Skinning maps

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> On this side he fell down from Heaven, and the land which before stood out here made a veil of the sea for fear of him and came to our hemisphere; and perhaps to escape from him that which appears on this side left here the empty space and rushed upwards.
>
> —Dante's *Inferno* XXXIV 121–126 (J. D. Sinclair's translation [7])

Let M be a compact oriented irreducible atoroidal 3–manifold with connected incompressible boundary that is not empty nor a torus. By W. Thurston's Geometrization Theorem for Haken Manifolds (see [61, 67, 68, 52]), the interior M° of M admits a geometrically finite hyperbolic structure with no cusps—G. Perelman has announced a proof of Thurston's Geometrization Conjecture in its entirety [70, 71, 72]. By the deformation theory of L. Ahlfors and L. Bers and stability theorems of A. Marden and D. Sullivan, the space of all such structures $\mathrm{GF}(M)$ may be identified with the Teichmüller space $\mathcal{T}(\partial M)$ of ∂M by identifying a manifold with its conformal boundary. The cover of M° corresponding to ∂M is a quasifuchsian manifold and we obtain a map

$$\mathrm{GF}(M) \cong \mathcal{T}(\partial M) \to \mathcal{T}(\partial M) \times \mathcal{T}(\overline{\partial M}) \cong \mathrm{GF}(\partial M \times \mathbb{R}),$$

whose first coordinate function is the identity map on $\mathcal{T}(\partial M)$. The second coordinate is Thurston's **skinning map**

$$\sigma_M \colon \mathcal{T}(\partial M) \to \mathcal{T}(\overline{\partial M}),$$

which reveals a surface obscured by the topology of M.

W. Haken proved [43] that a Haken manifold may be cut progressively along incompressible surfaces until a union of finitely many balls is attained, and Thurston's proof of the Geometrization Theorem proceeds by induction. The interior of such a union admits a hyperbolic metric, and this is the first step. Thurston reduces each subsequent step to the final one by the so–called Orbifold Trick, turning the part of the boundary one is not gluing into an array of mirrors.

At the final step you have two manifolds M and N, say, each with incompressible boundary homeomorphic to S, and a gluing map $\tau\colon \partial M \to \partial N$ that produces the manifold under study—there is another case where parts of the boundary of a connected

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manifold are identified. Suppose further that M and N are not both interval bundles over a surface—in that case, Thurston's Double Limit Theorem [86] is used, see [68]. The manifolds M and N have geometrically finite hyperbolic structures given by Riemann surfaces X and Y, and the Maskit Combination Theorems [57] say that the gluing map produces a hyperbolic structure if $\tau_*^{-1}(Y) = \sigma_M(X)$ *and* $\tau_*(X) = \sigma_N(Y)$, where τ_* is the induced map on Teichmüller spaces. Such a pair (X,Y) is a **solution to the gluing problem**.

When the pieces M and N are acylindrical, a solution is provided by the following remarkable theorem—when one of the pieces is cylindrical, something else must be done, see Section 7.

Bounded Image Theorem (Thurston [84])**.** *If M is acylindrical, the image of σ_M has compact closure.*[†]

Thurston proved that the space $\mathrm{AH}(M)$ of all hyperbolic structures on M° with the algebraic topology is compact [85], and it can be shown that σ_M admits a continuous extension to all of $\mathrm{AH}(M)$—as the proof that σ_M admits such an extension has never appeared in print, we supply a proof in Section 7. Compactness of $\mathrm{AH}(M)$ was subsequently proven by arboreal methods by J. Morgan and P. Shalen [62, 63, 64], and C. McMullen solved the gluing problem via an analytic study of σ_M [58].

Aside from the conclusion of the Bounded Image Theorem, little has been known about the image of σ_M. Y. Minsky has asked the following question.

Question (Minsky)**.** Given a topological description of M, can one give a quantitative bound on the diameter of the skinning map?

Notice that it is not obvious that zero does not serve as an upper bound, though D. Dumas and the author have shown that it will never do:

Theorem 1 (Dumas–Kent [36])**.** *Skinning maps are never constant.* □

The map $\mathrm{GF}(M) \to \mathrm{GF}(\partial M \times \mathbb{R})$ is the restriction of a map $\mathfrak{X}(M) \to \mathfrak{X}(\partial M)$ at the level of $\mathrm{SL}_2\mathbb{C}$–character varieties. We may restrict this map to the irreducible component $\mathfrak{X}_0(M)$ containing the Teichmüller space $\mathcal{T}(\partial M)$, whose complex dimension is that of $\mathcal{T}(\partial M)$, namely $-\frac{3}{2}\chi(\partial M)$. If the skinning map σ_M were constantly equal to Y, then the image of $\mathfrak{X}_0(M)$ would contain the Bers slice $\mathcal{T}(\partial M) \times \{Y\} \subset \mathrm{QF}(\partial M)$. This is forbidden by the following theorem.

Theorem (Dumas–Kent [36])**.** *A Bers slice in* $\mathrm{QF}(S)$ *is not contained in any subvariety of* $\mathfrak{X}(S)$ *of dimension* $-\frac{3}{2}\chi(S)$. □

A sufficiently effective proof of Thurston's fixed point theorem has the potential of transforming his Geometrization Theorem from simply the assertion of the existence of a hyperbolic structure into a statement containing geometric information about that structure, and Minsky's question points to the simplest scenario. With this in mind, S. Kerckhoff has put forth the following problem [54].

[†] The statement appears in [84], Peter Scott's notes (see [77]) of Thurston's lectures at the conference *Low Dimensional Topology*, in Bangor, in 1979. The author is grateful to Dick Canary for pointing this out.

Problem (Kerckhoff)**.** Find an effective proof of Thurston's fixed point theorem.

Our reply to Minsky follows—see Section 4.

Volume Estimate. *Let M be a finite volume hyperbolic 3–manifold with nonempty closed totally geodesic boundary. Then there are positive constants A, B, and ε depending only on the volume of M such that the image of σ_M is ε–thick and*

$$B \leq \operatorname{diam}(\sigma_M) \leq A.$$

A set in $\mathcal{T}(S)$ is **ε–thick** if each of its points has injectivity radius at least ε. A theorem of T. Jørgensen states that the only manner in which a collection of volumes of compact hyperbolic manifolds with totally geodesic boundaries can accumulate is via a sequence of hyperbolic Dehn fillings, see Section 4. The estimates are then obtained as a consequence of the Bounded Image Theorem, Theorem 1, and a uniform filling theorem (proven in Section 3).

Filling Theorem. *Let M be a finite volume hyperbolic manifold with nonempty closed totally geodesic boundary and let $\mathfrak{e} > 0$. There is an $\hbar > 0$ such that if the normalized length of each component of a Dehn filling slope α is at least $\hbar$, then*

$$\mathrm{d}_{\mathcal{T}(\overline{\partial M})}\big(\sigma_M(X), \sigma_{M(\alpha)}(X)\big) < \mathfrak{e}$$

for all X in $\mathcal{T}(\partial M)$.

In other words, as one performs higher and higher Dehn fillings on M, the skinning maps of the filled manifolds converge *uniformly* on all of Teichmüller space to the skinning map of M.

The reader will notice that we only require the components of α to be long in the cusp cross section of M, rather than in all of the geometrically finite manifolds M_X as X ranges over $\mathcal{T}(\partial M)$. As it turns out, a "flat" version of the Bounded Image Theorem tells us that the normalized lengths of a curve in all of the M_X are comparable, and so the normalized length condition is in fact a topological one. With the Universal Hyperbolic Dehn Filling Theorem of K. Bromberg, C. Hodgson, and S. Kerckhoff, this shows that when M has a single cusp, there is a finite set of filling slopes outside of which hyperbolic Dehn filling may be performed on *all* of the M_X. There is a similar statement in the presence of a number of cusps, though we warn the reader that it is not always the case that the set of filling slopes exceptional for hyperbolic Dehn filling is finite. For example, the Whitehead link complement admits infinitely many Lens space fillings. See Section 1.5.

If one simply desires the skinning maps to be close on a compact set, one may use J. Brock and K. Bromberg's Drilling Theorem [19], and make a geometric limit argument. This argument proceeds as follows. Given a geometrically finite hyperbolic manifold M with a rank–two cusp and nonempty connected conformal boundary, consider the quasifuchsian group corresponding to the latter. The image of its convex core in M misses a neighborhood of the cusp. By work of Bromberg, hyperbolic Dehn fillings may be performed while fixing the conformal boundary and higher and higher fillings produce bilipschitz maps outside a neighborhood of the cusp with better and better constants, by the Drilling Theorem. Since the core of the quasifuchsian group misses this neighborhood, the generators of this group move less and less in $\mathrm{PSL}_2\mathbb{C}$. This means that the quasifuchsian group at ∂M is close in the algebraic topology to the quasifuchsian group at the boundary of the filled manifold. Since the Teichmüller metric on the space of quasifuchsian groups induces the same topology, we know that the skinning surfaces are close. Notice that this proves that the skinning maps converge.

This argument passes from bilipschitz control to the algebraic topology to the Teichmüller metric. If one attempts the argument as the conformal boundary changes, the intermediate step is an obstacle: as the conformal boundary diverges, one may in fact need better and better quality bilipschitz maps in order to ensure reasonable quality estimates in the Teichmüller metric.

To gain uniform control over Teichmüller space we must make a more universal argument, which is roughly as follows. We have the arrangement above, now with a varying ideal conformal structure X. Rather than attempting passage through the algebraic topology, we estimate the Teichmüller distance between the skinning surfaces directly.

By the work of C. Epstein, there is a smooth strictly convex surface $\mathfrak{F}$ in the end of the quasifuchsian manifold $\mathrm{QF}(X,Y)$ facing Y. The curvatures of this surface are well behaved and the Hausdorff distance between $\mathfrak{F}$ and the convex core is bounded, both independent of X.

There is a universal Margulis tube about the cusp in M that misses the image of $\mathfrak{F}$. Now, in [19], Brock and Bromberg estimate the strain that the filling process places on M and $\mathfrak{F}$ and it follows that for high enough fillings, the principal curvatures of $\mathfrak{F}$ in the filled manifolds are uniformly close to those of $\mathfrak{F}$, and so $\mathfrak{F}$ is eventually strictly convex in the quasifuchsian covers of these manifolds—if $\hbar$ is large enough, the surface $\mathfrak{F}$ lifts to a surface embedded in these quasifuchsian manifolds, see Section 3.

Since $\mathfrak{F}$ is convex, there is a normal projection from $\sigma_M(X)$ to $\mathfrak{F}$. Since the image of $\mathfrak{F}$ is convex, there is a normal projection from the skinning surface of the filled manifold to the image of $\mathfrak{F}$. So we obtain a map between skinning surfaces. As we do higher and higher fillings, the principal curvatures of the image of $\mathfrak{F}$ converge in a controlled way to those of $\mathfrak{F}$, and since the derivatives of the projections depend only on these curvatures, they "cancel in the limit," and we conclude that our map is very close to conformal. This yields the desired estimate on the Teichmüller distance.

With some additional work, these techniques yield the following theorem due jointly to K. Bromberg and the author—see Section 5.

Pinched Manifolds (Bromberg–Kent)**.** *Let S be a closed hyperbolic surface. For each $\varepsilon > 0$ there is a $\delta > 0$ such that if M is an orientable hyperbolic 3–manifold with totally geodesic boundary Σ homeomorphic to S containing a pants decomposition $\mathcal{P}$ each component of which has length less than δ, then the diameter of the skinning map of M is less than ε.*

In particular, the lower bound in our Volume Estimate tends to zero as the volume grows.

The proof of the theorem is roughly as follows. Let Γ be the uniformizing Kleinian group for M. Since the pants decomposition $\mathcal{P}$ is very short, the Drilling Theorem allows it to be drilled from $\mathbb{H}^3/\Gamma$. This drilled manifold N contains an isometrically embedded copy of the convex core $\mathcal{C}$ of the so–called maximal cusp hyperbolic structure on M° corresponding to $\mathcal{P}$. The conformal boundary of the maximal cusp is a union of thrice–punctured spheres, and by a theorem of C. Adams [1], these spheres may be taken to be totally geodesic in all of the complete hyperbolic structures on N. As thrice–punctured spheres have no moduli, the manifold $\mathcal{C}$ isometrically embeds in every hyperbolic structure on N.

The meridional Dehn filling slopes of N have large normalized lengths in every hyperbolic structure on N, provided $\mathcal{P}$ is short enough, and so Dehn filling may be performed there over all of $\mathrm{GF}(N)$. So, by drilling, performing a quasiconformal deformation, and filling, we may convert $\mathbb{H}^3/\Gamma$ to any point in $\mathrm{GF}(M)$.

Now a surface $\mathfrak{G}$ with principal curvatures strictly greater than -1 is constructed in our copy of $\mathcal{C}$ in N. Unlike the surface used in the proof of the filling theorem, the surface $\mathfrak{G}$ is not convex.

Pushing $\mathfrak{G}$ back into $\mathbb{H}^3/\Gamma$ yields a surface whose principal curvatures are still above -1, which allows a normal projection to the skinning surface. As our rigid $\mathcal{C}$ in N is unaffected by any quasiconformal deformation of N, so unaffected is our surface $\mathfrak{G}$. Filling again still has little affect on $\mathfrak{G}$, and as in the proof of the Filling Theorem, we obtain a map between the skinning surface at Σ and the skinning surface at any other point that is very close to being conformal. This is the statement that the diameter of σ_M is small.

There is a strong form of Minsky's question:

Question (Minsky)**.** Is the diameter of σ_M bounded above by a constant depending only on the topology of ∂M?

A theorem of A. Basmajian [11] implies that a bound on the volume of a manifold with totally geodesic boundary yields a bound on the sum of the genera of the boundary components, and so an affirmative answer implies the upper bound in our Volume Estimate. Note that by the previous theorem, and M. Fujii and T. Soma's Density Theorem [39], there is no lower bound depending only on the topology of ∂M.

