula for what the twisting needs to be. This will be used in the proof of Theorem 1.1 to identify the obstruction to the existence of a universal sheaf by a deformation argument, but the results in this section may be of independent interest.

4.1. Let's first set up the context. We start with  $f : X \rightarrow S$ , a proper, smooth morphism of analytic spaces, and with  $0$  a closed point of  $S$ . The Brauer group we consider is  $\mathrm{Br}'_{\mathrm{an}}(X)_{\mathrm{tors}}$ , which is the natural generalization to the analytic setting of the étale Brauer group used in the algebraic case. Throughout this section we'll be loose in our notation and refer to  $\mathrm{Br}'_{\mathrm{an}}(X)_{\mathrm{tors}}$  as the Brauer group of  $X$ , or  $\mathrm{Br}'(X)$ .

Let  $X_0$  be the fiber of  $f$  over  $0$ . We consider an element  $\alpha \in \mathrm{Br}'(X)$ , such that  $\alpha|_{X_0}$  is trivial as an element of  $\mathrm{Br}'(X_0)$ , and we assume we are given a locally free  $\alpha$ -twisted sheaf  $\mathcal{E}$  on  $X$ . Since  $\alpha|_{X_0} = 0$ , we can modify the transition functions of  $\mathcal{E}|_{X_0}$  by a coboundary in such a way that we get an untwisted locally free sheaf  $\mathcal{E}_0$  on  $X_0$ . We want to understand what happens to  $c_1(\mathcal{E}_0)$  in the neighboring fibers. The actual value of  $c_1(\mathcal{E}_0)$  depends on the choice of coboundary used to trivialize  $\alpha|_{X_0}$  ( $\mathcal{E}_0$  could change by the twist by a line bundle), but the image of  $c_1(\mathcal{E}_0)/\mathrm{rk}(\mathcal{E}_0)$  in  $H^2(X_0, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$  is independent of these choices.

Since the morphism  $f$  is smooth, by possibly restricting first the base  $S$  to a smaller, simply connected one, the restriction homomorphisms provide identifications

$$H^i(X, \mathbf{Z}) \cong H^i(X_t, \mathbf{Z})$$

for all  $i \geq 0$  and all  $t \in S$ .

For any space  $X$  we have

$$H^2(X, \mathbf{Q}/\mathbf{Z}) = H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z} \oplus H^3(X, \mathbf{Z})_{\mathrm{tors}},$$

from the universal coefficient theorem. We saw (2.4) that  $\mathrm{Br}'(X)$  is the quotient of  $H^2(X, \mathbf{Q}/\mathbf{Z})$  by the image of  $\mathrm{Pic}(X) \otimes \mathbf{Q}/\mathbf{Z}$ . But classes in  $H^3(X, \mathbf{Z})_{\mathrm{tors}}$  cannot become zero in this quotient, so (in view of the fact that cohomology groups of the

fibers are locally constant over the base) the only way an element of  $\text{Br}'(X)$  can become zero in a central fiber  $X_0$  without being zero in the neighboring fibers is if it belongs to  $H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$ , and in the central fiber it is also in the image of  $\text{Pic}(X_0) \otimes \mathbf{Q}/\mathbf{Z}$ .

4.2. For an element  $w \in H^2(X, \mathbf{Q})$ , we'll denote by  $[w]$  the image of  $w$  in

$$(H^2(X, \mathbf{Z})/\text{NS}(X)) \otimes \mathbf{Q}/\mathbf{Z} \subseteq \text{Br}'(X).$$

Because of the considerations in (4.1), we can write  $\alpha$  as  $[c/n]$ , for some class  $c \in H^2(X, \mathbf{Z})$  and  $n \in \mathbf{Z}$ . The fact that  $\alpha|_{X_0} = 0$  means that  $c|_{X_0}$  belongs to  $\text{Pic}(X_0)$ . Our goal is to identify  $c|_{X_0}$  and  $n$ , in terms of the locally free sheaf  $\mathcal{E}_0$ . This is the content of the following theorem:

