NOTES ON INFINITE SEQUENCES AND SERIES OF FUNCTIONS

1. Sequences and Series of Functions

1.1. Pointwise and Uniform convergence of functions.

Let $\{f_n\}$ be a sequence of functions defined on a subset $E \subset \mathbb{C}$. We say that the sequence converges *pointwise* to a function g if for every $z \in E$, $\lim_{n\to\infty} f_n(z) = g(z)$. Precisely, this means:

$$\big(\forall z \in E\big)\big(\forall \epsilon > 0\big)\big(\exists N\big)\big(n \geq N \Longrightarrow |f_n(x) - g(x)| < \epsilon\big). \tag{1.1}$$

We say that the sequence converges *uniformly* to a function g if the N can depend on ϵ, but not on x:

$$\big(\forall \epsilon > 0\big)\big(\exists N\big)\big(\forall z \in E\big)\big(n \geq N \Longrightarrow |f_n(x) - g(x)| < \epsilon\big). \tag{1.2}$$

Although the definitions in (1.1) and (1.2) look very similar, there is an important difference. In the first definition for pointwise convergence, the choice of N can depend on both the quantity $\epsilon > 0$ and also on the point $z \in E$. For uniform convergence, the choice of N depends *only* on ϵ, and is independent of z.

Example 1. Let $f_n(x) = x^n$, and let

$$f_0(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

Then

(A) $\lim_{n\to\infty} f_n = f_0$ pointwise on the interval $[0, 1]$, but this convergence is NOT uniform.

(B) If $0 < r < 1$, then $\lim_{n\to\infty} f_n = f_0$ uniformly on the interval $[0, r]$.

In particular, this shows

The pointwise limit of a sequence of continuous functions need not be continuous. (1.3)

Example 2. Let $g_n(x) = \dfrac{1}{n}\sin(nx)$. Then $\lim_{n\to\infty} g_n = 0$ uniformly on the whole real line. Note that the limit is continuous. However $g_n'(x) = \cos(nx)$ does not converge anywhere except at $x = 0$.

In particular, this shows

The derivative of the uniform limit of a sequence of differentiable functions need not be equal to the limit of the sequence of derivatives. (1.4)

Example 3. Let

$$h_n(x) = \begin{cases} 0 & \text{if } -\infty < x \leq n, \\ n(x-n) & \text{if } n \leq x \leq n+1, \\ n & \text{if } x \geq n+1. \end{cases}$$

Then $\lim_{n\to\infty} h_n = 0$ pointwise.

Example 4. For $x \geq 0$ let

$$k_n(x) = n\left(\frac{e}{n}\right)^n x^n e^{-x}.$$

Note that $k_n(0) = 0$ and for each fixed n, $\lim\limits_{x\to\infty} k_n(x) = 0$. Clearly $k_n(x) \geq 0$ for all $x \geq 0$. We find the maximum value of k_n:

$$k_n'(x) = n\left(\frac{e}{n}\right)^n \left[nx^{n-1}e^{-x} - x^n e^{-x}\right] = n\left(\frac{e}{n}\right)^n x^{n-1}e^{-x}\left[n-x\right].$$

Thus $k_n'(x) = 0$ if and only if $x = 0$ or $x = n$. The maximum must occur at $x = n$, and we have

$$k_n(n) = n\left(\frac{e}{n}\right)^n n^n e^{-n} = n.$$

Thus $\lim_{n\to\infty} \sup_{x\geq 0} k_n(x) = +\infty$. However, for any *fixed* $x \geq 0$, we have

$$n > 2ex \quad \Longrightarrow \quad \left(\frac{x}{n}\right) < \frac{1}{2e} \quad \Longrightarrow \left(\frac{x}{n}\right)^n < \left(\frac{1}{2e}\right)^n.$$

Thus if $n > 2ex$ we ahve

$$k_n(x) = n\left(\frac{e}{n}\right)^n x^n e^{-x} = n\,e^n\left(\frac{x}{n}\right)^n e^{-x} < n\,e^n\left(\frac{1}{2e}\right)^n e^{-x} \leq \frac{n}{2^n}.$$

It follows that $\lim_{n\to\infty} k_n(x) \to 0$ pointwise on $[0,\infty)$.

