1. Holomorphic functions

1.1. **Complex-valued functions.**

At the start of the study of calculus, we usually consider real-valued functions f of a real variable x. Now we want to replace *real*-valued functions f by *complex*-valued functions F, and we want to replace the *real* variable x by a *complex* variable $z = x + iy$. Thus we consider functions F defined on the complex plane (or a subregion of the complex plane) which takes on complex values. Examples of such functions are:

$$\begin{aligned}
&F_1(x+iy) = (x+iy)^2, && \text{or equivalently} && F_1(z) = z^2 \\
&F_2(x+iy) = e^x\cos(y) + ie^x\sin(y), && \text{or equivalently} && F_2(z) = e^z \\
&F_3(x+iy) = x - iy, && \text{or equivalently} && F_3(z) = \bar{z}.
\end{aligned}$$

A complex-valued function has a real part and an imaginary part. Thus every complex-valued function F can be written

$$F(x+iy) = u(x+iy) + iv(x+iy)$$

where u and v are *real*-valued functions of a complex variable. For example

$$F_1(x+iy) = (x+iy)^2 = (x^2-y^2) + i\,(2xy),$$

so in this case

$$u(x+iy) = x^2 - y^2$$

and

$$v(x+iy) = 2xy.$$

It follows that a complex-valued function F of a complex variable is really the same as a pair of real-valued functions (u, v) of a complex variable. Also, since a complex number z is determined by giving its real part x and its imaginary part y, we can think of a real-valued functions u and v of a complex variable as the same as a pair of real-valued functions of two real variables (x, y).

> A complex-valued function F of a complex variable z is really the same as a pair (u, v) of real-valued functions of a pair (x, y) of real variables. We can write $F(x+iy) = u(x+iy) + iv(x+iy)$, or equivalently, $F(x, y) = \big(u(x, y), v(x, y)\big)$.

1.2. **Derivatives.**

In the differential calculus we define the *derivative* of a real-valued function f of a real variable x. We think of f as defined on an interval of the real axis with values on the real axis, and we write:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

We would like to do the same thing for *complex*-valued functions of a *complex* variable. The key point is that if F is a complex-valued function of a complex variable, and if h is a non-zero complex number, the expression

$$\frac{F(z+h) - F(z)}{h}$$

is the quotient of two complex numbers, and hence is itself a complex number.

It is important to realize that this situation is very special to the case of complex-valued functions of a complex variable. If we consider functions F which map $\mathbb{R}^3$ to $\mathbb{R}^3$, and if $\bar{x}$ and $\bar{h}$ are vectors in $\mathbb{R}^3$, then $F(\bar{x}+\bar{h}) - F(\bar{x})$ is a vector in $\mathbb{R}^3$, and the quotient

$$\frac{F(\bar{x}+\bar{h}) - F(\bar{x})}{\bar{h}}$$

makes no sense because division of vectors is not defined!

Now just as with real-valued functions of a real variable, we say that the function F is (complex) differentiable at the point z if the limit as $h \to 0$ of this expression exists, and we write

$$F'(z) = \lim_{h\to 0} \frac{F(z+h) - F(z)}{h}.$$

Definition 1.1. *A complex-valued function F of a complex variable is holomorphic in a region $\Omega \subset \mathbb{C}$ if F is complex differentiable at each point of Ω.*

Examples:

(i) Consider $F_1(z) = z^2$. We have

$$\frac{F_1(z+h) - F_1(z)}{h} = \frac{(z+h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = 2z + h$$

and so

$$\lim_{h\to 0} \frac{F_1(z+h) - F_1(z)}{h} = \lim_{h\to 0}[2z+h] = 2z.$$

It follows that $F_1(z) = z^2$ is a holomorphic function, and its derivative is $F_1'(z) = 2z$.

(ii) Consider $F_3(z) = \bar{z}$. We have

$$\frac{F_3(z+h) - F_3(z)}{h} = \frac{\overline{(z+h)} - \bar{z}}{h} = \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \frac{\bar{h}}{h}.$$

Now if h is real, $\bar{h} = h$, and so

$$\lim_{\substack{h\to 0\\ h \text{ real}}} \frac{F_3(z+h) - F_3(z)}{h} = \lim_{\substack{h\to 0\\ h \text{ real}}} \frac{h}{h} = 1.$$

On the other hand, if h is imaginary, $\bar{h} = -h$, and so

$$\lim_{\substack{h\to 0\\ h \text{ imaginary}}} \frac{F_3(z+h) - F_3(z)}{h} = \lim_{\substack{h\to 0\\ h \text{ imaginary}}} \frac{-h}{h} = -1.$$

It follows that

$$\lim_{h\to 0} \frac{F_3(z+h) - F_3(z)}{h} \text{ does not exist,}$$

and so $F_3(z) = \bar{z}$ is not a holomorphic function.

