Atoroidal surface bundles lacking structure

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Abstract

We prove that the closed atoroidal surface bundles constructed in [5] often do not admit real hyperbolic metrics. We also show that they typically do not admit complex structures.

In [5], we constructed infinitely many closed atoroidal surface bundles. The first topological invariants of these bundles were computed in [7], where J. Lafont, N. Miller, and L. Ruffoni proved that many of them have signature zero. Having nonzero signature provides an obstruction to admitting a real hyperbolic structure and so it is natural to ask if the bundles could be hyperbolic manifolds. Our first result shows that we should not expect this in general.

To state the theorem, we note that one interpretation of the construction in [5] is first to build a special family $\mathscr{B}$ of torus bundles, each with three distinguished sections, and then to consider certain *allowable* branched covers of the bundles in $\mathscr{B}$ to produce examples with nonzero Euler characteristic—see §1 for the precise description.

Theorem 1. *Given any E in $\mathscr{B}$, then among all of the allowable regular branched covers $\widehat{E} \to E$, at most finitely many commensurability classes can admit a real hyperbolic structure.*

The key idea in the proof of Theorem 1 is similar to that used by M. Gromov and W. Thurston [2] in their construction of nonhyperbolic negatively curved 4–manifolds. In particular, we show that if an allowable regular branched cover $\widehat{E} \to$

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E has sufficiently high branching degree, then $\widehat{E}$ does not admit a real hyperbolic structure.

While Theorem 1 does not rule out the possibility that *some* allowable regular branched cover $\widehat{E} \to E$ of E in $\mathscr{B}$ could be *real* hyperbolic, work of M. Kapovich [4], building on work of K. Liu [8], shows that surface bundles over surfaces are *never* complex hyperbolic. We take this one step further here and show that allowable regular branched covers $\widehat{E} \to E$ admit no complex structures at all.

Theorem 2. *Let E be a torus bundle in $\mathscr{B}$, and let $\widehat{E} \to E$ be an allowable regular branched cover. Then $\widehat{E}$ does not admit a complex structure. In particular, $\widehat{E}$ is not an algebraic surface.*

A theorem of D. Kotschick [6, Theorem 4] implies that if $\widehat{E}$ admits a complex structure *and* has signature 0, then the monodromy has finite image. Thus Theorem 2 also follows from Kotschick's Theorem when the signature of $\widehat{E}$ is zero (in particular, in all cases where the main result of [7] applies), since every atoroidal surface bundle necessarily has injective monodromy.

1 Construction

Let M_8 denote the complement of the figure eight knot in S^3, which we assume is endowed with its complete hyperbolic structure—see [9]. In [5] we constructed a homomorphism $\Delta\colon \pi_1(M_8) \to \mathrm{PMod}(S_{1,3})$ that is *type preserving*, meaning that every nonperipheral element of $\pi_1(M_8)$ is sent to a pseudo-Anosov mapping class. Given any quasi-Fuchsian closed surface $\Sigma \to M_8$, the restriction

$$\Delta_\Sigma = \Delta|_{\pi_1(\Sigma)}\colon \pi_1(\Sigma) \to \mathrm{PMod}(S_{1,3})$$

of Δ is injective with purely pseudo-Anosov image. Viewing $S_{1,3}$ as a torus with three (ordered) marked points, the homomorphism $\Delta|_\Sigma$ is the monodromy of an $S_{1,3}$–bundle

$$S_{1,3} \to E_\Sigma \to \Sigma.$$

over Σ. The three marked points in the fibers determine three sections of E_Σ. The set of torus bundles with sections mentioned in the introduction is then

$$\mathscr{B} = \{E_\Sigma\}_\Sigma,$$

where Σ ranges over all quasi-Fuchsian surfaces $\Sigma \to M_8$.

