PATTERSON–SULLIVAN THEORY IN HIGHER RANK

ANDREW ZIMMER

Abstract. These notes are based on lectures at the SLMath summer school "Geometry and Dynamics in Higher Rank Lie Groups" at UC Berkeley in Summer 2025. This was the second minicourse at the summer school and hence students had already seen lectures on Anosov groups, which motivates the (lack of) exposition here.

The main aim is to introduce Patterson–Sullivan measures in higher rank. These measures were introduced and studied by Quint for Zariski dense discrete subgroups. In recent years, the theory of such measures for Anosov, relatively Anosov, and transverse groups has been extensively developed and these notes will focus on these recent developments. To keep the Lie theory prerequisites at a minimum, I will only consider the Lie group $\mathsf{SL}(d, \mathbb{R})$.

There are probably many typos, imprecise references, and other mistakes. Any comments are very welcome!

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University of Wisconsin-Madison
E-mail address: amzimmer2@wisc.edu.
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1. Notations and Reminders

We fix the notation we will use in these lectures and also recall some facts from the first minicourse.

1.1. **Notation.** Let $\mathfrak{a}$ be the vector space of diagonal matrices with trace zero. The *Cartan projection* $\kappa : \mathsf{SL}(d,\mathbb{R}) \to \mathfrak{a}$ is the map

$$\kappa(g) = \operatorname{diag}(\log \sigma_1(g), \dots, \log \sigma_d(g))$$

where $\sigma_1(g) \geq \cdots \geq \sigma_d(g)$ are the singular values of g.

We will frequently use the functionals $\alpha_j, \omega_{\alpha_j} : \mathfrak{a} \to \mathbb{R}$ defined by

$$\alpha_j(A) = A_{j,j} - A_{j+1,j+1}$$

and

$$\omega_{\alpha_j}(A) = A_{1,1} + \cdots + A_{j,j}.$$

Often we will just write $\omega_j := \omega_{\alpha_j}$.

Observation 1.1. $\alpha_j(\kappa(g)) = \log \frac{\sigma_j(g)}{\sigma_{j+1}(g)}$ and $\omega_j(\kappa(g)) = \log\Big(\sigma_1(g)\cdots\sigma_j(g)\Big)$.

Let $\Delta := \{\alpha_1, \dots, \alpha_{d-1}\}$ and given a subset $\theta = \{\alpha_{j_1}, \dots, \alpha_{j_m}\} \subset \Delta$ with $j_1 < \cdots < j_m$, let $\mathcal{F}_\theta$ denote the partial flag manifold consisting of flags of the form

$$x^{j_1} \subset \cdots \subset x^{j_m}, \quad \dim x^{j_i} = j_i.$$

Let P_θ be the stabilizer of the standard flag $x_\theta \in \mathcal{F}_\theta$ defined by

$$x_\theta^j := \langle e_1, \dots, e_j \rangle \quad \text{for all } j \in \theta.$$

Also let $\theta^{\text{opp}} := \{\alpha_{d-j} : \alpha_j \in \theta\}$. Then two flags $x \in \mathcal{F}_\theta$ and $y \in \mathcal{F}_{\theta^{\text{opp}}}$ are *transverse* if

$$x^j \oplus y^{d-j} = \mathbb{R}^d$$

whenever $\alpha_j \in \theta$.

Unfortunately, the orthogonal matrices in singular value decompositions are in general not unique, but we do have the following.

Proposition 1.2. *If $g \in \mathsf{SL}(d,\mathbb{R})$, $\alpha_j(\kappa(g)) > 0$, and $g = k_1 e^{\kappa(g)} \ell_1 = k_2 e^{\kappa(g)} \ell_2$ are two singular value decompositions of g, then*

$$k_1 \langle e_1, \dots, e_j \rangle = k_2 \langle e_1, \dots, e_j \rangle .$$

Proof. See Problem 1. □

For each $g \in \mathsf{SL}(d,\mathbb{R})$, **fix** a singular value decomposition

$$g = k_g e^{\kappa(g)} \ell_g.$$

Due to the lack of uniqueness in the definition, the maps $g \mapsto k_g, \ell_g$ are not continuous. Then define

$$U_\theta(g) := k_g x_\theta \in \mathcal{F}_\theta\,.$$

We have the following connection between g and $U_{\theta^{\mathrm{opp}}}(g^{-1})$.

Proposition 1.3. *If $\min_{\alpha_j \in \theta} \alpha_j(\kappa(g)) > 0$, then*

$$U_{\theta^{\mathrm{opp}}}(g^{-1})^{d-j} = \ell_g^{-1} \langle e_{j+1}, \dots, e_d \rangle$$

for all $\alpha_j \in \theta$.

Proof. Note that

$$g^{-1} = \left(\ell_g^{-1} w_0^{-1}\right)\left(w_0 e^{-\kappa(g)} w_0^{-1}\right)\left(w_0 k_g^{-1}\right)$$

is a singular value decomposition if $w_0 \in \mathsf{SL}(d,\mathbb{R})$ satisfies $w_0(e_i) = \pm e_{d-i+1}$. Further, if $\alpha_j \in \theta$, then

$$\alpha_{d-j}(\kappa(g^{-1}) = \log \frac{\sigma_{d-j}(g^{-1})}{\sigma_{d-j+1}(g^{-1})} = \log \frac{\frac{1}{\sigma_{j+1}(g)}}{\frac{1}{\sigma_j(g)}} = \alpha_j(\kappa(g)) > 0.$$

Hence by the uniqueness in Proposition 1.2 we have

$$U_{\theta^{\mathrm{opp}}}(g^{-1})^{d-j} = \ell_g^{-1} w_0 \langle e_1, \dots, e_{d-j} \rangle = \ell_g^{-1} \langle e_d, \dots, e_{j+1} \rangle\,.$$

□

These maps have the following dynamical interpretation.

Proposition 1.4. *Suppose $\{g_n\} \subset \mathsf{SL}(d,\mathbb{R})$, $x^+ \in \mathcal{F}_\theta$, and $x^- \in \mathcal{F}_{\theta^{\mathrm{opp}}}$. Then the following are equivalent:*

1. *$\min_{\alpha\in\theta} \alpha(\kappa(g_n)) \to +\infty$, $U_\theta(g_n) \to x^+$, and $U_{\theta^{\mathrm{opp}}}(g_n^{-1}) \to x^-$.*
2. *$g_n x \to x^+$ for all $x \in \mathcal{F}_\theta$ transverse to x^- and the convergence is uniform on compact subsets of $\{x \in \mathcal{F}_\theta : x \text{ transverse to } x^-\}$.*

Problem 4 asks you to prove the proposition when $\theta = \{\alpha_1\}$. A proof in full generality can be found in [CZZ24, Appendix A].

1.2. **Divergent, Transverse, and (relatively Anosov) groups.** Fix a nonempty subset $\theta \subset \Delta$.

A subgroup $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is P_θ-*divergent* if

$$\min_{\alpha\in\theta} \alpha(\kappa(\gamma_n)) \to +\infty$$

for all sequences $\{\gamma_n\}$ of distinct elements in Γ. In this case, the P_θ-*limit set* is

$$\Lambda_\theta(\Gamma) := \{x \in \mathcal{F}_\theta : \exists \{\gamma_n\} \subset \Gamma \text{ distinct with } U_\theta(\gamma_n) \to x\}.$$

Using Proposition 1.4, one can show that for divergent groups, the limit set compactifies the group as follows.

Proposition 1.5. *If $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is P_θ-divergent, then $\overline{\Gamma}^\theta := \Gamma \sqcup \Lambda_\theta(\Gamma)$ has the topology of a compact metric space where*

1. *The inclusions $\Gamma \hookrightarrow \overline{\Gamma}^\theta$ and $\Lambda_\theta(\Gamma) \hookrightarrow \overline{\Gamma}^\theta$ are topological embeddings (here Γ has the discrete topology)*

(2) *The natural (left)* Γ *action is continuous.*
(3) $\{\gamma_n\} \subset \Gamma$ *converges to* $x \in \Lambda_\theta(\Gamma)$ *if and only if* $\{\gamma_n\}$ *is escaping and* $U_\theta(\gamma_n) \to x$.

Proof. Fix a distance d_θ on $\mathcal{F}_\theta$ which generates the standard topology and let $m_\theta(\gamma) = e^{-\min_{\alpha\in\theta}\alpha(\kappa(\theta))}$. Then define a distance on $\overline{\Gamma}^\theta$ by

- If $\gamma_1, \gamma_2 \in \Gamma$ are distinct, then
$$\mathrm{d}(\gamma_1, \gamma_2) = \max\{m_\theta(\gamma_1), m_\theta(\gamma_2)\} + \mathrm{d}_\theta(U_\theta(\gamma_1), U_\theta(\gamma_2)).$$
- If $\gamma \in \Gamma$ and $F \in \Lambda_\theta(\Gamma)$, then
$$\mathrm{d}(\gamma, F) = m_\theta(\gamma) + \mathrm{d}_\theta(U_\theta(\gamma), F).$$
- If $F_1, F_2 \in \Lambda_\theta(\Gamma)$, then
$$\mathrm{d}(F_1, F_2) = \mathrm{d}_\theta(F_1 F_2).$$

Then one has to just check that this defines a distance with the desired properties (see Problem 7). □

When $\theta = \theta^{\mathrm{opp}}$, a discrete subgroup $\Gamma < \mathsf{SL}(d, \mathbb{R})$ is

(1) P_θ*-transverse* if it is P_θ-divergent and distinct flags in $\Lambda_\theta(\Gamma)$ are transverse.
(2) P_θ*-Anosov* if it is P_θ-transverse, word hyperbolic as an abstract group, and there is an equivariant homeomorphism $\partial\Gamma \to \Lambda_\theta(\Gamma)$.
(3) *relatively* P_θ*-Anosov* if it is P_θ-transverse, relatively hyperbolic as an abstract group, and there is an equivariant homeomorphism between $\Lambda_\theta(\Gamma)$ and the associated Bowditch boundary.

For these groups, the group acts as a convergence group on its limit set.

Proposition 1.6. *If* $\Gamma < \mathsf{SL}(d, \mathbb{R})$ *is* P_θ*-transverse, then* Γ *acts on its limit set as a convergence group. Moreover, if* $\theta' \subset \theta$ *is non-empty, then the projection* $\Lambda_\theta(\Gamma) \to \Lambda_{\theta'}(\Gamma)$ *is a homeomorphism.*

Proof. Suppose $\{\gamma_n\}$ are distinct. Passing to a subsequence we can assume that $U_\theta(\gamma_n) \to x$ and $U_\theta(\gamma_n^{-1}) \to y$. Then by Proposition 1.4, $\gamma_n F \to x$ for all $F \in \mathcal{F}_\theta$ transverse to y. Since $\Lambda_\theta(\Gamma)$ is transverse, this means that $\gamma_n F \to x$ for all $F \in \Lambda_\theta(\Gamma) \smallsetminus \{y\}$. Hence Γ acts as a convergence group.

The "Moreover" part is Problem 8.

□

1.3. **Simplifying indices.** To avoid cumbersome notation we often just use numbers as indices, e.g.
$$\mathcal{F}_{1,2} = \mathcal{F}_{\{\alpha_1,\alpha_2\}}.$$
Likewise, we write things like "P_1-Anosov" instead of "$\mathsf{P}_{\{\alpha_1,\alpha_{d-1}\}}$-Anosov."

1.4. **Hyperbolic Geometry.** Recall that $\mathsf{PSL}(2, \mathbb{R})$ can be identified with the orientation preserving isometries of hyperbolic 2-space and $\mathsf{PSL}(2, \mathbb{C})$ can be identified with the orientation preserving isometries of hyperbolic 3-space. The definitions of P_θ-transverse, P_θ-Anosov, and relatively P_θ-Anosov given above extend verbatim to $\mathsf{PSL}(d, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then for a subgroup $\Gamma < \mathsf{PSL}(2, \mathbb{K})$ we have the following equivalences:

- Γ is discrete if and only if Γ is P_1-transverse.

- Γ is convex cocompact in the hyperbolic geometry sense if and only if Γ is P_1-Anosov.
- Γ is geometrically finite in the hyperbolic geometry sense if and only if Γ is relatively P_1-Anosov.

1.5. **The exterior algebra.** We can often reduce to considering α_1 or ω_1 using the representation $\tau_j : \mathsf{SL}(d,\mathbb{R}) \to \mathsf{SL}(\wedge^j \mathbb{R}^d)$ defined by

$$\tau_j(g)(v_1 \wedge \cdots \wedge v_j) = (gv_1) \wedge \cdots \wedge (gv_j).$$

In this subsection we recall some properties of this representation.

We identify $\wedge^j \mathbb{R}^d$ with $\mathbb{R}^{D_j}$, $D_j := \dim \wedge^j \mathbb{R}^d$, using the standard basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_j} : i_1 < \cdots < i_j\}$$

in lexographical order. Then endow $\wedge^j \mathbb{R}^d$ with the standard norm and inner product on $\mathbb{R}^{D_j}$. One can show that τ_j maps $\mathsf{O}(d)$ to $\mathsf{O}(D_j)$, diagonal matrices to diagonal matrices, and upper triangular matrices to upper triangular matrices.

Notice that if $g = ka\ell$ is a singular value decomposition, then

$$\tau_j(g) = (\tau_j(k)Q^{-1})(Q\tau_j(a)Q^{-1})(Q\tau_j(\ell))$$

is a singular value decomposition where Q is some signed permutation matrix which arranges the entries of $\tau_j(a)$ in decreasing order (we may need to allow Q to be signed to have $Q \in \mathsf{SL}(d,\mathbb{R})$). Hence the singular values of $\tau_j(g)$ are

$$\{\sigma_{i_1}(g) \cdots \sigma_{i_j}(g) : i_1 < \cdots < i_j\}.$$

Further

$$\|\tau_j(g)\| = \sigma_1(\tau_j(g)) = \sigma_1(g) \cdots \sigma_j(g) \quad \text{and} \quad \sigma_2(\tau_j(g)) = \sigma_1(g) \cdots \sigma_{j-1}(g)\sigma_{j+1}(g).$$

Hence

$$\alpha_1(\kappa(\tau_j(g))) = \log \frac{\sigma_1(g) \cdots \sigma_j(g)}{\sigma_1(g) \cdots \sigma_{j-1}(g)\sigma_{j+1}(g)} = \log \frac{\sigma_j(g)}{\sigma_{j+1}(g)} = \alpha_j(\kappa(g))$$

and

$$\omega_1(\kappa(\tau_j(g))) = \log \sigma_1(\tau_j(g)) = \log \sigma_1(g) \cdots \sigma_j(g) = \omega_j(\kappa(g)).$$

2. Problems

2.1. **Singular Value Decompositions.**

Problem 1.

(1) If $g \in \mathsf{SL}(d,\mathbb{R})$, $\alpha_j(\kappa(g)) > 0$, and $g = k_1 e^{\kappa(g)} \ell_1 = k_2 e^{\kappa(g)} \ell_2$ are singular value decompositions, then

$$k_1 \langle e_1, \ldots, e_j \rangle = k_2 \langle e_1, \ldots, e_j \rangle .$$

(2) Find a matrix $g \in \mathsf{SL}(d,\mathbb{R})$ where there exist two singular value decompositions $g = k_1 e^{\kappa(g)} \ell_1 = k_2 e^{\kappa(g)} \ell_2$ where

$$k_1 \langle e_1 \rangle \neq k_2 \langle e_1 \rangle .$$

Problem 2. Suppose $\theta \subset \Delta$. Recall that for each $g \in \mathsf{SL}(d,\mathbb{R})$ we fixed a singular value decomposition $g = k_g a_g \ell_g$ then defined $U_\theta(g)$ to be the flag with $U_\theta(g)^j = k_g \langle e_1, \ldots, e_j \rangle$.

(1) If $\alpha_j(\kappa(g)) > 0$ for all $\alpha_j \in \theta$ and $g = ke^{\kappa(g)}\ell$ is **some** singular value decomposition, then

$$U_\theta(g)^j = k\,\langle e_1, \dots, e_j\rangle \quad \text{and} \quad U_{\theta^{\mathrm{opp}}}(g^{-1})^{d-j} = \ell^{-1}\,\langle e_{j+1}, \dots, e_d\rangle$$

for all $\alpha_j \in \theta$.

(2) Let $\mathcal{O} := \{g \in \mathsf{SL}(d,\mathbb{R}) : \alpha_j(\kappa(g)) > 0 \text{ for all } \alpha_j \in \theta\}$. Show that

$$g \mapsto U_\theta(g)$$

is continuous.

Problem 3. ($\star$) Show that there does not exist a continuous map $f : \mathsf{SL}(d,\mathbb{R}) \to \mathsf{SO}(d)$ such that each g has a singular value decomposition of the form

$$g = f(g)e^{\kappa(g)}\ell.$$

2.2. **Dynamics on flag manifolds.**

Problem 4. Prove Proposition 1.4 in the special case where $\theta = \{\alpha_1\}$.

Hint: In this case $\mathcal{F}_\theta = \mathbb{P}(\mathbb{R}^d)$ and $\mathcal{F}_{\theta^{\mathrm{opp}}} = \mathrm{Gr}_{d-1}(\mathbb{R}^d)$. Fix singular value decompositions $g_n = k_n a_n \ell_n$ and reduce to the case where $k_n \to k$, $\ell_n \to \ell$, and $\frac{a_n}{\|a_n\|} \to P$ where P is a projection.

Problem 5. Fix a distance d_θ on $\mathcal{F}_\theta$ induced by a Riemannian metric. Using Proposition 1.4 prove: If $g \in \mathsf{SL}(d,\mathbb{R})$, $\{h_n\} \subset \mathsf{SL}(d,\mathbb{R})$, and $\alpha_j(\kappa(h_n)) \to +\infty$ for all $\alpha_j \in \theta$, then

$$\lim_{n\to\infty} \mathrm{d}_\theta\left(U_\theta(gh_n), gU_\theta(h_n)\right) = 0.$$

Hint: Reduce to the case where $U_\theta(gh_n), U_\theta(h_n)$, $U_{\theta^{\mathrm{opp}}}(h_n^{-1})$, $U_{\theta^{\mathrm{opp}}}(h_n^{-1}g^{-1})$ all converge and study the dynamics of $\{g_n\}$, $\{hg_n\}$ on $\mathcal{F}$.

Problem 6. Prove: If $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is P_θ-divergent, then the limit set $\Lambda_\theta(\Gamma)$ is Γ-invariant.

Hint: Use the previous problem.

Problem 7. Check the details in the proof of Proposition 1.5.

Problem 8. Prove the "moreover" part of Proposition 1.6.

Lecture #1 : Regularity of limit sets

In this lecture we discuss regularity results for limit sets of Anosov groups.

3. (lack of) Regularity in rank one

Given a compact metric space (X, d), the *δ-dimensional Hausdorff measure* of a subset $A \subset X$ is

$$\mathcal{H}^\delta(A) := \lim_{r\searrow 0} \inf\left\{\sum_i \mathrm{diam}(U_i)^\delta : A \subset \bigcup_i U_i,\ \mathrm{diam}(U_i) \le r\right\}$$

and the *Hausdorff dimension* is

$$\mathrm{H.Dim}(A) = \sup\{\delta \ge 0 : \mathcal{H}^\delta(A) < +\infty\} = \inf\{\delta \ge 0 : \mathcal{H}^\delta(A) = +\infty\}.$$

Example 3.1. A compact Riemannian manifold M of dimension n has Hausdorff dimension n and the n-dimensional Hausdorff measure is the same as the Riemannian volume measure, see Problem 24.