Exercise: Show that $F_2(z) = e^z$ and $F_4(z) = z^{-1}$ are holomorphic functions.

1.3. **The Cauchy-Riemann equations.**

We now examine the meaning of complex differentiablity in terms of the real and imaginary parts of a complex valued function. Thus suppose that $F(z) = u(z)+iv(z)$ where u and v are real valued functions, and suppose that F is complex differentiable. Then if h is a complex number,

$$\frac{F(z+h)-F(z)}{h} = \frac{\big[u(z+h)-u(z)\big]+i\big[v(z+h)-v(z)\big]}{h}.$$

If we let $z = x+iy = (x,y)$, and we take $h = (h,0)$ to be a (small) real number, it follow that

$$\begin{aligned} F'(z) &= \lim_{h\to 0}\frac{u(x+h,y)-u(x,y)}{h} + i\lim_{h\to 0}\frac{v(x+h,y)-v(x,y)}{h} \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y). \end{aligned}$$

On the other hand, if we take $h = ik = (0,k)$ to be a small imaginary number, it follows that

$$\begin{aligned} F'(z) &= \lim_{k\to 0}\frac{u(x,y+k)-u(x,y)}{ik} + i\lim_{h\to 0}\frac{v(x,y+k)-v(x,y)}{ik} \\ &= \frac{1}{i}\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y). \end{aligned}$$

It follows that we must have the following equalities: $\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$ and $\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y)$. These are called the *Cauchy-Riemann equations.* We have established

Lemma 1.2. *If $F(x+iy) = u(x+iy)+iv(x+iy)$ is a holomorphic function of a complex variable $z = x+iy$, then the real and imaginary parts u and v must satisfy*

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= +\frac{\partial v}{\partial y}(x,y), \\ \frac{\partial v}{\partial x}(x,y) &= -\frac{\partial u}{\partial y}(x,y). \end{aligned}$$

1.4. **Line integrals.**

We now consider line integrals of complex-valued functions. We need to introduce some notation. We let

$$\begin{aligned} F(z) &= u(x,y)+iv(x,y), \\ dz &= dx+idy, \end{aligned}$$

so that formally,

$$\begin{aligned} F(z)\,dz &= \big(u(x,y)+iv(x,y)\big)\big(dx+idy\big) \\ &= \big[u(x,y)\,dx - v(x,y)\,dy\big] + i\big[u(x,y)\,dy + v(x,y)\,dx\big]. \end{aligned}$$

Then if C is a curve in the complex plane, we define the line integral $\displaystyle\int_C F(z)\,dz$ to be:

$$\int_C F(z)\,dz = \int_C u(x,y)\,dx - v(x,y)\,dy + i\int_C u(x,y)\,dy + v(x,y)\,dx.$$

Theorem 1.3. *Suppose $\Omega \subset \mathbb{C} = \mathbb{R}^2$ is a region to which we can apply Green's theorem. Suppose that F is holomorphic and continuously differentiable on Ω and its boundary. Then*

$$\oint_{\partial\Omega} F(z)\,dz = 0.$$

Proof. From the definition, we have

$$\oint_{\partial\Omega} F(z)\,dz = \oint_{\partial\Omega} u(x,y)\,dx - v(x,y)\,dy + i \oint_{\partial\Omega} u(x,y)\,dy + v(x,y)\,dx.$$

Using Green's theorem, we show that

(a) $$\oint_{\partial\Omega} u(x,y)\,dx - v(x,y)\,dy = 0,$$

(b) $$\oint_{\partial\Omega} u(x,y)\,dy + v(x,y)\,dx = 0.$$

For (a), we have

$$\begin{aligned}\oint_{\partial\Omega} u(x,y)\,dx - v(x,y)\,dy &= \iint_{\Omega} \Big[-\frac{\partial u}{\partial y}(x,y) - \frac{\partial v}{\partial x}(x,y)\Big]\,dx\,dy \\ &= \iint_{\Omega} 0\,dx\,dy = 0\end{aligned}$$

from the second Cauchy-Riemann equation $\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y)$. For (b), we have

$$\begin{aligned}\oint_{\partial\Omega} u(x,y)\,dy + v(x,y)\,dx &= \iint_{\Omega} \Big[\frac{\partial u}{\partial x}(x,y) - \frac{\partial v}{\partial y}(x,y)\Big]\,dx\,dy \\ &= \iint_{\Omega} 0\,dx\,dy = 0\end{aligned}$$

from the first Cauchy-Riemann equation $\frac{\partial u}{\partial x}(x,y) = +\frac{\partial v}{\partial y}(x,y)$. This completes the proof. ∎

