A fake Schottky group in Mod(S)

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Abstract. We use the classical construction of Schottky groups in hyperbolic geometry to produce non-Schottky subgroups of the mapping class group.

1. Introduction

In hyperbolic geometry, a Schottky group is a free convex cocompact Kleinian group, classically constructed as follows. Pick four pairwise disjoint closed balls B_1^-, B_2^-, B_1^+, B_2^+ in S_∞^{n-1}, the ideal boundary of hyperbolic n–space. Suppose there are isometries f_1 and f_2 so that

$$f_i(B_i^-) = \overline{S_\infty^{n-1} - B_i^+}.$$

Then $\langle f_1, f_2 \rangle$ is a Schottky group isomorphic to the free group F_2 of rank two.

Now let S be a closed surface of genus $g \geq 2$ and let $\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S))$ be its mapping class group. By way of analogy with the theory of Kleinian groups, B. Farb and L. Mosher defined [**FM**] a notion of convex cocompactness for subgroups of Mod(S). In this setting, a *Schottky group* is a free convex cocompact subgroup of Mod(S). In [**KL1, KL2**], we extended Farb and Mosher's analogy, providing several characterizations of convex cocompactness borrowed from the Kleinian setting (see also Hamenstädt [**H**]). The analogy is an imperfect one, see [**KL3**] and the references there, and we point out some new imperfections here.

Theorem 1.1. *There exist pseudo-Anosov elements f_1 and f_2 in* Mod(S) *and pairwise disjoint closed balls $B_1^-, B_2^-, B_1^+, B_2^+$ in $\mathbb{P}\mathcal{ML}(S)$ for which*

$$f_i(B_i^-) = \overline{\mathbb{P}\mathcal{ML}(S) - B_i^+}$$

and yet $\langle f_1, f_2 \rangle \cong F_2$ is not a Schottky group.

The construction is based on work of N. Ivanov, and it is clear from his work in [**I**] that he was aware of this construction (see also McCarthy [**Mc**]). The group $G = \langle f_1, f_2 \rangle$ contains reducible elements and so fails to be convex cocompact. It is worth noting that there are sufficiently high powers of the f_i that generate a Schottky group, as proven by Farb and Mosher [**FM**], see also [**KL1, H**].

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Part of the analogy between Kleinian groups and mapping class groups was developed by J. McCarthy and A. Papadopoulos [**MP**], who constructed a limit set Λ_G and domain of discontinuity $\Delta_G \subset \mathbb{P}\mathcal{ML}(S) - \Lambda_G$ for any subgroup $G < \mathrm{Mod}(S)$, see Section 4. Unlike in the Kleinian setting, $\Delta_G \neq \mathbb{P}\mathcal{ML}(S) - \Lambda_G$ in general. While examples illustrate the necessity of taking an open set strictly smaller than $\mathbb{P}\mathcal{ML}(S) - \Lambda_G$ as a domain of discontinuity, it is not clear that Δ_G is an optimal choice. In [**KL1**], we asked whether or not Δ_G is the largest open set on which G acts properly discontinuously—see Question 3 there. Here, we answer this in the negative.

There is an obvious open set on which our group $G = \langle f_1, f_2 \rangle$ acts properly discontinuously and cocompactly, namely

$$\Omega = \bigcup_{g \in G} g \cdot \Theta$$

where Θ is the closure of the complement of our four balls. To see that Ω is open, note that Θ is contained in the interior $\mathcal{U}$ of

$$f_1(\Theta) \cup f_1^{-1}(\Theta) \cup f_2(\Theta) \cup f_2^{-1}(\Theta)$$

and that

$$\Omega = \bigcup_{g \in G} g \cdot \mathcal{U}$$

If the f_i are chosen carefully, the set Ω will contain Δ_G properly, and we have the following theorem.

Theorem 1.2. *There are irreducible subgroups $G < \mathrm{Mod}(S)$ for which Δ_G is* not *the largest open set on which G acts properly discontinuously.*

Asymmetry of the construction provides another domain Ω' on which G acts properly discontinuously, and we will also show that G does not act properly discontinuously on the union $\Omega \cup \Omega'$.

Though Δ_G is not a maximal domain of discontinuity, we show in Section 5 that, for the groups in Theorem 1.2, it is nonetheless the intersection of all such maximal domains.

2. Surface dynamics

If $\mathcal{X}$ is a subset of $\mathbb{P}\mathcal{ML}(S)$, we let

$$Z\mathcal{X} = \{[\nu] \in \mathbb{P}\mathcal{ML}(S) \,|\, i(\nu, \mu) = 0 \text{ for some } [\mu] \in \mathcal{X}\}$$

be the *zero locus of* $\mathcal{X}$. If $\mathcal{X} = \{[x]\}$ we sometimes write Zx for $Z\mathcal{X}$.