In their solution of Thurston's Ending Lamination Conjecture [60, 21, 20], Brock, R. Canary, and Minsky show that Minsky's model manifold associated to ending laminations λ_- and λ_+ on a surface S is L–bilipschitz to any hyperbolic manifold homeomorphic to $S\times\mathbb{R}$ with those ending laminations—this is part of the Bilipschitz Model Theorem. It follows from D. Sullivan's Rigidity Theorem [79] that there is a unique hyperbolic manifold homeomorphic to $S\times\mathbb{R}$ with ending laminations λ_- and λ_+. The model is particularly good in this case as the constant L only depends on the topology of S.

In the acylindrical case, the proof of the Model Theorem should proceed as follows. We have an acylindrical manifold M and a hyperbolic structure on its interior. The model is to be built by passing to the cover corresponding to ∂M, constructing the model for the end invariants λ and $\sigma_M(\lambda)$, cutting off the end facing the Riemann surface $\sigma_M(\lambda)$, and capping off with M equipped with an arbitrary metric. A uniform bound on the diameter of σ_M would be useful in promoting this model to one whose bilipschitz constant depends only on the topology of ∂M.

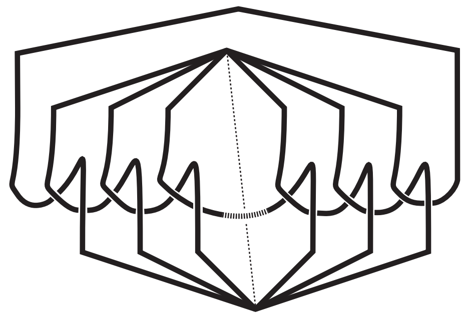

Figure 1: Suzuki's Brunnian graph $\mathcal{G}_n$ on n edges.

As we prove in Section 6, any uniformity in an upper bound will necessarily depend on the topology of ∂M:

Flexible Manifolds. *There are hyperbolic* 3*–manifolds with connected totally geodesic boundary whose skinning maps have diameter as large as you like.*

The exterior M_n of S. Suzuki's Brunnian graph $\mathcal{G}_n$ in Figure 1 [82] admits a hyperbolic structure with totally geodesic boundary, see [69] and [88]. As n tends to infinity, the length of the fine arc depicted tends to zero—one may see this directly from the explicit hyperbolic structure given in [69] and [88]; and indirectly by observing that the M_n are n–fold branched covers of $\frac{n}{0}$–orbifold fillings on the complement of a fixed tangle, one strand of which lifts to the fine arc. This means that we may normalize the domain of discontinuity to appear as in Figure 2, where the disk containing infinity unifomizes the boundary of M_n. The large disk in the center leads one to suspect that σ_{M_n} has large diameter, as a deformation of ∂M_n is carried by the skinning map to a deformation of $\overline{\partial M_n}$ that seems dominated by its effect in the central disk. For a particular deformation, this intuition is justified by E. Reich and K. Strebel's First Fundamental Inequality.

In fact, a manifold will have a skinning map of large diameter provided it possesses only shallow collars about its totally geodesic boundary, see Section 6; conversely, we have the following theorem, proven in Section 2:

Focused Manifolds. *Let S be a closed hyperbolic surface. Let $\varepsilon, \delta, R > 0$. There is a constant $d > 0$ such that if M is a hyperbolic* 3*–manifold with totally geodesic boundary Σ homeomorphic to S of injectivity radius at least δ possessing a collar of depth d, then* $\operatorname{diam}\big(\sigma_M\big(B(\Sigma,R)\big)\big) < \varepsilon$.

A theorem of Fujii and Soma [39] says that the surfaces appearing as the totally geodesic boundary of a hyperbolic 3–manifold without cusps form a dense subset of the Teichmüller space—this theorem depends on a theorem of R. Brooks [26] that says that the set of surfaces appearing as nonseparating totally geodesic surfaces in closed hyperbolic manifolds is dense. It is not difficult to see from the proof that for any $d > 0$, the totally geodesic boundaries of manifolds possessing a collar of depth d about the

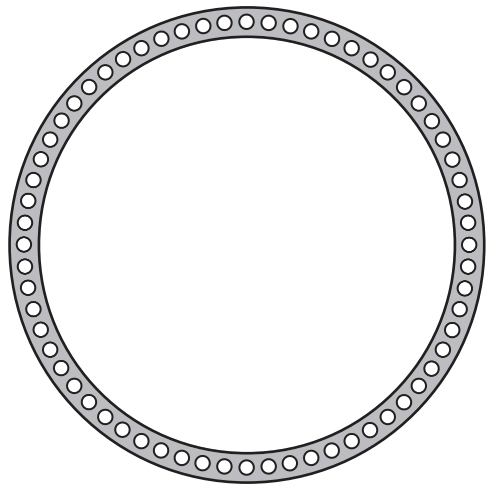

Figure 2: The domain of discontinuity of the uniformizing Kleinian group for $S^3 - \mathcal{G}_n$.

boundary are also dense, see Section 2.1, and so the manifolds in the theorem exist in great abundance.

To prove the theorem, the skinning surface is compared directly to the totally geodesic boundary as in the proof of the Filling Theorem. The necessary control is again guaranteed by work of Brock and Bromberg, namely their Geometric Inflexibility Theorem and the estimates of strain in its proof.

A corollary of the above theorem is the following quasifuchsian version of Fujii and Soma's density theorem, proven in Section 2.2—again this has the flavor of a theorem of Brooks: the set of quasifuchsian groups induced by nonseparating surfaces in closed hyperbolic manifolds in dense in the space of all such groups, see [26].

Quasifuchsian Density. *The subset of* $\mathrm{QF}(S)$ *consisting of quasifuchsian groups that appear as the only boundary subgroup of a geometrically finite acylindrical hyperbolic* 3*–manifold is dense.*

Another corollary is our Capping Theorem, proven in Section 2.3, which allows certain properties of skinning maps of manifolds with a number of boundary components to be promoted to those of manifolds with connected boundary.

We close the introduction by emphasizing that our manifolds with large diameter skinning maps have *only* very shallow collars about their totally geodesic boundaries, while those with highly contracting maps have very deep collars. With Basmajian's theorem [11] that totally geodesic surfaces have collars whose depth depends only on the surface's area, this is perhaps an indication that Minsky's second question has an affirmative answer.

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1 The landscape

1.1 Teichmüller spaces

We refer the reader to [6], [40], and [41] for more on quasiconformal mappings and Teichmüller theory.

A homeomorphism $f\colon X \to Y$ between Riemann surfaces is **K–quasiconformal** if it has locally integrable distributional derivatives that satisfy

$$|f_{\bar{z}}| \leq k|f_z|$$

where $k < 1$ is the number $(K-1)/(K+1)$. The **dilatation** $\mathrm{K}[f]$ of f is the infimum of all K for which f is K–quasiconformal. A quasiconformal map has a **Beltrami coefficient** μ_f given by the **Beltrami equation**

$$f_{\bar{z}} = \mu_f f_z \tag{1.1}$$

Let Γ be a torsion free Fuchsian group acting on $\mathbb{H}^2$—we consider $\mathbb{H}^2$ as a round disk in $\widehat{\mathbb{C}}$ of area π. An element μ of $L^\infty(\mathbb{H}^2)$ is a **Beltrami differential for Γ** if it satisfies

$$\mu(g(z))\frac{\overline{g'(z)}}{g'(z)} = \mu(z)$$

for all g in Γ—so that μ is the lift of a differential $\mu(z)\mathrm{d}\bar{z}/\mathrm{d}z$ on $X_\Gamma = \mathbb{H}^2/\Gamma$. The set of all such differentials is written $L^\infty(\mathbb{H}^2,\Gamma)$ or $L^\infty(X_\Gamma)$. Given such a μ with $\|\mu\|_\infty < 1$, there is a unique quasiconformal map $f_\mu\colon\ \mathbb{H}^2 \cup S^1_\infty \to \mathbb{H}^2 \cup S^1_\infty$ fixing 0, 1, and ∞ whose Beltrami coefficient is μ. Two differentials μ and ν are **Teichmüller equivalent** if f_μ and f_ν are identical on S^1_∞—we also say that f_μ and f_ν are equivalent in this case. The quotient of the open unit ball of $L^\infty(\mathbb{H}^2,\Gamma)$ by this equivalence relation is the **Teichmüller space $\mathcal{T}(\Gamma)$ of Γ**—also referred to as the Teichmüller space $\mathcal{T}(X_\Gamma)$ of X_Γ. O. Teichmüller's metric on $\mathcal{T}(\Gamma)$ is given by

$$\mathrm{d}_{\mathcal{T}(\Gamma)}\big([\mu],[\nu]\big) = \frac{1}{2}\inf \log \mathrm{K}[f_{\widetilde{\mu}} \circ f_{\widetilde{\nu}}^{-1}]$$

where the infimum is over all $\widetilde{\mu}$ in $[\mu]$ and $\widetilde{\nu}$ in $[\nu]$.

A quasiconformal f is **extremal** if the L^∞–norm of its Beltrami coefficient is less than or equal to that of any map equivalent to f. Every quasiconformal map is equivalent to an extremal one. Given a Beltrami differential μ in the unit ball of $L^\infty(X_\Gamma)$, we let $\mathrm{K}[\mu]_{X_\Gamma}$ denote the dilatation of an extremal quasiconformal map equivalent to f_μ, called its **extremal dilatation**, and we drop the subscript X_Γ when no confusion can arise.

We write $\mathcal{Q}(X_\Gamma)$ for the space of integrable holomorphic quadratic differentials on X_Γ, equipped with the norm

$$\|\cdot\| = \int_{X_\Gamma} |\cdot|$$

On $\mathbb{H}^2$, these are the integrable holomorphic functions φ satisfying

$$\varphi(g(z))g'(z)^2 = \varphi(z)$$

for all g in Γ and all z in $\mathbb{H}^2$—these are the **holomorphic cusp forms for** Γ.

1.1.1 The Fundamental Inequality

The inequality here due to Reich and Strebel estimates the extremal dilatation of a Beltrami differential from below. We refer the reader to Section 4.9 of [41] for a proof.

Theorem 2 (First Fundamental Inequality [41])**.** *Let $f\colon X \to Y$ be a quasiconformal map with Beltrami coefficient μ. Then the extremal dilatation satisfies*

$$\frac{1}{\mathrm{K}[\mu]_X} \leq \iint_X |\varphi| \frac{\left|1-\mu\frac{\varphi}{|\varphi|}\right|^2}{1-|\mu|^2} \tag{1.2}$$

for all unit norm holomorphic quadratic differentials φ on X. □

Note that if μ is a Teichmüller differential $k|\varphi|/\varphi$, we have equality in (1.2).

1.2 3–manifolds

Let M be a smooth compact 3–manifold with boundary. The manifold M is **irreducible** if every smoothly embedded 2–sphere bounds a ball. It is **atoroidal** if every π_1–injective map $S^1 \times S^1 \to M$ is homotopic into ∂M, and **acylindrical** if every homotopically essential map of pairs $(A, \partial A) \to (M, \partial M)$ is homotopic as a map of pairs into ∂M whenever A is an annulus $S^1 \times I$. A noncompact 3–manifold is **tame** if it is homeomorphic to the interior of a compact manifold.

1.3 Kleinian groups

The group of orientation preserving isometries of hyperbolic 3–space $\mathbb{H}^3$ is $\mathrm{PSL}_2\mathbb{C}$, and a **Kleinian group** is a discrete subgroup of this group. We will assume throughout that our Kleinian groups are **nonelementary**, meaning that they are not virtually abelian.

The **limit set** Λ_Γ of a Kleinian group Γ is the unique minimal nonempty closed Γ–invariant subset of $\widehat{\mathbb{C}}$. Its complement, $\Omega_\Gamma = \widehat{\mathbb{C}} - \Lambda_\Gamma$ is the **domain of discontinuity** of Γ.

We let $M_\Gamma = \mathbb{H}^3/\Gamma$ and let $\dot{M}_\Gamma = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$ denote the **Kleinian manifold**. The Riemann surface Ω_Γ/Γ is the **conformal boundary** of M_Γ. By the Ahlfors Finiteness Theorem [4, 5, 42], see also [51], the conformal boundary of a finitely generated Kleinian group has finite type.

The quotient $\mathcal{C}_\Gamma = \mathfrak{H}_\Gamma/\Gamma$ of the convex hull $\mathfrak{H}_\Gamma$ of Λ_Γ is the **convex core** of M_Γ. We sometimes write $\mathcal{C}(M_\Gamma)$ for $\mathcal{C}_\Gamma$. We say that Γ and M_Γ are **geometrically finite** if the 1–neighborhood of $\mathcal{C}_\Gamma$ has finite volume.

Given a hyperbolic manifold M, we write Γ_M for the uniformizing Kleinian group.

We let ε_3 be the 3–dimensional Margulis constant.

1.3.1 The algebraic topology

Let M be a compact oriented 3–manifold with *incompressible boundary* and let P be a π_1–injective 2–dimensional submanifold of ∂M of Euler characteristic zero containing all of the tori in ∂M. The **space of hyperbolic structures** $\mathrm{H}(M,P)$ is the space of discrete faithful representations $\rho\colon\ \pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$ up to conjugacy such that for each component P_0 of P, the image $\rho\big(\pi_1(P_0)\big)$ is purely parabolic. Such representations are then holonomy representations of complete hyperbolic manifolds homotopy equivalent to M and so we may think of $\mathrm{H}(M,P)$ as a space of hyperbolic manifolds. We let $\mathrm{AH}(M,P)$ denote this set equipped with the topology induced by the inclusion

$$\mathrm{AH}(M,P) \subset \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2\mathbb{C})\big/\mathrm{PSL}_2\mathbb{C},$$

called the **algebraic topology**—this is the same topology that $\mathrm{AH}(M,P)$ inherits from the inclusion into the $\mathrm{SL}_2\mathbb{C}$–character variety $\mathfrak{X}(M,P)$, which is a particular birational representative of the quotient by $\mathrm{SL}_2\mathbb{C}$ of the space of representations with parabolics at P in the sense of geometric invariant theory (we refer the reader to [35] and [78] for detailed discussions of the $\mathrm{SL}_2\mathbb{C}$–character variety). When P is the union of all of the tori in ∂M, we write $\mathrm{AH}(M) = \mathrm{AH}(M,P)$. When given a geometrically finite hyperbolic 3–manifold N, possibly with geodesic boundary, we often write $\mathrm{AH}(N)$ for the space $\mathrm{AH}(M,P)$ where M is a compact manifold whose interior is homeomorphic to that of N and P is the submanifold of ∂M corresponding to the cusps of N. A hyperbolic structure in $\mathrm{AH}(M,P)$ has **totally geodesic boundary** if the subgroups of $\pi_1(M)$ corresponding to $\partial M - P$ are all Fuchsian, so that the representation is the holonomy representation of a hyperbolic metric on $M - P$ with totally geodesic boundary.