Theorem 1.4 (Cauchy's integral formula)**.** *Suppose $\Omega \subset \mathbb{C} = \mathbb{R}^2$ is a region to which we can apply Green's theorem. Suppose that F is holomorphic and continuously differentiable on Ω and its boundary. If $z \in \Omega$, then*

$$F(z) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{F(w)}{w-z}\,dw.$$

Proof. We first consider the line integral $\displaystyle\oint_{C_\epsilon} \frac{F(w)}{w-z}\,dw$ where C_ϵ is the circle centered at $z = x+iy$ of radius ϵ. We can parameterize C_ϵ by

$$w(t) = z + \epsilon e^{it} = \big(x + \epsilon\cos(t), y + \epsilon\sin(t)\big), \quad 0 \le t \le 2\pi.$$

Then

$$\begin{aligned} w - z &= \epsilon e^{it}, \\ dw &= i\epsilon e^{it}\,dt,\end{aligned}$$

and so

$$\begin{aligned}
\oint_{C_\epsilon} \frac{F(w)}{w-z}\,dw &= i\int_0^{2\pi} F(z+\epsilon e^{it})\,dt \\
&= i\int_0^{2\pi} F(z)\,dt + i\int_0^{2\pi} [F(z+\epsilon e^{it}) - F(z)]\,dt \\
&= 2\pi i F(z) + i\int_0^{2\pi} [F(z+\epsilon e^{it}) - F(z)]\,dt.
\end{aligned}$$

Since F is continuous at z, we can make $|F(z+\epsilon e^{it}) - F(z)|$ as small as we like (for all t) by making ϵ sufficiently small. It follows that

$$\lim_{\epsilon\to 0} \oint_{C_\epsilon} \frac{F(w)}{w-z}\,dw = 2\pi i F(z).$$

But now it follows from Theorem 1.3 that for any $\epsilon > 0$,

$$\oint_{\partial\Omega} \frac{F(w)}{w-z}\,dw = \oint_{C_\epsilon} \frac{F(w)}{w-z}\,dw.$$

This completes the proof. □

1.5. **Evaluation of definite integrals.**

The Cauchy integral formula can be used to evaluate some definite integrals. We consider several examples.

Example 1: Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos(\theta)}$ with $0 \le a < 1$.

We can write $\cos(\theta) = \dfrac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right]$, and so

$$\begin{aligned}
\frac{1}{1+a\cos(\theta)} &= \frac{1}{1+\frac{a}{2}\left[e^{i\theta}+e^{-i\theta}\right]} \\
&= \frac{2e^{i\theta}}{2e^{i\theta} + a\left[e^{2i\theta}+1\right]}.
\end{aligned}$$

If we put $z = e^{i\theta}$, then $dz = ie^{i\theta}\,d\theta$, and so

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1+a\cos(\theta)} &= \frac{2}{i}\int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{ae^{2i\theta}+2e^{i\theta}+a} \\
&= \frac{2}{i}\oint \frac{dz}{az^2+2z+a} \\
&= (4\pi)\,\frac{1}{2\pi i}\oint \frac{dz}{az^2+2z+a}
\end{aligned}$$

where the line integral is take around the boundary of the unit circle. But the polynomial az^2+2z+a has two roots: $r_1 = -1-\sqrt{1-a^2}$ and $r_2 = -1+\sqrt{1-a^2}$. Moreover, $r_1 < -1 < r_2 < 0$. Thus one root r_2 is inside the circle, and the other is outside the circle. We can write

$$az^2+2z+a = a(z-r_1)(z-r_2).$$

Thus

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{1+a\cos(\theta)} &= (4\pi)\frac{1}{2\pi i}\oint \frac{dz}{(z-r_1)(z-r_2)}\\ &= (4\pi)\frac{1}{r_2-r_1}\\ &= \frac{2\pi}{\sqrt{1-a^2}}.\end{aligned}$$

Example 2: Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+4)(x^2+9)}$

The function

$$f(z) = \frac{1}{(z^2+4)(z^2+9)} = \frac{1}{(z+2i)(z-2i)(z+3i)(z-3i)}$$

is holomorphic everywhere except at the four points $\pm 2i$ and $\pm 3i$. We consider

$$\oint_{C(R_1,R_2,N)} \frac{dz}{(z^2+4)(z^2+9)}$$

where $C(R_1, R_2, N)$ is the square with vertices at $\{R_1, R_1+iN, -R_2+iN, -R_2\}$, where $N > 3$ Then the points $2i$ and $3i$ are inside this square, and the points $-2i$ and $-3i$ are outside.