If f is pseudo-Anosov, then it acts with *north–south* dynamics on $\mathbb{P}\mathcal{ML}(S)$, meaning that it has unique attracting and repelling fixed points $[\mu_f^+]$ and $[\mu_f^-]$, respectively—all other points are attracted to $[\mu_f^+]$ under iteration of f. In fact, for any neighborhood U of $[\mu_f^+]$ and any compact set $K \subset \mathbb{P}\mathcal{ML}(S) - \{[\mu_f^-]\}$, there is a natural number N so that

$$f^n(K) \subset U \tag{2.1}$$

for any $n \geq N$.

Ivanov proves that there is a similar situation for most pure reducible elements (see the Appendix of [**I**]). In particular, suppose α is a nonseparating simple closed curve in S preserved by a mapping class ϕ that is pseudo-Anosov when restricted to $S-\alpha$. Let $[\mu_\phi^+]$ and $[\mu_\phi^-]$ be the stable and unstable laminations for ϕ in $S-\alpha$ considered laminations on S, and note that

$$Z\mu_\phi^- = \big\{[s\mu_\phi^- + (1-s)\alpha] \in \mathbb{P}\mathcal{ML}(S) \,|\, s \in [0,1]\big\}.$$

If $K \subset \mathbb{P}\mathcal{ML}(S) - Z\mu_\phi^-$ is a compact set and U a neighborhood of $[\mu_\phi^+]$, then there is an $N > 0$ such that for all $n \geq N$ we have

$$\phi^n(K) \subset U. \tag{2.2}$$

Given a mapping class g of either type above, let $\lambda(g)$ denote the *expansion factor* of g, the number such that

$$g(\mu_g^+) = \lambda(g)\mu_g^+.$$

3. The construction

Let α be a nonseparating curve fixed by a mapping class ϕ that is pseudo-Anosov on $S-\alpha$, and let $[\mu_\phi^+]$ and $[\mu_\phi^-]$ be as in the previous section.

Let $\mathbb{S}_\phi \subset \mathbb{P}\mathcal{ML}(S)$ be a bicollared $(6g-8)$–dimensional sphere dividing $\mathbb{P}\mathcal{ML}(S)$ into two closed balls $\mathcal{A}_\phi$ and $\mathcal{B}_\phi$ containing $Z\mu_\phi^-$ and $[\mu_\phi^+]$, respectively.

According to (2.2), there is an $N > 0$ so that for all $n \geq N$ we have

$$\phi^n(\mathcal{B}_\phi) \subset \mathrm{int}(\mathcal{B}_\phi).$$

So we choose an $n \geq N$, let $h = \phi^n$, $B_h^- = \mathcal{A}_\phi$, and $B_h^+ = h(\mathcal{B}_\phi)$.

Recall H. Masur's theorem [**Ma**] that the set

$$\{([\mu_\psi^+],[\mu_\psi^-]) \,|\, \psi \in \mathrm{Mod}(S) \text{ pseudo-Anosov}\}$$

is dense in $\mathbb{P}\mathcal{ML}(S) \times \mathbb{P}\mathcal{ML}(S)$. So we choose a pseudo-Anosov ψ whose fixed points $[\mu_\psi^+]$ and $[\mu_\psi^-]$ lie in $\mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$. We let $\mathbb{S}_\psi \subset \mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$ be a bicollared $(6g-8)$–sphere which bounds two balls: $\mathcal{A}_\psi \subset \mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$ containing $[\mu_\psi^-]$ and $\mathcal{B}_\psi$ containing $[\mu_\psi^+]$. As $\mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$ is a neighborhood of $[\mu_\psi^+]$, (2.1) provides an $M > 0$ such that for all $m \geq M$, we have $\psi^m(\mathcal{B}_\psi) \subset \mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$. Arguing as in [**I**], we may choose m so that $\psi^m h$ is pseudo-Anosov, and we do so. We let $f = \psi^m$, $B_f^- = \mathcal{A}_\psi$, and $B_f^+ = f(\mathcal{B}_\psi)$.

We now have elements f, h, and pairwise disjoint closed balls

$$B_h^-, B_h^+, B_f^-, B_f^+$$

with

$$h(B_h^-) = \overline{\mathbb{P}\mathcal{ML}(S) - B_h^+} \text{ and } f(B_f^-) = \overline{\mathbb{P}\mathcal{ML}(S) - B_f^+}.$$

See Figure 1.