In the next several results, we endow $\mathbb{P}(\mathbb{C}^2)$ with a distance induced by a Riemannian metric.

In rank one, limit sets are often fractal like objects and in particular have Hausdorff dimension strictly greater than their topological dimension. This phenomena is perhaps best demonstrated by the following classical result of Bowen.

Theorem 3.2 (Bowen [Bow79])**.** *Suppose $\Gamma < \mathsf{PSL}(2,\mathbb{C})$ is convex cocompact (i.e. P_1-Anosov) and $\Lambda_1(\Gamma)$ is homeomorphic to a circle. Then*

$$\mathrm{H.Dim}(\Lambda_1(\Gamma)) \geq 1$$

with equality if and only if Γ is conjugate into $\mathsf{PSL}(2,\mathbb{R})$.

Remark 3.3. For similar results see [Tuk84, Yue96, BK04, Xie08, Kap09]

Corollary 3.4. *Suppose $\Gamma < \mathsf{PSL}(2,\mathbb{C})$ is convex cocompact (i.e. P_1-Anosov) and $\Lambda_1(\Gamma)$ is homeomorphic to a circle. If Γ is Zariski dense, then*

$$\mathrm{H.Dim}(\Lambda_1(\Gamma)) > 1.$$

Notice that if $\Gamma < \mathsf{PSL}(2,\mathbb{R})$ and $\Lambda_1(\Gamma)$ is homeomorphic to a circle, then $\Lambda_1(\Gamma) = \mathbb{P}(\mathbb{R}^2)$. Hence, in this case, the limit set is $\mathcal{C}^\infty$-smooth. So we have the following dichotomy.

Corollary 3.5. *If $\Gamma < \mathsf{PSL}(2,\mathbb{C})$ is convex cocompact and $\Lambda_1(\Gamma)$ is homeomorphic to a circle, then either*

1. *$\Lambda_1(\Gamma)$ is a "fractal," or*
2. *$\Lambda_1(\Gamma)$ is a $\mathcal{C}^\infty$-submanifold.*

4. Examples of Anosov groups with regular limit sets

Surprisingly higher rank allows more regularity in limit sets and there are many examples of Zariski dense Anosov groups with $\mathcal{C}^1$-smooth limit sets.

4.1. **Hitchin representations.** Suppose Σ is a closed hyperbolic surface. Then we can write $\Sigma = \Gamma\backslash\mathbb{H}^2$ where $\Gamma \simeq \pi_1(\Sigma)$, $\Gamma < \mathsf{PSL}(2,\mathbb{R})$, and $\mathbb{H}^2$ is the hyperbolic plane.

Let $\Phi_d : \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$ be the standard irreducible representation: if we identify $\mathbb{R}^d$ with polynomials $P : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$P(x,y) = \sum_{k=0}^{d-1} c_k x^k y^{d-k},$$

then Φ_d is defined by

$$\Phi_d(g)[P] = [P \circ g^{-1}].$$

Definition 4.1. A representation $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is *Hitchin* if it is in the same connected component of $\Phi_d|_\Gamma$ in $\mathrm{Hom}(\Gamma, \mathsf{PSL}(d,\mathbb{R}))$.

Theorem 4.2 (Labourie [Lab06])**.** *If $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is Hitchin, then $\rho(\Gamma)$ is P_k-Anosov for $k = 1,\dots,d-1$. Moreover, each limit set $\Lambda_k(\rho(\Gamma)) \subset \mathrm{Gr}_k(\mathbb{R}^d)$ is a $\mathcal{C}^{1,\epsilon}$-submanifold.*

Remark 4.3. For details on the moreover part, see [PSW21].

There exist Hitchin representations with Zariski dense image and hence the above theorem provides examples of Zariski dense Anosov groups with $\mathcal{C}^1$-smooth limit set.

4.2. **Convex divisible domains.** A subset $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is called *properly convex* if there exists an affine chart where Ω is a bounded convex open set. Such a set is *divisible* if there exists a discrete subgroup $\Gamma < \mathsf{PSL}(d,\mathbb{R})$ which preserves Ω and acts co-compactly.

Theorem 4.4 (Benoist [Ben04])**.** *If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma < \mathsf{PSL}(d,\mathbb{R})$ is a discrete group which preserves Ω and acts co-compactly, then the following are equivalent:*

1. *Γ is word hyperbolic.*
2. *$\partial\Omega$ is a $\mathcal{C}^1$-submanifold.*
3. *Ω is strictly convex, i.e. if $x, y \in \partial\Omega$ are distinct, then $(x,y) \subset \Omega$.*

Moreover, in this case, Γ is P_1-Anosov and $\Lambda_1(\Gamma) = \partial\Omega$.

Remark 4.5. For details on the moreover part, see [GW12].

There exist groups satisfying Theorem 4.4 with Zariski dense image and hence the above theorem provides examples of Zariski dense Anosov groups with $\mathcal{C}^1$-smooth limit set.

5. General results

In this section we state some general results about P_1-Anosov groups with $\mathcal{C}^1$-smooth limit set. For simplicity, we only consider the case where the Gromov boundary is a circle.

For the rest of the section let

$$\theta := \{\alpha_1, \alpha_2, \alpha_{d-2}, \alpha_{d-1}\}.$$

Definition 5.1.

1. A subgroup $\Gamma < \mathsf{PSL}(d,\mathbb{R})$ is *hyperconvex* if it is P_θ-Anosov and
$$x^1 \oplus y^1 \oplus z^{d-2} = \mathbb{R}^d$$
for all pairwise distinct $x, y, z \in \Lambda_\theta(\Gamma)$.
2. A representation $\rho : \Gamma_0 \to \mathsf{PSL}(d,\mathbb{R})$ is *hyperconvex* if it has hyperconvex image and finite kernel.

Example 5.2 (see [PSW21] or [ZZ24])**.** If $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is Hitchin, then $\wedge^k\rho : \Gamma \to \mathsf{PSL}(\wedge^k \mathbb{R}^d)$ is hyperconvex for $k = 1, \dots, d-1$.

Proposition 5.3. *The set of hyperconvex representations is open in* $\mathrm{Hom}(\Gamma_0, \mathsf{PSL}(d,\mathbb{R}))$.

Proof. See Problem 9. □

Theorem 5.4 (Pozzetti–Sambarino–Wienhard [PSW21])**.** *If $\Gamma < \mathsf{PSL}(d,\mathbb{R})$ is hyperconvex and $\Lambda_\theta(\Gamma) \simeq \partial_\infty\Gamma$ is a circle, then $\Lambda_1(\Gamma)$ is a $\mathcal{C}^1$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ and*

$$T_{x^1}\Lambda_1(\Gamma) = x^2$$

for all $x \in \Lambda_\theta(\Gamma)$.

A similar result is also true when $\partial_\infty\Gamma$ is a higher dimensional sphere, see [PSW21]. Further, hyperconvexity actually implies that the limit set is a $\mathcal{C}^{1,\epsilon}$-submanifold of $\mathbb{P}(\mathbb{R}^d)$, see [ZZ24].

Proof idea. See for Problem 10 for a guided proof. Here we will just sketch the main idea.

Given $x \in \Lambda_\theta(\Gamma)$, the projective limit set $\Lambda_1(\Gamma)$ being $\mathcal{C}^1$-smooth at x^1 with tangent plane x^2 essentially means

$$\lim_{y\to x} \angle(x^1 \oplus y^1, x^2) = 0$$

(where $y \in \Lambda_\theta(\Gamma)$). We will verify this when x is the attracting fixed point of a loxodromic element γ. Changing coordinates we can assume that $x^1 = \langle e_1\rangle$, $x^2 = \langle e_1, e_2\rangle$ and the repelling fixed point z of γ satisfies $z^{d-2} = \langle e_3, \dots, e_d\rangle$, $z^{d-1} = \langle e_2, \dots, e_d\rangle$. Then

$$\gamma = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & A \end{pmatrix}$$

where $|\lambda_1| > |\lambda_2|$ and $|\lambda_2| > \lambda_1(A)$.

Fix $y_n \in \Lambda_\theta(\Gamma) \smallsetminus \{x, z\}$ converging to x. Then there exists $m_n \to \infty$ such that $\{\hat{y}_n := \gamma^{-m_n} y_n\}$ is relatively compact in $\Lambda_\theta(\Gamma) \smallsetminus \{x, z\}$. Since each flag in $\overline{\{\hat{y}_n\}}$ is transverse to z and $z^{d-1} = \langle e_2, \dots, e_d\rangle$, we can write

$$\hat{y}_n^1 = [e_1 + c_n e_2 + v_n]$$

where $v_n \in \langle e_3, \dots, e_d\rangle$ and $\{c_n\}, \{v_n\}$ are bounded sets. Since $x^1 + z^{d-2} = \langle e_1, e_3, \dots, e_d\rangle$ and $x, z \notin \overline{\{\hat{y}_n\}}$, hyperconvexity implies that $\inf |c_n| > \epsilon$ for some $\epsilon > 0$. Then

$$y_n^1 = \gamma^{m_n}\hat{y}_n^1 = \left[e_1 + c_n \frac{\lambda_2^{m_n}}{\lambda_1^{m_n}} e_2 + \frac{A^{m_n}}{\lambda_1^{m_n}} v_n\right].$$

Since $|\lambda_2| > \lambda_1(A)$,

$$\lim_{n\to\infty} \frac{A^{m_n}}{\lambda_2^{m_n}} = 0$$

and hence

$$\begin{aligned} \angle(x^1 \oplus y_n^1, x^2) &= \angle(\langle e_1\rangle \oplus y_n^1, \langle e_1, e_2\rangle) \\ &= \angle\left(\left\langle e_1, e_2 + \frac{A^{m_n}}{c_n\lambda_2^{m_n}} v_n\right\rangle, \langle e_1, e_2\rangle\right) \to 0. \end{aligned}$$ □

For irreducible groups, the following converse is true.

Theorem 5.5 (Zhang–Z. [ZZ24])**.** *If $\Gamma < \mathsf{PSL}(d, \mathbb{R})$ is an irreducible P_1-Anosov group, $\Lambda_1(\Gamma) \simeq \partial_\infty\Gamma$ is a circle, and $\Lambda_1(\Gamma)$ is a $\mathcal{C}^{1,\epsilon}$-submanifold of $\mathbb{P}(\mathbb{R}^d)$, then Γ is hyperconvex.*

6. Problems

6.1. **Hyperconvex groups.**

Problem 9. Suppose Γ is word hyperbolic. Prove that the set of hyperconvex representations is open in $\mathrm{Hom}(\Gamma, \mathsf{PSL}(d, \mathbb{R}))$.

Hint: You need two facts

- *The set of Anosov representations is open and the boundary maps $\partial_\infty\Gamma \to \mathcal{F}_\theta$ depend continuously on the representation.*
- *A word hyperbolic group Γ acts co-compactly on the space of distinct triples $\partial_\infty\Gamma^{(3)}$.*

Problem 10. Suppose $\Gamma < \mathsf{PSL}(d,\mathbb{R})$ is hyperconvex, $\theta = \{\alpha_1, \alpha_2, \alpha_{d-2}, \alpha_{d-1}\}$, and $\Lambda_\theta(\Gamma) \simeq \partial_\infty\Gamma$ is a circle. The goal of this problem is to prove that $\Lambda_1(\Gamma)$ is a $\mathcal{C}^1$-submanifold of $\mathbb{P}(\mathbb{R}^d)$.

Let d_θ, $\mathrm{d}_{\mathbb{P}}$ be a distance of $\mathcal{F}_\theta$, $\mathbb{P}(\mathbb{R}^d)$ respectively induced by a Riemannian metric.

(a) Prove the following lemma:

Lemma 6.1. *For any $\epsilon > 0$ there exists $\delta > 0$ such that: if $x, y \in \Lambda_\theta(\Gamma)$ are distinct and $\mathrm{d}_\theta(x,y) < \delta$, then*

$$\angle(x^1 \oplus y^1, x^2) < \epsilon.$$

Hint: Suppose not. Then for each n there exist distinct $x_n, y_n \in \Lambda_\theta(\Gamma)$ such that

$$\mathrm{d}_\theta(x_n, y_n) < 1/n \quad \text{and} \quad \angle(x_n^1 \oplus y_n^1, x_n^2) > \epsilon.$$

Passing to a subsequence $x_n, y_n \to a$. Fix $b \in \Lambda_\theta(\Gamma) \smallsetminus \{a\}$. For n sufficiently large x_n, y_n, b are pairwise distinct and hence there exists γ_n such that $\gamma_n(x_n, y_n, b)$ is contained in a compact subset of $\Lambda_\theta(\Gamma)^{(3)}$, the space of distinct triples. Without loss of generality, $\gamma_n(x_n, y_n, b) \to (x', y', b')$. Show that $\lim U_\theta(\gamma_n) = b'$ and $\lim U_\theta(\gamma_n^{-1}) = a$. Then try to understand $x_n = \gamma_n^{-1}(\gamma_n x_n)$ and $y_n = \gamma_n^{-1}(\gamma_n y_n)$.

(b) For $x, y \in \Lambda_\theta(\Gamma)$ distinct, let $\pi_{y,x} : \Lambda_1(\Gamma) \smallsetminus \{y^1\} \to \mathbb{P}(x^2)$ be induced by the projection

$$\mathbb{R}^d = y^{d-2} \oplus x^2 \to x^2.$$

Using hyperconvexity, prove that $\pi_{y,x}$ is injective.

(c) Using (a), prove that for any compact subset $K \subset \Lambda_\theta(\Gamma) \smallsetminus \{y\}$ there exists $C = C(x,y,K) > 0$ such that: if $z \in K$, then

$$\mathrm{d}(x^1, z^1) \geq C\, \mathrm{d}(x^1, \pi_{x,y}(z^1)).$$

(d) Prove that $\pi_{x,y}$ is locally biLipschitz (i.e. biLipschitz on any compact subset of $\Lambda_1(\Gamma) \smallsetminus \{y^1\}$).

There exists a neighborhood V of x in $\Lambda_\theta(\Gamma)$ and a continuous map $z \in V \mapsto A_z \in \mathrm{Lin}(z^2, x^2)$ such that $\pi_{x,y} = A_z \pi_{z,y}$. Then use (c).

(e) Prove that $\Lambda_1(\Gamma)$ is a $\mathcal{C}^1$-submanifold of $\mathbb{P}(\mathbb{R}^d)$.

Hint: Use (d) to show that $\Lambda_1(\Gamma)$ has finite length and hence can be parametrized by arc length. Use (a) to show that this parametrization is $\mathcal{C}^1$.

Lecture #2 : Patterson–Sullivan theory in rank one

In this lecture we discuss some classical results in the theory of Patterson–Sullivan theory in real hyperbolic geometry.

7. Motivation

In general the action of a discrete group of $\mathsf{Isom}(\mathbb{H}^d)$ does not preserve a measure on $\partial\,\mathbb{H}^d$.

Proposition 7.1. *If $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is a discrete non-elementary subgroup, then there does not exist an Γ-invariant probability measure on $\partial\,\mathbb{H}^d$.*

Proof. See Problem 11. □

However, there are invariant measure classes.

Example 7.2. Identify $\mathbb{H}^d = \mathbb{B}^d \subset \mathbb{R}^d$ with the Klein–Beltrami model. Then $\partial\,\mathbb{H}^d = \partial\,\mathbb{B}^d$ and $\mathsf{Isom}(\mathbb{H}^d)$ acts by diffeomorphisms on $\partial\,\mathbb{B}^d$. Hence $\mathsf{Isom}(\mathbb{H}^d)$ preserves the Lebesgue measure **class**.

Patterson–Sullivan measures are invariant measure classes associated to any discrete subgroup.

8. The Busemann cocycle and Patterson–Sullivan measures

Fixing a basepoint $o \in \mathbb{H}^n$, the Busemann function based at $x \in \partial\,\mathbb{H}^n$ is defined by

$$b_x(p) = \lim_{q \to x} \mathrm{d}(q,p) - \mathrm{d}(q,o)$$

(one can use formulas for the distance to show that b_x exists).

Then the *Busemann cocycle* $B : \mathsf{Isom}(\mathbb{H}^n) \times \partial\,\mathbb{H}^n \to \mathbb{R}$ is defined by

$$B(g,x) = b_x(g^{-1}o).$$

This satisfies the cocycle identity:

Observation 8.1. $B(g_1g_2,x) = B(g_1,g_2x) + B(g_2,x)$.

Proof. Check! □

Example 8.2. Identify $\mathbb{H}^d = \mathbb{B}^d \subset \mathbb{R}^d$ with the Klein–Beltrami model. Then $\partial\,\mathbb{H}^d = \partial\,\mathbb{B}^d$ and the stabilizer of the origin in $\mathsf{Isom}(\mathbb{H}^d)$ is $\mathsf{O}(d)$. Let ν be the smooth probability measure on $\partial\,\mathbb{B}^d$ which is invariant under $\mathsf{O}(d)$. If $o = 0$ and $B : \mathsf{Isom}(\mathbb{H}^d) \times \partial\,\mathbb{B}^d \to \mathbb{R}$ is the Busemann cocycle, then

$$B(g^{-1},x) = -\frac{1}{d-1} \log \frac{dg_*\nu}{d\nu}(x)$$

for all $g \in \mathsf{Isom}(\mathbb{H}^d)$.

Given a subgroup $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ and $\delta \geq 0$, a Borel probability measure μ on $\partial\,\mathbb{H}^d$ is called a *Patterson–Sullivan measure of dimension δ for* Γ if for any $\gamma \in \Gamma$ the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-\delta B(\gamma^{-1},x)} = e^{-\delta b_x(\gamma o)} \tag{1}$$

for μ-a.e. $x \in \partial\,\mathbb{H}^n$.

Example 8.3. The measure introduced in Example 8.2 is a Patterson–Sullivan measure of dimension $(d-1)$ for $\mathsf{Isom}(\mathbb{H}^d)$.

Given a discrete subgroup $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ we will show that there exists a Patterson–Sullivan measure with dimension equal to the *critical exponent* of the subgroup, which is

$$\delta(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma : \mathrm{d}(o, \gamma o) \le R\}.$$

Remark 8.4. One can show that

$$\delta(\Gamma) \le \limsup_{R\to\infty} \frac{1}{R} \log \mathrm{Vol}_{\mathbb{H}^d}(B(o, R)) = d - 1$$

and hence $\delta(\Gamma)$ is always finite. One can also show that $\delta(\Gamma) > 0$ when Γ is non-elementary.

One often considers the Poincaré series

$$Q(s) := \sum_{\gamma\in\Gamma} e^{-s\,\mathrm{d}(o,\gamma o)}.$$

One can show that $Q(s) = +\infty$ when $s < \delta(\Gamma)$ and $Q(s) < +\infty$ when $s > \delta(\Gamma)$, see Problem 16. The convergence/divergence at $s = \delta(\Gamma)$ is a delicate issue.