Now suppose that $\widehat{S} \to S_{1,3}$ is any connected cover branched over the three marked points such that the local degree is at least 2 at each branch point. We may choose a finite cover $\widehat{\Sigma} \to \Sigma$ so that the restriction $\Delta_{\widehat{\Sigma}}$ to $\pi_1\widehat{\Sigma}$ lifts to a homomorphism

$$\widehat{\Delta}_{\widehat{\Sigma}} \colon \pi_1\widehat{\Sigma} \to \mathrm{Mod}(\widehat{S}),$$

which is the monodromy of a $\widehat{S}$–bundle

$$\widehat{S} \to \widehat{E}_{\widehat{\Sigma}} \to \widehat{\Sigma}.$$

The fiberwise branched cover $\widehat{S} \to S_{1,3}$ and *bona fide* cover $\widehat{\Sigma} \to \Sigma$ give us a branched cover

$$\widehat{E}_{\widehat{\Sigma}} \to E_\Sigma.$$

Conversely, any cover $\widehat{E} \to E_\Sigma$ branched over the three sections of $E_\Sigma \to \Sigma$ with local branching of degree at least 2 everywhere is given by this construction. We refer to these branched covers $\widehat{E}_{\widehat{\Sigma}} \to E_\Sigma$ as *allowable*.

Theorem 3 (Kent-Leininger [5])**.** *If E_Σ lies in $\mathscr{B}$ and $\widehat{E}_{\widehat{\Sigma}} \to E_\Sigma$ is an allowable branched cover, then $\widehat{E}_{\widehat{\Sigma}}$ is atoroidal.* □

Let E_Σ be a bundle in $\mathscr{B}$, let Θ_1, Θ_2, and Θ_3 be its three distinguished sections, and suppose that $\widehat{E}_{\widehat{\Sigma}} \to E_\Sigma$ is a *regular* allowable branched cover. Regularity guarantees that the local degree over every point of Θ_i is a fixed constant $n_i \geq 2$ for each $i = 1, 2, 3$. Consequently, we can give E_Σ the structure of an orbifold with orbifold loci Θ_1, Θ_2, and Θ_3 with local groups $\mathbb{Z}/n_1\mathbb{Z}$, $\mathbb{Z}/n_2\mathbb{Z}$, and $\mathbb{Z}/n_3\mathbb{Z}$, respectively. The branched cover $\widehat{E}_{\widehat{\Sigma}} \to E_\Sigma$ is then a regular *orbifold* cover, and we let $G(\widehat{E}_{\widehat{\Sigma}})$ denote the orbifold covering group, so that $\widehat{E}_{\widehat{\Sigma}}/G(\widehat{E}_{\widehat{\Sigma}}) = E_\Sigma$.

2 No hyperbolic structure

We are now ready for the proof of Theorem 1.

Proof. Fix $\widehat{E} = \widehat{E}_{\widehat{\Sigma}} \to E_\Sigma = E$ a regular allowable branched cover where $\widehat{S}$ is the fiber of $\widehat{E} \to \widehat{\Sigma}$. As described above, we view E as an orbifold so that $\widehat{E} \to E$ is regular orbifold covering with orbifold group $G(\widehat{E})$, and let n_1, n_2, n_3 be the orders

of the orbifold loci $\Theta_1, \Theta_2, \Theta_3$, respectively. The bundle $E \to \Sigma$ is then an orbifold bundle, and the fiber is the orbifold $S_{1,3}(n_1, n_2, n_3)$, the torus with three orbifold points of order n_1, n_2, and n_3.

Since the orbifold Euler characteristic is multiplicative under covers and the Euler characteristic of a bundle is the product of that of the base and the fiber, we have

$$\begin{aligned}\chi(\widehat{E}) &= |G(\widehat{E})|\,\chi(E) \\ &= |G(\widehat{E})|\,\chi(\Sigma)\,\chi(S_{1,3}(n_1, n_2, n_3)) \\ &= |G(\widehat{E})|\,\chi(\Sigma)\sum_{i=1}^{3}\frac{1-n_i}{n_i}.\end{aligned}$$