Let $G = \langle f, h\rangle$, set

$$\Theta = \overline{\mathbb{P}\mathcal{ML}(S) - \Big(B_h^- \cup B_h^+ \cup B_f^- \cup B_f^+\Big)}.$$

and let

$$\Omega = \bigcup_{g \in G} g \cdot \Theta.$$

The group G acts on Ω properly discontinuously and cocompactly with fundamental domain Θ, and the usual ping–pong argument implies that $G \cong F_2$.

A slight modification now provides the desired example.

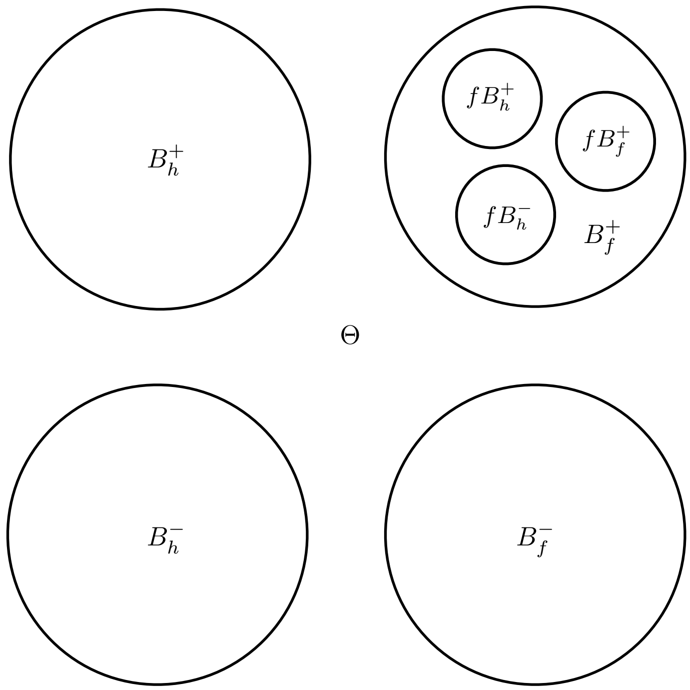

FIGURE 1

We let $f_1 = fh$ and $f_2 = f$, both pseudo-Anosov by construction. Of course, $G = \langle f_1, f_2 \rangle$, and we need only find balls $B_1^\pm$ and $B_2^\pm$ with

$$f_i(B_i^-) = \overline{\mathbb{P}\mathcal{ML}(S) - B_i^+}. \tag{3.1}$$

Set $B_1^- = B_h^-$ and $B_1^+ = f(B_h^+)$. The ball B_2^- is constructed as a regular neighborhood of $B_f^- \cup B_h^+ \cup \delta$ in $\mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_f^+)$, where δ is an arc in Θ from B_f^- to B_h^+. The ball B_2^+ is defined to be $\overline{\mathbb{P}\mathcal{ML}(S) - f(B_2^-)}$. See Figure 2.

One can now check (3.1).

4. Proper discontinuity

Let $G = \langle h, f \rangle$ be the group constructed in the previous section, and let ∂G be the Gromov boundary of G. By the work in [**MP**], the limit set

$$\Lambda_G = \overline{\{[\mu_g^+] \in \mathbb{P}\mathcal{ML}(S) \,|\, g \in G \text{ is pseudo-Anosov}\}}$$

is the unique minimal closed G–invariant subset of $\mathbb{P}\mathcal{ML}(S)$. In [**KL2**] we showed that one may choose h and f as above so that G has the following property.

PROPERTY 4.1. *There exists a continuous G–equivariant homeomorphism*

$$\mathfrak{I} \colon \partial G \to \Lambda_G.$$

Moreover, for each $x \in \partial G$ which is a fixed point of a conjugate ${}^g h$ of h, $\mathfrak{I}(x)$ is the stable or unstable lamination of that conjugate ${}^g h$ (respecting the dynamics).