Combining work of V. Chuckrow [33] and Jørgensen [48], it follows that $\mathrm{AH}(M,P)$ is closed in $\mathfrak{X}(M,P)$, and, as mentioned in the introduction, the key to the Bounded Image Theorem is the following compactness theorem of Thurston, see also [62, 63, 64].

Theorem 3 (Thurston [85, 87])**.** *If (M,P) admits a hyperbolic structure with totally geodesic boundary, then $\mathrm{AH}(M,P)$ is compact.* □

By a theorem of Marden [56] and Sullivan [81], one component of the interior of $\mathrm{AH}(M,P)$ is precisely the set of geometrically finite hyperbolic structures on M° compatible with the orientation on M with no parabolics other than those arising from P. We let $\mathrm{GF}(M,P)$ denote this component. Identifying a manifold with its conformal boundary, the Measurable Riemann Mapping Theorem of Ahlfors and Bers [3] then implies the following theorem, see [14] or [31].

Theorem 4 (Ahlfors–Bers–Marden–Sullivan)**.** *Quasiconformal conjugation induces a biholomorphic isomorphism* $\mathrm{GF}(M,P) \cong \mathcal{T}(\partial M - P)$. □

Given a point X in $\mathcal{T}(\partial M - P)$, we write M_X for the manifold M° equipped with the hyperbolic structure corresponding to X.

Let S be a closed surface. When M is homeomorphic to $S \times [0,1]$, we write $\mathrm{QF}(S) = \mathrm{GF}(M)$. In this case, Theorem 4 is Bers' Simultaneous Uniformization Theorem:

Theorem 5 (Simultaneous Uniformization [13])**.** $\mathrm{QF}(S) \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S})$. □

If X and Y are points in $\mathcal{T}(S)$ and $\mathcal{T}(\overline{S})$ we write $\mathrm{QF}(X,Y)$ for the hyperbolic manifold in $\mathrm{QF}(S)$ corresponding to the pair (X,Y). If A and B are sets in $\mathcal{T}(S)$ and $\mathcal{T}(\overline{S})$, we let $\mathrm{QF}(A,B)$ denote the set of all manifolds $\mathrm{QF}(X,Y)$ with X in A and Y in B.

1.3.2 The geometric topology

A sequence Γ_n of Kleinian groups converges in the **Chabauty topology** to a group $\Gamma < \mathrm{PSL}_2\mathbb{C}$ if the following conditions hold

1. For each γ in Γ, there are elements γ_n in Γ_n such that $\gamma = \lim \gamma_n$.
2. If Γ_{n_k} is a subsequence, and γ_{n_k} in Γ_{n_k} are such that $\lim \gamma_{n_k} = \gamma$, then γ lies in Γ.

Geometrically, this is formulated as follows. A **framed hyperbolic manifold** is simply a hyperbolic manifold equipped with an orthonormal frame at a basepoint. A sequence of framed hyperbolic manifolds M_n **converge geometrically** to a framed hyperbolic manifold M if for each smooth compact submanifold K of M containing the basepoint, there is a smooth frame preserving map $\varphi_n\colon K \to M_n$, and the φ_n converge to an isometry in the C^∞–topology—meaning that the lifted maps $\widetilde{\varphi}_n\colon \widetilde{K} \to \mathbb{H}^3$, normalized so that all frames are the standard frame at the origin, converge in the topology of C^∞–convergence on compact sets.

These two notions are equivalent, see Chapter 2 of [59] and Chapter E of [12].

1.3.3 Pleated surfaces

A map $f\colon \Sigma \to M$ from a hyperbolic surface Σ to a hyperbolic 3–manifold M is a **pleated surface** provided it is a path isometry (meaning that it sends rectifiable arcs to rectifiable arcs of the same length) and for each x in Σ, there is a geodesic segment γ through x that f carries to a geodesic segment in M.

Confusing a pleated surface with its image, the boundary $\partial\mathcal{C}_\Gamma$ of the convex core of a hyperbolic manifold M_Γ is a collection of pleated surfaces.

A **framed pleated surface** is a pleated surface $f\colon \Sigma \to M$ together with choices of orthonormal frames at points p and q in Σ and M, respectively, such that f carries the frame at p into the frame at q.

For $A, \varepsilon > 0$, let $\mathcal{PSF}(A,\varepsilon)$ be the set of all framed pleated surfaces $f\colon \Sigma \to M$ into hyperbolic 3–manifolds M such that Σ has area less than A and such that the injectivity radii of Σ and M are bounded below by ε at the base frame. Using the Chabauty topology on the uniformizing Fuchsian and Kleinian groups for the Σ and M, and the compact–open topology for the maps, we obtain a topology on $\mathcal{PSF}(A,\varepsilon)$.

We will need the following theorem of Thurston [85], see Theorem 5.2.2 of [29].

Theorem 6 (Pleated Surfaces Compact)**.** *The space $\mathcal{PSF}(A,\varepsilon)$ is compact.* □

We will also need the following standard lemma.

Lemma 7. *Let $\varepsilon_3 \geq \varepsilon > 0$ and let $f\colon \Sigma \to M$ be a pleated surface. There is a $\delta_\varepsilon < \varepsilon$ depending only on ε and the topological type of Σ such that if $f(\Sigma)$ intersects a component $\mathbf{T}^{\delta_\varepsilon}$ of the δ_ε–thin part of M, then there is an essential simple closed curve γ in Σ of length less than ε such that $f(\gamma) \subset \mathbf{T}^\varepsilon$, where $\mathbf{T}^\varepsilon$ is the component of the ε–thin part containing $\mathbf{T}^{\delta_\varepsilon}$.*

Proof. A theorem of Brooks and Matelski [27] says that the distance between $\partial\mathbf{T}^\varepsilon$ and $\partial\mathbf{T}^\delta$ tends to infinity as δ tends to zero. So, if $f(\Sigma)$ intersects $\mathbf{T}^\delta$ for some small δ, then, since f is 1–Lipschitz, the set $f^{-1}(\mathbf{T}^\varepsilon)$ either contains an essential simple closed curve of length less than ε or a disk of enormous area. The Gauss–Bonnet Theorem completes the proof. □

1.3.4 Deformations

A diffeomorphism $f\colon (M,g) \to (N,h)$ between Riemannian manifolds is ***L*–bilipschitz** if there is an $L \geq 1$ such that for every x in M,

$$\frac{1}{L}\|v\|_g \leq \|Df_x v\|_h \leq L\|v\|_g$$

for all v in T_xM. We will make use of the following well–known theorem—see Theorem 2.5 and Corollary B.23 of [59].

Theorem 8. *Let Γ be a Kleinian group.*

A Γ–equivariant L–bilipschitz map $\mathbb{H}^3 \to \mathbb{H}^3$ extends continuously to a Γ–equivariant $K(L)$–quasiconformal map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Moreover, $K(L)$ tends to 1 as L does.

A Γ–equivariant K–quasiconformal map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ extends continuously to a Γ–equivariant $K^{3/2}$–bilipschitz map $\mathbb{H}^3 \to \mathbb{H}^3$. □

And we will once mention the following proposition—see Proposition 2.16 in [59].

Proposition 9. *Let $\Psi\colon \mathbb{H}^3 \to \mathbb{H}^3$ be an L–bilipschitz map and let Λ be a closed set in $\widehat{\mathbb{C}}$ with convex hull $\mathfrak{H}_\Lambda$. Then there is a constant $h(L)$ such that $\Psi(\mathfrak{H}_\Lambda)$ lies in the $h(L)$–neighborhood of $\mathfrak{H}_{\Psi(\Lambda)}$.* □

1.4 Skinning

Let M be a hyperbolic manifold with connected totally geodesic boundary Σ. There is a map induced by inclusion

$$\iota\colon \mathcal{T}(\Sigma) \cong \mathrm{GF}(M) \longrightarrow \mathrm{QF}(\Sigma) \cong \mathcal{T}(\Sigma) \times \mathcal{T}(\overline{\Sigma})$$

whose first coordinate function is the identity on $\mathcal{T}(\Sigma)$. Composition with projection onto the second factor yields a map

$$\sigma_M\colon\ \mathcal{T}(\Sigma) \to \mathcal{T}(\overline{\Sigma}),$$

called the **skinning map** associated to M. The fact that the target of the map ι is indeed $\mathrm{QF}(\Sigma)$ follows from the Annulus Theorem [89, 32] and the fact, first observed by Thurston, that a geometrically finite Kleinian group with nonempty domain of discontinuity has the property that all of its finitely generated subgroups are geometrically finite, which itself follows from the Ahlfors Finiteness Theorem, see [61]. As the map $\iota\colon\ \mathrm{GF}(M) \to \mathrm{QF}(\Sigma)$ is the restriction of the regular map $\mathfrak{X}(M) \to \mathfrak{X}(\Sigma)$ between $\mathrm{SL}_2\mathbb{C}$–character varieties induced by inclusion, and the identifications $\mathcal{T}(\Sigma) \cong \mathrm{GF}(M)$ and $\mathrm{QF}(\Sigma) \cong \mathcal{T}(\Sigma) \times \mathcal{T}(\overline{\Sigma})$ are biholomorphic isomorphisms, the skinning map is holomorphic.

We henceforth denote the open unit disk in $\widehat{\mathbb{C}}$ by $\mathcal{L}$, and let $\mathcal{U}$ denote the interior of its complement.

After conjugating, we assume that the boundary Σ is uniformized by a Fuchsian subgroup $\Gamma_\Sigma \leq \Gamma_M$ acting on $\mathcal{U}$.

The skinning map is easily described at the level of equivariant Beltrami differentials by sending the Teichmüller class of a differential μ in $L^\infty(\mathcal{U}, \Gamma_\Sigma)$ to the class of the differential in $L^\infty(\mathcal{L}, \Gamma_{\overline{\Sigma}})$ obtained by transporting μ to a differential on $\widehat{\mathbb{C}}$ via the functional equation

$$\mu\big(g(w)\big)\frac{\overline{g'(w)}}{g'(w)} = \mu(w) \quad \text{when } g \in \Gamma_M \tag{1.3}$$

and restricting to $\mathcal{L}$.

Suppose that $\Sigma = \Sigma_0 \sqcup \Sigma_1 \sqcup \ldots \sqcup \Sigma_m$, and that Σ_0 is connected and uniformized by Γ_{Σ_0} in the disk $\mathcal{U}$. We define the **skinning map** σ_{M,Σ_0} **of** M **relative to** $\Sigma_1 \sqcup \ldots \sqcup \Sigma_m$ by transporting a differential in $L^\infty(\mathcal{U}, \Gamma_\Sigma)$ to $\Gamma_M\mathcal{U}$ using (1.3), declaring that the differential be identically zero on $\widehat{\mathbb{C}} - \Gamma_M\mathcal{U}$, and restricting to $\mathcal{L}$.

Given a skinning map σ, we will, to simplify the notation, write $\sigma(\mu)$ for the differential in $L^\infty(\mathcal{L}, \Gamma_{\overline{\Sigma}})$ constructed above, and $\sigma([\mu]) = [\sigma(\mu)]$ for its Teichmüller equivalence class—although the latter is the actual image of $[\mu]$ under the skinning map, it is useful to work with the representative $\sigma(\mu)$.

1.5 Dehn filling for families

Given a 3–manifold M, a **slope** is the isotopy class of an essential embedded 1–manifold on the union of tori in ∂M with at most one component on each torus. Given a slope α, we may perform **Dehn filling** along α to obtain a manifold $M(\alpha)$—if α is a union of circles α_j, then $M(\alpha)$ is obtained by gluing solid tori T_j to ∂M so that α_j bounds a disk in T_j.

If M is a geometrically finite hyperbolic manifold, **slope** will mean an isotopy class of an essential embedded 1–manifold on the boundary of the complement of a horospherical neighborhood of the cusps, and Dehn fillings will refer to fillings of this complement. If the α–filling of M admits a complete hyperbolic metric, we reserve $M(\alpha)$

to denote the hyperbolic structure *with the same conformal boundary* as M, and a filling is **hyperbolic** if the cores of the filling tori are geodesics—in Jørgensen's theorem we alter this convention, see Section 4.

In the finite volume setting, Thurston's Hyperbolic Dehn Filling Theorem provides a set of filling slopes outside of which any filling may be taken to be hyperbolic, see [73] for instance. Hodgson and Kerckhoff have shown [46] that as long as the normalized length of each component of a slope α in the horospherical cross sections of the cusps is at least $7.515\sqrt{2}$, then hyperbolic Dehn filling may be performed at α—the **normalized length** of a curve γ in a flat torus is its length divided by the positive square root of the area of the torus, the normalized length of an isotopy class the infimum of the normalized lengths of its representatives.

Given a geometrically finite hyperbolic manifold M, a generalization of the Hyperbolic Dehn Filling theorem—first used by F. Bonahon and J.-P. Otal [16]—applies to show that, at each cusp, there is a finite set of slopes that are exceptional for hyperbolic filling [34]. It is not *a priori* obvious that this set may be taken to be independent of the geometrically finite hyperbolic structure on M.[‡] Thanks to the work of Bromberg, Hodgson and Kerckhoff's Universality Theorem extends (with a different constant) to the geometrically finite setting, which gives to each cusp a uniform finite set of exceptional slopes for all hyperbolic structures in $\mathrm{GF}(M)$, as we now discuss.