If $\widehat{E}$ is hyperbolic, then by the Gauss–Bonnet Theorem, its volume is

$$\mathrm{Vol}(\widehat{E}) = \frac{4\pi^2}{3}\chi(\widehat{E}).$$

The group $G(\widehat{E})$ acts on $\widehat{E}$, and by Mostow Rigidity, every element is homotopic to an isometry. Thus the isometry group of $\widehat{E}$ contains a group $G_0 \cong G(\widehat{E})$. The quotient $\widehat{E}/G_0$ is a hyperbolic 4–orbifold with volume given by

$$\mathrm{Vol}(\widehat{E}/G_0) = \frac{\mathrm{Vol}(\widehat{E})}{|G_0|} = \frac{4\pi^2}{3}\chi(\Sigma)\sum_{i=1}^{3}\frac{1-n_i}{n_i} \leq 4\pi^2|\chi(\Sigma)|.$$

By Wang's Finiteness Theorem [10], there are at most finitely many hyperbolic 4–orbifolds with volume at most $4\pi^2|\chi(\Sigma)|$, and hence, among all the atoroidal bundles $\widehat{E}$ that are regular branched covers of a fixed E in $\mathscr{B}$ and admit a hyperbolic metric, all of them are orbifold covers of one of finitely many hyperbolic 4–orbifolds. Consequently, there are only finitely many commensurability classes of such atoroidal bundles. □

Remark. For any integers $n_1, n_2, n_3 \geq 2$, one can in fact construct regular, allowable branched covers $\widehat{E} \to E$ so that the branching over Θ_i is n_i, and by the volume formulae above, for at most finitely many choices of n_1, n_2, n_3 can such $\widehat{E}$ be hyperbolic. To see that such branched covers exist, one first considers any branched cover $\widehat{S}' \to S_{1,3}$ so that the branching over the i^{th} marked point is n_i (which can easily be done explicitly). This determines a branched cover $\widehat{E}' \to E$ with the correct degree of branching, but which may not be regular. Deleting the branch loci

produces an unbranched cover $\widehat{E}_0' \to E_0$, and if we then pass to the finite cover corresponding to the normal core

$$\bigcap_{g \in \pi_1 E_0} g \pi_1 \widehat{E}_0' g^{-1} \lhd \pi_1 E_0,$$

we obtain a cover that extends to a branched cover of E with the same branching degree (since the peripheral elements of $\pi_1 \widehat{E}_0'$ determining the branching lie in all of the conjugates of $\pi_1 E_0$).

3 No complex structure

To prove Theorem 2, we need a little more machinery. Given a surface S, let $\mathscr{T}(S)$ denote the Teichmüller space of S and let $\mathscr{M}(S) = \mathscr{T}(S)/\mathrm{Mod}(S)$ be the moduli space of Riemann surfaces homeomorphic to S. If a bundle

$$S \to E \to \Sigma$$

admits a complex structure, then a result of M. Kapovich [4, Theorem 2], independently proved by J. Hillman [3] and Kotschick [6, Proposition 1], tells us that Σ admits a complex structure for which there is a holomorphic map $\phi\colon \Sigma \to \mathscr{M}(S)$ with $\phi_* = \rho_E$, where ρ_E is the monodromy and ϕ_* is the induced homomorphism

$$\phi_*\colon \pi_1(\Sigma) = \pi_1^{orb}(\Sigma) \to \mathrm{Mod}(S) = \pi_1^{orb}(\mathscr{M}(S)).$$

The bundles we are considering have injective monodromy, and so we may pass to a finite index torsion free subgroup $\mathrm{Mod}'(S) < \mathrm{Mod}(S)$ and a finite index subgroup of $\pi_1(\Sigma)$ mapping into $\mathrm{Mod}'(S)$. The action of $\mathrm{Mod}'(S)$ on $\mathscr{T}(S)$ is then an action by covering transformations and thus isomorphic to the actual fundamental group of the corresponding quotient of $\mathscr{T}(S)$. In particular, this allows us to avoid the discussion of orbifold fundamental groups.

We begin with the following fact.