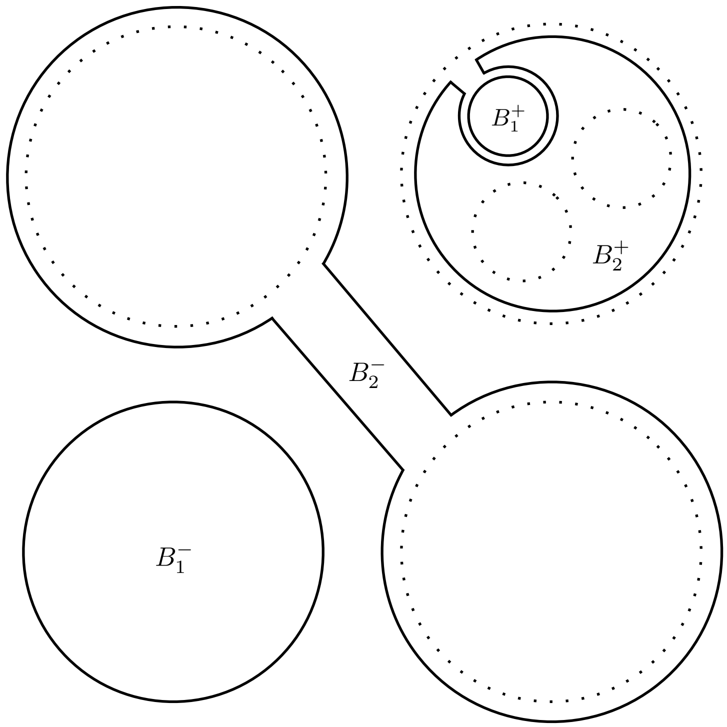

FIGURE 2

Otherwise $\mathfrak{I}(x)$ *is a uniquely ergodic filling lamination. In particular, every element* $g \in G$ *not conjugate to a power of* h *is pseudo-Anosov.*

We henceforth assume that G satisfies Property 4.1.
The domain of discontinuity is defined to be

$$\Delta_G = \mathbb{P}\mathcal{ML}(S) - Z\Lambda_G.$$

This is an open set on which G acts properly discontinuously [**MP**], which justifies the name.

We may describe the zero locus $Z\Lambda_G$ for G explicitly. For each conjugate ${}^g h$ of h, we have the attracting and repelling fixed points $x^{\pm}_{{}^g h}$ in ∂G. By Property 4.1, the map $\mathfrak{I}$ sends these to the stable and unstable laminations

$$\mathfrak{I}(x^{\pm}_{{}^g h}) = [\mu^{\pm}_{{}^g h}] = g[\mu^{\pm}_h].$$

For any such point $g[\mu^{\pm}_h] \in \Lambda_G$, the set $Zg\mu^{\pm}_h = gZ\mu^{\pm}_h$ is a 1–simplex in $Z\Lambda_G$. Since $\mathfrak{I}(x)$ is uniquely ergodic and filling for every other point $x \in \partial G$, it follows that $Z\Lambda_G$ is the union of Λ_G and all of these intervals.

The intervals $Z\mu^-_h$ and $Z\mu^+_h$ intersect each other at α, and so the union

$$\mathbb{J}_h = Z\mu^-_h \cup Z\mu^+_h$$

is an interval joining μ_h^- to μ_h^+. All in all, we have

$$Z\Lambda_G = \Lambda_G \cup \bigcup_{g\in G} g\,\mathbb{J}_h \tag{4.1}$$

We impose one further restriction on h and f—more precisely, on the balls $B_f^\pm$. Since the fixed points of f do not meet the interval $\mathbb{J}_h$, we may replace f with a power so that the balls $B_f^\pm$ are disjoint from this interval. This implies that

$$\operatorname{int} Z\mu_h^+ = Z\mu_h^+ \cap \bigcup_{n\in\mathbb{Z}} h^n\Theta$$

and so $Z\mu_h^+$ intersects the h^n translates of Θ, and no other G–translates. As $Z\mu_h^-$ does not intersect Ω, these are the only G–translates of Θ that $\mathbb{J}_h$ intersects. Write $\Sigma_h^\pm = \partial B_h^\pm$ and $\Sigma_f^\pm = \partial B_f^\pm$.

We claim that

$$\Sigma_f^+ \cap Z\Lambda_G = \emptyset.$$

To see this, note that if Σ_f^+ nontrivially intersected $Z\Lambda_G$, it would do so in some $g\mathbb{J}_h$, by (4.1); and then g must be a power of h, since Σ_f^+ lies in Θ. But $h\mathbb{J}_h = \mathbb{J}_h$, and so Σ_f^+ would intersect $\mathbb{J}_h$, contrary to our choice of f. The claim follows.

Now, Theorem 1.2 will follow from

THEOREM 4.2. *The set Δ_G is properly contained in Ω. In fact,*

$$\Omega = \mathbb{PML}(S) - \left(\Lambda_G \cup \bigcup_{g\in G} gZ\mu_h^-\right).$$

First note that $\Delta_G \neq \Omega$ as $\Sigma_h^- \subset \Theta \subset \Omega$ nontrivially intersects $\mathbb{J}_h \subset Z\Lambda_G = \mathbb{PML}(S) - \Delta_G$.