Theorem 8.5 (Patterson [Pat76], Sullivan)**.** *If $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is discrete, then there exists a Patterson–Sullivan measure of dimension $\delta(\Gamma)$ for Γ. Moreover, if Γ is non-elementary, then any Patterson–Sullivan measure for Γ has dimension at least $\delta(\Gamma)$.*

Proof. We first prove existence in the special case when

$$Q(\delta(\Gamma)) = \sum_{\gamma\in\Gamma} e^{-\delta(\Gamma)\,\mathrm{d}(o,\gamma o)} = +\infty.$$

For $s > \delta(\Gamma)$, define the probability measure

$$\mu_s := \frac{1}{Q(s)} \sum_{\gamma\in\Gamma} e^{-s\,\mathrm{d}(o,\gamma o)}\, \mathcal{D}_{\gamma o}$$

where $\mathcal{D}_{\gamma o}$ is the Dirac mass at γo.

Since $\mathbb{H}^d \cup \partial\,\mathbb{H}^d$ is compact, we can find $s_n \searrow \delta(\Gamma)$ such that μ_{s_n} converges to a probability measure μ on $\mathbb{H}^d \cup \partial\,\mathbb{H}^d$ in the weak-$*$ topology. Since $Q(s_n) \to +\infty$, the measure μ is supported on $\partial\,\mathbb{H}^d$.

Fix $\alpha \in \Gamma$. Then

$$\begin{aligned}
\alpha_*\mu_s &= \frac{1}{Q(s)} \sum_{\gamma\in\Gamma} e^{-s\,\mathrm{d}(o,\gamma o)} \alpha_*\, \mathcal{D}_{\gamma o} = \frac{1}{Q(s)} \sum_{\gamma\in\Gamma} e^{-s\,\mathrm{d}(o,\gamma o)}\, \mathcal{D}_{\alpha\gamma o} \\
&= \frac{1}{Q(s)} \sum_{\gamma\in\Gamma} e^{-s\,\mathrm{d}(o,\alpha^{-1}\gamma o)}\, \mathcal{D}_{\gamma o} \\
&= \frac{1}{Q(s)} \sum_{\gamma\in\Gamma} e^{-s(\mathrm{d}(\alpha o,\gamma o)-\mathrm{d}(o,\gamma o))} e^{-s\,\mathrm{d}(o,\gamma o)}\, \mathcal{D}_{\gamma o}\,.
\end{aligned}$$

Now define $w : \mathbb{H}^d \cup \partial\,\mathbb{H}^d \to \mathbb{R}$ by

$$w(x) = \begin{cases} \mathrm{d}(\alpha o, x) - \mathrm{d}(o, x) & x \in \mathbb{H}^d \\ b_x(\alpha o) & x \in \partial\,\mathbb{H}^d \end{cases}.$$

Then w is continuous. Further,

$$\int f d\alpha_* \mu_s = \int f e^{-sw} d\mu_s$$

for all continuous $f : \mathbb{H}^d \cup \partial\,\mathbb{H}^d \to \mathbb{R}$. Taking the limit $s_n \searrow \delta(\Gamma)$,

$$\int f d\alpha_* \mu = \int f e^{-\delta(\Gamma) w} d\mu$$

for all continuous $f : \mathbb{H}^d \cup \partial\,\mathbb{H}^d \to \mathbb{R}$. Since μ is supported on $\partial\,\mathbb{H}^d$ and $w(x) = b_x(\alpha o) = B(\alpha^{-1}, x)$ when $x \in \partial\,\mathbb{H}^d$, this implies that

$$\frac{d\alpha_* \mu}{d\mu}(x) = e^{-\delta(\Gamma) B(\alpha^{-1}, x)}.$$

Hence μ is a Patterson–Sullivan measure of dimension $\delta(\Gamma)$.

Now we consider the general case. By [Pat76, Lemma 3.1], there exists a non-decreasing function $\chi \colon \mathbb{R} \to \mathbb{R}_{\geq 1}$ such that

(1) For every $\epsilon > 0$ there exists $R > 0$ such that $\chi(r + t) \leq e^{\epsilon t} \chi(r)$ for any $r \geq R$ and $t \geq 0$,
(2) $\sum_{\gamma \in \Gamma} \chi(\mathrm{d}(o, \gamma o)) e^{-\delta(\Gamma)\, \mathrm{d}(o, \gamma o)} = +\infty$

(when $\sum_{\gamma \in \Gamma} e^{-\delta(\Gamma)\, \mathrm{d}(o, \gamma o)} = +\infty$, we can take $\chi \equiv 1$).

For $s > \delta(\Gamma)$, define a Borel probability measure on $\mathbb{H}^d \cup \partial\,\mathbb{H}^d$ by

$$\mu_s := \frac{1}{Q_\chi(s)} \sum_{\gamma \in \Gamma} \chi(\mathrm{d}(o, \gamma o)) e^{-s\, \mathrm{d}(o, \gamma o)} \mathcal{D}_{\gamma o},$$

where $Q_\chi(s) := \sum_{\gamma \in \Gamma} \chi(\mathrm{d}(o, \gamma o)) e^{-s\, \mathrm{d}(o, \gamma o)}$, which is finite by property (1). Then fix $s_n \searrow \delta(\Gamma)$ such that $\mu_{s_n} \to \mu$ in the weak-$*$ topology. Since $Q_\chi(s_n) \to +\infty$, the measure μ is supported on $\partial\,\mathbb{H}^d$. Arguing as in the special case shows that μ is a Patterson–Sullivan measure of dimension $\delta(\Gamma)$.

The moreover part requires the shadow lemma and will be an exercise, see Problem 17.

□

9. Shadows and the shadow lemma

One of the primary tools for studying Patterson–Sullivan measures is the Shadow Lemma, which provides an estimate for certain geometrically defined sets.

Given $p, q \in \mathbb{H}^d$ distinct and $R > 0$, the associated *R-shadow* is

$$\mathcal{O}_R(p, q) := \left\{ x \in \partial\,\mathbb{H}^n : \begin{array}{c} \text{the geodesic starting at } p \text{ and limiting to } x \\ \text{intersects the ball of radius } R \text{ about } q \end{array} \right\}.$$

Theorem 9.1. *Suppose $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is non-elementary and μ is a Patterson–Sullivan measure of dimension δ for Γ. For any $R > 0$ sufficiently large, there exists $C = C(R) > 0$ such that*

$$C^{-1} e^{-\delta\, \mathrm{d}(o, \gamma o)} \leq \mu(\mathcal{O}_R(o, \gamma o)) \leq C e^{-\delta\, \mathrm{d}(o, \gamma o)}$$

for all $\gamma \in \Gamma$.

The proof requires two lemmas.

Lemma 9.2. *If $x \in \mathcal{O}_R(o,p)$, then*

$$-\mathrm{d}(o,p) \leq b_x(p) \leq 2R - \mathrm{d}(o,p).$$

Hence, if $x \in \mathcal{O}_R(\gamma^{-1}o, o)$, then

$$-2R + \mathrm{d}(o,\gamma o) \leq B(\gamma, x) \leq 2R + \mathrm{d}(o,\gamma o).$$

Proof. Notice that b_x, being a limit of 1-Lipschitz functions, is 1-Lipschitz. Further, $b_x(o) = 0$. Hence

$$-\mathrm{d}(o,p) \leq b_x(p).$$

Fix $q_n \to x$. Then for n sufficiently large, the geodesic joining o to q_n intersects $B_{\mathbb{H}^d}(p,R)$ at some point u_n. Then

$$\begin{aligned} b_x(p) &= \lim_{n\to\infty} \mathrm{d}(q_n,p) - \mathrm{d}(q_n,o) = \lim_{n\to\infty} \mathrm{d}(q_n,p) - \mathrm{d}(q_n,u_n) - \mathrm{d}(u_n,o) \\ &\leq 2R + \lim_{n\to\infty} \mathrm{d}(q_n,p) - \mathrm{d}(q_n,p) - \mathrm{d}(p,o) = 2R - \mathrm{d}(o,p). \end{aligned}$$

For the "hence" part, notice that

$$\begin{aligned} B(\gamma,x) = b_x(\gamma^{-1}o) &= \lim_{q\to x} \mathrm{d}(\gamma^{-1}o,q) - \mathrm{d}(q,o) = \lim_{q\to x} \mathrm{d}(o,\gamma q) - \mathrm{d}(\gamma q,\gamma o) \\ &= -\lim_{q\to \gamma x} \mathrm{d}(q,\gamma o) - \mathrm{d}(o,q) = -b_{\gamma x}(\gamma o). \end{aligned}$$

Further, if $x \in \mathcal{O}_R(\gamma^{-1}o,o)$, then $\gamma x \in \mathcal{O}_R(o,\gamma o)$. □

Lemma 9.3. *With Γ and μ as in the Shadow Lemma, for $R > 0$ sufficiently large,*

$$\inf_{\gamma\in\Gamma} \mu(\mathcal{O}_R(\gamma^{-1}o,o)) > 0.$$

Proof. Suppose not. Then there exist $\{\gamma_n\} \subset \Gamma$ and $R_n \to +\infty$ such that

$$\mu(\mathcal{O}_{R_n}(\gamma_n^{-1}o,o)) < 1/n.$$

Passing to a subsequence we can suppose that $\gamma_n^{-1}o \to x$. If $x \in \mathbb{H}^d$, then

$$\mathcal{O}_{R_n}(\gamma_n^{-1}o,o) = \partial\,\mathbb{H}^d$$

when n is sufficiently large, which implies that $1 = \mu(\partial\,\mathbb{H}^d) \leq \lim_{n\to\infty} 1/n = 0$. Thus we must have $x \in \partial\,\mathbb{H}^d$. If U is a neighborhood of x in $\partial\,\mathbb{H}^n$, then

$$\partial\,\mathbb{H}^d \smallsetminus U \subset \mathcal{O}_{R_n}(\gamma_n^{-1}o,o)$$

for n large. Hence

$$\mu(\partial\,\mathbb{H}^d \smallsetminus U) \leq \lim_{n\to\infty} \mu(\mathcal{O}_{R_n}(\gamma_n^{-1}o,o)) = 0.$$

Since U is an arbitrary neighborhood of x, then $\mu = \mathcal{D}_x$ is a Dirac mass at x. Since Γ is non-elementary, there exists $\gamma \in \Gamma$ with $\gamma x \neq x$. Then $\gamma_*\mu = \mathcal{D}_{\gamma x}$ is singular to $\mu = \mathcal{D}_x$, which is a contradiction. □

Proof of the Shadow Lemma. Fix $R > 0$ so that

$$m := \inf_{\gamma\in\Gamma} \mu(\mathcal{O}_R(\gamma^{-1}o,o))$$

is positive. Then if $\gamma \in \Gamma$, we have

$$\begin{aligned} \mu(\mathcal{O}_R(o,\gamma o)) &= \mu(\gamma\,\mathcal{O}_R(\gamma^{-1}o,o)) = (\gamma^{-1})_*\mu(\mathcal{O}_R(\gamma^{-1}o,o)) \\ &= \int_{\mathcal{O}_R(\gamma^{-1}o,o)} \frac{d(\gamma^{-1})_*\mu}{d\mu} d\mu = \int_{\mathcal{O}_R(\gamma^{-1}o,o)} e^{-\delta B(\gamma,x)} d\mu(x). \end{aligned}$$

Thus by the Lemmas,

$$\mu(\mathcal{O}_R(o,\gamma o)) \leq e^{2\delta R} e^{-\delta\, \mathrm{d}(o,\gamma o)}$$

and

$$\mu(\mathcal{O}_R(o,\gamma o)) \geq m e^{-\delta\, \mathrm{d}(o,\gamma o)}. \qquad \square$$

10. The Ergodic Dichotomy

Let

$$\partial^{(2)}\,\mathbb{H}^d \subset \partial\,\mathbb{H}^d \times \partial\,\mathbb{H}^d$$

denote the set of distinct pairs. Then the *Gromov product* $G : \partial^{(2)}\,\mathbb{H}^d \to [0,+\infty)$ is defined by

$$G(x,y) = \lim_{p\to x, q\to y} \mathrm{d}(p,o) + \mathrm{d}(o,q) - \mathrm{d}(p,q)$$

(one can use formulas for the distance to show that G exists).

Observation 10.1. If $g \in \mathsf{Isom}(\mathbb{H}^d)$ and $(x,y) \in \partial^{(2)}\,\mathbb{H}^d$, then

$$G(gx,gy) - G(x,y) = B(g,x) + B(g,y).$$

Proof. Notice that

$$\begin{aligned} G(gx,gy) - G(x,y) &= \lim_{p\to x, q\to y} \mathrm{d}(gp,o) + \mathrm{d}(o,gq) - \mathrm{d}(gp,gq) - \mathrm{d}(p,o) - \mathrm{d}(o,q) + \mathrm{d}(p,q) \\ &= \lim_{p\to x, q\to y} \mathrm{d}(p,g^{-1}o) - \mathrm{d}(p,o) + \mathrm{d}(q,g^{-1}o) - \mathrm{d}(o,q) \\ &= b_x(g^{-1}o) + b_y(g^{-1}o) = B(g,x) + B(g,y). \qquad \square \end{aligned}$$

The next result provides a dynamical dichotomy for Patterson–Sullivan measures.

Theorem 10.2 (The Hopf–Tsuji–Sullivan Ergodic Dichotomy)**.** *Suppose $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is a non-elementary discrete group. Let μ be a Patterson–Sullivan measure of dimension β for Γ. If*

$$d\nu(x,y) := e^{\beta G(x,y)} d\mu(x) \otimes d\mu(y),$$

then ν is a Γ-invariant measure on $\partial^{(2)}\,\mathbb{H}^d$. Moreover, we have the following dichotomy:

1. *If $\sum_{\gamma\in\Gamma} e^{-\beta\, \mathrm{d}(o,\gamma o)} = +\infty$, then:*
   - (a) *$\beta = \delta(\Gamma)$.*
   - (b) *$\mu(\Lambda^{\mathrm{con}}(\Gamma)) = 1$ and μ has no atoms.*
   - (c) *The Γ action on $(\partial^{(2)}\,\mathbb{H}^d, \nu)$ is ergodic and conservative.*
2. *If $\sum_{\gamma\in\Gamma} e^{-\beta\, \mathrm{d}(o,\gamma o)} < +\infty$, then:*
   - (a) *$\beta \geq \delta(\Gamma)$.*
   - (b) *$\mu(\Lambda^{\mathrm{con}}(\Gamma)) = 0$.*
   - (c) *The Γ action on $(\partial^{(2)}\,\mathbb{H}^d, \nu)$ is non-ergodic and dissipative.*

The theorem in full generality was established by Sullivan [Sul79, Sul81]. For a proof in the CAT(-1) setting, see [Rob03].

Let us just check the Γ-invariance of ν. Notice that

$$\begin{aligned} d\gamma_*\nu(x,y) &= e^{\beta G(\gamma^{-1}x,\gamma^{-1}y)} d\gamma_*\mu(x) \otimes d\gamma_*\mu(y) \\ &= e^{\beta G(\gamma^{-1}x,\gamma^{-1}y)-\beta B(\gamma^{-1},x)-\beta B(\gamma^{-1},y)} d\mu(x)\otimes d\mu(y) \\ &= e^{\beta G(x,y)} d\mu(x)\otimes d\mu(y) = d\nu(x,y). \end{aligned}$$

In the convex cocompact case, the ergodic dichotomy implies the following.

Corollary 10.3. *If $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is convex cocompact, then there is a unique Patterson–Sullivan measure μ of dimension $\delta(\Gamma)$ for Γ. Moreover, the Γ action on $(\partial\,\mathbb{H}^d, \mu)$ is ergodic.*

Proof. Let $\delta := \delta(\Gamma)$. Since Γ is convex cocompact, $\Lambda^{\mathrm{con}}(\Gamma) = \Lambda(\Gamma)$. By Theorem 8.5, there exists a Patterson–Sullivan measure μ of dimension δ for Γ supported on $\Lambda(\Gamma)$. Now

$$\mu(\Lambda^{\mathrm{con}}(\Gamma)) = \mu(\Lambda(\Gamma)) = 1$$

and so by the ergodic dichotomy, $\sum_{\gamma\in\Gamma} e^{-\delta\,\mathrm{d}(o,\gamma o)} = +\infty$ and the Γ action on $(\partial^{(2)}\,\mathbb{H}^d, e^{\delta G}\mu\otimes\mu)$ is ergodic. We claim that the Γ action on $(\partial\,\mathbb{H}^d,\mu)$ is ergodic. Suppose $A\subset\partial\,\mathbb{H}^d$ is Γ-invariant, then

$$A' := \partial^{(2)}\,\mathbb{H}^d \cap (\partial\,\mathbb{H}^d \times A)$$

is a Γ-invariant set. So A' has either full or zero measure. Since μ has no atoms, this implies that A has either full or zero measure. Hence the Γ action on $(\partial\,\mathbb{H}^d,\mu)$ is ergodic.

Now suppose that μ' is a Patterson–Sullivan measure of dimension δ for Γ. Then for $t\in[0,1]$ the measure $\mu_t := t\mu' + (1-t)\mu$ is also a Patterson–Sullivan measure of dimension δ for Γ. Since $\sum_{\gamma\in\Gamma} e^{-\delta\,\mathrm{d}(o,\gamma o)} = +\infty$, arguing as above shows that the Γ action on each $(\partial\,\mathbb{H}^d,\mu_t)$ is ergodic. Now fix $0<s,t<1$ distinct. Then μ_s and μ_t are absolutely continuous and so

$$f = \frac{d\mu_s}{d\mu_t}$$

exists. If $\gamma\in\Gamma$, then

$$e^{-\delta B(\gamma^{-1},x)}\mu_s = \gamma_*\mu_s = \gamma_*(f\mu_t) = f\circ\gamma^{-1}\gamma_*\gamma_*\mu_t = f\circ\gamma^{-1}e^{-\delta B(\gamma^{-1},x)}\mu_t.$$

So

$$\mu_s = f\circ\gamma^{-1}\mu_t.$$

Then, by uniqueness of Radon–Nikodym derivatives,

$$f\circ\gamma^{-1} = f \quad \mu_t \text{ a.e.}$$

Since $\gamma\in\Gamma$ is arbitrary, f is Γ-invariant. Then by ergodicity, f is constant a.e. Since both μ_s, μ_t are probability measures, we must have $\mu_s=\mu_t$. Hence $\mu=\mu'$. Thus μ is the unique Patterson–Sullivan measure of dimension δ for Γ. □

11. Hausdorff dimension of the limit set

In this section we consider the Klein–Beltrami model of $\mathbb{H}^d$, in which case we can identify $\partial\,\mathbb{H}^d = \mathbb{S}^{d-1}$.

Let $\mathrm{d}_{\mathbb{E}}$ be the Euclidean metric restricted to $\mathbb{S}^{d-1} = \partial\,\mathbb{H}^d$ and for $x \in \mathbb{S}^{d-1}$ let

$$D(x,r) := \{y \in \mathbb{S}^{d-1} : \mathrm{d}_{\mathbb{E}}(x,y) < r\}$$

be the associated metric balls.