**Theorem 4.1.** *Let  $\mathcal{E}$  be a locally free  $\alpha$ -twisted sheaf on  $X$ , and let  $\mathcal{E}_0 = \mathcal{E}|_{X_0}$ . Assume that  $S$  is small enough (say, contractible), so that we have an identification  $H^i(X, \mathbf{Z}) \cong H^i(X_t, \mathbf{Z})$  for all  $i$  and all  $t \in S$ . We assume that  $\alpha|_{X_0} = 0$ , and therefore we can modify the transition functions of  $\mathcal{E}_0$  so that it is an usual sheaf on  $X_0$ . Then we have*

$$\alpha = \left[ -\frac{1}{\text{rk}(\mathcal{E}_0)} c_1(\mathcal{E}_0) \right].$$

The interpretation of this theorem is the following: if we try to deform a vector bundle  $\mathcal{E}_0$ , given on the central fiber  $X_0$ , in a family in which the class  $c_1(\mathcal{E}_0)$  is not algebraic in neighboring fibers  $X_t$ , the only hope to be able to do this is to deform  $\mathcal{E}_0$  as a twisted sheaf, and then the twisting should be precisely

$$\left[ -\frac{1}{\text{rk}(\mathcal{E}_0)} c_1(\mathcal{E}_0) \right].$$

4.3. The idea of the proof is quite straightforward: given a locally free sheaf  $\mathcal{E}$  (twisted or not) on a space  $X$ , we consider its associated projective bundle. For any projective bundle we define a topological invariant (the *topological twisting characteristic*), which is an element of  $H^2(X, \mathbf{Z}/n\mathbf{Z})$  (where  $n = \text{rk}(\mathcal{E})$ ). This topological invariant behaves well with respect to restriction, and it is related to  $c_1(\mathcal{E})/\text{rk}(\mathcal{E})$  when  $\mathcal{E}$  is not twisted, and to  $\alpha$  when  $\mathcal{E}$  is  $\alpha$ -twisted. This enables us to compare  $\alpha$  and  $c_1(\mathcal{E}_0)/\text{rk}(\mathcal{E}_0)$  in the situation we are interested in.

4.4. Consider the two short exact sequences:

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \text{GL}(n) \rightarrow \text{PGL}(n) \rightarrow 0$$

and

$$0 \rightarrow \mathbf{Z}/n\mathbf{Z} \rightarrow \text{SL}(n) \rightarrow \text{PGL}(n) \rightarrow 0.$$

For an element  $p$  of  $H^1(X, \text{PGL}(n))$ , let  $a(p)$  and  $t(p)$  be the images of  $p$  under the two coboundary maps

$$H^1(X, \text{PGL}(n)) \rightarrow H^2(X, \mathcal{O}_X^*)$$

and

$$H^1(X, \mathrm{PGL}(n)) \rightarrow H^2(X, \mathbf{Z}/n\mathbf{Z}),$$

respectively. We'll call  $a(p)$  the *analytic twisting class* of  $p$  and  $t(p)$  the *topological twisting class* of  $p$ . The first one belongs to  $\mathrm{Br}'(X)$ , and as such depends on the complex structure of  $X$ , while the second one is in  $H^2(X, \mathbf{Z}/n\mathbf{Z})$  and depends only on the topology of  $X$ .

If  $Y \rightarrow X$  is a  $\mathbf{P}^{n-1}$ -bundle over  $X$ , then the analytic and topological twisting classes of  $Y/X$ ,  $t(Y/X)$  and  $a(Y/X)$ , are the classes of the element of  $H^1(X, \mathrm{PGL}(n))$  associated to the bundle  $Y \rightarrow X$ .

Note that if  $\mathcal{E}$  is an  $\alpha$ -twisted locally free sheaf on  $X$ , we can consider its associated projective bundle  $Y \rightarrow X$  (which makes sense even in the twisted case), and then its analytic twisting class satisfies

$$a(Y/X) = \alpha.$$

4.5. The following proposition computes  $t(Y/X)$  when  $Y/X$  is the projectivization of a locally free (untwisted) sheaf of rank  $n$  on  $X$ :

**Proposition 4.2.** *Let  $X$  be a scheme or analytic space,  $\mathcal{E}$  a rank  $n$  locally free sheaf on  $X$ , and let  $Y = \mathbf{Proj}(\mathcal{E}) \rightarrow X$  be the associated projective bundle. Then*

$$t(Y/X) = -c_1(\mathcal{E}) \bmod n.$$

(Here, and in the sequel, by reducing mod  $n$  we mean applying the natural map  $H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathbf{Z}/n\mathbf{Z})$ ).