Let M be a compact orientable irreducible acylindrical 3–manifold. Let T be the union of the tori in ∂M. So, $\mathrm{GF}(M) \cong \mathcal{T}(\partial M - T)$. Let $\mathcal{T}(T)$ be the Teichmüller space of T—the space of marked flat metrics on T such that each component of T has area one. There is a map

$$\tau_M \colon \mathrm{GF}(M) \to \mathcal{T}(T)$$

that assigns the shapes of the cusp cross sections to a geometrically finite hyperbolic structure. We have the following "flat" version of the Bounded Image Theorem.

Theorem 10. *The image of τ_M has compact closure.*

Proof. The space $\mathrm{AH}(M)$ is compact, and τ_M extends continuously to the closure $\overline{\mathrm{GF}(M)}$ of $\mathrm{GF}(M)$ in $\mathrm{AH}(M)$. □

While continuity in the proof is apparent *here*, it is not in the proof of the Bounded Image Theorem, see Section 7.

The following theorem is a version of Hodgson and Kerckhoff's Universal Hyperbolic Dehn Filling Theorem, extended via work of Bromberg, that allows the manifolds to have infinite volume, and is stated in [22].

Theorem 11 (Bromberg–Hodgson–Kerckhoff [46, 23, 24])**.** *There is a universal constant h such that the following holds. If M is as above and the normalized length of each component of a filling slope α is at least h on T, then hyperbolic Dehn filling may*

[‡]A slope α that is nonexceptional for hyperbolic Dehn filling on M may be exceptional for hyperbolic Dehn filling on M_X. To see this, note that while the manifold $M(\alpha)$ does admit a hyperbolic structure $M(\alpha)_X$ with conformal boundary X, thanks to Thurston's Geometrization Theorem for Haken Manifolds, it is not clear that the core of the filling torus may be taken to be geodesic in $M(\alpha)_X$. So α may be exceptional for hyperbolic Dehn filling on M_X.

be performed at α. Moreover, for each ℓ, there is an h_ℓ such that if these normalized lengths are at least h_ℓ, then the lengths of the cores of the filling tori are at most ℓ. □

The hyperbolic Dehn filling is achieved by a cone deformation that pushes the cone angle from zero up to 2π, see Section 1.8.

Theorem 11 has the following topological corollary.

Theorem 12 (Universal Exceptions)**.** *Let M be as above and let $\ell > 0$. Let $\mathbf{P}_1, \ldots, \mathbf{P}_n$ be the rank–two cusps of M, and let $\mathfrak{S}_i$ be the set of slopes on $\partial\mathbf{P_i}$. Then there are finite sets $\mathcal{E}_i \subset \mathfrak{S}_i$ such that for all M_X in $\mathrm{GF}(M)$ and all*

$$\alpha \in \{(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \notin \mathcal{E}_i \text{ for all } i\},$$

one may perform hyperbolic Dehn filling on M_X at α. Moreover, the lengths of the cores of the filling tori are less than ℓ in $M_X(\alpha)$ for all M_X in $\mathrm{GF}(M)$.

Proof. The theorem is simply a corollary of Theorem 11, Theorem 10, and the fact that the length spectrum of a flat torus is discrete with finite multiplicities. □

1.6 Strain fields

Deformations of hyperbolic manifolds such as cone deformations and quasiconformal conjugacies induce bilipschitz maps between manifolds. In these cases, the pointwise bilipschitz constant may be controlled by estimating norms of certain strain fields. These estimates often provide control of other geometric information, such as various curvatures of embedded surfaces, which we will soon desire and so now discuss.

The material in the next few sections is drawn mostly from [19], [18], and [59], where the reader will find more detailed discussions—see also [45].

In the following, g_t will denote a smooth family of hyperbolic metrics on a 3–manifold M. There are vector valued 1–forms η_t defined by

$$\frac{\mathrm{d}g_t(x,y)}{\mathrm{d}t} = 2g_t\big(x, \eta_t(y)\big)$$

and, choosing an orthonormal frame field $\{e_1, e_2, e_3\}$ for the g_t metric, we define the norm

$$\|\eta_t\|^2 = \sum_{i,j=1}^{3} g_t\big(\eta_t(e_i), \eta_t(e_j)\big). \tag{1.4}$$

We consider metrics g_t that are the pullback of a fixed metric g under the flow φ_t of a time–dependent vector field v_t. Letting ∇^t be the Levi–Civita connection for g_t, we have

$$\eta_t = \operatorname{sym} \nabla^t v_t$$

when $g_t = \varphi_t^* g$. The v_t will be **harmonic**, meaning that they are divergence free and $\operatorname{curl}\operatorname{curl} v_t = -v_t$ (where the curl is half of the usual one). On vector–valued k–forms we define the operator

$$D_t = \sum_{i=1}^{3} \omega^i \wedge \nabla^t_{e_i}$$

where $\{\omega^1, \omega^2, \omega^3\}$ is a coframe field dual to $\{e_1, e_2, e_3\}$. The formal adjoint of this operator is then

$$D_t^* = \sum_{i=1}^{3} \iota(e_i)\nabla^t_{e_i}$$

where ι is contraction. In [44] it is shown that $D_t^* \eta_t = 0$.

Proposition 13 (Brock–Bromberg, Proposition 6.4 of [19])**.** *Let $\gamma(s)$ be a smooth curve in M and let $C(t)$ be the geodesic curvature of γ at $\gamma(0) = p$ in the g_t–metric. For each $\varepsilon > 0$ there exists a $\delta > 0$ depending only on ε, a, and $C(0)$ such that $|C(a) - C(0)| < \varepsilon$ if $\|\eta_t(p)\| \leq \delta$, $\| * D_t \eta_t(p)\| \leq \delta$, and $D_t^* \eta_t = 0$ for all $t \in [0, a]$.* $\square$

The pointwise norm of $*D_t\eta_t$ is defined as in (1.4).

Given a vector field v, the symmetric traceless part of ∇v is the **strain** of v, which measures the conformal distortion of the metric pulled back by the flow of v. When v is divergence–free ∇v is traceless, and so $\eta = \operatorname{sym}\nabla v$ is a strain field. We say that η is **harmonic** if v is.

If η is a harmonic strain field, so is $*D\eta$, by Proposition 2.6 of [44].

We will need the following mean–value inequality for harmonic strain fields, see Theorem 9.9 of [23].

Theorem 14 (Hodgson–Kerckhoff)**.** *Let η be a harmonic strain field on a ball B_R of radius R centered at p. Then*

$$\|\eta(p)\| \leq \frac{3\sqrt{2\,\mathrm{vol}(B_R)}}{4\pi f(R)} \sqrt{\int_{B_R} \|\eta\|^2 \mathrm{d}V}\,,$$

where

$$f(R) = \cosh(R)\sin\left(\sqrt{2}R\right) - \sqrt{2}\sinh(R)\cos\left(\sqrt{2}R\right)$$

and $R < \pi/\sqrt{2}$. $\square$

1.7 Cone manifolds

Let M be a compact manifold and let $\mathbf{c} \subset M$ be a compact 1–manifold. A **hyperbolic cone metric** on $(M, \mathbf{c})$ is a hyperbolic metric on the interior of $M - \mathbf{c}$ whose metric completion is a metric on the interior of M that in a neighborhood of a component of $\mathbf{c}$ has the form

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\theta^2 + \cosh^2 r\,\mathrm{d}z^2$$

where θ is measured modulo a **cone angle** a.

The hyperbolic metric on the universal cover of the interior of $M - \mathbf{c}$ has a developing map to $\mathbb{H}^3$, and we say that a family g_t of hyperbolic cone metrics is smooth if its family of developing maps is smooth.

Our hyperbolic cone manifolds will be geometrically finite, meaning that the cone metric extends to a conformal structure on the portion of ∂M that contains no tori.

1.8 Drilling

The following theorem of Bromberg allows one to deform certain cone manifolds by pushing the cone angle all the way to zero while fixing the conformal boundary.

Theorem 15 (Bromberg, Theorem 1.2 of [23])**.** *Given $a > 0$ there exists an $\ell > 0$ such that the following holds. Let M_a be a geometrically finite hyperbolic cone manifold with no rank–one cusps, singular locus* $\mathbf{c}$ *and cone angle a at* $\mathbf{c}$*. If the tube radius R about each component of* $\mathbf{c}$ *is at least* $\operatorname{arcsinh}\sqrt{2}$ *and the total length $\ell_{M_a}(\mathbf{c})$ of* $\mathbf{c}$ *in M_a satisfies $\ell_{M_a}(\mathbf{c}) < \ell$, then there is a 1–parameter family M_t of cone manifolds with fixed conformal boundary and cone angle $t \in [0,a]$ at* $\mathbf{c}$*.* □

(In our work here, the tube radius condition is easily established, as we always begin or end with cone angle 2π, and a theorem of Brooks and J. P. Matelski [27] tells us that the radius of a tube is large when its core is short.) Together with the Hodge Theorem of Hodgson and Kerckhoff [44] and Bromberg's generalization of it to geometrically finite cone manifolds [23], this yields the following theorem.

Theorem 16 (Bromberg–Hodgson–Kerckhoff, Corollary 6.7 of [19])**.** *Let M_t be the 1–parameter family in Theorem 15. There exists a 1–parameter family of cone metrics g_t on M such that $M_t = (M, g_t)$ and η_t is a harmonic strain field outside a small tubular neighborhood of the singular locus and the rank–two cusps.* □

Brock and Bromberg establish an L^2–bound on the norm of this strain field:

Theorem 17 (Brock–Bromberg, Theorem 6.8 of [19])**.** *Given $\varepsilon > 0$, there are $\ell, K > 0$ such that if $\ell_{M_t}(\mathbf{c}) \leq \ell$, then*

$$\int_{M_t - \mathbf{T}_t^{\varepsilon}(\mathbf{c})} \|\eta_t\|^2 + \|*D_t\eta_t\|^2 \;\leq\; K^2 \ell_{M_t}(\mathbf{c})^2$$

where $\mathbf{T}_t^{\varepsilon}(\mathbf{c})$ is the ε–Margulis tube about $\mathbf{c}$ *in M_t.* □

This allows them to prove the following theorem.

Theorem 18 (Brock–Bromberg, Theorem 6.2 of [19])**.** *For each $\varepsilon_3 \geq \varepsilon > 0$ and $L > 1$, there is an $\ell > 0$ such that the following holds. If M is a geometrically finite hyperbolic 3–manifold and* $\mathbf{c}$ *is a geodesic in M with length $\ell_M(c) < \ell$, there is an L–bilipschitz diffeomorphism of pairs*

$$h\colon\; \big(M - \mathbf{T}^{\varepsilon}(\mathbf{c}),\; \partial\mathbf{T}^{\varepsilon}(\mathbf{c})\big) \longrightarrow \big(M_0 - \mathbf{P}^{\varepsilon}(\mathbf{c}),\, \partial\mathbf{P}^{\varepsilon}(\mathbf{c})\big),$$

where $\mathbf{T}^{\varepsilon}(\mathbf{c})$ is the ε–Margulis tube about $\mathbf{c}$ *in M, the manifold M_0 is $M - \mathbf{c}$ equipped with the complete hyperbolic structure whose conformal boundary agrees with that of M, and $\mathbf{P}^{\varepsilon}(\mathbf{c})$ is the ε–Margulis tube at the cusp corresponding to* $\mathbf{c}$*.* □

In fact, Proposition 13, Theorem 14, and Theorem 17 yield the following theorem.

Theorem 19 (Brock–Bromberg, Corollary 6.10 of [19])**.** *For any $\varepsilon, \delta, C > 0$ and $L > 1$, there exists an $\ell > 0$ so that if $\ell_{M_a}(\mathbf{c}) < \ell$ then the following holds. Let W be a subset of M, $\gamma(s)$ a smooth curve in W and $C(t)$ the geodesic curvature of γ in the g_t–metric at*

$\gamma(0)$. *If W lies in the ε–thick part $M_t^{\geq \varepsilon}$ for all $t \in [t_0, a]$ and $C(0) \leq C$ then the identity map*

$$\mathrm{id}\colon (W, g_a) \longrightarrow (W, g_{t_0})$$

is L–bilipschitz and $|C(0) - C(a)| \leq \delta$. □

The map in Theorem 18 is obtained by letting W be the entire thick part and extending the map to the complement of a tubular neighborhood of the singular locus. This not only provides desirable bilipschitz maps, but will further allow us to control the change in the principal curvatures of certain surfaces under Dehn filling. We pause to formulate what we need.

Let $\mathfrak{F}$ be an oriented Riemannian surface and let M and N be oriented Riemannian 3–manifolds. Let

$$\mathfrak{F} \xrightarrow{f} M \xrightarrow{g} N$$

be smooth with f an immersion and g an orientation preserving embedding. There are unique unit normal fields on $f(\mathfrak{F})$ and $gf(\mathfrak{F})$ compatible with the orientations, and these are the unit normal fields that we use to define the normal curvatures of the two surfaces:

$$\kappa_f(v) = \frac{\mathrm{II}_f(v,v)}{\|v\|_M^2} \quad \text{and} \quad \kappa_{gf}(w) = \frac{\mathrm{II}_{gf}(w,w)}{\|w\|_N^2}$$

where II_h is the second fundamental form of $h(\mathfrak{F})$.

We say that **the normal curvatures of $gf(\mathfrak{F})$ are within δ of those of $f(\mathfrak{F})$** if for each v in $T_p f(\mathfrak{F})$,

$$\left| \kappa_f(v) - \kappa_{gf}\big(Dg(v)\big) \right| < \delta.$$

Given a family of such maps

$$\mathfrak{F}_j \xrightarrow{f_j} M_j \xrightarrow{g_j} N_j \ ,$$

we say that **the normal curvatures of $g_j f_j(\mathfrak{F}_j)$ tend to those of $f_j(\mathfrak{F}_j)$** if they are within δ_j of each other with δ_j tending to zero. We will need the following statement.