We can replace this line integral with the sum of two line integrals, one going around a small circle C_2 centered at $2i$ and the other around a small circle C_3 centered at $3i$. Using the Cauchy integral formula, we have

$$\oint_{C_2} \frac{dz}{(z^2+4)(z^2+9)} = 2\pi i\big[(2i+2i)(2i+3i)(2i-3i)\big]^{-1} = \frac{2\pi i}{-20i^3} = \frac{\pi}{10};$$

$$\oint_{C_3} \frac{dz}{(z^2+4)(z^2+9)} = 2\pi i\big[(3i+2i)(3i-2i)(3i+3i)\big]^{-1} = \frac{2\pi i}{30i^3} = -\frac{\pi}{15}.$$

Thus

$$\oint_{C(R_1,R_2,N)} \frac{dz}{(z^2+4)(z^2+9)} = \frac{\pi}{10} - \frac{\pi}{15} = \frac{\pi}{30}.$$

But $\oint_{C(R_1,R_2,N)} \frac{dz}{(z^2+4)(z^2+9)}$ is the sum of four integrals $I_1 + I_2 + I_3 + I_4$. The first is parameterized by $z = x$ with $-R_2 \le x \le R_1$, and gives

$$I_1 = \int_{-R_2}^{R_1} \frac{dx}{(x^2+4)(x^2+9)}.$$

The second is parameterized by $z = R_1 + it$ with $0 \le t \le N$, and gives

$$I_2 = \int_0^N \frac{i\,dt}{(R_1+it)^2+4)(R_1+it)^2+9)}.$$

The third is paramterized by $z = t + iN$ with $R_1 \ge t \ge -R_2$, and gives

$$I_3 = \int_{R_1}^{-R_2} \frac{dt}{(t+iN)^2+4)(t+iN)^2+9)}.$$

The fourth is parameterized by $z = -R_2 + it$ with $N \ge t \ge 0$, and gives

$$I_4 = \int_N^0 \frac{i\,dt}{(-R_1+it)^2+4)(-R_2+it)^2+9)}.$$

Now we need to make estimates of the absolute values of I_2, I_3, and I_4. We can show

$$\begin{aligned}|I_2| &\le \frac{N}{(R_1^2-4)(R_1^2-9)}\\ |I_3| &\le \frac{1}{N^2}\int_{-\infty}^{+\infty}\frac{dt}{t^2+9} \le \frac{\pi}{2N^2}\\ |I_4| &\le \frac{N}{(R_2^2-4)(R_2^2-9)}.\end{aligned}$$

We first let R_1 and R_2 go to positive infinity, which makes $|I_2|$ and $|I_4|$ go to zero. We then let N go to positive infinity, which makes $|I_3|$ go to zero. In this process, I_1 goes to the desired integral, and so we get

$$\int_{-\infty}^{+\infty}\frac{dx}{(x^2+4)(x^2+9)} = \frac{\pi}{30}.$$

Example 3: Show that $\displaystyle\int_{-\infty}^{+\infty}\frac{\sin(x)}{x}\,dx = \pi.$

1.6. Holomorphic functions and power series.

We want to prove:

Theorem 1.5. *Suppose $F(z)$ is holomorphic in an open set containing the closed disk $D(R) = \left\{z \in \mathbb{C}\,\middle|\,|z| \le R\right\}$. Then for $|z| < R$ we can represent $F(z)$ by a convergent power series*

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Moreover, the coefficients are given by

$$a_n = \frac{1}{n!}F^{(n)}(0) = \frac{1}{2\pi i}\oint_{\partial D(R)}\frac{F(w)}{w^{n+1}}\,dw.$$

Proof. For any $|z| < R$, we can use the Cauchy integral formula to write

$$\begin{aligned}F(z) &= \frac{1}{2\pi i}\oint_{\partial D(R)}\frac{F(w)}{w-z}\,dw\\ &= \frac{1}{2\pi i}\oint_{\partial D(R)}\frac{F(w)}{w}\left(1-\frac{z}{w}\right)^{-1} t\,dw\\ &= \frac{1}{2\pi i}\oint_{\partial D(R)}\frac{F(w)}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^{n}dw\\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_{\partial D(R)}\frac{F(w)}{w^{n+1}}\,dw\right]z^n.\end{aligned}$$

□