1.2. **Uniform convergence and continuity.**

The difficulty encountered in the statement in (1.4) is removed if we replace pointwise convergence with uniform convergence. We have the following important result.

Theorem 1.1. *Suppose that $\{f_n\}$ is a sequence of functions defined on a set $E \subset \mathbb{C}$. Suppose that each f_n is continuous at a point $p \in E$, and that $\lim_{n\to\infty} f_n = f_0$ uniformly on E. Then f_0 is also continuous at p. In particular, if each f_n is continuous on all of E, then the uniform limit f_0 is also continuous on E.*

Proof. Choose $\epsilon > 0$. Since $f_n \to f_0$ uniformly on E, there exists N so that for all $n \geq N$ we have $|f_n(t) - f_0(t)| < \frac{1}{3}\epsilon$ for all $z \in E$. Now the function f_N is continuous at p, so there exists $\delta > 0$ so that if $|z-p| < \delta$ then $|f_N(p) - f_N(z)| < \frac{1}{3}\epsilon$.

Now suppose $|p - z| < \delta$. Then

$$\begin{aligned}|f_0(p) - f_0(x)| &= |f_0(p) - f_N(p) + f_N(p) - f_N(x) + f_N(x) - f_0(x)| \\ &\leq |f_0(p) - f_N(p)| + |f_N(p) - f_N(x)| + |f_N(x) - f_0(x)| \\ &\leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon.\end{aligned}$$

Thus f_0 is continuous at p. □

Corollary 1.2. *Suppose $\{u_k\}$ is a sequence of continuous functions on an set $E \subset \mathbb{C}$. If the infinite series $\sum_{n=1}^{\infty} u_n$ converges uniformly on E, then the sum is also a continuous function on I.*

1.3. **Uniform convergence and integration.**

The following theorem contains the main results about convergence of power series.

Theorem 1.3. *Suppose that $\{f_n\}$ is a sequence of continuous functions on an interval $[a,b]$ and suppose $f_n \to f_0$ uniformly on $[a,b]$. Define*

$$g_n(x) = \int_a^x f_n(t)\,dt,$$
$$g_0(x) = \int_a^x f_0(t)\,dt.$$

Then the sequence $\{g_n\}$ converges uniformly to g_0 on the interval $[a,b]$.

Proof. Let $\epsilon > 0$. Since $f_n \to f_0$ uniformly on I, there exists a positive integer N so that for all $n \geq N$ we have $|f_n(t) - f_0(t)| < \frac{\epsilon}{(b-a)}$ for all $t \in [a,b]$. Then for any $x \in [a,b]$, if $n \geq N$ we have

$$\begin{aligned}|g_n(x) - g_0(x)| &= \Big| \int_a^x f_n(t)\,dt - \int_a^b f_0(t)\,dt \Big| \\ &= \Big| \int_a^x (f_n(t) - f_0(t))\,dt \Big| \\ &\leq \int_a^x \big| f_n(t) - f_0(t) \big|\,dt \\ &\leq \int_a^b \frac{\epsilon}{(b-a)}\,dt = \epsilon.\end{aligned}$$

It follows that g_n converges uniformly to g_0 on the interval [a,b]. □

1.4. **Uniform convergence and differentiation.**

Theorem 1.4. *Suppose that $\{f_n\}$ is a sequence of continuously differentiable functions on an interval $[a,b]$. Suppose that f_n converges uniformly to f_0 and f_n' converges uniformly to g_0. Then f_0 is differentiable, and $f_0'(x) = g\prime_0(x)$.*

Proof. We have

$$f_n(x) = f_n(x) + \int_a^x f_n'(t)\,dt.$$

Taking the limit as $n \to \infty$, it follows that

$$\begin{aligned}f_0(x) &= \lim_{n\to\infty} f_n(x) \\ &= \lim_{n\to\infty} f_n(a) + \lim_{n\to\infty} \int_a^x f_n'(t)\,dt \\ &= f_0(a) + \int_a^x \lim_{n\to\infty} f_n'(t)\,dt \\ &= f_0(a) + \int_a^x g_0(t)\,dt.\end{aligned}$$

It follows from the Fundamental Theorem of Calculus that f_0 is differentiable, and that $f_0'(x) = g_0(x)$. □