Proposition 20. *For any $\varepsilon, \delta, C > 0$ and $L > 1$, there exists an $\ell > 0$ so that if $\ell_{M_a}(\mathbf{c}) < \ell$ the following holds. Let $\mathfrak{F}$ be a smooth surface in $M = M_{t_0}$ such that $\mathfrak{F}$ lies in the ε–thick part $M_t^{\geq \varepsilon}$ for all $t \in [t_0, a]$ and the principal curvatures of $\mathfrak{F}$ are bounded by C. Then if $f_a\colon M \to M_a$ is the L–bilipschitz map given by Theorem 18, the normal curvatures of $f_a(\mathfrak{F})$ are within δ of those of $\mathfrak{F}$.*

In particular, the principal curvatures of $f_a(\mathfrak{F})$ are within δ of those of $\mathfrak{F}$. □

1.9 Geometric inflexibility

In addition to the geometric effect of cone deformations, we will need to control the effect of quasiconformal conjugacies of Kleinian groups deep in the hyperbolic manifolds. McMullen's Geometric Inflexibility Theorem says that in the convex core of a purely loxodromic Kleinian group, the effect of a quasiconformal deformation decays exponentially as one moves away from the boundary of the core. Precisely,

Theorem 21 (McMullen, Theorem 2.11 of [59])**.** *Let Γ be a finitely generated purely loxodromic Kleinian group and let $M_\Gamma = \mathbb{H}^3/\Gamma$. A K–quasiconformal conjugacy of Γ induces a map f between hyperbolic manifolds whose bilipschitz constant at a point p in the convex core $\mathcal{C}(M_\Gamma)$ satisfies*

$$\log \operatorname{bilip}(f)(p) \leq Ce^{-A\mathrm{d}(p,\partial\mathcal{C}(M_\Gamma))} \log K$$

where C is a universal constant and A depends only on a lower bound on the injectivity radius at p and an upper bound on the injectivity radius in $\mathcal{C}(M_\Gamma)$. □

Typically this theorem is used in the geometrically infinite setting, where one is free to move perpetually deeper into the core. We will be studying sequences of geometrically finite manifolds whose cores are getting deeper and deeper still while the dilatations of our deformations remain bounded. Unfortunately, our sequences will have convex cores of larger and still larger injectivity radii, and so McMullen's theorem will not be applicable. Fortunately, the following Geometric Inflexibility Theorem of Brock and Bromberg trades this bound for a bound on the area of the boundary and is sufficient for our purpose here.

Theorem 22 (Brock–Bromberg [18])**.** *Let Γ be a finitely generated purely loxodromic Kleinian group and let $M_\Gamma = \mathbb{H}^3/\Gamma$. A Γ–invariant Beltrami differential μ with $\|\mu\|_\infty < 1$ on $\widehat{\mathbb{C}}$ induces a map f_μ between hyperbolic manifolds whose bilipschitz constant at a point p in the convex core $\mathcal{C}(M_\Gamma)$ satisfies*

$$\log \operatorname{bilip}(f_\mu)(p) \leq Ce^{-A\mathrm{d}(p,\partial\mathcal{C}(M_\Gamma))}$$

where C and A depend only on $\|\mu\|_\infty$, a lower bound on the injectivity radius at p, and the area of $\partial\mathcal{C}(M_\Gamma)$. □

The map f_μ is constructed in a natural way that we now recall—see [18] and Appendix B of [59].

By a visual averaging process, any vector field v on $\widehat{\mathbb{C}}$ has a visual extension to a vector field $\operatorname{ex}(v)$ on $\mathbb{H}^3$, and any Beltrami differential μ on $\mathbb{C}$ has such an extension to a strain field $\operatorname{ex}(\mu)$.

Theorem 23 (The Beltrami Isotopy)**.** *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a K–quasiconformal map normalized to fix 0, 1, and ∞. Then there is a quasiconformal isotopy*

$$\varphi\colon \widehat{\mathbb{C}} \times [0,1] \to \widehat{\mathbb{C}}$$

fixing 0, 1, and ∞ obtained by integrating a family v_t of continuous $\frac{1}{2}\log K$–quasiconformal vector fields, such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = f$.

The isotopy is natural in the sense that given γ_0 and γ_1 in $\mathrm{PSL}_2\mathbb{C}$ such that $\gamma_1 \circ f = f \circ \gamma_0$, there are interpolating Möbius transformations γ_t such that $\gamma_t \circ \varphi_t = \varphi_t \circ \gamma_0$. □

The Beltrami isotopy has a natural visual extension

$$\Phi\colon \mathbb{H}^3 \times [0,1] \to \mathbb{H}^3$$

that is the integral of the harmonic vector field $\mathrm{ex}(v_t)$. The maps Φ_t are bilipschitz, and letting η_t denote the harmonic strain field $\mathrm{ex}(\overline{\partial} v_t)$, the pointwise bilipschitz constants satisfy

$$\log \mathrm{bilip}(\Phi_t)(p) \leq \int_0^t \left\| \eta_s\big(\Phi_s(p)\big) \right\| \, \mathrm{d}s \, .$$

Taking our differential μ in Theorem 22, we begin with the normalized quasiconformal mapping with Beltrami coefficient μ and obtain an equivariant discussion, a 1–parameter family of hyperbolic manifolds $M_t = (M, g_t)$, and bilipschitz $\Psi_t \colon M_0 \to M_t$ obtained by pushing the Φ_t down to the quotient. The harmonic strain fields $\mathrm{ex}(\overline{\partial} v_t)$ descend to those obtained from the g_t by the construction in Section 1.6, and at time one, this is the descendant of the strain field $\mathrm{ex}(\mu)$.

Our map f_μ equals Ψ_1, and its bilipschitz constant satisfies

$$\log \mathrm{bilip}(f_\mu)(p) \leq \int_0^1 \left\| \eta_t\big(\Phi_t(p)\big) \right\| \, \mathrm{d}t \, .$$

The integral on the right is now estimated using

Theorem 24 (Brock–Bromberg [18]). *Let Γ be a finitely generated purely loxodromic Kleinian group and μ a Γ–invariant Beltrami differential on $\widehat{\mathbb{C}}$. Let K be defined by $\|\mu\|_\infty = (K-1)/(K+1)$ and let $b = \frac{1}{2}\log K$. Let the harmonic strain field $\eta = \mathrm{ex}(\mu)$ be the visual extension of μ. Then*

$$\|\eta(p)\| + \|*D\eta(p)\| \leq 3Ab\sqrt{\mathrm{area}(\partial\mathcal{C}(M))}\, e^{-\mathrm{d}(p,\partial\mathcal{C}(M))}$$

where p lies in the ε–thick part of the convex core $\mathcal{C}(M)$ and A depends only on ε. $\square$

Theorem 8 and Proposition 9 allow the transfer of this estimate to the desired estimate of $\log \mathrm{bilip}(f_\mu)(p)$, see [18].

As Theorem 24 is not explicitly stated in [18], we derive it from the work there.

Derivation of Theorem 24. The proof of Theorem 3.8 of [18] and the fact that $*D\eta$ is a harmonic strain field establish the following theorem.

Theorem 25 (Brock–Bromberg [18]). *Let N be a complete hyperbolic 3–manifold with compact boundary and let η be a harmonic strain field on N. Then*

$$\|\eta(p)\| + \|*D\eta(p)\| \leq A e^{-\mathrm{d}(p,\partial N)} \sqrt{\int_N \|\eta\|^2 + \|*D\eta\|^2}$$

where p lies in the ε–thick part of N and A depends only on ε.

The term

$$\int_N \|\eta\|^2 + \|*D\eta\|^2$$

is bounded by Lemma 5.2 of [18]:

Lemma 26 (Brock–Bromberg, Lemma 5.2 of [18])**.** *Let N be a complete hyperbolic 3–manifold such that $\pi_1(N)$ is finitely generated and assume that N has no rank–one cusps. Let η be a harmonic strain field on N such that $\|\eta\|_\infty$ and $\|*D\eta\|_\infty$ are bounded by B. Then*

$$\int_{\mathcal{C}(N)} \|\eta\|^2 + \|*D\eta\|^2 \leq B^2 \operatorname{area}(\partial\mathcal{C}(M)).$$

When η is the strain field in Theorem 24, a theorem of H. M. Reimann [75], Part 1 of Theorem 5.1 of [18], says that $\|\eta\|_\infty$ and $\|*D\eta\|_\infty$ are bounded by $3b$. Theorem 24 now follows by letting $N = \mathcal{C}(M)$ in Theorem 25 and Lemma 26. □

The estimate in Theorem 24 provides the following Proposition, thanks to Proposition 13.

Proposition 27. *For any $\varepsilon, \delta, C > 0$ and $0 < k < 1$, there is a d such that the following holds. Let μ be a Beltrami differential for a finitely generated purely loxodromic Kleinian group Γ with $\|\mu\|_\infty = k$ and let $M = M_\Gamma$. Let $\mathfrak{F}$ be a smooth surface in $M^{\geq\varepsilon} \cap \mathcal{C}(M)$ at a distance at least d from $\partial\mathcal{C}(M)$, and suppose that the principal curvatures of $\mathfrak{F}$ are bounded by C. Then if $f_\mu\colon M \to M_\mu$ is the map in Theorem 22, the normal curvatures of $f_\mu(\mathfrak{F})$ are within δ of those of $\mathfrak{F}$.*

In particular, the principal curvatures of $f_\mu(\mathfrak{F})$ are within δ of those of $\mathfrak{F}$. □

1.10 Useful surfaces

Let $\mathfrak{F}$ be a smooth surface in $\mathbb{H}^3$. Pick a point p in $\mathfrak{F}$ and normalize so that p sits at $(0,0,1)$ in the upper half-space model with unit normal $-\frac{\partial}{\partial z}$ such that the principal directions at p are $(1,0,0)$ and $(0,1,0)$. Let Π be the normal projection of $\mathfrak{F}$ to $\mathbb{C}$. Picking an orthonormal basis for $T_p\mathfrak{F}$ along its principal directions and the usual basis for $T_0\mathbb{C}$, the derivative of Π at p is given by the matrix

$$D\Pi_p = \begin{pmatrix} \frac{1+\kappa_1}{2} & 0 \\ 0 & \frac{1+\kappa_2}{2} \end{pmatrix} \tag{1.5}$$

where the κ_i are the principal curvatures of $\mathfrak{F}$—the reader will find it a nice exercise in plane geometry to verify the formula using the fact that, with our normalization, the hyperbolic curvature at $(0,0,1)$ in a vertical hyperbolic plane is one plus the Euclidean curvature (a proof of *this* fact may be found in [8]).

1.10.1 The Epstein surface

Given a quasifuchsian manifold $\mathrm{QF}(X,Y)$ with convex core $\mathcal{C}(X,Y)$ possessing boundaries $\mathcal{O}_X$ and $\mathcal{I}_Y$ facing X and Y respectively, we let $\mathcal{E}_X$ be the component of $\mathrm{QF}(X,Y) - \mathcal{C}(X,Y)$ facing X, $\mathcal{E}_Y$ the component facing Y.

Lemma 28. *Let $\mathcal{K}$ be a compact set in $\mathcal{T}(\overline{\Sigma})$. There is a number $d > 0$ such that for any quasifuchsian manifold $\mathrm{QF}(X,Y)$ in $\mathrm{QF}(\mathcal{T}(\Sigma),\mathcal{K})$ there is a smooth convex surface in $\mathcal{E}_Y$ that lies in the d–neighborhood of $\mathcal{I}_Y$ and whose principal curvatures are greater than $\frac{1}{2}$ and no more than 2.*

Given a conformal metric m on a simply connected domain Ω in $\widehat{\mathbb{C}}$, there is a unique map $E\colon (\Omega, m) \to \mathbb{H}^3$ with the property that the visual metric at $E(z)$ agrees with m at z in Ω. This map was discovered by C. Epstein [37], and the following information concerning it may be found in the work of Epstein [37] and C. G. Anderson [8]. A nice summary of the information we need here may be found in [23].

We will only be interested in scalar multiples of the Poincaré metric ρ on Ω, and we write $E_t\colon (\Omega, e^t\rho) \to \mathbb{H}^3$.

We define the norm

$$\|\cdot\|_{\infty,\rho} = \left\| \cdot (1-|z|^2)^{-2} \right\|_{L^\infty(\mathcal{L})}$$

Let f_Ω be a Riemann mapping carrying the unit disk $\mathcal{L}$ to Ω. By Theorem 7.5 of [37], there is a T depending only on an upper bound for the weighted L^∞–norm $\|Sf_\Omega\|_{\infty,\rho}$ of the Schwarzian derivative Sf_Ω of f_Ω such that E_s is an embedding with strictly convex image whenever $s \geq T$—the surface is curving *away* from Ω. Thanks to a theorem of Z. Nehari and W. Kraus, the norm $\|Sf_\Omega\|_{\infty,\rho}$ is bounded by six, see Section 5.4 of [40]. So we may take T universal.

We now restrict our attention to Ω that arise as a component of the domain of discontinuity of a quasifuchsian group $\mathrm{QF}(X,Y)$—we will always think of Ω as the domain uniformizing Y. The principal curvatures of the surface $E_t(\Omega)$ tend to one independently of Ω as t tends to infinity, see Section 3 of [8] or Proposition 6.3 of [23], and so we assume as we may that for all s greater than or equal to T, these curvatures are at least $1/2$ and no more than 2.

All of the above works $\Gamma_{\mathrm{QF}(X,Y)}$–equivariantly and so, for any $\mathrm{QF}(X,Y)$, we have a smoothly embedded convex copy $E_T(Y)$ of Y in $\mathcal{E}_Y$ with principal curvatures trapped between $1/2$ and 2. The map $E_T\colon Y \to \mathrm{QF}(X,Y)$ is Lipschitz with constant

$$\|Sf_\Omega\|_{\infty,\rho} + \frac{1}{2}(e^T + e^{-T}) \leq 6 + \frac{1}{2}(e^T + e^{-T}),$$

see [8].