Lemma 11.1. *For any $R > 0$ there exist $c_1, c_2 > 0$ such that*

$$D\left(\frac{p}{\|p\|}, c_1 e^{-\mathrm{d}_{\mathbb{H}^d}(o,p)}\right) \subset \mathcal{O}_R(o,p) \subset D\left(\frac{p}{\|p\|}, c_2 e^{-\mathrm{d}_{\mathbb{H}^d}(o,p)}\right)$$

for all non-zero $p \in \mathbb{H}^d$

Proof. See Problem 15. □

Theorem 11.2 (Sullivan [Sul79])**.** *If $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is convex co-compact and μ is the Patterson–Sullivan measure of dimension $\delta(\Gamma)$ for Γ, then there exist $A > 1$, $r_0 > 0$ such that*

$$A^{-1} r^{\delta(\Gamma)} \leq \mu(D(x,r)) \leq A r^{\delta(\Gamma)}$$

for all $x \in \Lambda(\Gamma)$ and $r \in [0, r_0]$. Hence, the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ with respect to the metric $\mathrm{d}_{\mathbb{E}}$ is $\delta(\Gamma)$.

Proof. By conjugating we can assume that 0 is in the convex hull $\mathcal{C}$ of $\Lambda(\Gamma)$ in $\mathbb{H}^d$. Fix $r > 0$ such that

$$\mathcal{C} \subset \Gamma \cdot B_{\mathbb{H}^d}(0,r)$$

and any number at least r satisfies the Shadow Lemma for μ

Using Lemma 11.1, fix $c_1, c_2 > 0$ such that

$$D\left(x, c_1 e^{-\mathrm{d}_{\mathbb{H}^d}(o,tx)}\right) \subset \mathcal{O}_r(o,tx) \subset \mathcal{O}_{2r}(o,tx) \subset D\left(x, c_2 e^{-\mathrm{d}_{\mathbb{H}^d}(o,tx)}\right)$$

for all $x \in \partial\,\mathbb{H}^d$ and $0 < t < 1$. Also fix $C > 1$ satisfying the Shadow Lemma for both $r, 2r$.

Now suppose $x \in \Lambda(\Gamma)$. Then $tx \in \mathcal{C}$ for any $0 < t < 1$ and hence there exists $\gamma_t \in \Gamma$ such that

$$\mathrm{d}_{\mathbb{H}^d}(tx, \gamma_t(0)) < r.$$

Then

$$D\left(x, c_1 e^{-\mathrm{d}_{\mathbb{H}^d}(o,tx)}\right) \subset \mathcal{O}_r(o,tx) \subset \mathcal{O}_{2r}(o,\gamma_t(0))$$

and

$$D\left(x, c_2 e^{-\mathrm{d}_{\mathbb{H}^d}(o,tx)}\right) \supset \mathcal{O}_{2r}(o,tx) \supset \mathcal{O}_r(o,\gamma_t(0)).$$

Hence by the Shadow Lemma

$$\begin{aligned}\mu\left(D\left(x, c_1 e^{-\mathrm{d}_{\mathbb{H}^d}(o,tx)}\right)\right) &\leq \mu(\mathcal{O}_{2r}(o,\gamma_t(0))) \leq C e^{-\delta(\Gamma)\,\mathrm{d}(0,\gamma_t(0))} \\ &\leq \frac{C e^{\delta(\Gamma) r}}{c_1^{\delta(\Gamma)}} \left(c_1 e^{-\mathrm{d}_{\mathbb{H}^d}(0,tx)}\right)^{\delta(\Gamma)}\end{aligned}$$

and

$$\mu\left(D\left(x, c_2 e^{-\mathrm{d}_{\mathbb{H}^d}(o,tx)}\right)\right) \geq \mu(\mathcal{O}_r(o, \gamma_t(0))) \geq C^{-1} e^{-\delta(\Gamma)\,\mathrm{d}(0,\gamma_t(0))}$$
$$\geq \frac{1}{C c_2^{\delta(\Gamma)} e^{\delta(\Gamma) r}} \left(c_2 e^{-\mathrm{d}_{\mathbb{H}^d}(0,tx)}\right)^{\delta(\Gamma)}$$

So there exist $A > 1$, $r_0 > 0$ such that: if $x \in \Lambda(\Gamma)$ and $0 \leq r \leq r_0$, then

$$A^{-1} r^{\delta(\Gamma)} \leq \mu(D(x, r)) \leq A r^{\delta(\Gamma)}.$$

The "hence" part is a consequence of the following general theorem (which you will prove in Problem 23). □

Theorem 11.3. *Suppose X is a compact metric space and μ is a Borel probability measure on X. If there exist $A \geq 1, r_0 > 0$ such that*

$$A^{-1} r^{\delta} \leq \mu(B_X(x, r)) \leq A r^{\delta}$$

for all $x \in X$ and $r \in [0, r_0]$, then $\mathrm{H.dim}(X) = \delta$ and the δ-dimensional Hausdorff measure satisfies

$$C^{-1}\mu \leq \mathcal{H}^{\delta} \leq C\mu$$

for some $C \geq 1$ and $\mathrm{H.dim}(X) = \delta$.

12. Problems

12.1. **Odds and ends.**

Problem 11. Suppose $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is a non-elementary discrete subgroup. Show that there is no Γ-invariant probability measure on $\partial\,\mathbb{H}^d$.

Problem 12. Suppose $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is discrete, $o \in \mathbb{H}^d$, and $x \in \partial\,\mathbb{H}^d$. Prove that the following are equivalent:

(1) x is a conical limit point of Γ in the convergence group sense: there exist $\{\gamma_n\} \subset \Gamma$ and $a, b \in \partial\,\mathbb{H}^d$ distinct such that $\gamma_n x \to a$ and $\gamma_n y \to b$ for all $y \in \partial\,\mathbb{H}^n \smallsetminus \{x\}$.
(2) x is a conical limit point of Γ in the hyperbolic geometry sense: if $\ell : [0, \infty) \to \mathbb{H}^n$ is the geodesic ray with $\ell(0) = o$ and $\lim_{t\to\infty} \ell(t) = x$, then there exist $\{\gamma_n\} \subset \Gamma$ and $t_n \to +\infty$ such that
$$\sup_{n\geq 1} \mathrm{d}(\ell(t_n), \gamma_n o) < +\infty.$$
(3) There exist $\{\gamma_n\} \subset \Gamma$ distinct and $R > 0$ such that $x \in \mathcal{O}_R(o, \gamma_n o)$ for all $n \geq 1$.

Problem 13. Prove Observation 8.1.

12.2. **Shadows in the Klein–Beltrami model.**

In the next two problems, identify $\mathbb{H}^d$ with the Klein–Beltrami model $\mathbb{B}^d \subset \mathbb{R}^d$ and let $\|\cdot\|$ denote the standard norm on $\mathbb{R}^d$. The following two facts will be helpful:

(1) Euclidean line segments can be parametrized to be hyperbolic geodesics.
(2) If $x, y \in \mathbb{B}^d$ are distinct, then the hyperbolic distance satisfies

$$\mathrm{d}_{\mathbb{H}^d}(p, q) = \frac{1}{2} \log \frac{\|x - b\|\,\|y - a\|}{\|x - a\|\,\|y - b\|}$$

where the affine line through x, y intersects $\partial\,\mathbb{B}^d$ at a, b and the points have order a, x, y, b along the line.

Problem 14. For $g \in \mathsf{Isom}(\mathbb{H}^d)$ and $\epsilon > 0$, let

$$\mathcal{S}_\epsilon(g) := g\{y \in \partial\,\mathbb{H}^d : \left\|y - g^{-1}(0)\right\| > \epsilon\}.$$

Prove:

(1) For any $R > 0$, there exists $\epsilon > 0$ such that $\mathcal{O}_R(0, \gamma(0)) \subset \mathcal{S}_\epsilon(\gamma)$.
(2) For any $\epsilon > 0$, there exists $R > 0$ such that $\mathcal{S}_\epsilon(\gamma) \subset \mathcal{O}_R(0, \gamma(0))$.

For $x \in \partial\,\mathbb{H}^d$, let

$$D(x, r) = \{y \in \partial\,\mathbb{H}^d : \|y - x\| < r\}.$$

Problem 15. For any $R > 0$ there exist $c_1, c_2 > 0$ such that

$$D\left(x, c_1 e^{-\,\mathrm{d}(o,tx)}\right) \subset \mathcal{O}_R(o, tx) \subset D\left(x, c_2 e^{-\,\mathrm{d}(o,tx)}\right)$$

for all $x \in \partial\,\mathbb{H}^d$ and $0 \leq t < 1$.

Hint: It suffices to consider $x = (1, 0, \ldots, 0)$.

12.3. **Coarse counting using Patterson–Sullivan measures.**

Problem 16. Suppose $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is a discrete group. Consider the function $Q : (0, \infty) \to (0, +\infty]$ given by

$$Q(s) = \sum_{\gamma \in \Gamma} e^{-s\,\mathrm{d}(o,\gamma o)}.$$

Prove that $Q(s) = +\infty$ if $s < \delta(\Gamma)$ and $Q(s) < +\infty$ if $s > \delta(\Gamma)$.

Problem 17. Suppose $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is a non-elementary discrete group and μ is a Patterson–Sullivan measure of dimension β for Γ. The goal of this problem is to show that

$$\beta \geq \delta(\Gamma).$$

Let

$$\mathcal{A}_n := \{\gamma \in \Gamma : \mathrm{d}(o, \gamma o) \in [n, n+1)\}.$$

(a) Prove: If $\gamma_1, \gamma_2 \in \mathcal{A}_n$ and $\mathcal{O}_R(o, \gamma_1 o) \cap \mathcal{O}_R(o, \gamma_2 o) \neq \emptyset$, then

$$\mathrm{d}(\gamma_1 o, \gamma_2 o) \leq T + 2R.$$

(b) Prove: there exists $N \in \mathbb{N}$ such that: if $x \in \partial\,\mathbb{H}^d$, then

$$\#\{\gamma \in \mathcal{A}_n : x \in \mathcal{O}_R(o, \gamma o)\} \leq N$$

and hence

$$1 = \mu(\partial\,\mathbb{H}^d) \geq \frac{1}{N} \sum_{\gamma \in \mathcal{A}_n} \mu(\mathcal{O}_R(o, \gamma o)).$$

(c) Use the Shadow Lemma to prove: there exists $C > 0$ such that

$$\#\mathcal{A}_n \leq C e^{\beta n}$$

and deduce that $\beta \geq \delta(\Gamma)$.

Problem 18. Prove: If $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is discrete, then $\delta(\Gamma) \leq d - 1$. *Hint: Use the last problem and Example 8.3.*

Problem 19. Prove: If $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is convex cocompact, then there exists $C > 1$ such that

$$C^{-1}e^{\delta(\Gamma)n} \leq \#\{\gamma \in \Gamma : \mathrm{d}(o, \gamma o) \leq n\} \leq Ce^{\delta(\Gamma)n}.$$

Hint: Let μ be a Patterson–Sullivan measure of dimension $\delta(\Gamma)$. A previous problem provides the upper bound. For the lower bound, show that there exists $R, k > 1$ such that

$$\partial\,\mathbb{H}^d \subset \bigcup_{\substack{\gamma\in\Gamma \\ \mathrm{d}(o,\gamma o)\in[n-k,n]}} \mathcal{O}_R(o, \gamma o)$$

for all $n \geq k$. Then use the Shadow Lemma to prove a lower bound on

$$\#\{\gamma \in \Gamma : \mathrm{d}(o, \gamma o) \in [n - k, n]\}.$$

12.4. **Lebesgue Differentiation.** The next two problems show that Shadows and Patterson–Sullivan measures satisfy a version of the Lebesgue Differentiation theorem.

Problem 20. ($\star$) Prove:

(1) If $\mathrm{d}(o, p) \leq \mathrm{d}(o, q)$ and $\mathcal{O}_R(o, p) \cap \mathcal{O}_R(o, q) \neq \emptyset$, then $\mathcal{O}_R(o, q) \subset \mathcal{O}_{3R}(o, p)$.
(2) If $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is discrete and $I \subset \Gamma$, then there exists $J \subset I$ such that the shadows $\{\mathcal{O}_R(o, \gamma o)\}_{\gamma\in J}$ are disjoint and

$$\bigcup_{\gamma\in I} \mathcal{O}_R(o, \gamma o) \subset \bigcup_{\gamma\in J} \mathcal{O}_{3R}(o, \gamma o).$$

Hint for (2): Enumerate $I = \{\gamma_n\}$ so that $\mathrm{d}(o, \gamma_1 o) \leq \mathrm{d}(o, \gamma_2 o) \leq \mathrm{d}(o, \gamma_3 o) \cdots$. Then look up and adapt the proof of the Vitali covering lemma.

Problem 21. ($\star$) Suppose $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is discrete, μ is a Patterson–Sullivan measure of dimension δ, and $R > 0$ satisfies the Shadow Lemma. Prove: If $f \in L^1(\partial\,\mathbb{H}^d, \mu)$, then for μ-a.e. x we have

$$f(x) = \lim_{n\to\infty} \frac{1}{\mu(\mathcal{O}_R(o, \gamma_n o))} \int_{\mathcal{O}_R(o,\gamma_n o)} f d\mu$$

whenever $\{\gamma_n\}$ are distinct and $x \in \mathcal{O}_R(o, \gamma_n o)$ for all $n \geq 1$.

Hint: Look up and adapt the proof of the Lebesgue differentiation theorem.

The above result is quite useful when the conical limit points have full measure.

Problem 22. ($\star$) Suppose $\Gamma < \mathsf{Isom}(\mathbb{H}^d)$ is discrete, μ is a Patterson–Sullivan measure of dimension δ, and $\mu(\Lambda^{\mathrm{con}}(\Gamma)) = 1$. Prove, using Problem 21, that the Γ action on $(\partial\,\mathbb{H}^d, \mu)$ is ergodic.

Hint: Fix a Γ-invariant set $E \subset \partial\,\mathbb{H}^d$ with $\mu(E) > 0$. Then fix $R_j \nearrow +\infty$ such that each satisfies the Shadow lemma. Show that there exist $x \in E$ and $\{\gamma_n\}$ distinct such that $x \in \mathcal{O}_{R_j}(o, \gamma_n o)$ for all $n \geq 1$ and

$$1 = 1_E(x) = \lim_{n\to\infty} \frac{1}{\mu(\mathcal{O}_{R_j}(o, \gamma_n o))} \int_{\mathcal{O}_{R_j}(o,\gamma_n o)} 1_E d\mu = \lim_{n\to\infty} \frac{\mu(E \cap \mathcal{O}_{R_j}(o, \gamma_n o))}{\mu(\mathcal{O}_{R_j}(o, \gamma_n o))}.$$

By studying

$$1 = \lim_{n\to\infty} \frac{\mu(E \cap \mathcal{O}_{R_j}(o, \gamma_n o))}{\mu(\mathcal{O}_{R_j}(o, \gamma_n o))} = \lim_{n\to\infty} \frac{(\gamma_n^{-1})_*\mu(E \cap \mathcal{O}_{R_j}(\gamma_n^{-1} o, o))}{(\gamma_n^{-1})_*\mu(\mathcal{O}_{R_j}(\gamma_n^{-1} o, o))}$$

show that $\mu(E) = 1$ (what can you say about $\frac{d(\gamma_n^{-1})_*\mu}{d\mu}$ on $\mathcal{O}_{R_j}(\gamma_n^{-1} o, o)$?).

12.5. **Facts about Hausdorff dimension.**

Problem 23. Prove Theorem 11.3.

Hint: To prove $\mathcal{H}^\delta \leq C\mu$ you can use the Vitali covering lemma.

Problem 24. ($\star$) Suppose (M, g) is a compact Riemannian n-manifold endowed with its Riemannian distance. Prove that $\mathcal{H}^n$ coincides with the measure induced by the Riemannian volume form.

Hint: Recall, in a chart the Riemannian volume form ν_g satisfies

$$\int f\, d\nu_g = \int f\sqrt{\det(g_{i,j})}dx^1 \cdots dx^n.$$

Fix $\epsilon > 0$ and cover M by charts $\{U_i\}$ where on each chart $(1-\epsilon)\mathrm{Id} \leq g_{i,j} \leq (1+\epsilon)\mathrm{Id}$. Then, on each chart, show that

$$(1-\epsilon)^{n/2}\nu_g \leq \mathcal{H}^n \leq (1+\epsilon)^{n/2}\nu_g.$$

Lectures #3-5 : Patterson–Sullivan theory in higher rank

13. The Iwasawa cocycle

Let $\mathsf{U} \subset \mathsf{SL}(d, \mathbb{R})$ be the subgroup of upper triangular matrices with ones on the diagonal. Recall that any matrix $g \in \mathsf{SL}(d, \mathbb{R})$ can be written uniquely as

$$g = \ell e^H u$$

where $\ell \in \mathsf{O}(d)$, $H \in \mathfrak{a}$, and $u \in \mathsf{U}$. This is called the *Iwasawa decomposition of g*.

Also, let $x_0 \in \mathcal{F}_\Delta$ be the standard flag given by

$$x_0^j = \langle e_1, \ldots, e_j \rangle .$$

Using the Iwasawa decomposition we define the *Iwasawa cocycle* $B : \mathsf{SL}(d, \mathbb{R}) \times \mathcal{F}_\Delta \to \mathfrak{a}$ by letting $B(g, x)$ be the $\mathfrak{a}$ part of the Iwasawa decomposition of gk where $x = kx_0$, i.e.

$$gk = \ell e^{B(g,kx_0)}u.$$

Lemma 13.1. *B is well defined and is a cocycle:*

$$B(g_1g_2, x) = B(g_1, g_2x) + B(g_2, x).$$

Proof. To prove that B is well defined, we suppose $k_1x_0 = k_2x_0$ and show that gk_1, gk_2 have the same $\mathfrak{a}$ part in their Iwasawa decomposition. Since $k_2^{-1}k_1x_0 = x_0$, we have $k_1 = k_2m$ where m is a diagonal matrix with ± 1 on the diagonal. Suppose $gk_2 = \ell e^H u$ is the Iwasawa decomposition of gk_2. Then

$$gk_1 = gk_2m = \ell e^H um = \ell e^H m(m^{-1}um) = (\ell m)e^H(m^{-1}um).$$

Since $\ell m \in \mathsf{O}(d)$ and $m^{-1}um \in \mathsf{U}$, this is the Iwasawa decomposition of gk_1. Hence gk_1, gk_2 have the same $\mathfrak{a}$ part in their Iwasawa decomposition.

Next we verify the cocycle identity. Suppose $x = kx_0$. Let $g_1g_2k = \ell_1 e^{B(g_1g_2,x)}u_1$ and $g_2k = \ell_2 e^{B(g_2,x)}u_2$ be the Iwasawa decompositions. Then

$$g_2x = g_2kx_0 = \ell_2 e^{B(g_2,x)}u_2x_0 = \ell_2x_0.$$

Let $g_1\ell_2 = \ell_3 e^{B(g_1,g_2x)}u_3$ be the Iwasawa decomposition. Then

$$\begin{aligned} g_1g_2k &= g_1\ell_2 e^{B(g_2,x)}u_2 = \ell_3 e^{B(g_1,g_2x)}u_3 e^{B(g_2,x)}u_2 \\ &= \ell_3 e^{B(g_1,g_2x)+B(g_2,x)}\left(e^{-B(g_2,x)}u_3 e^{B(g_2,x)}u_1\right) \end{aligned}$$

is the Iwasawa decomposition. Hence

$$B(g_1g_2,x) = B(g_1,g_2x) + B(g_2,x). \qquad \square$$

When a fundamental weight is applied to the Iwasawa cocycle we obtain the following formula.