Definition 1.5. *Let $\{u_n\}$ be an infinite sequence of functions defined on a set E of real or complex numbers. Let $S_N(x) = \sum_{n=1}^N u_n(x)$.*

(A) *The infinite series $\displaystyle\sum_{n=1}^{\infty} u_n(x)$ converges* ***pointwise*** *to $S(x)$ if the sequence of functions $\{S_N(x)\}$ converges pointwise on E to $S(x)$. That if, for every $x \in E$ and for every $\epsilon > 0$ there exists a positive integer N so that if $k \geq n$ then $|S_k(x) - S(x)| < \epsilon$.*

(B) *The infinite series $\displaystyle\sum_{n=1}^{\infty} u_n(x)$ converges* ***uniformly*** *to $S(x)$ if the sequence of functions $\{S_N(x)\}$ converges uniformly on E to $S(x)$. That is, if for every $\epsilon > 0$ there exists a positive integer N so that if $k \geq N$ and if $x \in E$ then $|S_k(x) - S(x)| < \epsilon$.*

Suppose $\{u_n\}$ is an infinite sequence of continuous functions defined on an interval I. We then have the following facts:

(1) If $\displaystyle\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on I, then the infinite sum defines a continuous function on I.

(2) If $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on I, and if $v_n(x) = \int_a^x u_n(t)\,dt$, then

$$\int_a^x \Big[\sum_{n=1}^{\infty} u_n(t)\Big]\,dt = \sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt = \sum_{n=1}^{\infty} v_n(x).$$

(3) $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on I, and if $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on I, then the function $\sum_{n=1}^{\infty} u_n(x)$ is differentiable and

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty} \frac{du_n}{dx}(x) = \sum_{n=1}^{\infty} u_n'(x).$$

We need a criterion for deciding when an infinite series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly.

Theorem 1.6 (The Weierstrass M-test)**.** *Suppose an infinite series of functions $\sum_{n=1}^{\infty} u_n$ converges pointwise on a set E. Suppose also that there is an infinite sequence of non-negative constants $\{M_n\}$ such that*

(a) The series $\sum_{n=1}^{\infty} M_n$ converges.

(b) For every $x \in E$ and all $n \geq 1$ we have

$$0 \leq |u_n(x)| \leq M_n.$$

Then the series $\sum_{n=1}^{\infty} u_n(x)$ converges absolutely and uniformly on the set E.

Proof. We hgave

$$\begin{aligned}\left|\sum_{n=1}^{\infty} u_n(x) - \sum_{n=1}^{N} u_n(x)\right| &= \left|\sum_{n=N+1}^{\infty} u_n(x)\right| \\ &\leq \sum_{n=N+1}^{\infty} |u_n(x)| \\ &\leq \sum_{n=N+1}^{\infty} M_n.\end{aligned}$$

This last can be made as small as we like since the series $\sum_{n=1}^{\infty} M_n$ converges. □

2. Power Series

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z-c)^n \tag{2.1}$$

where the coefficients $\{a_n\}$ and the 'center' c are complex numbers. We discuss the following questions about such series:

(1) For which values of z does this series converge?

(2) If we define a function f by setting $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ for those complex numbers z for which the series converges, what are the properties of the function f? In particular, is f continuous, and can one integrate or differentiate f by integrating or differentiating the infinite series in (2.1) term by term?

(3) Given a function g, can we write g as a power series? In particular, can one expand familiar functions such as e^x, $\cos(x)$, $\sin(x)$, $\log(1+x)$, $\arctan(x)$ in power series?

2.1. **Convergence.**

The following theorem contains the main results about convergence:

Theorem 2.1. *Let $\sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series. Then there is a "number" $r \in [0,+\infty]$, called the **radius of convergence**, with the following properties.*

(a) *The series converges absolutely for every z such that $|z-c| < r$.*

(b) *The series diverges for every z such that $|z-c| > r$; in fact, if $|z-c| > r$, the terms $\{a_n(z-c)^n\}$ do not go to zero.*

(c) *For any real number r_1 such that $0 < r_1 < r$, the series converges uniformly on the closed disk*

$$\{z \in \mathbb{C} \,\big|\, |z-c| \le r_1\}.$$

If $r = 0$, the series converges if and only if $z = c$, while if $r = +\infty$, the series converges absolutely for all z and converges uniformly on any bounded set of complex numbers.