To prove the containment, we must gather some information about the complement of Ω. Let $\mathfrak{X} = \mathbb{PML}(S) - \Omega$.

LEMMA 4.3. *There is a continuous G–equivariant map*

$$\mathfrak{K}\colon\ \mathfrak{X} \to \partial G.$$

PROOF. The spheres $\Sigma_h^\pm$ and $\Sigma_f^\pm$ are bicollared with collars $N(\Sigma_h^\pm)$ and $N(\Sigma_f^\pm)$. We assume as we may that

$$h(N(\Sigma_h^-)) = N(\Sigma_h^+) \text{ and } f(N(\Sigma_f^-)) = N(\Sigma_f^+)$$

and that all of the G–translates of these collars are pairwise disjoint.

Let $\mathcal{G}$ be the Cayley graph of G and identify $\partial G = \partial\mathcal{G}$. We define a continuous G–equivariant map

$$\mathfrak{K}_0\colon\ \Omega \to \mathcal{G}$$

by identifying $\mathcal{G}$ with the tree dual to the hypersurface

$$\bigcup_{g\in G} g(\Sigma_h^-) \cup \bigcup_{g\in G} g(\Sigma_f^-)$$

in Ω and projecting in the usual manner, see [**Sh**].

The map $\mathfrak{K}_0$ extends continuously to a G–equivariant map

$$\mathfrak{K}\colon\ \mathbb{PML}(S) \to \overline{\mathcal{G}} = \mathcal{G} \cup \partial G$$

whose restriction to $\mathfrak{X}$ is the map we desire. The extension is described concretely as follows.

First note that given any point $[\eta] \in \mathfrak{X}$, there is a unique sequence of elements

$$x_1^{\epsilon_1}, x_2^{\epsilon_2}, x_3^{\epsilon_3}, \ldots$$

where $x_i \in \{f, h\}$ and $\epsilon_i \in \{\pm 1\}$ with the property that $[\eta]$ is contained in the nested intersection

$$\bigcap_{i=1}^{\infty} x_1^{\epsilon_1} \cdots x_i^{\epsilon_i}(B_{x_i}^{\epsilon_i}).$$

where $B_h^{\pm 1} = B_h^{\pm}$ and $B_f^{\pm 1} = B_f^{\pm}$. Identifying ∂G with the set of infinite reduced words, our map is given there by

$$\mathfrak{K}([\eta]) = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots .$$

To see that $\mathfrak{K}$ is continuous, let $\mathcal{U}_g \subset \overline{\mathcal{G}}$ be the open set consisting of all infinite reduced words in ∂G with prefix g together with the union of the open tails of the corresponding paths in $\mathcal{G}$. Now if g ends in $x_0^{\epsilon_0}$ with $x_0 \in \{h, f\}$ and $\epsilon_0 \in \{\pm 1\}$, then

$$\mathfrak{K}^{-1}(\mathcal{U}_g) = g x_0^{-\epsilon_0}(\operatorname{int} B_{x_0}^{\epsilon_0})$$

which is open. □

LEMMA 4.4. *$\mathfrak{K}$ is a one-sided inverse to $\mathfrak{I}$. That is, $\mathfrak{K} \circ \mathfrak{I} = \mathrm{id}_{\partial G}$.*

PROOF. Since $\mathfrak{X}$ is a G–invariant closed set, it contains Λ_G, and so $\mathfrak{K} \circ \mathfrak{I}$ is well-defined. Next, suppose that x_f^+ is the attracting fixed point of f. Then $\mathfrak{I}(x_f^+) = [\mu_f^+]$ is the attracting fixed point in $\mathbb{P}\mathcal{ML}(S)$ of f, and hence $\mathfrak{K}(\mathfrak{I}(x_f^+)) = x_f^+$. The same is true for any conjugate of f, and hence $\mathfrak{K} \circ \mathfrak{I}$ is the identity on the set of attracting fixed points of conjugates of f. Being G–invariant, this set is dense in ∂G, and so, by continuity, $\mathfrak{K} \circ \mathfrak{I}$ is the identity on all of ∂G. □

Theorem 4.2 follows easily from the following lemma.