Lemma 13.2. $\omega_j(B(g,x)) = \log \frac{\|g(v_1\wedge\cdots\wedge v_j)\|}{\|v_1\wedge\cdots\wedge v_j\|}$ *where* $x^j = \langle v_1,\dots,v_j\rangle$.

Proof. Suppose $x = kx_0$. Then $x^j = k\langle e_1,\dots,e_j\rangle$ and so

$$\frac{\|g(v_1\wedge\cdots\wedge v_j)\|}{\|v_1\wedge\cdots\wedge v_j\|} = \frac{\|gk(e_1\wedge\cdots\wedge e_j)\|}{\|e_1\wedge\cdots\wedge e_j\|}.$$

Further, if $gk = \ell e^{B(g,x)}u$ is the Iwasawa decomposition, then

$$\begin{aligned} gk(e_1\wedge\cdots\wedge e_j) &= \ell e^{B(g,x)}u(e_1\wedge\cdots\wedge e_j) = \ell e^{B(g,x)}(e_1\wedge\cdots\wedge e_j) \\ &= \ell\left(e^{B(g,x)_{1,1}}e_1\right)\wedge\cdots\wedge\left(e^{B(g,x)_{j,j}}e_j\right) \\ &= \ell e^{B(g,x)_{1,1}+\cdots+B(g,x)_{j,j}}(e_1\wedge\cdots\wedge e_j) \\ &= e^{\omega_j(B(g,x))}\ell(e_1\wedge\cdots\wedge e_j). \end{aligned}$$

Hence

$$\frac{\|g(v_1\wedge\cdots\wedge v_j)\|}{\|v_1\wedge\cdots\wedge v_j\|} = \frac{\|gk(e_1\wedge\cdots\wedge e_j)\|}{\|k(e_1\wedge\cdots\wedge e_j)\|} = e^{\omega_j(B(g,x))},$$

since $\mathsf{O}(d)$ acts orthogonally on $\wedge^j\mathbb{R}^d$. $\square$

14. Partial Iwasawa cocycles

Using the fact that $\omega_{\alpha_j}(B(g,x))$ only depends on x^j, we will define a cocycle on $\mathcal{F}_\theta$.

For $\theta\subset\Delta$ non-empty, we let

$$\mathfrak{a}_\theta := \{H\in\mathfrak{a} : \alpha(H) = 0 \text{ if } \alpha\notin\theta\}.$$

Observation 14.1. $\{\omega_\alpha|_{\mathfrak{a}_\theta} : \alpha\in\theta\}$ spans $\mathfrak{a}_\theta^*$ and hence there exists a unique projection $p_\theta : \mathfrak{a}\to\mathfrak{a}_\theta$ which satisfies

$$\omega_\alpha(p_\theta(H)) = \omega_\alpha(H)$$

for all $\alpha\in\theta$ and $H\in\mathfrak{a}$.

Proof. See Problem 25. $\square$

For $\theta\subset\Delta$, the *partial Iwasawa cocycle* $B_\theta : \mathsf{SL}(d,\mathbb{R})\times\mathcal{F}_\theta\to\mathfrak{a}_\theta$ is defined by

$$B_\theta(g,x) = p_\theta(B(g,\tilde{x}))$$

where $\tilde{x}\in\mathcal{F}_\Delta$ is any lift of $x\in\mathcal{F}_\theta$.

Lemma 14.2. *B_θ is a well-defined cocycle.*

Proof. Using the fact that $\{\omega_\alpha|_{\mathfrak{a}_\theta} : \alpha \in \theta\}$ is a basis for $\mathfrak{a}_\theta^*$, it suffices to fix $\alpha_j \in \theta$ and show that $\omega_{\alpha_j} B_\theta$ is a well-defined cocycle. By the previous lemma,

$$\omega_{\alpha_j}(p_\theta(B(g,\tilde{x}))) = \omega_{\alpha_j}(B(g,\tilde{x})) = \log \frac{\|g(v_1 \wedge \cdots \wedge v_j)\|}{\|v_1 \wedge \cdots \wedge v_j\|}$$

where $\tilde{x}^j = \langle v_1, \ldots, v_j \rangle$. So $\omega_{\alpha_j}(p_\theta(B(g,\tilde{x})))$ only depends on the image of $\tilde{x}$ in $\mathcal{F}_\theta$ and hence $\omega_{\alpha_j} B_\theta$ is well-defined. Since

$$\log \frac{\|g_1 g_2(v_1 \wedge \cdots \wedge v_j)\|}{\|v_1 \wedge \cdots \wedge v_j\|} = \log \frac{\|g_1(g_2(v_1 \wedge \cdots \wedge v_j))\|}{\|g_2(v_1 \wedge \cdots \wedge v_j)\|} + \log \frac{\|g_2(v_1 \wedge \cdots \wedge v_j)\|}{\|v_1 \wedge \cdots \wedge v_j\|},$$

we see that $\omega_{\alpha_j} B_\theta$ is a cocycle.

□

Next we show that the Iwasawa cocycles have a Busemann-like property. Let

$$\kappa_\theta := p_\theta \circ \kappa : \mathsf{SL}(d,\mathbb{R}) \to \mathfrak{a}_\theta \,.$$

Then $w_j \kappa_\theta = w_j \kappa$ for all $\alpha_j \in \theta$.

Proposition 14.3. *If $\alpha(\kappa(h_n)) \to +\infty$ for all $\alpha \in \theta$ and $U_\theta(h_n) \to x$, then*

$$\kappa_\theta(g h_n) - \kappa_\theta(h_n) \to B_\theta(g,x).$$

We start with a lemma.

Lemma 14.4. *If $\frac{\sigma_1(h_n)}{\sigma_2(h_n)} \to +\infty$ and $U_1(h_n) \to \langle v \rangle$, then*

$$\log \|g h_n\| - \log \|h_n\| \to \log \frac{\|gv\|}{\|v\|}.$$

Proof. It suffices to consider the case where

$$\log \|g h_n\| - \log \|h_n\| \to L \in [-\infty, +\infty]$$

and prove that $L = \log \frac{\|gv\|}{\|v\|}$. Suppose $h_n = k_n a_n \ell_n$ is a singular value decomposition of h_n with $U_1(h_n) = k_n \langle e_1 \rangle$. We can assume that $k_n \to k$, $\ell_n \to \ell$. Then

$$\langle v \rangle = \lim_{n\to\infty} U_1(h_n) = \lim_{n\to\infty} k_n \langle e_1 \rangle = k \langle e_1 \rangle \,.$$

Further, since $\frac{\sigma_1(h_n)}{\sigma_2(h_n)} \to +\infty$,

$$\frac{h_n}{\|h_n\|} \to kP\ell$$

where P is the orthogonal projection onto $\langle e_1 \rangle$. Then

$$L = \lim_{n\to\infty} \log \frac{\|g h_n\|}{\|h_n\|} = \lim_{n\to\infty} \log \left\| g \frac{h_n}{\|h_n\|} \right\| = \log \|gkP\ell\| = \log \|gke_1\| = \log \frac{\|gv\|}{\|v\|}$$

since $ke_1 = \pm \frac{v}{\|v\|}$. □

Proof. Using the fact that $\{\omega_\alpha|_{\mathfrak{a}_\theta} : \alpha \in \theta\}$ is a basis for $\mathfrak{a}_\theta^*$, it suffices to fix $\alpha_j \in \theta$ and show that

$$\omega_{\alpha_j} \kappa(g h_n) - \omega_{\alpha_j} \kappa(h_n) \to \omega_{\alpha_j} B(g,x). \tag{2}$$

(recall that $\omega_{\alpha_j} p_\theta = \omega_{\alpha_j}$). Notice that

$$\omega_{\alpha_j} \kappa(g h_n) - \omega_{\alpha_j} \kappa(h_n) = \log \|\tau_j(g)\tau_j(h_n)\| - \log \|\tau_j(h_n)\| \,.$$

and

$$\omega_{\alpha_j} B(g,x) = \log \frac{\|g(v_1 \wedge \cdots \wedge v_j)\|}{\|v_1 \wedge \cdots \wedge v_j\|}$$

where $x^j = \langle v_1, \ldots, v_j \rangle$. Further,

$$\frac{\sigma_1(\tau_j(h_n))}{\sigma_2(\tau_j(h_n))} = e^{\alpha_j(\kappa(h_n))} \to +\infty$$

and

$$U_1(\tau_j(h_n)) = k_{h_n} \langle e_1 \wedge \cdots \wedge e_j \rangle = U_\theta(h_n)^j \to x = \langle v_1 \wedge \cdots \wedge v_j \rangle .$$

So Equation (2) follows from the Lemma.

□

15. Patterson–Sullivan measures

Following Quint [Qui02], given a subgroup $\Gamma < \mathsf{SL}(d,\mathbb{R})$, $\delta \geq 0$, and $\phi \in \mathfrak{a}_\theta^*$, a Borel probability measure μ on $\mathcal{F}_\theta$ is called *ϕ-Patterson–Sullivan measure of dimension δ for Γ* if for any $\gamma \in \Gamma$ the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-\delta \phi B_\theta(\gamma^{-1},x)} \tag{3}$$

for μ-a.e. $x \in \mathcal{F}_\theta$.

Example 15.1. Let

$$\psi := \sum_{i<j} \delta_i - \delta_j \in \mathfrak{a}^*$$

where $\delta_i(\mathrm{diag}(a_1, \ldots, a_d)) = a_i$. If μ is the unique $\mathsf{O}(d)$-invariant probability measure on $\mathcal{F}_\theta$, then μ is a $\psi|_{\mathfrak{a}_\theta}$-Patterson–Sullivan measure of dimension one for $\mathsf{SL}(d,\mathbb{R})$.

Question. When do Patterson–Sullivan measures exist?

A complete answer is unknown. Quint has studied this problem for Zariski dense groups [Qui02] and in this mini-course we will focus on transverse groups.

Given a discrete subgroup $\Gamma < \mathsf{SL}(d,\mathbb{R})$ and $\phi \in \mathfrak{a}_\theta^*$, the associated *critical exponent* is

$$\delta^\phi(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma : \phi(\kappa_\theta(\gamma)) \leq R\}.$$

As in the hyperbolic case, the critical exponent is the critical exponent of a series

$$Q(s) = \sum_{\gamma\in\Gamma} e^{-s\phi(\kappa_\theta(\gamma))},$$

i.e. $Q(s) = +\infty$ for $s < \delta^\phi(\Gamma)$ and $Q(s) < +\infty$ for $s > \delta^\phi(\Gamma)$.

Proposition 15.2. *If Γ is a P_θ-divergent group and $\delta^\phi(\Gamma) < +\infty$, then there exists a ϕ-Patterson–Sullivan measure of dimension $\delta^\phi(\Gamma)$ for Γ.*

Proof. Using Propositions 1.5 and 14.3, the argument is **exactly the same** as in the hyperbolic case (Theorem 8.5)!

Let $\delta := \delta^\phi(\Gamma)$. By [Pat76, Lemma 3.1], there exists a non-decreasing function $\chi\colon \mathbb{R} \to \mathbb{R}_{\geq 1}$ such that

(1) For every $\epsilon > 0$ there exists $R > 0$ such that $\chi(r+t) \leq e^{\epsilon t}\chi(r)$ for any $r \geq R$ and $t \geq 0$,

(2) $\sum_{\gamma\in\Gamma}\chi(\phi\kappa_\theta(\gamma))e^{-\delta\phi\kappa_\theta(\gamma)}=+\infty$

(when $\sum_{\gamma\in\Gamma}e^{-\delta\,\mathrm{d}(o,\gamma o)}=+\infty$, we can take $\chi\equiv 1$).

Recall that $\Gamma\sqcup\Lambda_\theta(\Gamma)$ has the topology of a compact metric space (see Proposition 1.5). Then for $s>\delta^\phi(\Gamma)$ consider the measure μ_s on $\Gamma\sqcup\Lambda_\theta(\Gamma)$ defined by

$$\mu_s:=\frac{1}{Q_\chi(s)}\sum_{\gamma\in\Gamma}\chi(\phi\kappa_\theta(\gamma))e^{-s\phi\kappa_\theta(\gamma)}\mathcal{D}_\gamma,$$

where $Q_\chi(s):=\sum_{\gamma\in\Gamma}\chi(\phi\kappa_\theta(\gamma))e^{-s\phi\kappa_\theta(\gamma)}$.

Arguing as in Theorem 8.5, one can show that any limit point of μ_s as $s\searrow\delta$ is a Patterson–Sullivan measure.

□

16. Anosov groups with smooth limit set

For non-zero $v\in\mathbb{R}^d$, consider the map $\phi_v:\mathbb{R}^d\to T_{\langle v\rangle}\,\mathbb{P}(\mathbb{R}^d)$ given by

$$\phi_v(w)=\left.\frac{d}{dt}\right|_{t=0}\langle v+tw\rangle\,.$$

Then endow $\mathbb{P}(\mathbb{R}^d)$ with the $\mathsf{O}(d)$-invariant Riemannian metric satisfying

$$\|\phi_v(w)\|_{\langle v\rangle}=\frac{\|v\wedge w\|}{\|v\|^2}$$

(note $\phi_{cv}(cw)=\phi_v(w)$, which explains the power of 2).

Proposition 16.1. *Suppose $\Gamma<\mathsf{SL}(d,\mathbb{R})$ is P_1-Anosov, $\Lambda_1(\Gamma)\simeq\partial_\infty\Gamma$ is a circle, and $\Lambda_1(\Gamma)$ is a $\mathcal{C}^1$-submanifold of $\mathbb{P}(\mathbb{R}^d)$. Let ν denote the arc-length measure on $\Lambda_1(\Gamma)$ and let $\mu=\iota_*\nu$ where $\iota:\Lambda_1(\Gamma)\to\mathcal{F}_{1,2}$ is the map*

$$\iota(x)=(x,V_x),$$

where $T_x\,\mathbb{P}(V_x)=T_x\Lambda_1(\Gamma)$. Then μ is a α_1-Patterson–Sullivan measure of dimension 1 for Γ.

Remark 16.2. A similar construction is true when the limit set is a Lipschitz submanifold of any dimension [PSW23].

Proof. Notice that if $g\in\mathsf{SL}(d,\mathbb{R})$, $v\in\mathbb{R}^d$ is non-zero, and $w\in\mathbb{R}^d$, then

$$d(g)\phi_v(w)=\left.\frac{d}{dt}\right|_{t=0}[g(v+tw)]=\phi_{gv}(gw).$$

So

$$\frac{\|d(g)\phi_v(w)\|_{\langle gv\rangle}}{\|\phi_v(w)\|_{\langle v\rangle}}=\frac{\|gv\wedge gw\|}{\|gv\|^2}\frac{\|v\|^2}{\|v\wedge w\|}.$$

So if $\gamma\in\Gamma$ and $x\in\Lambda_1(\Gamma)$, then

$$\frac{d\gamma_*^{-1}\nu}{d\nu}(x)=\frac{\|\gamma(v_x\wedge w_x)\|}{\|\gamma v_x\|^2}\frac{\|v_x\|^2}{\|v_x\wedge w_x\|}$$

where $x=[v_x]$ and $\phi_{v_x}(w_x)\in T_x\Lambda_1(\Gamma)$ is non-zero. Then Lemma 13.2 implies that

$$\frac{d\gamma_*^{-1}\nu}{d\nu}(x)=e^{-(2\omega_1-\omega_2)B_{1,2}(\gamma,\iota(x))}=e^{-\alpha_1 B_{1,2}(\gamma,\iota(x))}.$$

Hence $\mu=\iota_*\nu$ is a α_1-Patterson–Sullivan measure of dimension 1 for Γ. □

17. The Gromov Product

In this section we introduce an analogue of the Gromov product from hyperbolic geometry.

Fix $\theta \subset \Delta$ symmetric and let

$$\mathcal{F}_\theta^{(2)} := \{(x, y) : x \text{ transverse to } y\}.$$

Recall that $\{\omega_j|_{\mathfrak{a}_\theta}\}_{\alpha_j \in \theta}$ is a basis for $\mathfrak{a}_\theta^*$. Hence there is a unique function $G_\theta : \mathcal{F}_\theta^{(2)} \to \mathfrak{a}_\theta$, called the *Gromov product*, satisfying

$$\omega_j G_\theta(x, y) = -\log \frac{\|v_1 \wedge \cdots \wedge v_j \wedge w_1 \wedge \cdots \wedge w_{d-j}\|}{\|v_1 \wedge \cdots \wedge v_j\| \, \|w_1 \wedge \cdots \wedge w_{d-j}\|}$$

for all $\alpha_j \in \theta$, where $x^j = \langle v_1, \ldots, v_j \rangle$ and $y^{d-j} = \langle w_1, \ldots, w_{d-j} \rangle$.

The Gromov product measures how transverse two flags are.

Observation 17.1. If $\{(x_n, y_n)\} \subset \mathcal{F}_\theta^{(2)}$ and $x_n \to x$, $y_n \to y$ with x, y not transverse, then

$$\|G_\theta(x_n, y_n)\| \to +\infty.$$

We first show that uniform bounds on the Gromov product provide uniform bounds on the Iwasawa cocycle.

