1. Using integration by part to estimate integrals

Recall that the formula for integration by parts is

$$\int_a^b f(x)\,g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\,g(x)\,dx.$$

If $u = f(x)$ and $dv = g'(x)\,dx$, we have $du = f'(x)\,dx$ and $v = g(x)$, so this formula is sometimes written

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du,$$

We will see how this can be used to estimate certain integrals whose integrand involves a term like $\sin(x)$ or $\cos(x)$ which oscillates. As an example, consider the function defined for $x > 0$ by

$$F(x) = \int_0^x \frac{\sin(t)}{\sqrt{t}}\,dt.$$

We want to see how big $F(x)$ gets as $x \to +\infty$ gets large. Without integration by parts, we can argue as follows:

$$|F(x)| = \left|\int_0^x \frac{\sin(t)}{\sqrt{t}}\,dt\right| \le \int_0^x \left|\frac{\sin(t)}{\sqrt{t}}\right|\,dt \le \int_0^x \frac{1}{\sqrt{t}}\,dt = 2\,t^{\frac{1}{2}}\Big|_0^x = 2\sqrt{x}.$$

Here, in the second inequality, we have used the fact that $|\sin(t)| \le 1$ for all t. Note that, according to this estimate, it is still possible that $|F(x)|$ gets arbitrarily large as $x \to +\infty$. However, we now show

Theorem 1: $\displaystyle \left|\int_0^x \frac{\sin(t)}{\sqrt{t}}\,dt\right| \le \frac{8}{3}$ for all $x \ge 0$.

Proof: For $x \ge 1$, we can write

$$\begin{aligned}\left|\int_0^x \frac{\sin(t)}{\sqrt{t}}\,dt\right| &= \left|\int_0^1 \frac{\sin(t)}{\sqrt{t}}\,dt + \int_1^x \frac{\sin(t)}{\sqrt{t}}\,dt\right| \\ &\le \left|\int_0^1 \frac{\sin(t)}{\sqrt{t}}\,dt\right| + \left|\int_1^x \frac{\sin(t)}{\sqrt{t}}\,dt\right|,\end{aligned} \tag{1}$$

and we shall study each integral separately. We have

$$\left|\int_0^1 \frac{\sin(t)}{\sqrt{t}}\,dt\right| = \left|\int_0^1 \frac{\sin(t)}{t}\,t^{\frac{1}{2}}\,dt\right| \le \int_0^1 \left|\frac{\sin(t)}{t}\right|\,t^{\frac{1}{2}}\,dt \le \int_0^1 t^{\frac{1}{2}}\,dt = \frac{2}{3}, \tag{2}$$

where we have use the inequality $\left|\frac{\sin(t)}{t}\right| \le 1$, which holds for all real numbers t. This takes care of the first integral in (1).

To deal with the second integral, we integrate by parts. Let $u = \frac{1}{\sqrt{t}} = t^{-\frac{1}{2}}$ and $dv = \sin(t)\,dt$ so that $du = -\frac{1}{2}t^{-\frac{3}{2}}$ and $v = -\cos(t)$. We have

$$\int_1^x \frac{\sin(t)}{\sqrt{t}}\,dt = \cos(1) - \frac{\cos(x)}{\sqrt{x}} - \frac{1}{2}\int_1^x \frac{\cos(t)}{t^{\frac{3}{2}}}\,dt.$$

It follows that

$$\begin{aligned}\left|\int_1^x \frac{\sin(t)}{\sqrt{t}}\,dt\right| &= \left|\cos(1) - \frac{\cos(x)}{\sqrt{x}} - \frac{1}{2}\int_1^x \frac{\cos(t)}{t^{\frac{3}{2}}}\,dt\right| \\ &\le \big|\cos(1)\big| + \left|\frac{\cos(x)}{\sqrt{x}}\right| + \frac{1}{2}\int_1^x \left|\frac{\cos(t)}{t^{\frac{3}{2}}}\right|\,dt \\ &\le 1 + \frac{1}{\sqrt{x}} + \frac{1}{2}\int_1^x t^{-\frac{3}{2}}\,dt \\ &\le 1 + \frac{1}{\sqrt{x}} - t^{-\frac{1}{2}}\Big|_1^x \\ &= 2.\end{aligned} \tag{3}$$

Putting the inequalities (1), (2), and (3) together, we see that

$$\left|\int_0^x \frac{\sin(t)}{\sqrt{t}}\,dt\right| \le \frac{8}{3}.$$

2. Evaluation of $\int_0^1 x^m(1-x)^n\,dx$

We want to explicitly find the value of

$$B(m,n) = \int_0^1 x^m(1-x)^n\,dx.$$

We could try to expand out the expression $(1-x)^n$ by using the binomial theorem, but this quickly gets very messy. Instead, we shall use integration by parts.

Proposition 1: If m and n are positive integers, we have

$$B(m,n) = \frac{m}{n+1}\,B(m-1,n+1).$$

Proof: In the integral defining $B(m,n)$, let $u = x^m$ and $dv = (1-x)^n\,dx$ so that $du = m\,x^{m-1}\,dx$ and $v = -\dfrac{1}{n+1}(1-x)^{n+1}$. Then integration by parts gives

$$\begin{aligned}B(m,n) &= \int_0^1 x^m(1-x)^n\,dx \\ &= -\frac{1}{n+1}x^m(1-x)^{n+1}\Big|_0^1 - \int_0^1 -\frac{1}{n+1}(1-x)^{n+1}\,m\,x^{m-1}\,dx \\ &= \frac{m}{n+1}\int_0^1 x^{m-1}\,(1-x)^{n+1}\,dx \\ &= \frac{m}{n+1}\,B(m-1,n+1),\end{aligned}$$

as we wanted to show.

Proposition 2: If $N \ge 0$ is an integer, then $B(0,N) = \dfrac{1}{N+1}$.

Proof: We have

$$B(0,N) = \int_0^1 (1-x)^N\,dx = -\frac{1}{N+1}(1-x)^{N+1}\Big|_0^1 = \frac{1}{N+1},$$

as asserted.

We can now prove

Theorem 2: If m and n are positive integers, then $B(m,n) = \dfrac{m!\,n!}{(m+n+1)!}$.

Proof: Suppose that m and n are non-negative integers. Using Proposition 1 over and over again, we have

$$\begin{aligned}
B(m,n) &= \frac{m}{n+1}\,B(m-1,n+1)\\
&= \frac{m}{n+1}\cdot\frac{m-1}{n+2}\,B(m-2,n+2)\\
&= \frac{m}{n+1}\cdot\frac{m-1}{n+2}\cdot\frac{m-2}{n+3}\,B(m-3,n+3)\\
&= \cdots\\
&= \frac{m}{n+1}\cdot\frac{m-1}{n+2}\cdot\frac{m-2}{n+3}\cdots\frac{3}{m+n-2}\cdot\frac{2}{m+n-1}\cdot\frac{1}{m+n}B(0,m+n)\\
&= \frac{m!\,n!}{(m+n)!}\,B(0,m+n).
\end{aligned}$$

But then Proposition 2 shows that

$$B(m,n) = \frac{m!\,n!}{(m+n)!}\,\frac{1}{(m+n+1)} = \frac{m!\,n!}{(m+n+1)!}.$$

This proves the theorem.