LEMMA 4.5. *For all $x \in \partial G$, we have $\mathfrak{K}^{-1}(x) \subset Z\mathfrak{I}(x)$. In fact, if x is the repelling fixed point $x_{{}^g h}^-$ of a conjugate ${}^g h$ of h, then $\mathfrak{K}^{-1}(x) = gZ\mu_h^-$. Otherwise, the set $\mathfrak{K}^{-1}(x)$ is a singleton contained in Λ_G.*

PROOF OF THEOREM 4.2 ASSUMING LEMMA 4.5. By the first statement, $\mathfrak{X} \subset Z\Lambda_G$ since

$$Z\Lambda_G = \bigcup_{x \in \partial G} Z\mathfrak{I}(x).$$

So $\Omega \supset \Delta_G$ as required. Again, the containment is proper as $Z\mu_h^+$ nontrivially intersects Ω.

The description of Ω follows from the second and third statements. □

We need the following general fact about sequences of laminations.

LEMMA 4.6. *Suppose $\mathfrak{S} \subset \mathcal{ML}(S)$ is a compact set, $\{f_k\} \subset \mathrm{Mod}(S)$ is an infinite sequence of distinct pseudo-Anosov mapping classes with*

$$\mu_{f_k}^{\pm} \to \mu^{\pm}$$

in $\mathcal{ML}(S)$, and that $\{\nu_k\}_{k=1}^{\infty} \subset \mathfrak{S}$ and $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ are sequences with

$$t_k f_k(\nu_k) \to \eta$$

in $\mathcal{ML}(S)$.

If there is an $r > 0$ *such that*

$$i(\nu, \mu^{\pm}) > r$$

for all $\nu \in \mathfrak{S}$, *then* $t_k \to 0$.

Proof. Note that continuity of i and compactness of $\mathfrak{S}$ imply that there exist $K > 0$ and $R > 1$ such that for all $k \geq K$ and all $\nu \in \mathfrak{S}$

$$\frac{1}{R} < i\big(\nu, \mu^{\pm}_{f_k}\big) < R.$$

By the continuity of i we have

$$\lim_{k\to\infty} i\big(t_k f_k(\nu_k), \mu^{-}_{f_k}\big) = i(\eta, \mu^{-}),$$

and so, for sufficiently large k, we have

$$i(\eta, \mu^{-}) - 1 < i\big(t_k f_k(\nu_k), \mu^{-}_{f_k}\big) < i(\eta, \mu^{-}) + 1.$$

The central term of this inequality is also given by

$$\begin{aligned} i\big(t_k f_k(\nu_k), \mu^{-}_{f_k}\big) &= i\big(t_k \nu_k, f_k^{-1}(\mu^{-}_{f_k})\big) \\ &= t_k i\big(\nu_k, \lambda(f_k^{-1})\mu^{-}_{f_k}\big) \\ &= t_k \lambda(f_k) i\big(\nu_k, \mu^{-}_{f_k}\big) \end{aligned}$$

where $\lambda(f_k)$ is the expansion factor of f_k, and so, for all sufficiently large k, we have

$$\frac{i(\eta, \mu^{-}) - 1}{R} < t_k \lambda(f_k) < R(i(\eta, \mu^{-}) + 1).$$

Since the f_k are all distinct, and their fixed points converge in $\mathbb{P}\mathcal{ML}(S)$, it follows that $\lambda(f_k) \to \infty$. So $t_k \to 0$ as required. □

Proof of Lemma 4.5. First assume that $x \in \partial G$ is the fixed point of a conjugate of h. By the G–equivariance of $\mathfrak{K}$, it suffices to consider the case of h itself. Then, we have $x = x_h^{+}$ or $x = x_h^{-}$. In this case, the sequences of balls nesting to $\mathfrak{K}^{-1}(x_h^{+})$ and $\mathfrak{K}^{-1}(x_h^{-})$ are given by

$$\{h^k(B_h^{+})\}_{k=1}^{\infty} \text{ and } \{h^{-k}(B_h^{-})\}_{k=1}^{\infty},$$

respectively.

From the discussion in Section 2, we already know that

$$\mathfrak{K}^{-1}(x_h^{+}) = \bigcap_{k=1}^{\infty} h^k(B_h^{+}) = \{[\mu_h^{+}]\} \subset Z\mathfrak{I}(x_h^{+})$$

and

$$\mathfrak{K}^{-1}(x_h^{-}) = \bigcap_{k=1}^{\infty} h^{-k}(B_h^{-}) = Z\mu_h^{-} = Z\mathfrak{I}(x_h^{-}).$$

If $g \in G$ is any other element not conjugate to a power of h, then, by Property 4.1, g is pseudo-Anosov, and the dynamical properties of pseudo-Anosov mapping classes discussed in Section 2 implies

$$\mathfrak{K}^{-1}(x^{\pm}(g)) = \{[\mu^{\pm}(g)]\} = Z\mathfrak{I}(x^{\pm}(g)).$$

Therefore, to complete the proof of the lemma, we assume that $x \in \partial G$ is not a fixed point of any element of G.