*Proof.* Consider the commutative diagram with exact rows and diagonals

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \searrow & & & & \nearrow \\
 0 & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \mathrm{SL}(n) & \longrightarrow & \mathrm{PGL}(n) \longrightarrow 0 \\
 & & \parallel & & \searrow & & \nearrow \\
 & & \mathbf{Z}/n\mathbf{Z} & & & & \mathrm{GL}(n) \\
 & & \parallel & & \nearrow & \searrow & \det \\
 0 & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \mathcal{O}_X^* & \xrightarrow{\cdot n} & \mathcal{O}_X^* \longrightarrow 0 \\
 & & \parallel & & \nearrow & & \searrow \\
 & & 0 & & & & 0.
 \end{array}$$

By an easy exercise in homological algebra we get the anti-commutative diagram

$$\begin{array}{ccc}
 H^1(X, \mathrm{PGL}(n)) & \xrightarrow{t_1} & H^2(X, \mathbf{Z}/n\mathbf{Z}) \\
 \nearrow & & \parallel \\
 H^1(X, \mathrm{GL}(n)) & & \\
 \searrow \det & & \\
 H^1(X, \mathcal{O}_X^*) & \xrightarrow{c_1 \bmod n} & H^2(X, \mathbf{Z}/n\mathbf{Z}),
 \end{array}$$

which is precisely what we need.  $\square$

**Proposition 4.3.** *For any integer  $n$ , the following diagram commutes*

$$\begin{array}{ccc} H^2(X, \mathbf{Z}) & \xrightarrow{\text{mod } n} & H^2(X, \mathbf{Z}/n\mathbf{Z}) \\ & \searrow & \swarrow p \\ x \mapsto \left[\frac{1}{n}x\right] & & \text{Br}'(X), \end{array}$$

where the map  $p : H^2(X, \mathbf{Z}/n\mathbf{Z}) \rightarrow \text{Br}'(X)$  is obtained from the natural inclusion  $\mathbf{Z}/n\mathbf{Z} \hookrightarrow \mathcal{O}_X^*$ , and the map  $H^2(X, \mathbf{Z}) \rightarrow \text{Br}'(X)$  is taking  $x \in H^2(X, \mathbf{Z})$  to  $\frac{1}{n}x \in H^2(X, \mathbf{Z}) \otimes \mathbf{Q}/\mathbf{Z}$ , which then maps to  $\text{Br}'(X)$ .

Furthermore, if  $Y \rightarrow X$  is a  $\mathbf{P}^{n-1}$ -bundle over  $X$ , we have

$$p(t(Y/X)) = a(Y/X).$$

*Proof.* Trivial chase through the definitions. The last statement follows from the commutativity of the diagram

$$\begin{array}{ccc} H^1(X, \text{PGL}(n)) & \xrightarrow{t} & H^2(X, \mathbf{Z}/n\mathbf{Z}) \\ \parallel & & \downarrow p \\ H^1(X, \text{PGL}(n)) & \xrightarrow{a} & \text{Br}'(X), \end{array}$$

which is deduced from the map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \text{SL}(n) & \longrightarrow & \text{PGL}(n) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & \text{GL}(n) & \longrightarrow & \text{PGL}(n) \longrightarrow 0. \end{array}$$

$\square$

*Proof of Theorem 4.1.* Let  $n = \text{rk}(\mathcal{E})$ , and consider the projective bundle associated to  $\mathcal{E}$ ,  $Y = \mathbf{Proj}(\mathcal{E}) \rightarrow X$ . Using the naturality of the topological twisting class we get

$$\alpha = a(Y/X) = p(t(Y/X)) = p(t(Y_0/X_0)) = p(-c_1(\mathcal{E}_0) \text{ mod } n) = \left[ -\frac{1}{n}c_1(\mathcal{E}_0) \right],$$

where the last three equalities are to be understood via the identification

$$H^2(X, \mathbf{Z}/n\mathbf{Z}) \cong H^2(X_0, \mathbf{Z}/n\mathbf{Z}),$$

in other words  $t(Y_0/X_0)$  and  $c_1(\mathcal{E}_0)$  are considered as classes in  $H^2(X, \mathbf{Z}/n\mathbf{Z})$  and  $H^2(X, \mathbf{Z})$ , respectively.  $\square$

## 5. THE PROOF OF THE MAIN THEOREMS

5.1. We consider again the setup of Theorem 3.1:  $X$  is a K3 surface,  $v$  is a primitive, isotropic Mukai vector on  $X$ , and  $M$  is the moduli space of stable sheaves on  $X$  whose Mukai vector is  $v$ . We assume that  $M$  is computed with respect to a polarization of  $X$  such that  $M$  is compact and non-empty, so that  $M$  is again a K3. The integer  $n$  is the one defined in (3.4), and  $\varphi$  is any of the correspondences defined in (3.7).