Proposition 17.2. *For any $R > 0$ there exists $C = C(R) > 0$ such that: If $g, h \in \mathsf{SL}(d, \mathbb{R})$ satisfy $\left\|G_\theta(U_\theta(g^{-1}), U_\theta(h))\right\| \leq R$, then*

$$\|k_\theta(gh) - k_\theta(g) - k_\theta(h)\| \leq C.$$

Proof. Since $\{\omega_j|_{\mathfrak{a}_\theta}\}_{\alpha_j \in \theta}$ is a basis for $\mathfrak{a}_\theta^*$, it suffices to fix $\alpha_j \in \theta$ and bound

$$\left|\omega_j k_\theta(gh) - \omega_j k_\theta(g) - \omega_j k_\theta(h)\right|.$$

Notice that

$$\omega_j k_\theta(gh) = \log \|\tau_j(gh)\| \leq \log \|\tau_j(g)\| \, \|\tau_j(h)\| = \omega_j k_\theta(g) + \omega_j k_\theta(h).$$

The other bound is more challenging. We can fix a singular value decomposition $g = k e^{\kappa(g)} \ell$ such that

$$U_\theta(g^{-1})^{d-j} = \ell^{-1} \left\langle e_{j+1}, \ldots, e_d \right\rangle.$$

Then

$$\begin{aligned}
\|\tau_j(gh)\| &= \left\|\tau_j(e^{\kappa(g)} \ell k_h e^{\kappa(h)})\right\| \geq \left|\left\langle e_1 \wedge \cdots \wedge e_j, e^{\kappa(g)} \ell_g k_h e^{\kappa(h)} (e_1 \wedge \cdots \wedge e_j)\right\rangle\right| \\
&= e^{\omega_j \kappa_\theta(g) + \omega_j \kappa_\theta(h)} \left|\langle e_1 \wedge \cdots \wedge e_j, \ell k_h (e_1 \wedge \cdots \wedge e_j)\rangle\right| \\
&= e^{\|\tau_j(g)\| + \|\tau_j(h)\|} \left|e_1 \wedge \cdots \wedge e_j \text{ coefficient of } \ell k_h (e_1 \wedge \cdots \wedge e_j)\right| \\
&= e^{\|\tau_j(g)\| + \|\tau_j(h)\|} \left\|e_{j+1} \wedge \cdots \wedge e_d \wedge \ell k_h (e_1 \wedge \cdots \wedge e_j)\right\| \\
&= e^{\|\tau_j(g)\| + \|\tau_j(h)\|} \left\|\ell^{-1}(e_{j+1} \wedge \cdots \wedge e_d) \wedge k_h (e_1 \wedge \cdots \wedge e_j)\right\|.
\end{aligned}$$

Now $U_\theta(g^{-1})^{d-j} = \ell^{-1}(e_{j+1} \wedge \cdots \wedge e_d)$ and $U_\theta(h)^j = k_h(e_1 \wedge \cdots \wedge e_j)$. So

$$\begin{aligned}
\omega_j k_\theta(gh) = \log \|\tau_j(gh)\| &\leq \log \|\tau_j(g)\| + \log \|\tau_j(h)\| - \omega_j G_\theta(U_\theta(g^{-1}), U_\theta(h)) \\
&\geq \omega_j k_\theta(g) + \omega_j k_\theta(h) - \|\omega_j\| \, R.
\end{aligned}$$

□

Proposition 17.3. *For any $R > 0$ there exists $C = C(R) > 0$ such that: If $g \in \mathsf{SL}(d,\mathbb{R})$, $x \in \mathcal{F}_\theta$, and $\left\|G_\theta(U_\theta(g^{-1}), x)\right\| \leq R$, then*

$$\|B_\theta(g,x) - \kappa_\theta(g)\| \leq C.$$

Proof. Fix $\{h_n\} \subset \mathsf{SL}(d,\mathbb{R})$ with $U_\theta(h_n) \to x$ and $\min_{\alpha_j \in \theta} \alpha_j(\kappa(h)) \to \infty$. Then

$$\|B_\theta(g,x) - \kappa_\theta(g)\| = \lim_{n\to\infty} \|\kappa_\theta(gh_n) - \kappa(h_n) - \kappa(g)\|$$

and

$$\left\|G_\theta(U_\theta(g^{-1}), x)\right\| = \lim_{n\to\infty} \left\|G_\theta(U_\theta(g^{-1}), U_\theta(h))\right\|.$$

So the result follow immediately form the last proposition.

□

Finally, we note that the Gromov product is linked with the Iwasawa cocycle. Let $\iota : \mathfrak{a} \to \mathfrak{a}$ be the opposite involution

$$\iota(\mathrm{diag}(a_1,\dots,a_d)) = \mathrm{diag}(-a_d,-a_{d-1},\dots,-a_1).$$

Notice that

$$\begin{aligned}\omega_j \iota(\mathrm{diag}(a_1,\dots,a_d)) &= -(a_d + a_{d-1} + \cdots + a_{d-j-1}) = a_1 + \cdots + a_{d-j} \\ &= \omega_{d-j}(\mathrm{diag}(a_1,\dots,a_d)).\end{aligned}$$

Proposition 17.4. *If $g \in \mathsf{SL}(d,\mathbb{R})$ and $(x,y) \in \mathcal{F}_\theta^{(2)}$, then*

$$G_\theta(gx,gy) - G_\theta(x,y) = B_\theta(g,x) + \iota B_\theta(g,y).$$

Proof. It suffices to fix $\alpha_j \in \theta$ and show that

$$\omega_j G_\theta(gx,gy) - \omega_j G_\theta(x,y) = \omega_j B_\theta(g,x) + \omega_{d-j} B_\theta(g,y).$$

Let $x^j = \langle v_1,\dots,v_j\rangle$ and $y^{d-j} = \langle w_1,\dots,w_{d-j}\rangle$. Notice that

$$\begin{aligned}&\omega_j G_\theta(gx,gy) - \omega_j G_\theta(x,y) \\ &\quad = -\log \frac{\|g(v_1\wedge\cdots\wedge v_j\wedge w_1\wedge\cdots\wedge w_{d-j})\|}{\|g(v_1\wedge\cdots\wedge v_j)\|\,\|g(w_1\wedge\cdots\wedge w_{d-j})\|} \frac{\|v_1\wedge\cdots\wedge v_j\|\,\|w_1\wedge\cdots\wedge w_{d-j}\|}{\|v_1\wedge\cdots\wedge v_j\wedge w_1\wedge\cdots\wedge w_{d-j}\|} \\ &\quad = -\log \frac{\|(\det g)(v_1\wedge\cdots\wedge v_j\wedge w_1\wedge\cdots\wedge w_{d-j})\|}{\|v_1\wedge\cdots\wedge v_j\wedge w_1\wedge\cdots\wedge w_{d-j}\|} + \omega_j B_\theta(g,x) + \omega_{d-j} B_\theta(g,y) \\ &\quad = \omega_j B_\theta(g,x) + \omega_j \iota B_\theta(g,y).\end{aligned}$$

□

18. Shadows and the Shadow Lemma

Suppose $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is a non-elementary P_θ-transverse group. Recall that $\overline{\Gamma} := \Gamma \sqcup \Lambda_\theta(\Gamma)$ admits a topology of a compact metrizable space where $\gamma_n \to x \in \Lambda_\theta(\Gamma)$ if and only if $\{\gamma_n\}$ is escaping and $U_\theta(\gamma_n) \to x$. Fix a distance d_θ that generates this topology.

For $\gamma \in \Gamma$ and $R > 0$, let

$$\mathcal{O}_R(\gamma) := \gamma\{x \in \Lambda_\theta(\Gamma) : \mathrm{d}_\theta(\gamma^{-1}, x) > 1/R\}.$$

Theorem 18.1. *Suppose μ is a ϕ-Patterson–Sullivan measure of dimension δ for Γ supported on $\Lambda_\theta(\Gamma)$. For any $R > 0$ sufficiently large, there exists $C = C(R) > 1$ such that*

$$C^{-1} e^{-\delta\phi(\kappa_\theta(\gamma))} \leq \mu(\mathcal{O}_R(\gamma)) \leq C e^{-\delta\phi(\kappa_\theta(\gamma))}$$

for all $\gamma \in \Gamma$.

Notice that if $\gamma \in \Gamma$, we have

$$\begin{aligned}\mu(\mathcal{O}_R(\gamma)) &= \mu(\gamma\gamma^{-1}\,\mathcal{O}_R(\gamma)) = (\gamma^{-1})_*\mu(\gamma^{-1}\,\mathcal{O}_R(\gamma)) \\ &= \int_{\gamma^{-1}\,\mathcal{O}_R(\gamma)} \frac{d(\gamma^{-1})_*\mu}{d\mu} d\mu = \int_{\gamma^{-1}\,\mathcal{O}_R(\gamma)} e^{-\delta\phi B_\theta(\gamma,x)} d\mu(x).\end{aligned}$$

So we need to estimate the Iwasawa cocycle on $\gamma^{-1}\,\mathcal{O}_R(\gamma)$ and also estimate the measure of $\gamma^{-1}\,\mathcal{O}_R(\gamma)$.

We need two lemmas.

Lemma 18.2. *For any $R > 0$ there exists $D = D(R) > 0$ such that: If $x \in \gamma^{-1}\,\mathcal{O}_R(\gamma)$, then*

$$\|\kappa_\theta(\gamma) - B_\theta(\gamma, x)\| \leq D.$$

Proof. Suppose not. Then for each n there exist $\gamma_n \in \Gamma$ and $x_n \in \gamma_n^{-1}\,\mathcal{O}_R(\gamma_n)$ with

$$\|\kappa_\theta(\gamma_n) - B_\theta(\gamma_n, x_n)\| > n.$$

Since

$$(g, x) \in \mathsf{SL}(d,\mathbb{R}) \times \mathcal{F}_\theta \mapsto \kappa_\theta(g) - B_\theta(g, x) \in \mathfrak{a}_\theta$$

is continuous, the sequence $\{\gamma_n\}$ is escaping. Passing to subsequences we can suppose that $\gamma_n^{-1} \to y$ in $\overline{\Gamma}^\theta$ and $x_n \to x$. Then $U_\theta(\gamma_n^{-1}) \to y \in \Lambda_\theta(\Gamma)$ and $\mathrm{d}_\theta(y, x) \geq R^{-1}$. So x and y are transverse.

Using the Busemann-like formula for the Iwasawa cocycle, we can fix $\{h_n\} \subset \mathsf{SL}(d,\mathbb{R})$ such that $U_\theta(h_n) \to x$ and

$$+\infty = \lim_{n\to\infty} \|\kappa_\theta(\gamma_n) - B_\theta(\gamma_n, x_n)\| = \lim_{n\to\infty} \|\kappa_\theta(\gamma_n) - (\kappa_\theta(\gamma_n h_n) - \kappa_\theta(h_n))\|.$$

However,

$$\lim_{n\to\infty} \left\|G_\theta(U_\theta(\gamma_n^{-1}), U_\theta(h_n))\right\| = \|G_\theta(y, x)\| < +\infty$$

and so Proposition 17.2 implies that

$$\lim_{n\to\infty} \|\kappa_\theta(\gamma_n) - (\kappa_\theta(\gamma_n h_n) - \kappa_\theta(h_n))\| < +\infty.$$

So we have a contradiction. □

Lemma 18.3. *For any $R > 0$ sufficiently large,*

$$\inf_{\gamma\in\Gamma} \mu(\gamma^{-1}\,\mathcal{O}_R(\gamma)) > 0.$$

Proof. Suppose not. Then for each $n \geq 1$ there exists $\gamma_n \in \Gamma$ with

$$\mu\big(\{x \in \Lambda_\theta(\Gamma) : \mathrm{d}_\theta(\gamma_n^{-1}, x) > 1/n\}\big) = \mu(\gamma_n^{-1}\,\mathcal{O}_n(\gamma_n)) < \frac{1}{n}.$$

Passing to a subsequence we can suppose that $\gamma_n^{-1} \to y \in \overline{\Gamma}^\theta$. Then

$$\mu(\Lambda_\theta(\Gamma) - \{y\}) = 0.$$

So $\mu = \mathcal{D}_y$ is a Dirac mass at y. Since Γ is non-elementary, there exists $\gamma \in \Gamma$ with $\gamma y \neq y$. Then $\gamma_*\mu = \mathcal{D}_{\gamma y}$ is singular to $\mu = \mathcal{D}_y$, which is a contradiction. □

With the two lemmas, the rest of proof is exactly the same as in the hyperbolic case.

Proof of the Shadow Lemma. Fix $R > 0$ so that

$$m := \inf_{\gamma \in \Gamma} \mu(\gamma^{-1} \mathcal{O}_R(\gamma))$$

is positive. Then if $\gamma \in \Gamma$, we have

$$\begin{aligned}\mu(\mathcal{O}_R(\gamma)) &= \mu(\gamma\gamma^{-1} \mathcal{O}_R(\gamma)) = (\gamma^{-1})_* \mu(\gamma^{-1} \mathcal{O}_R(\gamma)) \\ &= \int_{\gamma^{-1} \mathcal{O}_R(\gamma)} \frac{d(\gamma^{-1})_*\mu}{d\mu} d\mu = \int_{\gamma^{-1} \mathcal{O}_R(\gamma)} e^{-\delta\phi B_\theta(\gamma, x)} d\mu(x).\end{aligned}$$

Thus by the Lemmas,

$$\mu(\mathcal{O}_R(\gamma)) \leq e^{\delta D(R)\|\phi\|} e^{-\delta\phi\kappa_\theta(\gamma)}$$

and

$$\mu(\mathcal{O}_R(\gamma)) \geq m e^{-\delta D(R)\|\phi\|} e^{-\delta\phi\kappa_\theta(\gamma)}.$$

□

19. The Ergodic Dichotomy

In this section we state a version of the Hopf–Sullivan dichotomy for Patterson–Sullivan measures of transverse groups. Unlike the hyperbolic case, the conclusion is not in terms of the dynamics of the product of the Patterson–Sullivan measure with itself, but instead the product of the Patterson–Sullivan measure with a Patterson–Sullivan measure associated to the "opposite" functional.

Given $\phi \in \mathfrak{a}_\theta^*$, let $\bar{\phi} := \phi \circ \iota$ where ι is the opposite involution. Notice that if $\phi = \sum_{\alpha_j \in \theta} c_j \omega_j$, then

$$\bar{\phi} = \sum_{\alpha_j \in \theta} c_{d-j} \omega_j$$

and so $\bar{\phi} \in \mathfrak{a}_\theta^*$.

Observation 19.1. $\delta^\phi(\Gamma) = \delta^{\bar{\phi}}(\Gamma)$.

Proof. Notice that $\phi\kappa(\gamma^{-1}) = \phi\iota\kappa(\gamma) = \bar{\phi}\kappa(\gamma)$ and hence

$$\#\{\gamma \in \Gamma : \phi\kappa(\gamma) \leq R\} = \#\{\gamma \in \Gamma : \bar{\phi}\kappa(\gamma) \leq R\}.$$ □

Recall that a P_θ-transverse group $\Gamma < \mathsf{SL}(d, \mathbb{R})$ acts as a convergence group on its limit set $\Lambda_\theta(\Gamma)$ and so there is a well-defined notion of *conical limit points* $\Lambda_\theta^{\mathrm{con}}(\Gamma)$.

Theorem 19.2 (Canary–Zhang-Z. [CZZ24])**.** *Suppose $\Gamma < \mathsf{SL}(d, \mathbb{R})$ is a non-elementary P_θ-transverse group, $\phi \in \mathfrak{a}_\theta^*$, and $\delta^\phi(\Gamma) < +\infty$.*

Let μ be a ϕ-Patterson–Sullivan measure of dimension β for Γ and let $\bar{\mu}$ be a $\bar{\phi}$-Patterson–Sullivan measure of dimension β for Γ, both supported on $\Lambda_\theta(\Gamma)$. If

$$d\nu(x, y) := e^{\beta\phi G_\theta(x,y)} d\mu(x) \otimes d\bar{\mu}(y)|_{\mathcal{F}_\theta^{(2)}}$$

then ν is a Γ-invariant measure. Moreover, we have the following dichotomy:

1. *If $\sum_{\gamma \in \Gamma} e^{-\beta\phi(\kappa(\Gamma))} = +\infty$, then:*
   - (a) *$\beta = \delta^\phi(\Gamma)$.*
   - (b) *$\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1 = \bar{\mu}(\Lambda_\theta^{\mathrm{con}}(\Gamma))$ and $\mu, \bar{\mu}$ have no atoms.*
   - (c) *The Γ action on $(\mathcal{F}_\theta^{(2)}, \nu)$ is ergodic.*
2. *If $\sum_{\gamma \in \Gamma} e^{-\beta\phi(\kappa(\Gamma))} < +\infty$, then:*
   - (a) *$\beta \geq \delta^\phi(\Gamma)$.*

(b) $\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 0 = \bar{\mu}(\Lambda_\theta^{\mathrm{con}}(\Gamma))$.
(c) *The* Γ *action on* $(\mathcal{F}_\theta^{(2)}, \nu)$ *is non-ergodic.*

For relatively Anosov groups, the Poincaré series diverges at its critical exponent.

Theorem 19.3 (Canary–Zhang-Z. [CZZ25])**.** *Suppose* $\Gamma < \mathsf{SL}(d,\mathbb{R})$ *is a non-elementary relatively* P_θ*-Anosov group. If* $\phi \in \mathfrak{a}_\theta^*$ *and* $\delta^\phi(\Gamma) < +\infty$*, then*

$$\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi\kappa_\theta(\gamma))} = +\infty.$$

Combined with the ergodic dichotomy, we have the following existence/uniqueness result.

Corollary 19.4. *Suppose* $\Gamma < \mathsf{SL}(d,\mathbb{R})$ *is a non-elementary relatively* P_θ*-Anosov group. If* $\phi \in \mathfrak{a}_\theta^*$ *and* $\delta^\phi(\Gamma) < +\infty$*, then there is a unique* ϕ*-Patterson–Sullivan measure* μ *of dimension* $\delta^\phi(\Gamma)$ *supported on* $\Lambda_\theta(\Gamma)$*. Moreover, the* Γ *action on* $(\mathcal{F}_\theta, \mu)$ *is ergodic.*

Proof. The proof is identical to the analogous result for convex cocompact groups in hyperbolic geometry, see Corollary 10.3. □

For hyperconvex Anosov groups, we can use the ergodic dichotomy to determine the critical exponent for α_1.

Theorem 19.5. [PS17, PSW21] *If* $\Gamma < \mathsf{SL}(d,\mathbb{R})$ *is hyperconvex and* $\partial_\infty\Gamma$ *is a circle, then* $\delta^{\alpha_1}(\Gamma) = 1$*.*

Proof. Let $\theta = \{\alpha_1, \alpha_2, \alpha_{d-2}, \alpha_{d-1}\}$. Since $\alpha_1 = 2\omega_1 - \omega_2$, we have $\alpha_1 \in \mathfrak{a}_\theta^*$.

By definition Γ is P_θ-Anosov and by Theorem 5.4 the projective limit set $\Lambda_1(\Gamma)$ is a $\mathcal{C}^1$-smooth submanifold with tangent spaces

$$T_{x^1}\Lambda_1(\Gamma) = x^2$$

for $x \in \Lambda_\theta(\Gamma)$. Hence by Proposition 16.1, there exists a α_1-Patterson–Sullivan measure of dimension 1 for Γ supported on

$$\{(z, T_z\Lambda_1(\Gamma)) : z \in \Lambda_1(\Gamma)\} \subset \mathcal{F}_{1,2},$$

which coincides with $\Lambda_{1,2}(\Gamma)$. Let μ be the push-forward of this measure under the homeomorphism $\Lambda_{1,2}(\Gamma) \to \Lambda_\theta(\Gamma)$. Then μ is also a α_1-Patterson–Sullivan measure of dimension 1. Since Γ is Anosov, $\Lambda_\theta(\Gamma) = \Lambda_\theta^{\mathrm{con}}(\Gamma)$ and so

$$1 = \mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)).$$

Thus the ergodic dichotomy implies that the dimension of μ equals the critical exponent, i.e. $1 = \delta^{\alpha_1}(\Gamma)$.

□

20. Problems

20.1. **Partial Cartan algebras.** Recall that

$$\mathfrak{a}_\theta := \{H \in \mathfrak{a} : \alpha_j(H) = 0 \text{ for all } \alpha_j \notin \theta\}.$$

Problem 25.

(1) Prove that $\{\omega_j|_{\mathfrak{a}_\theta}\}_{\alpha_j\in\theta}$ is a basis of $\mathfrak{a}_\theta^*$.

(2) Prove that there exists a unique linear map $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ such that

$$\omega_j p_\theta = \omega_j$$

for all $\alpha_j \in \theta$.

