Math 275 **Assignment # 5**

Due Thursday, November 2, 2006

In the text, read Chapter 1, sections 6, 8 – 10, 12 – 14, and 16 – 25. Do the following problems:

Problem 1: Using the axioms for area given on pages 58 and 59 in the text, prove the following:

(a) The union of a finite number of straight line segments in the plane has zero area.

(b) Every right triangular region in the plane is measurable, and its area is one half the product of its base and its altitude.

(c) Every triangular region in the plane is measurable, and its area is one half the product of its base and its altitude.

Problem 2: A point (x, y) in the plane is called a lattice point if both coordinates x and y are integers. If P is a polygon in the plane all of whose vertices are lattice points, let I denote the number of lattice points inside the polygon P, let B denote the number of lattice points on the boundary of P, and let A denote the area of the polygon P. The object of this problem is to show that

$$A = I + \frac{1}{2}B - 1. \tag{1}$$

(a) Prove formula (1) if P is a rectangle with sides parallel to the coordinate axes.

(b) Prove formula (1) if P is a right triangle with sides parallel to the coordinate axes.

(c) Prove formula (1) if P is an arbitrary triangle.

(d) Prove formula (1) if P is an arbitrary convex polygon.

In the next problems, if x is a real number, then $[x]$ denote the greatest integer which is less than or equal to x.

Problem 3: Let $f(x) = [x]$ and let $g(x) = [2x]$. Draw the graph on the interval $[-2, 2]$ of each of the following functions h:

(a) $h(x) = 2f(x) + g(x)$;

(b) $h(x) = f(x) + g\left(\frac{x}{3}\right)$;

(c) $h(x) = f(3x)g\left(\frac{x}{3}\right)$.

Problem 4: Show that $[2x] = [x]+[x+\frac{1}{2}]$ and that $[3x] = [x]+[x+\frac{1}{3}]+[x+\frac{2}{3}]$. Then generalize these two results and find a formula for $[nx]$.

Problem 5: Evaluate each of the following integrals:

(a) $\displaystyle\int_{-1}^{3} [2x]\,dx$;

(b) $\displaystyle\int_{-1}^{3} 2[x]\,dx$;

(c) $\displaystyle\int_{-1}^{3} [2x]^2\,dx$;

(d) $\displaystyle\int_{-1}^{3} \left([x] + [x + \frac{1}{2}]\right)\,dx$;

Problem 6:

(a) Compute $\displaystyle\int_0^9 [\sqrt{t}]\,dt$;

(b) Prove that if n is a positive integer, $\displaystyle\int_0^{n^2} [\sqrt{t}]\,dt = \frac{1}{6}n(n-1)(4n+1)$.

Problem 7: Evaluate each of the following integrals:

(a) $\displaystyle\int_{-2}^{3} \left(x^4 - 9x^3 + 7x^2 - 3x + 1\right)\,dx$;

(b) $\displaystyle\int_{-2}^{3} x^3\left(x^2 - 1\right)^3\,dx$;

(c) $\displaystyle\int_{-2}^{3} (x-5)^{19}\,dx$;

(d) $\displaystyle\int_0^2 f(x)\,dx$ where $f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } 1 \le x \le 2 \end{cases}$;

(e) $\displaystyle\int_{-2}^{3} \left|x^2 - x - 2\right|\,dx$.

Some extra problems to think about:

(A) We say that a region in the plane is *Kakeya set* if it is possible to turn a line segment of length 1 completely around inside the region. What is the smallest possible area of a Kakeya set?

(B) We construct subset $\{E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots\}$ of the unit interval $[0,1]$ as follows:

(1) E_0 is the whole unit interval $[0,1]$.

(2) E_1 is obtained from E_0 by removing the open middle third interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Thus E_1 consists of the two closed intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$.

(3) E_2 is obtained from E_1 by removing the open middle third of each of the two intervals in E_1. Thus E_2 consists of the four closed intervals $\left[0, \frac{1}{9}\right]$, $\left[\frac{2}{9}, \frac{1}{3}\right]$, $\left[\frac{2}{3}, \frac{7}{9}\right]$, and $\left[\frac{8}{9}, 1\right]$.

(4) E_3 is obtained from E_2 by removing the open middle third of ech of the four intervals in E_2, *etc.*

Note that E_n consists of 2^n closed intervals, each of length 3^{-n}, and E_{n+1} is obtained from E_n by removing the open middle third of each of these 2^n intervals.

(a) What is the total length of the intervals that are removed, and what is the length of the set of points that are never removed?

(b) Which points are never removed? Can you find such a point which is not one of the endpoints?