The proof Theorem 2.1 is based on the following result.

Lemma 2.2. *Suppose that the power series $\sum_{n=0}^{\infty} a_n(z-c)^n$ converges for some complex number z_0. Let $\rho = |z_0 - c|$. Then the series converges absolutely for any z such that $|z-c| < \rho$, and if $0 < \rho_1 < \rho$, the series converges uniformly on the closed disk $\{z \in \mathbb{C} \,\big|\, |z-c| \le \rho_1\}$.*

Proof. If $\sum_{n=0}^{\infty} a_n(z_0-c)^n$ converges, then $\lim_{n\to\infty} a_n(z_0-c)^n = 0$. It follows that there exists N so that for all $n \ge N$ we have

$$|a_n(z_0-c)^n| \le 1.$$

Then

$$n \ge N,\ |z-c| \le \rho_1 < |z_0 - a| \quad\Longrightarrow\quad |a_n(z-c)^n| = |a_n(z_0-c)^n| \, \frac{|z-c|^n}{|z_0-c|^n} \le \left(\frac{\rho_1}{|z_0-c|}\right)^n.$$

Since $\dfrac{\rho_1}{|z_0-c|} < 1$, the geometric series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{\rho_1}{|z_0-c|}\right)^n$ converges. This shows that $\sum_{n=0}^{\infty} a_n(z-c)^n$ converges absolutely, and it follows from the Weierstrass M-test that that the series converges uniformly. □

The following can be useful in computing the radius of convergence

Lemma 2.3. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series, and suppose $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists. Then $r = \rho^{-1}$ is the radius of convergence.*

Proof. Suppose $|z| < \rho^{-1}$. Then

$$\sqrt[n]{|a_n z^n|} = \sqrt[n]{|a_n|}|z| \longrightarrow \rho|z| < 1,$$

and so there exists a positive integer N and a real number β with $\rho|z| < \beta < 1$ so that for $n \ge N$ we have $\sqrt[n]{|a_n z^n|} < \beta$. Then $|a_n z^n| < \beta^n$, and so $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

On the other hand, suppose $|z| > \rho^{-1}$. Then

$$\sqrt[n]{|a_n z^n|} = \sqrt[n]{|a_n|}|z| \longrightarrow \rho|z| > 1,$$

and so there exists a positive integer N and a real number β with $\rho|z| > \beta > 1$ so that for $n \ge N$ we have $\sqrt[n]{|a_n z^n|} > \beta$. Then $|a_n z^n| > \beta^n \to \infty$, and so $\sum_{n=0}^{\infty} a_n z^n$ diverges. This completes the proof. □

2.2. **Properties of functions given by power series.**

We first observe that a function given by a power series must be continuous insider the radius of convergence. If the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-c)^n$ is $r > 0$, then for any $0 < r_1 < r$, the series converges uniformly on the set $\{z \in \mathbb{C} \,\big|\, |z-c| \le r_1\}$. That is, the sequence of partial sums $S_N(z) = \sum_{k=0}^{N} a_k(z-c)^k$ converges uniformly to f on $\{z \in \mathbb{C} \,\big|\, |z-c| \le r_1\}$. But each partial sum is a continuous function, and it follows that the limit is also continuous. This give the following.

Theorem 2.4. *Suppose that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-c)^n$ is $r > 0$. If we define a function f by setting*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n,$$

then f is a continuous function on the disk $\{z \in \mathbb{C} \,\big|\, |z-c| < r\}$.

Next, we investigate termwise integration and differentiation. Given a power series $\displaystyle\sum_{n=0}^{\infty} a_n(z-\alpha)^n$, we can formally integrate or differentiate the series term by term to get two new power series:

$$\sum_{n=1}^{\infty} n a_n(z-\alpha)^{n-1} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-\alpha)^{n+1}.$$

Lemma 2.5. *The power series $\displaystyle\sum_{n=0}^{\infty} a_n(z-\alpha)^n$, $\displaystyle\sum_{n=1}^{\infty} n a_n(z-\alpha)^{n-1}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-\alpha)^{n+1}$ all have the same radius of convergence.*

Proof. Without loss of generality, we can take $\alpha = 0$. We shall consider only the case in which the radius of convergence is given by $\left(\lim_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}$. But then

$$\lim_{n\to\infty} \sqrt[n]{n|a_n|} = \lim_{n\to\infty} \sqrt[n]{n} \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|},$$
$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n+1}|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n+1}} \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

□

As a consequence, we have the following result.