We write x as an infinite reduced word

$$x = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots .$$

Since x is not the fixed point of any element of G, we can assume that $x_n = f$ and, say, $\epsilon_n = +1$ for infinitely many n (the case that $x_n = f$ and $\epsilon_n = -1$ for infinitely many n is similar). The G–equivariance of $\mathfrak{K}$ implies that we may also assume that $x_1 = f$ and $\epsilon_1 = 1$. Let $\{n_k\}_{k=1}^{\infty}$ be the increasing sequence of natural numbers for which $x_{n_k} = f$ and $\epsilon_{n_k} = +1$. Finally, set

$$f_k = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots x_{n_k}^{\epsilon_{n_k}} \in G.$$

Then, we have $\mathfrak{K}^{-1}(x)$ expressed as the nested intersection

$$\mathfrak{K}^{-1}(x) = \bigcap_{k=1}^{\infty} f_k(B_f^+).$$

Any point $[\eta]$ in the frontier of $\mathfrak{K}^{-1}(x)$ is a limit of a sequence in the frontiers

$$[\eta] = \lim_{k\to\infty} f_k([\nu_k])$$

where

$$[\nu_k] \in \mathrm{Fr}(B_f^+) = \Sigma_f^+.$$

We fix any such $[\eta] \in \mathrm{Fr}(\mathfrak{K}^{-1}(x))$ and such a sequence $\{[\nu_k]\}$.

We pass to a further subsequence so that $\mu_{f_k}^{\pm} \to \mu^{\pm} \in \mathcal{ML}(S)$. Since $[\mu_{f_k}^{\pm}] \in \Lambda_G$ for all k, we also have $[\mu^{\pm}] \in \Lambda_G$. In fact, since $f_k = x_1^{\epsilon_1} \cdots x_{n_k}^{\epsilon_{n_k}}$ is cyclically reduced, the axes for f_k in $\mathcal{G}$ all go through the origin and limit to a geodesic $\gamma \subset \mathcal{G}$ through $\mathbf{1}$ with positive ray ending at x. Therefore, $x_{f_k}^+ \to x$ as $k \to \infty$, and by continuity of $\mathfrak{I}$, it follows that

$$\mathfrak{I}(x) = [\mu^+] \in \Lambda_G.$$

Moreover, the negative ray of γ ends at some point $y \in \partial G$ and is described by an infinite word

$$y = y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3} \cdots$$

where $y_1^{\delta_1} \neq f$ since $x_1^{\epsilon_1} = f$ and γ is a geodesic. Therefore, again appealing to the continuity of $\mathfrak{I}$ we see that

$$\mathfrak{I}(y) = [\mu^-] \in \Lambda_G \cap \mathbb{P}\mathcal{ML}(S) - B_f^+.$$

By similar reasoning, for any $[\mu] \in \Lambda_G \cap B_f^+$, we have

$$f_k([\mu]) \to [\mu^+] = \mathfrak{I}(x).$$

In fact, it follows from [**MP**, Lemma 2.7] that there is a μ (a fixed point of a pseudo-Anosov in G) and a sequence s_k tending to zero such that

$$\lim_{k\to\infty} s_k f_k(\mu) = \mu^+ \in \mathcal{ML}(S).$$

We now let $\mathfrak{S} \subset \mathcal{ML}(S)$ be the image of Σ_f^+ under some continuous section of $\mathcal{ML}(S) \to \mathbb{P}\mathcal{ML}(S)$. Since $\Sigma_f^+ \cap Z\Lambda_G = \emptyset$, there is an $r > 0$ such that

$$i(\nu, \mu^{\pm}) > r$$

for every $\nu \in \mathfrak{S}$.

We take the representatives ν_k of $[\nu_k]$ to lie in $\mathfrak{S}$. Then, according to Lemma 4.6, the sequence t_k for which

$$\lim_{k\to\infty} t_k f_k(\nu_k) = \eta$$

must tend to zero. So

$$i(\eta, \mu^+) = \lim_{k\to\infty} i\big(t_k f_k(\nu_k), s_k f_k(\mu)\big) = \lim_{k\to\infty} t_k s_k i(\nu_k, \mu) = 0$$

since s_k and t_k tend to zero and $i(\nu_k, \mu)$ is uniformly bounded by compactness of $\mathfrak{S}$. Since μ^+ is uniquely ergodic, we conclude that $[\eta] = [\mu^+] = \mathfrak{I}(x)$.