(3) Show that $\mathrm{Image}(p_\theta) = \mathfrak{a}_\theta$ and $p_\theta^2 = p_\theta$ (i.e. p_θ is a projection).

Problem 26. Suppose $\theta = \{\alpha_1, \alpha_{d-1}\}$ and compute $\mathfrak{a}_\theta$ and p_θ.

20.2. **Patterson–Sullivan measures.**

Problem 27. Prove: If $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is P_θ-transverse, $\theta' \subset \theta$ is non-empty, $\phi \in \mathfrak{a}_{\theta'}^*$, and μ is a ϕ-Patterson-Sullivan measure supported on $\Lambda_\theta(\Gamma)$, then the pushforward of μ under the homeomorphism $\Lambda_\theta(\Gamma) \to \Lambda_{\theta'}(\Gamma)$ is also a ϕ-Patterson-Sullivan measure.

Problem 28. Write down a careful proof of Proposition 15.2 in the special case

$$\sum_{\gamma\in\Gamma} e^{-\delta^\phi(\Gamma)\phi\kappa_\theta(\gamma)} = +\infty.$$

Hint: In this case you can take $\chi \equiv 1$ and try to adapt the proof from the hyperbolic case (see Theorem 8.5).

Problem 29. If $g \in \mathsf{SL}(d,\mathbb{R})$ and $(x,y) \in \mathcal{F}_\theta^{(2)}$, then

$$G_\theta(gx,gy) - G_\theta(x,y) = B_\theta(g,x) + \iota B_\theta(g,y).$$

Problem 30. Prove that the measure $\nu = e^{\beta\phi G_\theta}\mu \otimes \bar{\mu}|_{\mathcal{F}_\theta^{(2)}}$ in the statement of Theorem 19.2 is Γ-invariant.

20.3. **Calculations in the Klein–Beltrami model.** In this set of problems, we will relate the classical Busemann cocycle, Patterson–Sullivan measures, and Gromov product from hyperbolic geometry to the ones introduced for $\mathsf{SL}(d,\mathbb{R})$.

Recall that

$$\langle x,y\rangle = x_1y_1 + \cdots + x_dy_d - x_{d+1}y_{d+1}$$

and we can view the Klein–Beltrami model as the subset

$$\mathbb{H}^d := \{[x:1] \in \mathbb{P}(\mathbb{R}^{d+1}) : \|x\| < 1\} = \{[v] \in \mathbb{P}(\mathbb{R}^d) : \langle v,v\rangle < 0\}$$

of real projective space. Then we can identify

$$\mathsf{Isom}(\mathbb{H}^d) = \mathsf{O}^+(d,1) < \mathsf{SL}(d+1,\mathbb{R}),$$

where $\mathsf{O}^+(d,1) < \mathsf{O}(d,1)$ is the subgroup preserving

$$\{v \in \mathbb{R}^{d+1} : \langle v,v\rangle = 1,\ v_{d+1} > 0\}.$$

We fix the base point $o = [0:\cdots:0:1] \in \mathbb{H}^d$. Recall:

- The distance satisfies

$$\mathrm{d}(p,q) = \cosh^{-1}\left(-\frac{\langle\tilde{p},\tilde{q}\rangle}{\sqrt{\langle\tilde{p},\tilde{p}\rangle\,\langle\tilde{q},\tilde{q}\rangle}}\right)$$

(notice that the formula is independent of which lifts we use, since any difference in scalar cancels).

- The Busemann function based at $x \in \partial\,\mathbb{H}^d$ is defined by

$$b_x(p) = \lim_{q\to x} \mathrm{d}(p,q) - \mathrm{d}(o,q).$$

- The Busemann cocycle $B : \mathsf{Isom}(\mathbb{H}^d) \times \partial\,\mathbb{H}^d \to \mathbb{R}$ is defined by
$$B(g, x) = b_x(g^{-1}o).$$
- The Gromov product of $(x, y) \in \partial^{(2)}\,\mathbb{H}^d$ is defined by
$$G(x, y) = \lim_{p\to x, q\to y} \mathrm{d}(p, o) + \mathrm{d}(o, q) - \mathrm{d}(p, q).$$

Given $x \in \overline{\mathbb{H}^d} \subset \mathbb{P}(\mathbb{R}^{d+1})$, we use $\tilde{x}$ to denote **some** representative of x in $\mathbb{R}^{d+1}$ with positive $(d+1)$-component.

Problem 31. For $x \in \partial\,\mathbb{H}^d$, let
$$V_x = \{v \in \mathbb{R}^{d+1} : \langle v, \tilde{x}\rangle = 0\}.$$
Convince yourself that $T_x\partial\,\mathbb{H}^d = T_x\,\mathbb{P}(V_x)$.

Hint: It may be easier to consider $x = [e_1 + e_{d+1}]$ first.

The main goal of this set of problems is the following.

Theorem 20.1. *Let $\theta = \{\alpha_1, \alpha_{d-1}\}$.*

(1) *If $g \in \mathsf{O}^+(d, 1)$ and $x \in \partial\,\mathbb{H}^d$, then*
$$B(g, x) = \omega_1 B_\theta(g, (x, V_x)).$$
(2) *If $x, y \in \partial\,\mathbb{H}^d$ are distinct, then*
$$G(x, y) = \omega_1 G_\theta((x, V_x), (y, V_y)).$$
(3) *If $\Gamma < \mathsf{O}^+(d, 1)$ is discrete, then Γ is P_θ-transverse and*
$$\Lambda_\theta(\Gamma) = \{(x, V_x) : x \in \Lambda_{\mathbb{H}^d}(\Gamma)\}.$$

Remark 20.2. Notice that $\mathsf{O}^+(d, 1)$ is a subgroup of $\mathsf{SL}^\pm(d+1, \mathbb{R}) = \{g \in \mathsf{GL}(d+1, \mathbb{R}) : \det g = \pm 1\}$, however all our definitions for $\mathsf{SL}(d+1, \mathbb{R})$ extend verbatim to $\mathsf{SL}^\pm(d+1, \mathbb{R})$.

Problem 32. Prove part (3) of Theorem 20.1.

Problem 33.

(1) Show that
$$b_x(p) = \log\left(\frac{\langle\tilde{p}, \tilde{x}\rangle}{\langle\tilde{o}, \tilde{x}\rangle}\sqrt{\frac{\langle\tilde{o}, \tilde{o}\rangle}{\langle\tilde{p}, \tilde{p}\rangle}}\right).$$
(2) Show that
$$G(x, y) = -\log\left(\frac{\langle\tilde{o}, \tilde{o}\rangle\,\langle\tilde{x}, \tilde{y}\rangle}{2\,\langle\tilde{x}, \tilde{o}\rangle\,\langle\tilde{y}, \tilde{o}\rangle}\right).$$
(3) Identify $\mathbb{H}^d$ with the unit ball $\mathbb{B}^d \subset \mathbb{R}^d$ via the map $[x : 1] \mapsto x$. Write down formulas for $b_x(p)$ and $G(x, y)$ in these coordinates.

Hint: Recall that
$$\cosh^{-1}(x) = \log\left(x + \sqrt{x^2 - 1}\right)$$
when $x \geq 1$. Hence if $x_n \to \infty$, then
$$\cosh^{-1}(x_n) - \log(2x_n) \to 0.$$

Problem 34. Show: If $x, y \in \partial\,\mathbb{H}^d$, then

$$\frac{\langle \tilde{x}, \tilde{o}\rangle}{\langle \tilde{y}, \tilde{o}\rangle} = \frac{\|\tilde{x}\|}{\|\tilde{y}\|}.$$

Problem 35. Prove part (1) of Theorem 20.1.

Problem 36. Show that

$$G(x,y) = -\log \frac{\|\tilde{x} \wedge w_1 \wedge \cdots \wedge w_d\|}{\|\tilde{x}\|\,\|w_1 \wedge \cdots \wedge w_d\|}$$

where $V_y = \langle w_1, \ldots, w_d\rangle$ (notice that the right hand side only depends on x and y).

Hint: Let

$$f(x,y) = -\log \frac{\|\tilde{x} \wedge w_1 \wedge \cdots \wedge w_d\|}{\|\tilde{x}\|\,\|w_1 \wedge \cdots \wedge w_d\|}.$$

Show that f and G are both invariant under the action of

$$\left\{ \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} : U \in \mathsf{O}(d) \right\}.$$

Then verify the identity in the special case when $y = [e_1 + e_{d+1}]$.

Problem 37. Prove part (2) of Theorem 20.1.

In the next problem, identify $\mathbb{H}^d$ with the unit ball in $\mathbb{R}^d$ via the map $[x,1] \in \mathbb{H}^d \mapsto x \in \mathbb{R}^d$ and let $\|\cdot\|_{\mathbb{E}}$ denote the standard norm on $\mathbb{R}^d$.

Problem 38. For $g \in \mathsf{Isom}(\mathbb{H}^d)$ and $R > 0$, let

$$\mathcal{O}_R(g) := g\{y \in \partial\,\mathbb{H}^d : \left\|y - g^{-1}(0)\right\|_{\mathbb{E}} > 1/R\}.$$

Compare these new Shadows to the original ones:

(1) For any $R > 0$, there exists $R' > 0$ such that $\mathcal{O}_R(0, g(0)) \subset \mathcal{O}_{R'}(g)$.
(2) For any $R > 0$, there exists $R' > 0$ such that $\mathcal{O}_R(g) \subset \mathcal{O}_{R'}(0, g(0))$.

Bonus Topic: Entropy Rigidity for Hitchin Representations

21. Properties of Shadows

Fix a P_θ-transverse group $\Gamma < \mathsf{SL}(d,\mathbb{R})$ and, as in Section 18, fix a distance d_θ on $\overline{\Gamma}^\theta$ that generates this topology. Also, as in that section, for $\gamma \in \Gamma$ and $R > 0$, let

$$\mathcal{O}_R(\gamma) := \gamma\{x \in \Lambda_\theta(\Gamma) : \mathrm{d}_\theta(\gamma^{-1}, x) > 1/R\}.$$

Lemma 21.1. *If $\{\gamma_n\} \subset \Gamma$ are distinct, then*

$$\lim_{n\to\infty} \operatorname{diam}(\mathcal{O}_R(\gamma_n)) = 0,$$

where the diameter is with respect to d_θ.

Proof. See Problem 39. □

Lemma 21.2. *For any $R > 0$ there exists $R' > 0$ such that:*

(1) *If $\gamma_1, \gamma_2 \in \Gamma$, $\|\kappa_\theta(\gamma_1)\| \le \|\kappa_\theta(\gamma_2)\|$, and $\mathcal{O}_R(\gamma_1) \cap \mathcal{O}_R(\gamma_2) \neq \emptyset$, then*

$$\mathcal{O}_R(\gamma_2) \subset \mathcal{O}_{R'}(\gamma_1).$$

(2) *For any $I \subset \Gamma$ there exists $J \subset I$ such that: the shadows $\{\mathcal{O}_R(\gamma) : \gamma \in J\}$ are disjoint and*

$$\bigcup_{\gamma \in I} \mathcal{O}_R(\gamma) \subset \bigcup_{\gamma \in J} \mathcal{O}_{R'}(\gamma).$$

Proof. Part (1) is surprisingly difficult, for a proof in a more general setting see [BCZZ24, Proposition 5.1]. Part (2) is an exercise (assuming part (1)), see Problem 40. □

Recall that $x \in \Lambda_\theta(\Gamma)$ is a *conical limit point* if there exist $\{\gamma_n\} \subset \Gamma$ and distinct $a, b \in \Lambda_\theta(\Gamma)$ such that $\gamma_n x \to a$ and $\gamma_n y \to b$ for all $y \in \Lambda_\theta(\Gamma) \smallsetminus \{x\}$. Let $\Lambda_\theta^{\mathrm{con}}(\Gamma)$ denote the set of conical limit points.

Then for $R > 0$, let $\Lambda_{\theta,R}^{\mathrm{con}}(\Gamma)$ denote the subset of conical limit points $x \in \Lambda_\theta^{\mathrm{con}}(\Gamma)$ where there exist $\{\gamma_n\} \subset \Gamma$ and $a, b \in \Lambda_\theta(\Gamma)$ such that $\gamma_n x \to a$, $\gamma_n y \to b$ for all $y \in \Lambda_\theta(\Gamma) \smallsetminus \{x\}$ and

$$\mathrm{d}_\theta(a, b) \geq 1/R.$$

Lemma 21.3.

(1) *$\Lambda_{\theta,R}^{\mathrm{con}}(\Gamma)$ is Γ-invariant.*
(2) *$\Lambda_\theta^{\mathrm{con}}(\Gamma) = \cup_{R>0} \Lambda_{\theta,R}^{\mathrm{con}}(\Gamma)$.*
(3) *If $x \in \Lambda_{\theta,R}^{\mathrm{con}}(\Gamma)$ and $R < R'$, then there exist $\{\gamma_n\} \subset \Gamma$ distinct such that*

$$x \in \bigcap_{n \geq 1} \mathcal{O}_{R'}(\gamma_n).$$

(4) *If $x \in \Lambda_\theta(\Gamma)$ and there exist $\{\gamma_n\} \subset \Gamma$ distinct such that*

$$x \in \bigcap_{n \geq 1} \mathcal{O}_R(\gamma_n),$$

then $x \in \Lambda_{\theta,R}^{\mathrm{con}}(\Gamma)$.

Proof. See Problem 41. □

Proposition 21.4. *Suppose $\phi \in \mathfrak{a}_\theta^*$, $\delta^\phi(\Gamma) < +\infty$, and*

$$\sum_{\gamma \in \Gamma} e^{-\delta^\phi(\Gamma)\phi\kappa_\theta(\gamma)} = +\infty.$$

If μ is a ϕ-Patterson–Sullivan measure of dimension $\delta(\Gamma)$ for Γ supported on $\Lambda_\theta(\Gamma)$, then

$$\mu(\Lambda_{\theta,R}^{\mathrm{con}}(\Gamma)) = 1$$

for $R > 0$ sufficiently large.

Proof. Using the ergodic dichotomy, one can show that Γ acts ergodically on $(\mathcal{F}_\theta, \mu)$ (see the proof of Corollary 10.3). The ergodic dichotomy also implies that

$$\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1$$

By Lemma 21.3,

$$\Lambda_\theta^{\mathrm{con}}(\Gamma) = \bigcup_{n=1}^{\infty} \Lambda_{\theta,n}^{\mathrm{con}}(\Gamma)$$

and so there exists some n with $\mu(\Lambda_{\theta,n}^{\mathrm{con}}(\Gamma)) > 0$. Then $\Lambda_{\theta,n}^{\mathrm{con}}(\Gamma)$ has positive measure and is Γ-invariant (by Lemma 21.3). Hence by ergodicity,

$$\mu(\Lambda_{\theta,n}^{\mathrm{con}}(\Gamma)) = 1.$$

□

22. Strict convexity of critical exponents

As an application of the Ergodic Dichotomy, it is possible to show that the critical exponent has a strict convexity property. In what follows, for $g \in \mathsf{SL}(d,\mathbb{R})$, let

$$\lambda(g) := \operatorname{diag}(\log \lambda_1(g), \dots, \log \lambda_d(g))$$

where $\lambda_1(g) \geq \cdots \geq \lambda_d(g)$ are the absolute values of the eigenvalues of g.

Theorem 22.1. [CZZ24] *Assume $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is non-elementary P_θ-transverse. Suppose $\phi_1, \phi_2 \in \mathfrak{a}_\theta^*$ and $\delta^{\phi_1}(\Gamma) = 1 = \delta^{\phi_2}(\Gamma)$. If $\phi := \lambda\phi_1 + (1-\lambda)\phi_2$ for some $\lambda \in (0,1)$, then*

$$\delta^{\phi}(\Gamma) \leq 1.$$

Moreover, if $\delta^{\phi}(\Gamma) = 1$ and $\sum_{\gamma\in\Gamma} e^{-\phi(\kappa(\Gamma))} = +\infty$, then

$$\phi_1(\lambda(\gamma)) = \phi_2(\lambda(\gamma))$$

for all $\gamma \in \Gamma$.

When Γ is Zariski dense, a result of Benoist [Ben04] implies that the set $\{\lambda(\gamma) : \gamma \in \Gamma\}$ spans $\mathfrak{a}$ (in the vector space sense). Further, when Γ is relatively Anosov, the Poincaré series diverges at its critical exponent (see Theorem 19.3). So we have the following corollary.

Corollary 22.2. *Assume $\Gamma < \mathsf{SL}(d,\mathbb{R})$ is a Zariski dense non-elementary relatively P_θ-Anosov. If $\phi_1, \phi_2 \in \mathfrak{a}_\theta^*$ are **distinct** and $\delta^{\phi_1}(\Gamma) = 1 = \delta^{\phi_2}(\Gamma)$, then*

$$\delta^{\lambda\phi_1+(1-\lambda)\phi_2}(\Gamma) < 1$$

for all $\lambda \in (0,1)$.

Now we start the proof of Theorem 22.1. Applying Hölder's inequality with $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$, we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} e^{-s\phi\kappa_\theta(\gamma)} &= \sum_{\gamma\in\Gamma} e^{-s\lambda\phi_1\kappa_\theta(\gamma) - s(1-\lambda)\phi_2\kappa_\theta(\gamma)} \\ &\leq \left(\sum_{\gamma\in\Gamma} e^{-s\phi_1\kappa_\theta(\gamma)}\right)^{\lambda} \left(\sum_{\gamma\in\Gamma} e^{-s\phi_2\kappa_\theta(\gamma)}\right)^{1-\lambda}.\end{aligned}$$

Hence

$$\delta^{\phi}(\Gamma) \leq \max\{\delta^{\phi_1}(\Gamma), \delta^{\phi_2}(\Gamma)\} = 1.$$

The "moreover" part is substantially more difficult. For the rest of the section suppose that

$$\delta^{\phi}(\Gamma) = 1 \quad \text{and} \quad \sum_{\gamma\in\Gamma} e^{-\phi(\kappa(\Gamma))} = +\infty.$$

Let $\bar\mu, \mu, \bar\mu_1, \mu_1, \bar\mu_2, \mu_2$ be 1-dimensional Patterson–Sullivan measures for $\bar\phi, \phi, \bar\phi_1, \phi_1, \bar\phi_2, \phi_2$ supported on $\Lambda_\theta(\Gamma)$, respectively. The key ideas in the proof are

(1) Use the Shadow Lemma to show that the measures

$$\mu \otimes \bar\mu \quad \text{and} \quad \mu_1 \otimes \bar\mu_1 + \mu_2 \otimes \bar\mu_2$$

are non-singular.

(2) Use ergodicity to show that after possibly relabelling
$$e^{\phi G_\theta}\mu\otimes\bar\mu = ce^{\phi_1 G_\theta}\mu_1\otimes\bar\mu_1$$
on $\mathcal{F}^{(2)}_\theta$ for some $c>0$.
(3) Use the Shadow Lemma again to show that
$$\sup_{\gamma\in\Gamma}|\phi\kappa(\gamma)-\phi_1\kappa(\theta)|<+\infty.$$
(4) Use the fact that $\lambda(\gamma)=\lim_{n\to\infty}\frac{1}{n}\kappa(\gamma^n)$ to the finish the proof.