**Theorem 1.1.** *Let  $u \in \tilde{H}(X, \mathbf{Z})$  be such that  $(u.v) = 1 \pmod{n}$ . Then  $[\varphi(u)] \in \text{Br}(M)$  is the obstruction to the existence of a universal sheaf on  $X \times M$ , as defined in (3.6).*

*Remark 5.1.* Note that since  $\tilde{H}(X, \mathbf{Z})$  is unimodular, a vector  $u$  with  $(u.v) = 1 \pmod{n}$  can always be found.

*Proof.* First, assume that the moduli problem is fine, so that  $\mathcal{P}$  and  $\mathcal{Q}$  are untwisted and  $n = 1$ . We can therefore consider the correspondence

$$\varphi = \varphi_{X \rightarrow M}^{\mathcal{Q}, \mathcal{E}_M} : \tilde{H}(X, \mathbf{Z}) \rightarrow \tilde{H}(M, \mathbf{Z}).$$

Since  $\varphi$  is an isometry, and it maps  $v$  to  $(0, 0, \omega)$ , it follows that the degree 0 part of  $\varphi(u)$  is precisely  $(u.v)$ . Using (2.9) and the projection formula, we get that for any locally free sheaf  $\mathcal{E}$  on  $M$  we have

$$\varphi_{X \rightarrow M}^{\mathcal{Q}, \mathcal{E}}(u) = \varphi(u) \cdot \frac{\text{ch}(\mathcal{E})}{\text{rk}(\mathcal{E})},$$

and therefore the  $H^2(M, \mathbf{Q})$ -component of  $\varphi_{X \rightarrow M}^{\mathcal{Q}, \mathcal{E}}(u)$  satisfies

$$\begin{aligned} \varphi_{X \rightarrow M}^{\mathcal{Q}, \mathcal{E}}(u)_2 &= \varphi(u)_0 \frac{c_1(\mathcal{E})}{\text{rk}(\mathcal{E})} + \varphi(u)_2 \\ &= (u.v) \frac{c_1(\mathcal{E})}{\text{rk}(\mathcal{E})} + \text{integral part}. \end{aligned}$$

Let's move on now to the case when the moduli problem is not fine. We want to proceed by deforming  $X$  until the problem becomes fine, and this can be done by the argument in [13, pp. 385-386]. More precisely, we can find a smooth family  $\mathcal{X} \rightarrow T_0$  over a small analytic disk  $T_0$ , with the following properties:

1. There is a distinguished point  $1 \in T_0$  such that  $\mathcal{X}_1$  is isomorphic to  $X$ ; we'll identify  $\mathcal{X}_1$  with  $X$  from here on.
2. The restriction homomorphisms  $H^i(\mathcal{X}, \mathbf{Z}) \rightarrow H^i(\mathcal{X}_t, \mathbf{Z})$  are isomorphisms for all  $t \in T_0$ , and so the cohomology groups of all the fibers are naturally identified.
3. The Mukai vector  $v$  from  $\mathcal{X}_1$  is algebraic in the Mukai lattice of each fiber  $\mathcal{X}_t$ .
4. The polarization of  $\mathcal{X}_1$  is algebraic and ample in each fiber  $\mathcal{X}_t$ , and therefore all the fibers are naturally polarized.
5. For each fiber  $\mathcal{X}_t$ , the moduli space  $M(\mathcal{X}_t, v)$  is compact and non-empty when computed with respect to this natural polarization, so it is a K3 surface. The family of relative moduli spaces,  $\mathcal{M} \rightarrow T_0$ , is smooth over  $T_0$ .

6. There is a distinguished point  $0 \in T_0$  such that  $M(\mathcal{X}_0, v)$  is a *fine* moduli space of sheaves on  $\mathcal{X}_0$ .
7. There exists a twisting  $\alpha$  on  $\mathcal{M}$ , and a  $\pi_{\mathcal{M}}^* \alpha^{-1}$ -twisted sheaf  $\mathcal{P}$  on  $\mathcal{X} \times_{T_0} \mathcal{M}$ , which is flat over  $\mathcal{M}$ , and which restricts to a twisted universal sheaf on  $\mathcal{X}_t \times \mathcal{M}_t$  for each  $t \in T_0$ .