We are now ready for the proof of the lemma.

Proof of Lemma 28. All that remains to be shown is that the Hausdorff distance between $\mathcal{I}_Y$ and $E_T(Y)$ depends only on T and the compact set $\mathcal{K}$. Since the hyperbolic surfaces in $\mathcal{K}$ have uniformly bounded diameter, and the Lipschitz constant of E_T is less than $6 + \frac{1}{2}(e^T + e^{-T})$, we obtain a bound A on the diameter of the surfaces $E_T(Y)$ that only depends on T and $\mathcal{K}$.

Now, there is a universal lower bound D to the diameter of any complete hyperbolic surface, and so the diameter of the boundary of the R–neighborhood $N_R(\mathcal{I}_Y)$ of $\mathcal{I}_Y$ in $\mathcal{E}_Y$ is at least $D\cosh R$. If $E_T(Y)$ is entirely contained in the complement of $N_R(\mathcal{I}_Y)$, its diameter is at least $D\cosh R$, as the projection $E_T(Y) \to \partial N_R(\mathcal{I}_Y)$ is 1–Lipschitz and has degree one. So $E_T(Y)$ must intersect the $\operatorname{arccosh}(2A/D)$–neighborhood of $\mathcal{I}_Y$. Letting $d = A + \operatorname{arccosh}(2A/D)$ completes the proof. $\square$

2 Focusing

Given a totally geodesic surface Σ in a hyperbolic 3–manifold M, a **metric collar** of **depth** d about Σ is a d–neighborhood of Σ in M homeomorphic to $\Sigma \times I$.

The following theorem says that the skinning map of a manifold with a deep collar about its boundary is highly contracting on a large compact set.

Theorem 29 (Focused Manifolds)**.** *Let S be a closed hyperbolic surface. Let $\varepsilon, \delta, R > 0$. There is a constant $d > 0$ such that if M is a hyperbolic 3–manifold with totally geodesic boundary Σ homeomorphic to S with* $\operatorname{injrad}(\Sigma) > \delta$ *possessing a collar of depth d, then* $\operatorname{diam}\big(\sigma_M\big(B(\Sigma,R)\big)\big) < \varepsilon$.

Proof. Fix $\delta, R > 0$ and let $\{M_d\}_{d=1}^{\infty}$ be *any* sequence of hyperbolic manifolds with totally geodesic boundary such that $\Sigma_d = \partial M_d$ is homeomorphic to S, has injectivity radius at least δ, and has a metric collar of depth d in M_d. We write $\sigma_d = \sigma_{M_d}$. We show that the diameter of $\sigma_d\big(B(\Sigma_d,R)\big)$ is less than ε for all sufficiently large d.

We consider $\pi_1(\dot{M}_d)$ with basepoint in Σ_d as a Kleinian group Γ_d normalized so that the complement $\mathcal{U}$ of the closed unit disk $\overline{\mathcal{L}}$ in $\widehat{\mathbb{C}}$ is the component of the domain of discontinuity Ω_{Γ_d} uniformizing Σ_d with uniformizing Fuchsian group Γ_{Σ_d}.

Let $\mu = \mu_d$ be a Beltrami differential of norm $k \le (e^{2R}-1)/(e^{2R}+1)$ for Γ_{Σ_d} in $\mathcal{U}$, so that $[\mu]$ lies in $B(\Sigma_d,R)$.

Let $\mathfrak{F}_d$ be the component of the boundary of the $d/2$–neighborhood of Σ_d contained in the convex core of M_d. Since Σ_d is totally geodesic, this is a smooth surface. Moreover, it is easy to see that the principal curvatures of the $\mathfrak{F}_d$ tend to one as d grows. Since the injectivity radius of Σ_d is at least δ, we may assume that the injectivity radius of M_d is at least δ at $\mathfrak{F}_d$, by choosing d larger than δ.

The differential μ induces a bilipschitz map $f_{d,\mu}\colon M_d \to M_{d,\mu}$, and we write $\mathfrak{F}_{d,\mu} = f_{d,\mu}(\mathfrak{F})$. As d grows, the bilipschitz constants of the $f_{d,\mu}$ near $\mathfrak{F}_d$ tend to one in a manner depending only on d, R, δ, and the topological type of S, by Theorem 22.

Since the norm of μ is bounded, Proposition 27 tells us that for any small $\eta > 0$, there is a large d such that the principal curvatures of the $\mathfrak{F}_{d,\mu}$ are within η of those of $\mathfrak{F}_d$. As the principal curvatures of $\mathfrak{F}_d$ are universally bounded away from zero, the $\mathfrak{F}_{d,\mu}$ are strictly convex for large d, and so there are diffeomorphic normal projections

$$\Pi_{d,\mu}\colon\ \sigma_d([\mu]) \to \mathfrak{F}_{d,\mu} \quad \text{and} \quad \Pi_d\colon\ \Sigma_d \to \mathfrak{F}_d.$$

Consider the composition

$$F_{d,\mu} = \Pi_{d,\mu}^{-1} \circ f_{d,\mu} \circ \Pi_d\colon\ \Sigma_d \longrightarrow \sigma_d([\mu])$$

and its derivative

$$DF_{d,\mu} = D\Pi_{d,\mu}^{-1} \circ Df_{d,\mu} \circ D\Pi_d$$

at a point p in Σ_d (we suppress p in the notation).

By the discussion in Section 1.10, we may choose coordinates so that

$$DF_{d,\mu} = A^{-1}\begin{pmatrix} \frac{1+\kappa_1(\mathfrak{F}_{d,\mu})}{2} & 0 \\ 0 & \frac{1+\kappa_2(\mathfrak{F}_{d,\mu})}{2} \end{pmatrix} A \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix} \begin{pmatrix} \frac{2}{1+\kappa_1(\mathfrak{F}_d)} & 0 \\ 0 & \frac{2}{1+\kappa_2(\mathfrak{F}_d)} \end{pmatrix},$$

where A is the rotation carrying the principal directions of $\mathfrak{F}_{d,\mu}$ to $(1,0)$ and $(0,1)$.

To estimate the Teichmüller distance between Σ_d and $\sigma_d([\mu])$, we would like to understand the "norm" of the image of a vector under $DF_{d,\mu}$. Unfortunately, our skinning surfaces do not come to us equipped with tractable Riemannian metrics, and so we must be careful when using information about $DF_{d,\mu}$ to estimate the Teichmüller distance between them. This is achieved as follows.

Note that with our normalization the hyperbolic inner products on the tangent spaces to our surfaces $\mathfrak{F}_d$ and $\mathfrak{F}_{d,\mu}$ agree with the standard Euclidean ones, as we have chosen charts so that they are tangent to the horosphere of height one. Our coordinate patches for Σ_d and $\sigma_d([\mu])$ lie centered at 0 in $\mathbb{C}$, and we equip their tangent spaces at zero with the standard Euclidean inner products, which are compatible with the conformal structures on Σ_d and $\sigma_d([\mu])$.

Now, if $\lambda_+ \geq \lambda_-$ are the maximum and minimum values of $\|DF_{d,\mu}v\|$ as v ranges over the unit circle in $T_p\Sigma_d$, then the dilatation of $F_{d,\mu}$ at p is no more than λ_+/λ_-, see the first chapter of [6]. It follows that if the linear map $DF_{d,\mu}$ is ℓ–bilipschitz, the map $F_{d,\mu}$ is ℓ^2–quasiconformal at p.

Writing

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = A \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix}$$

and multiplying, we have

$$DF_{d,\mu} = A^{-1}B = A^{-1} \begin{pmatrix} w\,\frac{1+\kappa_1(\mathfrak{F}_{d,\mu})}{1+\kappa_1(\mathfrak{F}_d)} & x\,\frac{1+\kappa_1(\mathfrak{F}_{d,\mu})}{1+\kappa_2(\mathfrak{F}_d)} \\ y\,\frac{1+\kappa_2(\mathfrak{F}_{d,\mu})}{1+\kappa_1(\mathfrak{F}_d)} & z\,\frac{1+\kappa_2(\mathfrak{F}_{d,\mu})}{1+\kappa_2(\mathfrak{F}_d)} \end{pmatrix}.$$

The numbers w, x, y, and z are bounded above in absolute value by a constant depending only on the bilipschitz constant of $f_{d,\mu}$, which is bounded independent of μ.

Let $\eta > 0$ be very small. For large d, the principal curvatures of the surfaces involved are all within η of one. So the bilipschitz constant of the linear map B, and hence that of $A^{-1}B$, is very close to that of $Df_{d,\mu}$ in a manner depending only on d, R, δ, and the topological type of S. Since the $f_{d,\mu}$ are bilipschitz with constants tending to one as d grows, the same is true for the linear maps $DF_{d,\mu}$. So the $F_{d,\mu}$ are quasiconformal with constants tending to one as d grows, and Σ_d and $\sigma_d([\mu])$ are close in the Teichmüller metric independent of μ. □

2.1 Ubiquity of focus

To prove our Quasifuchsian Density Theorem in Section 2.2, and the Capping Theorem in Section 2.3, we will need the following theorem.

Theorem 30. *Let X be a closed connected hyperbolic surface, let $B(X,R)$ be the ball of radius R about X in $\mathcal{T}(X)$, and let $\varepsilon > 0$. Then there is a convex cocompact hyperbolic manifold M with totally geodesic boundary Σ homeomorphic to X satisfying*

$$\mathrm{d}(\Sigma, X) < \varepsilon \quad \textit{and} \quad \mathrm{diam}\big(\sigma_M\big(B(X,R)\big)\big) < \varepsilon.$$

Let $\mathfrak{C}_d$ be the class of convex cocompact hyperbolic manifolds with totally geodesic boundary whose boundaries have collars of depth d. By Theorem 29, to prove the theorem we need only prove the following mild generalization of Fujii and Soma's theorem that the set of totally geodesic boundaries of hyperbolic manifolds is dense in the Teichmüller space [39].

Theorem 31. *For any $d \geq 0$, the set of hyperbolic surfaces in $\mathcal{T}(S)$ appearing as totally geodesic boundaries of hyperbolic 3–manifolds in $\mathfrak{C}_d$ whose boundaries are homeomorphic to S is dense.*

A circle packing on a hyperbolic surface is a collection of circles with disjoint interiors whose union has the property that all of the complementary regions are curvilinear triangles. A hyperbolic structure on a surface is a **circle packing hyperbolic structure** if it admits a circle packing.

We have the following theorem of R. Brooks.

Theorem 32 (Brooks [26, 25])**.** *The set of circle packing hyperbolic structures on a surface is dense in the Teichmüller space.* □

Define the **radius** of a circle packing to be the maximum of the radii of its circles. The work in [25] and [26] in fact proves the following theorem.

Theorem 33 (Brooks)**.** *Let $\varepsilon > 0$. Then the set of hyperbolic structures on a surface admitting circle packings of radius at most ε is dense in the Teichmüller space.*

Sketch (see the proof of Theorem 3 of [26]). Let X be a hyperbolic surface. Any configuration of circles in X may be extended to a configuration whose complementary regions are curvilinear triangles and rectangles. We choose such a configuration consisting of circles of radius $\varepsilon/2$ whose complementary regions have diameter at most $\varepsilon/2$.

Consider the group Γ of Möbius transformations generated by the Fuchsian group uniformizing X and all of the reflections in the circles of our configuration. As in the proof of Theorem 3 of [26], we may choose an arbitrarily small quasiconformal deformation of Γ carrying our configuration of circles to a configuration that fits into a circle packing. Since the deformation may be taken as small as desired, we may take the radius of the resulting circle packing less than ε. The fact that this deformation induces a small quasiconformal deformation of X completes the proof. □

Proof of Theorem 31. We begin the proof as Fujii and Soma do, but use the Bounded Image Theorem (which Fujii and Soma avoid) and McMullen's Geometric Inflexibility Theorem to complete it.

Fix a manifold B with totally geodesic boundary homeomorphic to S for later use, and let $\mathcal{K}$ be a compact set in $\mathcal{T}(\overline{S})$ containing the image of σ_B.

Let $d, \delta > 0$.

We will show that for any surface X in $\mathcal{T}(S)$ there is a manifold in $\mathfrak{C}_d$ whose totally geodesic boundary is within δ of X.

Let Σ be a hyperbolic surface equipped with a circle packing $\mathcal{C}$ of radius ε. By Theorem 33, these are dense in $\mathcal{T}(S)$ for any ε. We prove that if ε is small enough, then there is a convex cocompact manifold with geodesic boundary within $\delta/2$ of Σ that possesses a collar of depth d. It follows that *any* point in $\mathcal{T}(S)$ is within δ of the boundary of such a manifold.

By inscribing each complementary triangle in a circle, we obtain a dual configuration of circles $\mathcal{D}$. Consider the group Δ of Möbius transformations generated by Γ_Σ and the reflections in the circles in $\mathcal{C} \cup \mathcal{D}$. This group has a finite index torsion free subgroup whose Kleinian manifold has four boundary components, each conformally or anticonformally equivalent to Σ. Moreover, the corresponding convex core has totally geodesic boundary.

This manifold may be constructed directly by "scalloping" the circle packing, as we now discuss. Lift the configurations $\mathcal{C}$ and $\mathcal{D}$ to the unit disk in $\widehat{\mathbb{C}} = \partial_\infty \mathbb{H}^3$. Each circle is the ideal boundary of a totally geodesic hyperplane in $\mathbb{H}^3$, and for each such circle we excise from $\mathbb{H}^3$ the open halfspace incident to the interior of the circle. We also excise the halfspace corresponding to the complement of the unit disk. What remains is a convex subset of $\mathbb{H}^3$ whose boundary is a union of totally geodesic polygons meeting at right angles, together with a single copy of $\mathbb{H}^2$.