Proposition 4. *The moduli space $\mathscr{M}(S_{1,3})$ admits no compact holomorphic curves. Precisely, if X is a compact Riemann surface and $\phi\colon X \to \mathscr{M}(S_{1,3})$ is holomorphic, then ϕ is constant.*

Proof. There is a smooth finite cover $\mathscr{N}$ of $\mathscr{M}(S_{1,3})$ that splits as an iterated holomorphic fibration as follows. The total space $\mathscr{N}$ splits as a holomorphic fibration

$$\mathscr{P} \to \mathscr{N} \to C$$

where C is a noncompact curve of finite type (a cover of the modular curve) and $\mathscr{P}$ itself splits as a holomorphic fibration

$$\mathscr{F} \to \mathscr{P} \to \mathscr{Q}$$

where $\mathscr{F}$ and $\mathscr{Q}$ are also noncompact Riemann surfaces.

Let Y be the cover of X admitting a lift $\psi\colon Y \to \mathscr{N}$ of ϕ. Since the iterated fibration is holomorphic, the composition of ψ and the projection down to C is constant, since C is noncompact. So the image of ψ must lie in the 2–dimensional fiber $\mathscr{P}$ of that projection. But now, projecting to $\mathscr{Q}$, we have another constant map, and so the image lies in $\mathscr{F}$, and so the map ψ, and hence ϕ, is constant. □

Now suppose we have a regular covering $p\colon \widehat{S} \to S_{1,3}$ branched over the three marked points. This induces a holomorphic embedding

$$p^{\#}\colon \mathscr{T}(S_{1,3}) \to \mathscr{T}(\widehat{S})$$

which is equivariant with respect to a lifting homomorphism

$$p^{*}\colon \Gamma \to \mathrm{Mod}(\widehat{S})$$

from a finite index subgroup $\Gamma < \mathrm{Mod}(S_{1,3})$.

Proposition 5. *For a regular branched cover $p\colon \widehat{S} \to S_{1,3}$ with branching degree at least 2 over every marked point, the image $p^{\#}(\mathscr{T}(S_{1,3})) \subset \mathscr{T}(\widehat{S})$ is totally geodesic with respect to the Weil–Petersson metric.*

Proof. Since the branching degree is at least 2, then we can view $p\colon \widehat{S} \to S_{1,3}$ as a regular orbifold cover with covering group $G(\widehat{S})$. In this case, $G(\widehat{S}) < \mathrm{Mod}(\widehat{S})$ is a finite subgroup which acts on $\mathscr{T}(\widehat{S})$ with fixed point set precisely $p^{\#}(\mathscr{T}(S_{1,3}))$. Given points X and Y in $p^{\#}(\mathscr{T}(S_{1,3}))$, consider the Weil–Petersson geodesic σ between them. If the image of σ is not entirely contained in $p^{\#}(\mathscr{T}(S_{1,3}))$, then there is an element g in $G(\widehat{S})$ and a point on σ which is not fixed by g. But then $g \circ \sigma$ is another geodesic between X and Y. Since the Weil–Petersson metric is negatively curved, geodesics are unique, which is a contradiction. Thus, the image of σ is contained in $p^{\#}(\mathscr{T}(S_{1,3}))$, as required. □

We are now ready for the proof of Theorem 2, which is inspired by Daskalopoulos and Wentworth's proof of [1, Theorem 5.7].

Proof of Theorem 2. Suppose E_Σ lies in $\mathscr{B}$ and

$$\widehat{E}_{\widehat{\Sigma}} \to E_\Sigma$$

is an allowable regular branched cover with monodromy $\Delta_{\widehat{\Sigma}}\colon \pi_1\widehat{\Sigma} \to \mathrm{Mod}(\widehat{S})$. Restriction to the fibers gives the branched cover $p\colon \widehat{S} \to S_{1,3}$, which we note is also regular. By passing to a further finite sheeted regular cover, we can assume that the image of the monodromy $\Delta_{\widehat{\Sigma}}$ lies in the image of a finite index subgroup $\Gamma < \mathrm{Mod}(S_{1,3})$ via a lifting homomorphism $p^*\colon \Gamma \to \mathrm{Mod}(\widehat{S})$. Thus, $\Delta_{\widehat{\Sigma}} = p^* \circ \Delta|_{\pi_1\widehat{\Sigma}}$.