On  $\mathcal{M}$  there exists an  $\alpha^{-1}$ -twisted locally free sheaf  $\mathcal{E}$ : one can take, for example,

$$\mathcal{E} = \pi_{\mathcal{M},*}(\pi_{\mathcal{X}}^* \mathcal{O}_{\mathcal{X}}(n) \otimes \mathcal{P}),$$

for a sufficiently high multiple  $\mathcal{O}_{\mathcal{X}}(n)$  of a relative polarization of  $\mathcal{X}/T_0$  (use the flatness of  $\mathcal{P}$  over  $\mathcal{M}$ ). Using  $\mathcal{E}$ , we can define a global correspondence

$$\varphi^{\mathcal{E}, \mathcal{E}} : \tilde{H}(\mathcal{X}, \mathbf{Q}) \rightarrow \tilde{H}(\mathcal{M}, \mathbf{Q}),$$

which restricts to the correspondence

$$\varphi_{\mathcal{X}_t \rightarrow \mathcal{M}_t}^{\mathcal{E}_t, \mathcal{E}_t} : \tilde{H}(\mathcal{X}_t, \mathbf{Q}) \rightarrow \tilde{H}(\mathcal{M}_t, \mathbf{Q})$$

for each  $t \in T_0$ . Note that since the groups in question are discrete, these correspondences are necessarily constant as  $t$  varies in  $T_0$ .

We are now in the situation of Theorem 4.1:  $\mathcal{E}$  is a locally free  $\alpha^{-1}$ -twisted sheaf on  $\mathcal{M}$ , and  $\alpha|_{\mathcal{M}_0} = 0$  because at  $t = 0$  the moduli problem is fine. Therefore, under the corresponding identifications,  $\alpha|_{\mathcal{M}_1}$  is the image of  $c_1(\mathcal{E}_0)/\text{rk}(\mathcal{E}_0)$  in  $\text{Br}(\mathcal{M}_1)$ , where  $\mathcal{E}_0$  is a gluing of  $\mathcal{E}|_{\mathcal{M}_0}$  to a locally free untwisted sheaf. By the calculation in the beginning of the proof, we have

$$\varphi_{\mathcal{X}_0 \rightarrow \mathcal{M}_0}^{\mathcal{E}_0, \mathcal{E}_0}(u)_2 = (u.v) \frac{c_1(\mathcal{E}_0)}{\text{rk}(\mathcal{E}_0)} + \text{integral part}.$$

But since the correspondences  $\varphi_{\mathcal{X}_t \rightarrow \mathcal{M}_t}^{\mathcal{E}_t, \mathcal{E}_t}$  are constant as a function of  $t$ , we also have

$$\varphi_{\mathcal{X}_1 \rightarrow \mathcal{M}_1}^{\mathcal{E}_1, \mathcal{E}_1}(u)_2 = (u.v) \frac{c_1(\mathcal{E}_0)}{\text{rk}(\mathcal{E}_0)} + \text{integral part}.$$

Mapping to  $\text{Br}(\mathcal{M}_1)$ , we get

$$[\varphi(u)] = [\varphi_{\mathcal{X}_1 \rightarrow \mathcal{M}_1}^{\mathcal{E}_1, \mathcal{E}_1}(u)_2] = \left[ (u.v) \frac{c_1(\mathcal{E}_0)}{\text{rk}(\mathcal{E}_0)} + \text{integral part} \right] = (u.v)\alpha,$$

and since  $\alpha$  is  $n$ -torsion, the assumption that  $(u.v) = 1 \pmod{n}$  implies that

$$[\varphi(u)] = \alpha,$$

which is what we wanted.

To finish the proof, we only need to prove that the choices that we have made in the above proof do not matter: the actual proof shows that the choice of  $u$  is irrelevant, and the fact that the choice of  $\mathcal{E}$  is irrelevant (for example, one may choose an  $\mathcal{E}$  on  $M$  that does not extend to the full family  $\mathcal{M}$ ) is (3.9).  $\square$

5.2. Now let's move on to the proof of Theorem 1.2. To prove that  $\alpha = [\varphi(u)]$  is in  $\text{Ker } \varphi^\vee$ , using (1.11) we need to show that

$$(\varphi(u).\varphi(t)) \in \mathbf{Z} \text{ for all } t \in T_X.$$

But since  $\varphi$  is an isometry, the above is equivalent to

$$(u.t) \in \mathbf{Z} \text{ for all } t \in T_X,$$

which is obvious.