Let $\mathcal{O}$ be the quotient of this convex set by Γ_Σ. The set $\partial \mathcal{O} - \Sigma$ may be decomposed into two families, a family of triangles T arising from the triangles complementary to $\mathcal{C}$, and the remaining polygons. Now, we double $\mathcal{O}$ along T, and then double this manifold along the double of the remaining polygons. The resulting manifold has four totally geodesic boundary components as described, and we glue two of them together to obtain a manifold M with $\partial M = \Sigma \cup \overline{\Sigma}$.

Notice that since the radius of $\mathcal{C}$ is less than ε, the boundary of M has a very large collar, as all of the totally geodesic planes in the universal cover must be far away from the one whose boundary is the unit circle.

The manifold M has cusps, and these may be filled using the Hyperbolic Dehn Filling Theorem so that the totally geodesic boundary of the resulting manifold is very close to that of M and so that the filled manifold still possesses a large collar about its boundary. More precisely, consider the double DM of M along its boundary. If we perform higher and higher hyperbolic Dehn fillings equivariantly with respect to the natural involution of DM, the resulting manifolds converge geometrically to DM, see Section E.1 of [12]. Moreover, these fillings are themselves doubles, and so the copies of ∂M in these manifolds are isotopic to totally geodesic surfaces. The geometric convergence implies that these Fuchsian groups are converging algebraically to the corresponding Fuchsian group in DM, and so the totally geodesic surfaces in these fillings are eventually within $\delta/4$ of Σ in $\mathcal{T}(S)$. The geometric convergence also guarantees a large collar about these surfaces, and cutting open again provides the desired manifolds.

We continue the argument with one of these filled manifolds M' with boundary $\Sigma' \cup \overline{\Sigma}'$.

For definiteness, we say that the depth of the collar about the boundary of M' is c. By the above, c tends to infinity as ε goes to zero.

Now attach B to the copy of $\overline{\Sigma}'$ in $\partial M'$ to obtain a manifold N. The manifold N is irreducible, acylindrical, and atoroidal, as is seen by considering potential essential spheres, annuli, and tori in N, which, if there, would result in a sphere, annulus, or torus in M' or B—see Lemma 2.1 of [66]. By Thurston's Hyperbolization Theorem for Haken manifolds, the manifold N admits a hyperbolic structure with totally geodesic boundary, and we equip N with this structure.

We claim that for large c, the totally geodesic boundary of N is within $\delta/4$ of Σ' (and hence within δ of Σ) with a collar of depth d.

To see this, consider the double DN of N. This manifold is obtained from a double DM' of M' by capping off with $B \sqcup \overline{B}$. We have a map

$$\iota\colon\ \mathrm{GF}(DN) \to \mathrm{GF}(DM') \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S})$$

where, of course, the domain is a single point, by G. D. Mostow's Rigidity Theorem [65]. The image of ι lies in $\overline{\mathcal{K}} \times \mathcal{K}$.

In DM', the surface Σ_0 that will be the boundary of N is at a distance c from $\partial DM'$. By McMullen's Geometric Inflexibility Theorem, the bilipschitz constant of the map between the hyperbolic structure on DM' with totally geodesic boundary and the image of ι is very close to one on a large neighborhood of Σ_0 when c is very large, as the diameter of $\mathcal{K}$ is fixed. We conclude that the totally geodesic boundary of N has a collar of depth d provided c is large enough. We also see that as c tends to infinity, the Fuchsian group in Γ_{DN} corresponding to Σ_0 is tending to the Fuchsian group $\Gamma_{\Sigma_0} = \Gamma_{\Sigma'}$ in the algebraic topology. We conclude that the totally geodesic boundary of N is within $\delta/4$ of Σ', and hence within $\delta/2$ of Σ. □

2.2 Quasifuchsian density

A theorem of Brooks' [26] says that any convex cocompact Kleinian group embeds in a cocompact Kleinian group after an arbitrarily small quasiconformal deformation. In case the Kleinian group is quasifuchsian, Brooks' argument produces a cocompact Kleinian group in which the quasifuchsian surface is nonseparating. We have the following.

Theorem 34 (Quasifuchsian Density)**.** *The subset of* $\mathrm{QF}(S)$ *consisting of quasifuchsian groups that appear as the only boundary subgroup of a geometrically finite hyperbolic* 3*–manifold is dense.*

Proof. Let $\mathrm{QF}(X,\overline{Y})$ be a quasifuchsian group in $\mathrm{QF}(S)$ and let $\varepsilon > 0$. Let R be such that X lies in $B(Y,R)$. By Theorem 30, there is a manifold M with totally geodesic boundary Z within $\varepsilon/2$ of Y such that $\sigma_M\big(B(Y,R)\big)$ has diameter less than $\varepsilon/2$. So $\sigma(X)$ is within ε of $\overline{Y}$. □

2.3 Capping off

The following theorem is useful in promoting properties of skinning maps of manifolds with many boundary components to those of manifolds with connected boundary.

Theorem 35 (Capping Theorem)**.** *Let M be a hyperbolic* 3*–manifold with totally geodesic boundary* $\Sigma_0 \sqcup \Sigma_1 \sqcup \ldots \sqcup \Sigma_m$*. Let* $\varepsilon > 0$*. Then* (M,Σ_0) *embeds as a pair into* *a hyperbolic manifold $(M_\varepsilon, \partial M_\varepsilon)$ with totally geodesic boundary homeomorphic to Σ_0 with the property that*

$$\mathrm{d}_{\mathcal{T}(\overline{\Sigma}_0)}\big(\sigma_{M,\Sigma_0}(X), \sigma_{M_\varepsilon}(X)\big) < \varepsilon$$

for all X in $\mathcal{T}(\Sigma_0)$, where σ_{M,Σ_0} is the skinning map of M relative to $\Sigma_1 \sqcup \ldots \sqcup \Sigma_m$.

Proof. By Theorem 30, for any ε and R greater than zero there are hyperbolic 3–manifolds $N_1, \ldots, N_m$ with totally geodesic boundaries $Y_1, \ldots, Y_m$ whose skinning maps satisfy

$$\operatorname{diam}\Big(\sigma_{N_j}\big(B(\Sigma_j, R)\big)\Big) < \frac{1}{4}\log(1+\varepsilon)^{2/3} \tag{2.1}$$

and

$$\mathrm{d}\Big(\overline{\Sigma}_j, \sigma_{N_j}\big(B(\Sigma_j, R)\big)\Big) < \frac{1}{4}\log(1+\varepsilon)^{2/3}. \tag{2.2}$$

Now, let $R = \operatorname{diam}\big(\sigma_M\big(\mathcal{T}(S)\big)\big)$ and form a 3–manifold M_ε by attaching N_j as above to the Σ_j in $\partial\dot{M}$. We have the maps

$$\iota^*\colon\ \mathrm{GF}(M_\varepsilon) \longrightarrow \mathrm{GF}(\dot{M}) \cong \mathcal{T}(\Sigma_0) \times \mathcal{T}(\Sigma_1) \times \cdots \times \mathcal{T}(\Sigma_m)$$

and

$$Q_0^*\colon\ \mathrm{GF}(\dot{M}) \to \mathrm{QF}(\Sigma_0)$$

induced by inclusion. By (2.1), (2.2), Theorem 8, and the triangle inequality, for any X, the hyperbolic metrics $\iota^*(X)$ and $(X, \Sigma_1, \ldots, \Sigma_m)$ are $(1+\varepsilon)$–bilipschitz. In particular, the quasifuchsian manifolds

$$Q_0^*\iota^*(X) = \mathrm{QF}\big(X, \sigma_{M_\varepsilon}(X)\big) \quad \text{and} \quad \mathrm{QF}\big(X, \sigma_{M,\Sigma_0}(X)\big)$$

are $(1+\varepsilon)$–bilipschitz. Theorem 8 completes the proof. □

3 Filling

The following theorem allows us to perform hyperbolic Dehn filling without terribly disturbing the skinning map.

Theorem 36 (Filling Theorem)**.** *Let M be a finite volume hyperbolic manifold with nonempty closed totally geodesic boundary and let $\mathfrak{e} > 0$. There is an $\hbar = \hbar_{\mathfrak{e}}(M) > 0$ such that if the normalized length of each component of α in M is at least $\hbar$, then*

$$\mathrm{d}_{\mathcal{T}(\overline{\partial M})}\big(\sigma_M(X), \sigma_{M(\alpha)}(X)\big) < \mathfrak{e}$$

for all X in $\mathcal{T}(\partial M)$.

Proof. For simplicity, we assume that ∂M is connected. We let $\ell(\alpha)$ be the minimum of the normalized lengths of the components of α.

Let M_X be M° equipped with the hyperbolic metric corresponding to X in $\mathcal{T}(\partial M)$. Let $\mathbf{P}_X^\varepsilon$ denote the ε–Margulis tube at the cusps of M_X, and let $\mathbf{T}_X^\varepsilon(\alpha)$ denote the union of the ε–Margulis tubes at the cusps of $M_X(\alpha)$ and about the components of the geodesic core $\mathbf{c}$ of the union of filling tori in $M_X(\alpha)$.

Let $\mathcal{I}_X$ be the component of the boundary of the convex core of $\mathrm{QF}\big(X,\sigma_M(X)\big)$ facing $\sigma_M(X)$, and let $\mathfrak{F}_X \subset \mathrm{QF}\big(X,\sigma_M(X)\big)$ be the smooth surface facing $\sigma_M(X)$ in the d–neighborhood of $\mathcal{I}_X$ given by the Bounded Image Theorem and Lemma 28. We have the restriction

$$f_X\colon\ \mathfrak{F}_X \to M_X$$

of the covering map $\mathrm{QF}\big(X,\sigma_M(X)\big) \to M_X$, and we claim that there is an ε such that, for all X, the surface $f_X(\mathfrak{F}_X)$ lies in the ε–thick part of M_X.

By the Bounded Image Theorem, there is an $\varepsilon_0 > 0$ such that the image of σ_M is ε_0–thick. By a theorem of Sullivan [80], there is a universal K for which $\mathcal{I}_X$ and $\sigma_M(X)$ are K–bilipschitz, see [38]. So the injectivity radius of $\mathcal{I}_X$ is at least ε_0/K. Taking ε_1 less than the constant $\delta_{\varepsilon_0/2K} = \delta_{\varepsilon_0/2K}(\partial M)$ given by Lemma 7 will ensure that $\mathcal{I}_X$ lies in the ε_1–thick part of M_X. (We also assume that ε_1 is less than the $\delta_{\varepsilon_3}(\partial M)$ given by Lemma 7, a fact we use in the claim below.) Since $\mathfrak{F}_X$ lies in the d–neighborhood of $\mathcal{I}_X$, this provides the desired $\varepsilon < \varepsilon_1 < \delta_{\varepsilon_0/2K}$, which we now fix.

Let h be the constant given by Theorems 11 and 10 such that the components of the geodesic core $\mathbf{c}$ have length less than ε when $\ell(\alpha) \geq h$, and assume that $\ell(\alpha)$ is this large.

Now let $L > 1$ and $\eta > 0$. Theorem 18 and Proposition 20 provide an $H \geq h$ and L–bilipschitz maps

$$F_{X,\alpha}\colon\ M_X - \mathbf{P}_X^{\varepsilon} \longrightarrow M_X(\alpha) - \mathbf{T}_X^{\varepsilon}(\alpha)$$

when $\ell(\alpha) \geq H$ such that the principal curvatures of the $\mathfrak{G}_{X,\alpha} = F_{X,\alpha}\big(f_X(\mathfrak{F}_X)\big)$ are within η of those of $f_X(\mathfrak{F}_X)$.

The map

$$F_{X,\alpha}\circ f_X\colon\ \mathfrak{F}_X \longrightarrow M_X(\alpha) - \mathbf{T}_X^{\varepsilon}(\alpha)$$

lifts to a map

$$G_{X,\alpha}\colon\ \mathfrak{F}_X \longrightarrow \mathrm{QF}\big(X,\sigma_{M(\alpha)}(X)\big),$$

whose image we call

$$\mathfrak{F}_{X,\alpha} = G_{X,\alpha}(\mathfrak{F}_X).$$

Claim. *There is an $\hbar \geq H$ depending only on H and the diameter and thickness of σ_M such that $G_{X,\alpha}$ is an embedding provided $\ell(\alpha) \geq \hbar$.*

Proof of claim. If γ is an element of $\pi_1\big(M(\alpha)\big)$ corresponding to a component of $\mathbf{c}$, then no power of γ is conjugate into $\pi_1\big(\partial M(\alpha)\big)$. To see this, consider the double DM. Filling equivariantly with respect to the involution, we find that $M(\alpha)$ admits a hyperbolic structure $\mathcal{C}\big(M_Y(\alpha)\big)$ with totally geodesic boundary Y, and that the geodesic cores of the filling tori lie in the interior of $\mathcal{C}\big(M_Y(\alpha)\big)$. Since $\partial\mathcal{C}\big(M_Y(\alpha)\big)$ is totally geodesic, no power of γ is conjugate into $\pi_1\big(\partial M(\alpha)\big)$.

Let

$$b\colon\ \mathrm{QF}\big(X,\sigma_M(X)\big) \to M_X$$

be the covering corresponding to $\pi_1(\partial \dot{M}_X)$ and let

$$b_\alpha\colon\ \mathrm{QF}\big(X,\sigma_{M(\alpha)}(X)\big) \to M_X(\alpha)$$

be the covering corresponding to $\pi_1\big(\partial \dot{M}_X(\alpha)\big)$. For any $\omega > 0$, write

$$\mathbf{B}^{\omega} = b^{-1}\big(\mathbf{P}^{\omega}_X\big) \quad \text{and} \quad \mathbf{B}^{\omega}_{\alpha} = b_{\alpha}^{-1}\big(\mathbf{T}^{\omega}_X(\alpha)\big).$$

Let $\mathcal{C}_\alpha$ be the convex core of $\mathrm{QF}\big(X, \sigma_{M(\alpha)}(X)\big)$, and let $\mathcal{I}_\alpha$ be the component of $\partial\mathcal{C}_\alpha$ facing $\sigma_{M(\alpha)}(X)$. By our choice of ε (smaller than the δ_{ε_3} in Lemma 7) and H (large enough to ensure that each component of $\mathbf{c}$ has length less than ε), if the image of $\mathcal{I}_\alpha$ in $M_X(\alpha)$ intersected $\mathbf{T}^{\varepsilon}_X(\alpha)$, then an essential closed curve in $\mathcal{I}_\alpha$ would be carried into $\mathbf{T}^{\varepsilon_3}_X(\alpha)$, contrary to the fact that no power of γ is conjugate into $\pi_1\big(\partial M(\alpha)\big)$. So $\mathcal{I}_\alpha$ is disjoint from $\mathbf{B}^{\varepsilon}_\alpha$.