This means that the frontier of $\mathfrak{K}^{-1}(x)$ is precisely $\{\mathfrak{I}(x)\}$, and hence

$$\mathfrak{K}^{-1}(x) = \{\mathfrak{I}(x)\} = Z\mathfrak{I}(x)$$

as required. □

5. Final comments

If we replace h with h^{-1} in our construction we obtain another G–invariant open set Ω' on which G acts properly discontinuously and cocompactly. By Lemma 4.5, we have descriptions

$$\Omega = \mathbb{P}\mathcal{ML}(S) - \left(\Lambda_G \cup \bigcup_{g\in G} gZ\mu_h^-\right)$$

and

$$\Omega' = \mathbb{P}\mathcal{ML}(S) - \left(\Lambda_G \cup \bigcup_{g\in G} gZ\mu_h^+\right),$$

and it follows that

$$\Omega \cup \Omega' = \mathbb{P}\mathcal{ML}(S) - \left(\Lambda_G \cup \bigcup_{g\in G} G\cdot\alpha\right).$$

The group G does not act properly discontinuously on $\Omega\cup\Omega'$, and in fact, we have the following.

Proposition 5.1. *If $\mathcal{U} \subset \mathbb{P}\mathcal{ML}(S)$ is any open set on which G acts properly discontinuously, then $\mathcal{U}\subset\Omega$ or $\mathcal{U}\subset\Omega'$.*

Proof. Let $\mathcal{U} \subset \mathbb{P}\mathcal{ML}(S)$ be a G–invariant open set. We will show that if $\mathcal{U}$ is not contained in either Ω or Ω', then G does not act properly discontinuously on $\mathcal{U}$.

If $\mathcal{U}\cap\Lambda_G \neq \emptyset$, then since G acts minimally on Λ_G and $\mathcal{U}$ is G–invariant, we must have $\Lambda_G\subset\mathcal{U}$. As G clearly fails to act properly discontinuously on $\mathcal{U}$ in this case, we assume that $\mathcal{U}\cap\Lambda_G = \emptyset$.

So if $\mathcal{U}$ fails to be contained in either Ω or Ω', there are points $[\eta^+]\in\mathcal{U}\cap Z\mu_h^+$ and $[\eta^-]\in\mathcal{U}\cap Z\mu_h^-$. Moreover, $[\eta^\pm]$ is in the interior of $Z\mu_h^\pm$. Let $\Upsilon^\pm$ be small compact balls contained in $\mathcal{U}$ containing $[\eta^\pm]$. Since $[\eta^+]\in\Omega$, we may assume that $\Upsilon^+\subset\Omega$. Moreover, G–invariance of $\mathcal{U}$ allows us to pick $[\eta^+]$ and Υ^+ to lie in B_h^-.

After passing to a subsequence, we can assume that the sequence of sets $\{h^{-k_j}(\Upsilon^+)\}_{j=1}^{\infty}$ converges in the Hausdorff topology. Moreover, we have

$$\lim_{j\to\infty} h^{-k_j}(\Upsilon^+) \subset \bigcap_{k=1}^{\infty} h^{-k}(B_k^-) = Z\mu_h^-.$$

Note that the Hausdorff limit must be connected since Υ^+ is. This limit contains α as the pointwise limit of $h^{-k}[\eta^+]$, and $[\mu_h^-]$ as the pointwise limit of any other point of Υ^+ under h^{-k}. Therefore,

$$\lim_{j\to\infty} h^{-k_j}(\Upsilon^+) = Z\mu_h^-.$$

Now, consider the compact set $\Upsilon = \Upsilon^+ \cup \Upsilon^-$. Since $\mathrm{int}(\Upsilon^-)$ is a neighborhood of $[\eta^-]$, we have

$$h^{-k_j}(\Upsilon) \cap \Upsilon \supset h^{-k_j}(\Upsilon^+) \cap \mathrm{int}(\Upsilon^-) \neq \emptyset$$

for all sufficiently large j. So G does not act properly discontinuously on $\mathcal{U}$. □

From this we deduce that Ω and Ω' are the only maximal open sets on which G acts properly discontinuously. By our descriptions of Ω and Ω' we also have

$$\Delta_G = \Omega \cap \Omega'.$$

It follows that Δ_G can be described purely in terms of the action of G on $\mathbb{P}\mathcal{ML}(S)$, *without referring to geometric structures on the surface.*

Though Δ_G may not be a *maximal* open set on which G acts nicely, it remains a *canonically defined* one, and we pose the following question.

QUESTION 5.2. If G is an irreducible subgroup of Mod(S), is Δ_G is the intersection of all maximal open sets on which G acts properly discontinuously?

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