Let  $\lambda \in T_X$  be such that  $v - \lambda$  is divisible by  $n$  in  $\tilde{H}(X, \mathbf{Z})$  (Theorem 3.4). Then  $(u.\lambda) = (u.v) = 1 \pmod n$ , and hence  $(\varphi(u).\varphi(\lambda)) = 1 \pmod n$ . But  $\varphi(\lambda)$  is divisible by  $n$  in  $T_M$ ,  $\varphi(\lambda) = n\lambda'$ , so we conclude that

$$(\varphi(u).\lambda') = \frac{1}{n} + \text{integer}.$$

This implies that  $\alpha = [\varphi(u)] \in \text{Br}(M)$  has order at least  $n$ . But by Theorem 3.4,  $\alpha$  is  $n$ -torsion, and the kernel of  $\varphi^\vee$  is cyclic of order  $n$ , so we conclude that  $\alpha$  generates  $\text{Ker } \varphi^\vee$  which is the first part of Theorem 1.2.

This implies at once the equality  $\text{Ker } \alpha = \varphi(T_X)$  (and not just  $\varphi(T_X) \subseteq \text{Ker } \alpha$ ). Since  $\varphi$  is a Hodge isometry  $\tilde{H}(X, \mathbf{Q}) \rightarrow \tilde{H}(M, \mathbf{Q})$ , it follows that  $\varphi$  restricts to a Hodge isometry  $T_X \cong \varphi(T_X) = \text{Ker } \alpha$ , which is the second statement of Theorem 1.2.

Finally, let  $\mathcal{P}$  be a  $\pi_M^* \alpha^{-1}$ -twisted universal sheaf on  $X \times M$ . To show that

$$\Phi_{M \rightarrow X}^{\mathcal{P}} : \mathbf{D}_{\text{coh}}^b(M, \alpha) \rightarrow \mathbf{D}_{\text{coh}}^b(X)$$

is an equivalence of categories, we need to verify the conditions of Theorem 2.1. For  $m \in M$ , let  $\mathcal{P}_m$  be the stable sheaf on  $X$  that corresponds to  $m$ . The condition

$$\text{Hom}_X(\mathcal{P}_m, \mathcal{P}_m) = \mathbf{C},$$

follows from the fact that a stable sheaf is simple. If  $m_1 \neq m_2$ ,  $\mathcal{P}_{m_1} \not\cong \mathcal{P}_{m_2}$ , and they are both stable, so

$$\text{Hom}_X(\mathcal{P}_{m_1}, \mathcal{P}_{m_2}) = 0.$$

By Serre duality

$$\text{Ext}_X^2(\mathcal{P}_{m_1}, \mathcal{P}_{m_2}) = 0,$$

and since

$$\chi(\mathcal{P}_{m_1}, \mathcal{P}_{m_2}) = -(v, v) = 0,$$

it follows that

$$\text{Ext}_X^1(\mathcal{P}_{m_1}, \mathcal{P}_{m_2}) = 0.$$

Since  $\mathcal{P}_m$  is a sheaf on  $X$  for all  $m \in M$  (and not a complex of sheaves), and  $\omega_X = \mathcal{O}_X$ , the remaining conditions of Theorem 2.1 are satisfied, and therefore  $\Phi_{M \rightarrow X}^{\mathcal{P}}$  is an equivalence of categories

