A short proof that composite twisted unknots are singly twisted unknots

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Abstract

We present a short proof of a theorem of Hayashi and Motegi and (independently) Goodman-Strauss that only singly twisted unknots are composite.
Keywords Knot, unknot, composite knot, twisting, Dehn surgery.
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Let K' be an unknotted circle in S^3, and let D be a disk in S^3 such that $K' \cup \partial D$ is a link L and $|K' \cap D| > 1$ and minimal in the isotopy class of K' in $S^3 - \partial D$. Performing $\frac{1}{n}$–Dehn surgery on ∂D yields a knot K. We will prove the following

Theorem (Hayashi and Motegi [4], Goodman-Strauss [1]). *If K is composite, then $n = \pm 1$.*

Proof. Let X_L denote the exterior of L. By $X_L(\alpha)$ we shall mean the manifold obtained from X_L by performing α–Dehn surgery on the component of ∂X_L corresponding to ∂D.

We assume the reader is familiar with the machinery surveyed in [2]. $X_L(\infty)$ is a solid torus and so contains a nonseparating disk $\widehat{B}$. Assuming that K is composite, $X_L(\frac{1}{n})$ contains an essential (non–boundary parallel) separating annulus $\widehat{A}$ whose boundary components are meridians of K. These two surfaces give rise to two labeled graphs G_B and G_A in the usual way [2], and after suitable isotopies we may assume the number of vertices in each minimal and each arc component of $A \cap B$ essential in both A and B. Let a be the number of vertices of G_A, b the number of vertices of G_B. Note that the number of boundary edges in each graph is at most two and that each boundary component of $\widehat{A}$ is incident to a boundary edge—since $\partial\widehat{B}$ is a longitude of K'.

We claim that neither G_A nor G_B represents all types.

Suppose that G_A represents all types. Let $\mathcal{D}$ be a set of (interior) faces of G_A representing all types. Since the edges of elements of $\mathcal{D}$ lie in $\widehat{B}$, Theorem 4.3 of [2] produces a summand M of S^3 with $H_1 M$ finite and nonzero. So G_A does not represent all types.

Suppose that G_B represents all types, and let $\mathcal{D}$ be a set of (interior) faces of G_B representing all types. The edges of the elements of $\mathcal{D}$ lie in $\widehat{A}$, and $\widehat{A}$ lies in a disk in S^3—just cap off a boundary component with a meridional disk. Again, Theorem 4.3 of [2]—using $\widehat{A}$ capped off rather than $\widehat{A}$—produces a summand of S^3 with torsion in its first homology group. So G_B does not represent all types.

Note that since a Scharlemann cycle represents all types, neither graph contains a Scharlemann cycle.

Suppose that $|n| \geq 2$. Since neither graph represents all types, G_A contains a $(b-1)$–web and G_B contains a great a–web, Λ—for G_B, the web is provided by Corollary 3.4 of [3]; For G_A, the web is obtained by interchanging the roles of A and B in the proof of Theorem 3.2 of [3]. Since $\widehat{A}$ is separating, a is even, hence greater than or equal to two. So there is a label x such that for any vertex v in Λ there is an edge in Λ labeled x at v. If $a \geq 4$ or $|n| \geq 3$, there are two such labels, x and y, say.

Suppose $a \geq 4$. If every label occurs at the endpoint (in Λ) of an edge not belonging to Λ, then there is a third label z possessing the property stated for x and y. If some label w does not occur in this fashion, then for every vertex v of Λ there are two edges in Λ labeled w at v. In either case, the fact that there are at most two boundary edges implies that there is a label u such that for every vertex v in Λ, there is an interior edge of Λ labeled u at v. This yields a great u–cycle, hence a Scharlemann cycle, in G_B. So $a = 2$.

Suppose that $|n| \geq 3$. In this case, one of the labels, x say, has the property that every vertex v in Λ is incident to two edges in Λ labeled x at v. Again, this gives rise to a Scharlemann cycle in G_B. So $a = 2$ and $|n| = 2$.

The two vertices v and w of G_A are of opposite sign since $\widehat{A}$ is separating, and so the $(b-1)$–web in G_A has a single vertex, v say. Since $|n| \geq 2$, there is a loop at v.

If this loop is inessential in $\widehat{A}$, then it must bound a disk containing w. So, every edge incident to w must be incident to v. Since there are $2b$ edges incident to each vertex, there are $2b$ edges joining w and v. But this contradicts the fact that G_A contains a $(b-1)$–web. *Note:* It is tempting to use the fact that there are boundary edges in G_A to obtain a contradiction here, but note that we have not bothered to exclude the possibility of a single boundary edge running from one component of $\partial\widehat{A}$ to the other.

So the loop at v separates $\widehat{A}$ into two annuli, A_1 and A_2. One of these, A_2 say, contains w. Since A_1 contains a component of $\partial\widehat{A}$, A_1 contains a single boundary edge. If the other boundary edge of G_A is incident to w, then G_A must appear

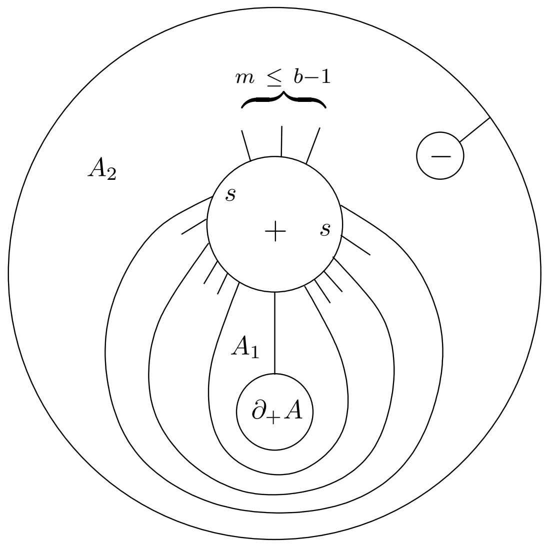

Figure 1: A great s–cycle is forced.

as in Figure 1. But this forces a great s–cycle in G_A, which must contain a Scharlemann cycle, a contradiction. So the other boundary edge is incident to v. But now cutting $\widehat{A}$ along the union of v and the boundary edges leaves a disk that contains w and all of its neighboring edges. So there are $2b$ edges joining v and w, again contradicting the fact that G_A contains a $(b-1)$–web. So $n = \pm 1$. $\square$

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