Suppose $\widehat{E}_{\widehat{\Sigma}}$ admits a complex structure. Let $\phi\colon \widehat{\Sigma} \to \mathscr{M}(\widehat{S})$ be the associated holomorphic map, where $\widehat{\Sigma}$ is given the associated complex structure. By [1], ϕ is energy minimizing with respect to the hyperbolic and Weil–Petersson metrics on the domain and range, respectively.

Write $\widetilde{\phi}\colon \widetilde{\Sigma} \to \mathscr{T}(\widehat{S})$ for the $\Delta_{\widehat{\Sigma}}$–equivariant lift to the universal covers. We claim that the image of $\widetilde{\phi}$ is contained in $p^\#(\mathscr{T}(S_{1,3}))$. Since $p^\#(\mathscr{T}(S_{1,3}))$ is totally geodesic, the closest point projection

$$r\colon \mathscr{T}(\widehat{S}) \to p^\#(\mathscr{T}(S_{1,3}))$$

is a $p^*(\Gamma)$–equivariant contraction. Consequently, if the image of $\widetilde{\phi}$ did not lie in $p^\#(\mathscr{T}(S_{1,3}))$, we could postcompose with the closest point projection to obtain a map $r \circ \widetilde{\phi}$ with smaller energy that is still $\Delta_{\widehat{\Sigma}}$–equivariant. This is a contradiction, and so the image of $\widetilde{\phi}$ lies in $p^\#(\mathscr{T}(S_{1,3}))$.

Since $\Delta_{\widehat{\Sigma}} = p^* \circ \Delta|_{\pi_1\widehat{\Sigma}}$, we see that $\widetilde{\phi}$ defines a $\Delta|_{\pi_1\widehat{\Sigma}}$–equivariant map

$$\bar{\phi} = p^\# \circ \bar{\phi}\colon \widetilde{\Sigma} \to \mathscr{T}(S_{1,3})$$

that descends to a holomorphic map

$$\widehat{\phi}\colon \widehat{\Sigma} \to \mathscr{M}(S_{1,3}).$$

Since $\Delta|_{\pi_1\widehat{\Sigma}}$ is injective, the map $\widehat{\phi}$ is nonconstant, and this contradicts Proposition 4. We conclude that $\widehat{E}$ admits no complex structure. □

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References

[1] Georgios D. Daskalopoulos and Richard A. Wentworth. Harmonic maps and Teichmüller theory. In *Handbook of Teichmüller theory. Vol. I*, volume 11 of *IRMA Lect. Math. Theor. Phys.*, pages 33–109. Eur. Math. Soc., Zürich, 2007.

[2] M. Gromov and W. Thurston. Pinching constants for hyperbolic manifolds. *Invent. Math.*, 89(1):1–12, 1987.

[3] Jonathan A. Hillman. Complex surfaces which are fibre bundles. *Topology Appl.*, 100(2-3):187–191, 2000.

[4] M. Kapovich. On normal subgroups in the fundamental groups of complex surfaces, 1998. `arXiv:math/9808085`.

[5] Autumn E. Kent and Christopher J. Leininger. Atoroidal surface bundles. Preprint, `arXiv:2405.12067`, 2024.

[6] D. Kotschick. On regularly fibered complex surfaces. In *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, volume 2 of *Geom. Topol. Monogr.*, pages 291–298. Geom. Topol. Publ., Coventry, 1999.

[7] Jean-François Lafont, Nicholas Miller, and Lorenzo Ruffoni. On signatures of the atoroidal bundles of Kent–Leininger. Preprint, `arXiv:2410.18029v1`, 2024.

[8] Kefeng Liu. Geometric height inequalities. *Math. Res. Lett.*, 3(5):693–702, 1996.

[9] Robert Riley. A quadratic parabolic group. *Math. Proc. Cambridge Philos. Soc.*, 77:281–288, 1975.

[10] Hsien Chung Wang. Topics on totally discontinuous groups. In *Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970)*, volume Vol. 8 of *Pure Appl. Math.*, pages 459–487. Dekker, New York, 1972.