$$\mathbf{D}_{\text{coh}}^b(M, \alpha) \cong \mathbf{D}_{\text{coh}}^b(X).$$

5.3. We conclude with a discussion of what the obstacles are to proving Conjecture 1.3. Given  $X, Y$ , and  $\alpha, \beta$  as in the statement of the conjecture, assume that  $\text{Ker } \alpha$  is Hodge isometric to  $\text{Ker } \beta$ . A naïve approach would be to try to find a third K3 surface  $Z$ , with  $T_Z \cong \text{Ker } \alpha \cong \text{Ker } \beta$ , and to try to realize  $X$  and  $Y$  as moduli spaces of stable sheaves on  $Z$ , with obstructions  $\alpha$  and  $\beta$ , respectively. This would yield  $\mathbf{D}_{\text{coh}}^b(Z) \cong \mathbf{D}_{\text{coh}}^b(X, \alpha)$  and  $\mathbf{D}_{\text{coh}}^b(Z) \cong \mathbf{D}_{\text{coh}}^b(Y, \beta)$ , which by transitivity would give the desired equivalence  $\mathbf{D}_{\text{coh}}^b(X, \alpha) \cong \mathbf{D}_{\text{coh}}^b(Y, \beta)$ . However, this approach is soon seen to be too simplistic: while any sublattice  $T$  of  $T_X$  such that  $T_X/T$  is cyclic can occur as  $\text{Ker } \alpha$  for some  $\alpha$ , not all such  $T$  can be primitively embedded in  $\mathcal{L}_{K3}$ . In other words not all  $\text{Ker } \alpha$  can occur as the transcendental lattice  $T_Z$  of some K3  $Z$ . In a certain sense, considering only moduli spaces of *untwisted* sheaves is too restrictive.

The solution to this seems to be the following: one would need to define a notion of stability for twisted sheaves, and consider moduli spaces of stable sheaves with arbitrary twisting, instead of just untwisted ones. In this case, the universal sheaf would be twisted in two directions, one from the space  $X$  where the stable sheaves live, and the other from the moduli space  $M$ , as the obstruction to the existence of a universal sheaf. With the extra flexibility available, one could hope to fully bypass the extra space  $Z$ , and to be able to view  $Y$  as a moduli space of stable  $\alpha$ -twisted sheaves on  $X$ , with  $\beta$  being the obstruction to the existence of a universal sheaf on  $X \times Y$ . This would fully reproduce Orlov's picture from the untwisted situation.

5.4. The first step thus seems to be to define an appropriate notion of stability for twisted sheaves. There is a natural way of doing this: given a space  $X$  with a chosen polarization  $H$ , and an  $\alpha$ -twisted locally free sheaf  $\mathcal{F}$  on  $X$ , we'll say that  $\mathcal{F}$  is  $\mu$ -stable if and only if for every proper  $\alpha$ -twisted subsheaf  $\mathcal{G}$  of  $\mathcal{F}$  we have

$$\mu(\underline{\text{Hom}}(\mathcal{G}, \mathcal{F})) > 0,$$

where the slope of the *untwisted* sheaf  $\underline{\text{Hom}}(\mathcal{G}, \mathcal{F})$  is computed as in [8, 1.2.11]. Note that if  $\mathcal{F}$  and  $\mathcal{G}$  were untwisted, we'd have

$$\mu(\underline{\text{Hom}}(\mathcal{G}, \mathcal{F})) = \mu(\mathcal{F}) - \mu(\mathcal{G}),$$

and thus the above definition would agree with the usual one ([8, 1.2.12]). We can give a similar definition for Gieseker stability of general twisted sheaves.

Another option would be to follow the ideas of Yoshioka [16] for defining stability; his ideas fit naturally in the context of twisted sheaves. To do that, fix an  $\alpha^{-1}$ -twisted locally free sheaf  $\mathcal{E}$ . Then we'll say that an  $\alpha$ -twisted sheaf  $\mathcal{F}$  is  $\mathcal{E}$ - $\mu$ -stable if and only if for every non-trivial subsheaf  $\mathcal{G} \subset \mathcal{F}$  we have

$$\mu_{\mathcal{E}}(\mathcal{G}) < \mu_{\mathcal{E}}(\mathcal{F}),$$

where

$$\mu_{\mathcal{E}}(\mathcal{F}) = \frac{\text{deg}_{\mathcal{E}}(\mathcal{F})}{\text{rk}(\mathcal{F})},$$

$\text{deg}_{\mathcal{E}}(\mathcal{F})$  being defined as  $c_{1, \mathcal{E}}(\mathcal{F}) \cdot H^{d-1}$ , with the Chern class defined by means of  $\mathcal{E}$  as in (2.9). Note that with this definition we have the extra freedom of the choice

of the locally free sheaf  $\mathcal{E}$ . This extra freedom may be what is needed in order to accommodate all possible choices of  $\alpha$  and  $\beta$  in Conjecture 1.3.

As a next step we need to study properties of stable twisted sheaves, construct moduli spaces, and redo the work of Mukai and Orlov in the twisted context. We leave this for a future paper.

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