Math 832 – Spring 2013

Homework 6

Due: Tuesday, April 23rd.

1. Let y be a recurrent state for an irreducible Markov chain. Let

$$N_n(y) = \sum_{m=1}^{n} 1(X_m = y) \quad \text{(Number of visits to } y \text{ in the first } n \text{ steps).}$$

Let $R(0) = 0$ and for $k \geq 1$,

$$R(k) \stackrel{def}{=} \min\{n \geq 1 : N_n(y) = k\} = \text{ time of } k\text{th return to } y$$

and $t_k \stackrel{def}{=} R(k) - R(k-1)$. Let μ be an arbitrary initial distribution. Prove that the random variables $\{t_k\}_{k=2}^{\infty}$ are i.i.d. with respect to P_μ and further that $\mathbb{E}_\mu t_k = \mathbb{E}_y R(1)$ for any $k \geq 2$.

2. Let p be a transition probability function for Markov chain on a countable state space S. We define a new MC, $Z = (X_n, Y_n)$, on $S \times S$ with the following transition function:

$$q((x_1, y_1), (x_2, y_2)) = \begin{cases} p(x_1, x_2)p(y_1, y_2) & \text{if } x_1 \neq y_1 \\ p(x_1, x_2) & \text{if } x_1 = y_1, x_2 = y_2 \\ 0 & \text{else} \end{cases}$$

This means that they move independently until they meet and then they stick together. Show that the projections are also Markov with transition probability p. That is, show that

$$P(X_{m+1} = x | \mathcal{F}_m^X) = p(X_m, x).$$

3. Let φ be a measure-preserving transformation on Ω. Recall that a set $A \in \mathcal{F}$ is invariant if $\varphi^{-1}A = A$ (up to null sets). Finally, we let the set of invariant sets be denoted $\mathcal{I}$. Show that $\mathcal{I}$ is a σ-field. Further, show that X is measurable wrt $\mathcal{I}$, if and only if X is invariant: $X \circ \varphi = X$ a.s.

4. Let $\{X_n, n \in \{0, 1, \dots\}\}$ be a Gaussian process, i.e., the finite dimensional distributions of X_n are multivariate random vectors. Then, if the means $\mathbb{E}X_k$ are constant and the covariances

$$Cov(X_j, X_k)$$

are a function $c(k - j)$ of the difference in their indices, then X_n is a stationary process.