Step 1. By Proposition 21.4 there exists $R_0\geq 1$ such that
$$\mu(\Lambda^{\mathrm{con}}_{\theta,R_0}(\Gamma))=\bar\mu(\Lambda^{\mathrm{con}}_{\theta,R_0}(\Gamma))=1. \tag{4}$$
Fix $R>R_0$ such that the Shadow Lemma holds for any number at least R.

Lemma 22.3. *There exists $C>0$ such that*
$$\mu\otimes\bar\mu\leq C(\mu_1\otimes\bar\mu_1+\mu_2\otimes\bar\mu_2).$$

Proof. It suffices to consider a measurable set of the form $A_1\times A_2\subset\mathcal{F}^2_\theta$. Fix $\eta>0$. By outer regularity, we can find an open neighborhood of the form $U_1\times U_2$ where
$$(\mu_1\otimes\bar\mu_1+\mu_2\otimes\bar\mu_2)(U_1\times U_2)<\eta+(\mu_1\otimes\bar\mu_1+\mu_2\otimes\bar\mu_2)(A_1\times A_2).$$
Let
$$I_i:=\{\gamma\in\Gamma:\mathcal{O}_R(\gamma)\subset U_i\}.$$
Then by Lemma 21.1 and Lemma 21.3 part (3),
$$A_1\times A_2\cap\Lambda^{\mathrm{con}}_{\theta,R_0}(\Gamma)\subset\bigcup_{(\gamma_1,\gamma_2)\in I_1\times I_2}\mathcal{O}_R(\gamma_1)\times\mathcal{O}_R(\gamma_2).$$
Let $R'>R$ and $J_i\subset I_i$ satisfy part (2) of Lemma 21.2. Then by Equation (4) and the Shadow Lemma,
$$\begin{aligned}\mu\otimes\bar\mu(A_1\times A_2)=\mu\otimes\bar\mu(A_1\times A_2\cap\Lambda^{\mathrm{con}}_{\theta,R_0}(\Gamma))&\leq\sum_{(\gamma_1,\gamma_2)\times J_1\times J_2}\mu(\mathcal{O}_{R'}(\gamma_1))\bar\mu(\mathcal{O}_{R'}(\gamma_2))\\
&\lesssim\sum_{(\gamma_1,\gamma_2)\in J_1\times J_2}e^{-\phi\kappa_\theta(\gamma_1)}e^{-\bar\phi\kappa_\theta(\gamma_2)}\\
&=\sum_{(\gamma_1,\gamma_2)\in J_1\times J_2}e^{-\lambda\phi_1\kappa_\theta(\gamma_1)}e^{-(1-\lambda)\phi_2\kappa_\theta(\gamma_1)}e^{-\lambda\bar\phi_1\kappa_\theta(\gamma_2)}e^{-(1-\lambda)\bar\phi_2\kappa_\theta(\gamma_2)}\\
&\lesssim\sum_{(\gamma_1,\gamma_2)\in J_1\times J_2}\mu_1(\mathcal{O}_R(\gamma_1))^\lambda\mu_2(\mathcal{O}_R(\gamma_1))^{1-\lambda}\bar\mu_1(\mathcal{O}_R(\gamma_2))^\lambda\bar\mu_2(\mathcal{O}_R(\gamma_2))^{1-\lambda}.\end{aligned}$$
Then by the weighted Arithmetic Mean-Geometric Mean Inequality,
$$\begin{aligned}\mu\otimes\bar\mu(A_1\times A_2)&\lesssim\sum_{(\gamma_1,\gamma_2)\in J_1\times J_2}\mu_1(\mathcal{O}_R(\gamma_1))\bar\mu_1(\mathcal{O}_R(\gamma_2))+\mu_2(\mathcal{O}_R(\gamma_1))\bar\mu_2(\mathcal{O}_R(\gamma_2))\\
&=\sum_{(\gamma_1,\gamma_2)\in J_1\times J_2}(\mu_1\otimes\bar\mu_1+\mu_2\otimes\bar\mu_2)(\mathcal{O}_R(\gamma_1)\times\mathcal{O}_R(\gamma_2))\\
&\leq(\mu_1\otimes\bar\mu_1+\mu_2\otimes\bar\mu_2)(U_1\times U_2)<\eta+(\mu_1\otimes\bar\mu_1+\mu_2\otimes\bar\mu_2)(A_1\times A_2).\end{aligned}$$
Since $\eta>0$ was arbitrary, the proof is complete. □

The above Lemma implies that $\nu := e^{\phi G_\theta}\mu \otimes \bar{\mu}|_{\mathcal{F}_\theta^{(2)}}$ is non-singular to at least one of $\nu_1 := e^{\phi_1 G_\theta}\mu_1 \otimes \bar{\mu}_1|_{\mathcal{F}_\theta^{(2)}}$ or $\nu_2 := e^{\phi_2 G_\theta}\mu_2 \otimes \bar{\mu}_2|_{\mathcal{F}_\theta^{(2)}}$. By relabelling, we can suppose that ν and ν_1 are non-singular.

Step 2. Since $\mu(\Lambda_\theta^{\mathrm{con}}(\Gamma)) = 1$, we must have $\mu_1(\Lambda_\theta^{\mathrm{con}}(\Gamma)) > 0$. Then, by the ergodic dichotomy, the Γ action on ν and ν_1 are both ergodic.

Lemma 22.4. *$\nu = c\nu_1$ for some $c > 0$.*

Proof. We first show that $\nu \ll \nu_1$. If not, there exists a measurable set $A \subset \mathcal{F}_\theta^{(2)}$ with $\nu(A) \neq 0$ and $\nu_1(A) = 0$. Then, since ν_1 is Γ-invariant and Γ is countable, we have $\nu_1(\Gamma \cdot A) = 0$. Also, by ergodicity, any Γ-invariant set has either full or zero ν-measure. Hence $\Gamma \cdot A$ has full measure. Thus ν and ν_1 are singular, which is a contradiction.

Now $\nu = f\nu_1$ for a unique $f \in L^1(\nu_1)$. By Γ-invariance, if $\gamma \in \Gamma$, then

$$\nu = \gamma_*\nu = \gamma_*(f\nu_1) = f \circ \gamma^{-1}\gamma_*\nu_1 = f \circ \gamma^{-1}\nu_1.$$

So by uniqueness, f is Γ-invariant. Hence, by ergodicity, f is constant almost everywhere. Hence $\nu = c\nu_1$ for some $c > 0$.

□

Step 3.

Lemma 22.5. $\sup_{\gamma\in\Gamma} |\phi\kappa_\theta(\gamma) - \phi_1\kappa_\theta(\gamma)| < +\infty$.

Proof. Suppose not. Then there exists a sequence of distinct elements $\{\gamma_n\} \subset \Gamma$ such that

$$|\phi\kappa_\theta(\gamma_n) - \phi_1\kappa_\theta(\gamma_n)| > n.$$

Fix $R > 0$ large enough to satisfy the Shadow Lemma. By Lemma 21.1,

$$\lim_{n\to\infty} \operatorname{diam} \mathcal{O}_R(\gamma_n) = 0.$$

So, after passing to a subsequence, we can fix $y \in \Lambda_\theta(\Gamma)$ and an open neighborhood U of y such that

$$\overline{U} \cap \overline{\cup_{n\geq 1} \mathcal{O}_R(\gamma_n)} = \emptyset.$$

Since $G_\theta : \mathcal{F}_\theta^{(2)} \to \mathfrak{a}_\theta$ is continuous, then there exists $C > 1$ such that

$$e^{\phi G_\theta(x,y)}, e^{\phi_1 G_\theta(x,y)} \in [C^{-1}, C]$$

when $(x, y) \in \cup_{n\geq 1} \mathcal{O}_R(\gamma_n) \times U$.

One can show that $\Gamma \cdot U \supset \Lambda_\theta(\Gamma)$ and hence we must have $\bar{\mu}(U) > 0$ and $\bar{\mu}_1(U) > 0$. Now by the Shadow Lemma,

$$e^{-\phi\kappa_\theta(\gamma_n)} \asymp \mu(\mathcal{O}_R(\gamma_n))\bar{\mu}(U) \asymp \nu(\mathcal{O}_R(\gamma_n) \times U)$$

and

$$e^{-\phi_1\kappa_\theta(\gamma_n)} \asymp \mu_1(\mathcal{O}_R(\gamma_n))\bar{\mu}_1(U) \asymp \nu_1(\mathcal{O}_R(\gamma_n) \times U).$$

Then since $\nu = c\nu_1$, we see that

$$|\phi\kappa_\theta(\gamma_n) - \phi_1\kappa_\theta(\gamma_n)|$$

is bounded, which is a contradiction. □

Step 4: Finishing the proof. Now we use the following general fact.

Theorem 22.6. *If $g \in \mathsf{SL}(d,\mathbb{R})$, then*

$$\lim_{n\to\infty} \frac{1}{n}\kappa(g^n) = \lambda(n).$$

Proof. Check! □

Now using Lemma 22.5 we have

$$\phi(\lambda(\gamma)) = \phi_1(\lambda(\gamma))$$

for all $\gamma \in \Gamma$. Since $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$, this implies that

$$\phi(\lambda(\gamma)) = \phi_1(\lambda(\gamma))$$

for all $\gamma \in \Gamma$.

23. Entropy Rigidity for Hitchin Representations

For notational convenience we identify

$$\mathfrak{a} = \left\{ (a_1,\dots,a_d) \in \mathbb{R}^d : \sum_{j=1}^d a_j = 0 \right\}.$$

Let

$$\vec{v}_d := \frac{1}{2}(d-1, d-3, \dots, 3-d, 1-d) = \sum_{j=1}^d \frac{d+1-2j}{2} e_j.$$

We consider the symmetric space

$$X_d := \mathsf{PSL}(d,\mathbb{R})/\mathsf{PSO}(d),$$

with base point $o = [\mathsf{PSO}(d)]$ and distance

$$\mathrm{d}_{X_d}(go, ho) = \frac{1}{\|\vec{v}_d\|_{\mathbb{E}}} \left\|\kappa(g^{-1}h)\right\|_{\mathbb{E}} = \frac{1}{\|\vec{v}_d\|} \sqrt{\sum_{j=1}^d [\log \sigma_j(g^{-1}h)]^2}.$$

Example 23.1. When $d = 2$, we have $\vec{v}_2 = (1/2, 1/2)$ and $\sigma_2(g) = \sigma_1(g)^{-1}$ for all $g \in \mathsf{PSL}(2,\mathbb{R})$. Hence $\|\vec{v}_2\|_{\mathbb{E}} = 2^{-1/2}$ and

$$\begin{aligned} \mathrm{d}_{X_2}(go, ho) &= \sqrt{2}\sqrt{[\log \sigma_1(g^{-1}h)]^2 + [\log \sigma_1(g^{-1}h)]^2} \\ &= 2\log \sigma_1(g^{-1}h). \end{aligned}$$

Further, one can show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot o \mapsto \frac{ai+b}{ci+d} \in \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

induces an isometry between X_2 and the hyperbolic plane.

Given a subgroup $\Gamma < \mathsf{PSL}(d,\mathbb{R})$, the *symmetric space critical exponent* is

$$\delta_{X_d}(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma : \mathrm{d}_{X_d}(\gamma o, o) \le R\}.$$

As before let $\Phi_d : \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$ denote the standard irreducible representation (see Section 4).

Theorem 23.2 (Potrie–Sambarino [PS17])**.** *If $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is Hitchin, then*

$$\delta_{X_d}(\rho(\Gamma)) \leq 1$$

with equality if and only if $\rho(\Gamma)$ can be conjugated into $\Phi_d(\mathsf{PSL}_d(\mathbb{R}))$.

Consider the function $\phi = \frac{1}{\|\vec{v}_d\|_{\mathbb{E}}^2} \langle \vec{v}_d, \cdot \rangle \in \mathfrak{a}^*$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$. Then

$$\phi(\kappa(g)) = \frac{1}{\|\vec{v}_d\|_{\mathbb{E}}^2} \langle \vec{v}_d, \kappa(g) \rangle \leq \frac{1}{\|\vec{v}_d\|_{\mathbb{E}}} \|\kappa(g)\|_{\mathbb{E}} = \mathrm{d}_{X_d}(go, o).$$

Hence

$$\delta_{X_d}(\Gamma) \leq \delta^{\phi}(\Gamma). \tag{5}$$

Thus Theorem 23.2 reduces to the following.

Theorem 23.3. *Suppose $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is Hitchin.*

1. *$\delta^{\phi}(\rho(\Gamma)) \leq 1$ with equality if and only if $\rho(\Gamma)$ can be conjugated into $\Phi_d(\mathsf{PSL}_d(\mathbb{R}))$.*
2. *$\delta_{X_2}(\rho(\Gamma)) = 1$ if and only if $\delta^{\phi}(\Gamma) = 1$.*

Suppose that $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is Hitchin.

23.1. **Proof of Theorem 23.3 part (1).** We start by expressing ϕ in terms of the basis $\alpha_1, \ldots, \alpha_{d-1}$.

Proposition 23.4.

$$\phi = \frac{1}{2\|\vec{v}_d\|_{\mathbb{E}}^2} \left(\sum_{j=1}^{\lfloor \frac{d-1}{2} \rfloor} (d+1-2j)(\alpha_j + \cdots + \alpha_{d-j}) + \left(d - 2\left\lfloor \frac{d-1}{2} \right\rfloor - 1 \right) \alpha_{\lfloor d/2 \rfloor} \right).$$

Moreover, the sum of the coefficients relative to the basis $\alpha_1, \ldots, \alpha_{d-1}$ is 1.

Proof. Check! □

Each representation $\wedge^j \rho : \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is hyperconvex (see Example 5.2) and hence by Theorem 19.5 we have

$$1 = \delta^{\alpha_1}(\wedge^j \rho(\Gamma)) = \delta^{\alpha_j}(\rho(\Gamma))$$

for all $j = 1, \ldots, d-1$.

Since $\phi = \sum c_j \alpha_j$ where $\sum_{j=1}^{d} c_j = 1$ and $c_1, \ldots, c_{d-1} > 0$, Theorem 22.1 implies that

$$\delta^{\phi}(\rho(\Gamma)) \leq 1$$

with equality if and only if

$$\alpha_j(\lambda(g)) = \alpha_i(\lambda(g))$$

for all i, j and $g \in \rho(\Gamma)$.

Now we use two theorems.

Theorem 23.5 (Sambarino [Sam24, Theorem A])**.** *The Zariski closure of $\rho(\Gamma)$ is conjugate to either $\Phi_d(\mathsf{PSL}(2,\mathbb{R}))$, $\mathsf{PSL}(d,\mathbb{R})$, $\mathsf{PSp}(2n,\mathbb{R})$ when $d = 2n$, $\mathsf{PSO}(n, n-1)$ when $d = 2n-1$, or G_2 when $n = 7$.*

Theorem 23.6 (Benoist [Ben97])**.** *Suppose $G < \mathsf{PSL}(d,\mathbb{R})$ is a subgroup with Zariski closure $\overline{G}^{\mathrm{Zar}}$. If there exist a function $\ell \in \mathfrak{a}^*$ such that*

$$\ell(\lambda(g)) = 0$$

for all $g \in G$, then

$$\ell(\lambda(g)) = 0$$

for all $g \in \overline{G}^{\mathrm{Zar}}$.

By Benoist's theorem and the discussion above, we see that $\delta^\phi(\rho(\Gamma)) = 1$ if and only if

$$\alpha_j(\lambda(g)) = \alpha_i(\lambda(g))$$

for all i, j and $g \in \overline{\rho(\Gamma)}^{\mathrm{Zar}}$. For each group in Sambarino's theorem not equal to $\Phi_d(\mathsf{PSL}(2,\mathbb{R}))$ one can explicitly construct an element g in the group and i, j such that

$$\alpha_j(\lambda(g)) \neq \alpha_i(\lambda(g)).$$

So we see that $\delta^\phi(\rho(\Gamma)) = 1$ if and only if $\overline{\rho(\Gamma)}^{\mathrm{Zar}}$ is conjugate to $\Phi_d(\mathsf{PSL}(2,\mathbb{R}))$.

23.2. **Proof of Theorem 23.3 part (2).** Since

$$\delta_{X_d}(\rho(\Gamma)) \leq \delta^\phi(\rho(\Gamma))$$

(see Equation (5)), we see that $\delta_{X_2}(\rho(\Gamma)) = 1$ implies that $\delta^\phi(\rho(\Gamma)) = 1$.

Suppose that $\delta^\phi(\rho(\Gamma)) = 1$. Then by part (1), there exists $h \in \mathsf{PSL}(d,\mathbb{R})$ such that

$$h\rho(\Gamma)h^{-1} < \Phi_d(\mathsf{PSL}(2,\mathbb{R})).$$

Since

$$\left|\mathrm{d}_{X_d}(hgh^{-1}o, o) - \mathrm{d}_{X_d}(go, o)\right| \leq 2\,\mathrm{d}_{X_d}(o, ho)$$

we have $\delta_{X_d}(\rho(\Gamma)) = \delta_{X_d}(h\rho(\Gamma)h^{-1})$. Thus we can assume that $\rho(\Gamma) < \Phi_d(\mathsf{PSL}(2,\mathbb{R}))$.

Lemma 23.7. *There exists $C_1 = C_1(d) > 1$ such that*

$$\frac{1}{C_1}\sigma_1(g)^{d+1-2j} \leq \sigma_j(\Phi_d(g)) \leq C_1\sigma_1(g)^{d+1-2j}$$

for all $g \in \mathsf{PSL}(2,\mathbb{R})$.

Proof. Check! □

Then

$$\begin{aligned}\|\kappa(\Phi_d(g)) - 2\log\sigma_1(g)\vec{v}_d\|_{\mathbb{E}} &= \sqrt{\sum_{j=1}^{d}[\log\sigma_j(\Phi_d(g)) - \log\sigma_1(g)^{d+1-2j}]^2} \\ &\leq \|(\log C_1, \ldots, \log C_1)\|_{\mathbb{E}} = \sqrt{d}\log C_1.\end{aligned}$$

Thus $\kappa(\Phi_d(g))$ is essentially in same the direction as $\vec{v}_d$. So there exist a constant $B > 0$ such that

$$\phi(\kappa(\Phi_d(g))) = \frac{1}{\|\vec{v}_d\|_{\mathbb{E}}^2}\langle\kappa(\Phi_d(g)), \vec{v}_d\rangle \geq \frac{1}{\|\vec{v}_d\|_{\mathbb{E}}^2}\|\kappa(\Phi_d(g))\|\,\|\vec{v}_d\|_{\mathbb{E}} - B = \mathrm{d}_{X_d}(\Phi_d(g)o, o) - B$$

for all $g \in \mathsf{PSL}(2,\mathbb{R})$. Finally, this implies that

$$1 = \delta^\phi(\rho(\Gamma)) \leq \delta_{X_d}(\rho(\Gamma)).$$

24. Problems

Problem 39. Prove Lemma 21.1.

Hint: use Proposition 1.4.

Problem 40. Prove Lemma 21.2 part (2).

Problem 41. Prove Lemma 21.3.

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