Problem 1 Evaluate each of the following integrals.

(a) $\displaystyle\int_0^{\pi} \big(1 - \sin(\theta)\big)\sqrt{\theta + \cos(\theta)}\, d\theta$

We make the substitution $u = \theta + \cos(\theta)$. Then $du = 1 - \sin(\theta)$, so

$$\int_0^{\pi} \big(1 - \sin(\theta)\big)\sqrt{\theta + \cos(\theta)}\, d\theta = \int_1^{\pi - 1} \sqrt{u}\, du = \frac{2}{3}\, u^{\frac{3}{2}}\Big|_1^{\pi-1} = \frac{2}{3}(\pi - 1)^{\frac{3}{2}} - \frac{2}{3}.$$

(b) $\displaystyle\int_0^1 (1 - y^2)^2\,(1 + y^2)\, dy$

We multiply out the integrand to get

$$\begin{aligned}\int_0^1 (1 - y^2)^2\,(1 + y^2)\, dy &= \int_0^1 (1 - 2y^2 + y^4)(1 + y^2)\, dy = \int_0^1 \left(1 + y^2 - 2y^2 - 2y^4 + y^4 + y^6\right)\, dy \\ &= \int_0^1 \left(1 - y^2 - y^4 + y^6\right)\, dy = \left(y - \frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7}\right)\Big|_0^1 \\ &= 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7}.\end{aligned}$$

(c) $\displaystyle\int_0^{\pi/2} \sin^2(t)\, \cos^3(t)\, dt$

Write $\cos^2(t) = 1 - \sin^2(t)$ and make the substitution $u = \sin(t)$. Then $du = \cos(t)\, dt$, so

$$\begin{aligned}\int_0^{\pi/2} \sin^2(t)\, \cos^3(t)\, dt &= \int_0^{\pi/2} \sin^2(t)\left(1 - \sin^2(t)\right)\, \cos(t)\, dt = \int_0^1 u^2(1 - u^2)\, du \\ &= \int_0^1 (u^2 - u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5}\Big|_0^1 = \frac{1}{3} - \frac{1}{5}.\end{aligned}$$

(d) (8 points) $\displaystyle\int_0^{\pi/4} (1 + x)\, \sin(2x)\, dx$

We write $\displaystyle\int_0^{\pi/4} (1 + x)\, \sin(2x)\, dx = \int_0^{\pi/4} \sin(2x)\, dx + \int_0^{\pi/4} x\, \sin(2x)\, dx$. The first of these integrals we can do directly: $\displaystyle\int_0^{\pi/4} \sin(2x)\, dx = -\frac{\cos(2x)}{2}\Big|_0^{\pi/4} = \frac{1}{2}$. We can do the second integral by using integration by parts. Let $u = x$ and $dv = \sin(2x)\, dx$. Then $du = dx$ and $v = -\dfrac{\cos(2x)}{2}$. Hence

$$\int_0^{\pi/4} x\, \sin(2x)\, dx = -\frac{x\, \cos(2x)}{2}\Big|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \cos(2x)\, dx = \left(-\frac{x\, \cos(2x)}{2} + \frac{\sin(2x)}{4}\right)\Big|_0^{\pi/4} = \frac{1}{4}.$$

Thus $\displaystyle\int_0^{\pi/4} (1 + x)\, \sin(2x)\, dx = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Problem 2

(a) Give a precise statement of the first fundamental theorem of the calculus.

Let f be Riemann integrable on the interval $[a,b]$, let $c \in [a,b]$, and for $x \in [a,b]$, set $F(x) = \int_c^x f(t)\,dt$. Then

(a) The function F is continuous at every point $x \in [a,b]$.
(b) The function F is differentiable at every point $x \in [a,b]$ at which f is continuous, and at such points $F'(x) = f(x)$.

(b) Give a brief but precise definition of what it means that a function $P(x)$ is a *primitive* of a function $f(x)$ on the interval $[a,b]$.

The function $P(x)$ is a primitive of a function $f(x)$ on the interval $[a,b]$ if and only if $P'(x) = f(x)$ for every $x \in [a,b]$.