Lemma 2.6. *Let $\alpha \in \mathbb{R}$, and suppose that the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-\alpha)^n$ is $r > 0$. For $|x-\alpha| < r$, define a function f by $f(x) = \sum_{n=0}^{\infty} a_n(x-\alpha)^n$. Then*

(1) *For any $|x| < r$ we have*

$$\int_{\alpha}^{x} f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-\alpha)^{n+1}.$$

(2) *The function f is continuously differentiable on the interval $|x-\alpha| < r$ and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x-\alpha)^{n-1}$$

for all $|x| < r$.

(3) *In particular, the function f is infinitely differentiable on the interval $|x-\alpha| < r$.*

2.3. **Taylor Expansions.**

We have already seen several examples where standard functions can be expanded in power series. We have

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots \qquad \text{for } |z| < 1; \tag{2.2}$$

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots + \frac{1}{n!}z^n + \cdots \qquad \text{for all } z; \tag{2.3}$$

$$\cos(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots \qquad \text{for all } z; \tag{2.4}$$

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots + \frac{(-1)^n}{(2n+1)!}z^{2n+1} + \cdots \qquad \text{for all } z. \tag{2.5}$$

According to the results of Section 1.2, we can integrate or differentiate these formulas term by term. Thus, for example, differentiation of the series in equation (2.2) gives

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots + nz^{n-1} + \cdots, \tag{2.6}$$

and integration term by term gives

$$-\log(1-z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \cdots + \frac{1}{n+1}z^{n+1} + \cdots. \tag{2.7}$$

The next two results give conditions under which the Taylor series of a function f actually converges to f.

Theorem 2.7. *Suppose that f is infinitely differentiable in the interval $(a-r, a+r)$, and assume there are positive constants C and A so that $|f^{(n)}(x)| \leq C\,A^n$ for all $|x-a| < r$. Then*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

for all $|x-a| < r$.

Proof. We know that the remainder term is given by

$$E_n[f](x;a) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$$

Assume that $x > a$. It follows that

$$\begin{aligned}
|E_n[f](x;a)| &= \left|\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt\right| \\
&\leq \frac{1}{n!}\int_a^x (x-t)^n |f^{(n+1)}(t)|\,dt \\
&\leq \frac{1}{n!}\int_a^x (x-t)^n C\,A^{n+1}\,dt \\
&= \frac{C\,A^{n+1}}{n!}\int_a^x (x-t)^n\,dt \\
&= \frac{C\,A^{n+1}}{(n+1)!}|x-a|^{n+1} \\
&\leq \frac{C\,A^{n+1} r^{n+1}}{(n+1)!}
\end{aligned}$$

But this goes to zero as $n \to \infty$, and completes the proof. □

Theorem 2.8 (Bernstein)**.** *Suppose that f and all its derivatives are non-negative on the inverval $[0, r]$. Then*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

for all $x \in [0, r)$.

Proof. Write

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0) x^k + E_n(x).$$

Since all derivatives are non-negative, we have $E_n(x) \leq f(x)$. Also, since each $f^{(k)}$ is monotone increasing, we have

$$\begin{aligned} E_n(x) &= \frac{1}{n!} \int_0^x (x-t)^n f^{n+1}(t)\, dt \\ &= \frac{x^{n+1}}{n!} \int_0^1 (1-s)^n f^{(n+1)}(xs)\, ds. \end{aligned}$$

It follows htat

$$\frac{E_n(x)}{x^{n+1}} = \frac{1}{n!} \int_0^1 (1-s)^n f^{(n+1)}(xs)\, ds.$$

Then it is clear that this last function is monotone increasing, and so

$$\frac{E_n(x)}{x^{n+1}} \leq \frac{E_n(r)}{r^{n+1}} \leq \frac{f(r)}{r^{n+1}}.$$

Hence

$$E_n(x) \leq \left(\frac{x}{r}\right)^{n+1} f(r) \longrightarrow 0$$

as $n \to \infty$. □