(c) If f is a continuous function on the interval $[a,b]$, how many primitives does f have? Give a clear explanation of your answer.

According to the first fundamental theorem of calculus, a continuous function on the interval $[a,b]$ always has a primitive. We can always add a constant to obtain another primitive. Thus a continuous function has infinitely many primitives. By the zero derivative theorem, any two primitives differ by a constant.

(d) Suppose that f is continuous on the interval $[a,b]$, and that P is a primitive for f. Using the results of parts (a), (b), and (c) of this problem, prove that $\int_a^b f(t)\,dt = P(b) - P(a)$.

Let P be a primitive for f on $[a,b]$, and let $F(x) = \int_a^x f(t)\,dt$. According to the first fundamental theorem of calculus, F is also a primitive for f. Thus by part (c), it follows that the two primitives P and F differ by a constant. Thus $F(x) = P(x) + C$. We have $F(a) = 0$, so if we plug in $x = a$, this formula shows that $0 = P(a) + C$, or $C = -P(a)$. Thus $F(x) = P(x) - P(a)$. In particular, $\int_a^b f(t)\,dt = F(b) = P(b) - P(a)$.

Problem 3 For each of the following, find the derivative of the function F:

(a) $F(x) = \int_x^\pi t^{10} \sin(t)\,dt$

Using the first fundamental theorem of calculus, we have $F'(x) = -x^{10}\sin(x)$.

(b) $F(x) = \int_{\sqrt{x}}^{x^2} \sqrt{1 + \sin^2(\theta)}\,d\theta$

Using the first fundamental theorem of calculus and the chain rule, we have

$$F'(x) = \sqrt{1 + \sin^2(x^2)}(2x) - \sqrt{1 + \sin^2(\sqrt{x})}\left(\frac{1}{2\sqrt{x}}\right).$$

(c) $F(x) = \int_{-x}^{+x} \sin(x+y)\,dy$

We have $\sin(x+y) = \sin(x)\,\cos(y) + \cos(x)\,\sin(y)$. Thus

$$F(x) = \sin(x)\int_{-x}^{+x} \cos(y)\,dy + \cos(x)\int_{-x}^{+x} \sin(y)\,dy.$$

Hence using the first fundamental theorem of calculus, the chain rule, and the product rule, we have

$$\begin{aligned} F'(x) &= \cos(x)\int_{-x}^{+x} \cos(y)\,dy + \sin(x)\,[\cos(x) + \cos(-x)] \\ &\qquad - \sin(x)\int_{-x}^{+x} \sin(y)\,dy + \cos(x)\,[\sin(x) + \sin(-x)] \\ &= \cos(x)\int_{-x}^{+x} \cos(y)\,dy + 2\sin(x)\cos(x) - \sin(x)\int_{-x}^{+x} \sin(y)\,dy \\ &= \cos(x)\big[\sin(x) - \sin(-x)\big] + 2\sin(x)\cos(x) - \sin(x)\big[-\cos(x) + \cos(-x)\big] \\ &= 4\sin(x)\cos(x). \end{aligned}$$

One can also integrate $F(x)$ directly to get $F(x) = -\cos(2x) + \cos(0)$, so that $F'(x) = 2\sin(2x) = 4\sin(x)\cos(x)$.

Problem 4

(a) State the formula for integration by parts.

The formula is:

$$\int_a^b f(x)\,g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\,g(x)\,dx.$$

(b) Explain how the formula in part (a) of this problem follows from the product rule for derivatives.

The product rule says that $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. We integrate both sides of this equation from a to b and get

$$\int_a^b (fg)'(x)\,dx = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx.$$

According to the second fundamental theorem of calculus, $\int_a^b (fg)'(x)\,dx = (fg)(b) - (fg)(a)$. Thus

$$f(b)g(b) - f(a)g(a) = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx.$$

If we put the term $\int_a^b f'(x)g(x)\,dx$ on the other side of the equation, we get the formula stated in part (a).