Let $\mathcal{E}_\alpha$ be the component of

$$\overline{\mathrm{QF}\big(X, \sigma_{M(\alpha)}(X)\big) - \mathcal{C}_\alpha}$$

that faces $\sigma_{M(\alpha)}(X)$. Let E_t be the component of the boundary of the t–neighborhood of $\mathcal{C}_\alpha$ that lies in $\mathcal{E}_\alpha$. The E_t are convex and foliate $\mathcal{E}_\alpha$, see [38]. Let Q_t be the submanifold of $\mathrm{QF}\big(X, \sigma_{M(\alpha)}(X)\big)$ facing X whose boundary is E_t.

Since the map $F_{X,\alpha} \circ f_X$ is L–lipschitz and the diameter of $\mathfrak{F}_X$ is bounded by a constant A depending only on the diameter and thickness of σ_M, the diameter of $\mathfrak{F}_{X,\alpha}$ is bounded by LA. There is a constant R depending only on LA such that $\mathfrak{F}_{X,\alpha}$ is contained in Q_R. To see this, let D be a universal lower bound on the diameter of all complete hyperbolic surfaces (as in the proof of Lemma 28). If $\mathfrak{F}_{X,\alpha}$ lies outside of Q_r, its diameter is at least that of E_r, which is at least $D\cosh r$. Taking $r = \operatorname{arccosh}(2LA/D)$, we see that $\mathfrak{F}_{X,\alpha}$ must intersect Q_r. Taking $R = LA + \operatorname{arccosh}(2LA/D)$ guarantees that $\mathfrak{F}_{X,\alpha}$ lies in Q_R.

Now, the image of E_R in $M_X(\alpha)$ penetrates $\mathbf{T}^{\varepsilon}_X(\alpha)$ to a depth no more than R. By Theorem 11 and Brooks and Matelski's theorem [27], there is an $\hbar \geq H$ depending only on R and H such that the depth of $\mathbf{T}^{\varepsilon}_X(\alpha)$ is at least $2R$ whenever $\ell(\alpha) \geq \hbar$. So when $\ell(\alpha) \geq \hbar$, we may find a tubular neighborhood $\mathbf{T}^{\omega}_X(\alpha) \subset \mathbf{T}^{\varepsilon}_X(\alpha)$ of $\mathbf{c}$ whose preimage $\mathbf{B}^{\omega}_\alpha$ lies outside of Q_R.

Finally, let

$$N^{\omega}_\alpha \longrightarrow \mathrm{QF}\big(X, \sigma_{M(\alpha)}(X)\big) - \mathbf{B}^{\omega}_\alpha$$

be the covering corresponding to

$$\pi_1(\mathcal{C}_\alpha) \ \subset \ \pi_1\Big(\mathrm{QF}\big(X, \sigma_{M(\alpha)}(X)\big) - \mathbf{B}^{\omega}_\alpha\Big)$$

and let $N^{\varepsilon} = \mathrm{QF}(X, \sigma_M(X)) - \mathbf{B}^{\varepsilon}$. The map $F_{X,\alpha}$ lifts to an embedding

$$\widetilde{F}_{X,\alpha}\colon\, N^{\varepsilon} \to N^{\omega}_\alpha,$$

and the composition

$$\mathfrak{F}_X \xrightarrow{\widetilde{F}_{X,\alpha}} N^{\omega}_\alpha \longrightarrow \mathrm{QF}\big(X, \sigma_{M(\alpha)}(X)\big) - \mathbf{B}^{\omega}_\alpha$$

is our map $G_{X,\alpha}$. As the tube $\mathbf{B}^{\omega}_\alpha$ misses it, the submanifold $Q_R \supset \mathfrak{F}_{X,\alpha}$ lifts homeomorphically to a submanifold of N^{ω}_α, since $\pi_1(Q_R) = \pi_1(\mathcal{C}_\alpha)$. Since $\mathfrak{F}_X$ is embedded in N^{ε}, we conclude that $G_{X,\alpha}$ is an embedding when $\ell(\alpha) \geq \hbar$. □

Let $\hbar$ be as in the claim and assume that $\ell(\alpha) \geq \hbar$.

Since the principal curvatures of $\mathfrak{F}_X$ are bounded away from zero by $1/2$, for sufficiently small η the $\mathfrak{F}_{X,\alpha}$ are strictly convex. So there are homeomorphic normal projections $\Pi\colon\ \sigma_M(X) \to \mathfrak{F}_X$ and $\Pi_\alpha\colon\ \sigma_{M(\alpha)}(X) \to \mathfrak{F}_{X,\alpha}$. Since $\mathfrak{F}_X$ and $\mathfrak{F}_{X,\alpha}$ are smooth, these maps are diffeomorphisms.

We obtain a map $s_{X,\alpha}\colon\ \sigma_M(X) \to \sigma_{M(\alpha)}(X)$ given by

$$s_{X,\alpha} = \Pi_\alpha^{-1} \circ G_{X,\alpha} \circ \Pi.$$

As in the proof of Theorem 29, we may write the derivative at a point p as

$$\begin{aligned}
Ds_{X,\alpha} &= A^{-1}B \\
&= A^{-1}\begin{pmatrix} \frac{1+\kappa_1(\mathfrak{F}_{X,\alpha})}{2} & 0 \\ 0 & \frac{1+\kappa_2(\mathfrak{F}_{X,\alpha})}{2} \end{pmatrix} A \begin{pmatrix} w' & x' \\ y' & z' \end{pmatrix} \begin{pmatrix} \frac{2}{1+\kappa_1(\mathfrak{F}_X)} & 0 \\ 0 & \frac{2}{1+\kappa_2(\mathfrak{F}_X)} \end{pmatrix} \\
&= A^{-1}\begin{pmatrix} w\,\frac{1+\kappa_1(\mathfrak{F}_{X,\alpha})}{1+\kappa_1(\mathfrak{F}_X)} & x\,\frac{1+\kappa_1(\mathfrak{F}_{X,\alpha})}{1+\kappa_2(\mathfrak{F}_X)} \\ y\,\frac{1+\kappa_2(\mathfrak{F}_{X,\alpha})}{1+\kappa_1(\mathfrak{F}_X)} & z\,\frac{1+\kappa_2(\mathfrak{F}_{X,\alpha})}{1+\kappa_2(\mathfrak{F}_X)} \end{pmatrix}
\end{aligned}$$

where A is the rotation carrying the principal directions of $\mathfrak{F}_{X,\alpha}$ to $(1,0)$ and $(0,1)$—recall that we have chosen coordinates so that the principal directions of $\mathfrak{F}_X$ are $(1,0)$ and $(0,1)$. As before, we equip the tangent spaces to $\sigma_M(X)$ and $\sigma_{M(\alpha)}(X)$ at our chosen points with the standard Euclidean inner products. Again, the numbers w, x, y, and z are bounded in absolute value by a constant depending only on L.

There are now two cases depending on whether our point of interest in $\mathfrak{F}_X$ is very nearly umbilic or not.

Let $\varepsilon' > 0$. There is a constant $\varepsilon'' > 0$ such that if the magnitude $|\,\kappa_1(\mathfrak{F}_X) - \kappa_2(\mathfrak{F}_X)\,|$ is less than ε'' at p, and η is small enough (so that the numbers $|\,\kappa_i(\mathfrak{F}_X) - \kappa_i(\mathfrak{F}_{X,\alpha})\,|$ are small), then the bilipschitz constant of B is within ε' of that of $DG_{X,\alpha}$—for the simple reason that the entries of B are then close to w, x, y, and z.

Suppose that $|\,\kappa_1(\mathfrak{F}_X) - \kappa_2(\mathfrak{F}_X)\,| \geq \varepsilon''$. Then, for any $\omega > 0$, we may choose η very small in comparison to ε'' so that the principal directions of $\mathfrak{F}_{X,\alpha}$ are within ω of the images under $DG_{X,\alpha}$ of the principal directions of $\mathfrak{F}_X$—this choice of η depends only on ω and ε'' by linearity of the second fundamental form and homogeneity of hyperbolic manifolds. In this case x and y are very close to zero. Since the $\kappa_i(\mathfrak{F}_X)$ are both less than 2, greater than $1/2$, and the numbers $|\,\kappa_i(\mathfrak{F}_X) - \kappa_i(\mathfrak{F}_{X,\alpha})\,|$ are small, the bilipschitz constant of B is within ε' of that of $DG_{X,\alpha}$.

In any case, the bilipschitz constant of B, and hence that of $Ds_{X,\alpha} = A^{-1}B$, is within ε' of that of $DG_{X,\alpha}$ provided $\ell(\alpha)$ is sufficiently large. But the map $G_{X,\alpha}$ is bilipschitz with constant L. We conclude that for L very close to one and η small, the Teichmüller distance between $\sigma_M(X)$ and $\sigma_{M(\alpha)}(X)$ is small in a manner independent of X. □

4 Volume estimate

In Section 1.5 we established the convention that given a manifold M with conformal boundary X, a filling $M(\alpha)$ is assumed to be equipped with conformal boundary X.

When working with manifolds with totally geodesic boundary, it is often useful to normalize so that the filled manifold has totally geodesic boundary. In the following theorem, a manifold with totally geodesic boundary is said to be obtained from a manifold M with totally geodesic boundary via hyperbolic Dehn filling if it is topologically obtained from M via Dehn filling and the core of the filling torus may be taken to be geodesic *in the hyperbolic metric with totally geodesic boundary*.

Theorem 37 (Jørgensen)**.** *Let $V > 0$. There is a finite list $N_1, \dots, N_n$ of hyperbolic manifolds with closed totally geodesic boundary such that if N is a hyperbolic manifold with totally geodesic boundary of volume less than V, then N may be obtained from one of the N_i by hyperbolic Dehn filling.*

Sketch. The theorem is typically stated for finite volume manifolds without boundary, as in Chapter 5 of [83], see also [49] and Chapter E.4 of [12].

Let N be a hyperbolic manifold of volume less than V with totally geodesic boundary. Let $\mathcal{X}$ be a subset of the thick part $N^{\geq \varepsilon}$ maximal with respect to the property that no two points of $\mathcal{X}$ have distance less than or equal to $\varepsilon/2$. The bound on the volume bounds the cardinality of $\mathcal{X}$, as the $\varepsilon/4$–neighborhood of $\mathcal{X}$ is embedded. Since $\mathcal{X}$ is maximal, the $\varepsilon/2$–balls about the points of $\mathcal{X}$ cover $N^{\geq \varepsilon}$.

Let U be the universal cover of $N^{\geq \varepsilon}$, pull back our collection of $\varepsilon/2$–balls to U, and consider the nerve $\widetilde{\mathcal{N}}$ of this covering of U—a convexity argument shows that the intersection of the elements of any subcollection of this covering is either empty or a topological cell. The group $\pi_1(N^{\geq \varepsilon})$ acts freely on $\widetilde{\mathcal{N}}$, so the quotient $\mathcal{N} = \widetilde{\mathcal{N}}/\pi_1(N^{\geq \varepsilon})$ is a simplicial complex and the quotient map $\widetilde{\mathcal{N}} \to \mathcal{N}$ is a covering. Since the balls in our original collection are bounded in number and size, there is a bound on the dimension and number of simplices in $\mathcal{N}$. By a theorem of A. Weil, see pages 466–468 of [17], the nerve $\widetilde{\mathcal{N}}$ is homotopy equivalent to U, which is contractible. By Whitehead's Theorem, $\mathcal{N}$ is homotopy equivalent to $N^{\geq \varepsilon}$.

We conclude that the number of fundamental groups of such $N^{\geq \varepsilon}$ is finite. As the $N^{\geq \varepsilon}$ are acylindrical (see below), they are determined up to homeomorphism by their fundamental groups, thanks to a theorem of K. Johannson [47]. It follows that $N^{\geq \varepsilon}$ belongs to a finite list of manifolds depending only on V.

If a Margulis tube in the thin part $N^{\leq \varepsilon}$ is incident to ∂N, there is only one topological Dehn filling there, and as the geometry of N depends only on its topology, by Mostow–Prasad Rigidity [74], we are free to fill these tubes to obtain a manifold N^*.

To see that this manifold admits a hyperbolic structure with totally geodesic boundary, note that the double DN is a hyperbolic manifold of finite volume. Now, the complement of a collection of simple geodesics in a finite volume hyperbolic 3–manifold admits a finite volume hyperbolic metric itself, see [55], for instance, and so DN^* admits such a metric. The involution on DN^* implies that ∂N^* is totally geodesic in this metric, and we obtain a metric with totally geodesic boundary on N^*. □

Theorem 38 (Volume Estimate)**.** *Let M be a finite volume hyperbolic 3–manifold with nonempty closed totally geodesic boundary. Then there are $A, B, \delta > 0$ depending only on the volume of M such that $\sigma_M\big(\mathcal{T}(\partial M)\big)$ is δ–thick and $B \leq \operatorname{diam}(\sigma_M) \leq A$.*

Proof. Given a finite set $\mathfrak{M}$ of finite volume manifolds, let $\mathfrak{C}(\mathfrak{M})$ be the maximum number of cusps possessed by any element of $\mathfrak{M}$.

Let $V > 0$.

Let $\mathfrak{M}_1 = \mathfrak{M}_1(V) = \{M_1, \ldots, M_{n_1}\}$ be the list of manifolds given by Jørgensen's theorem after discarding any manifolds with empty boundary.

For each i, let δ_i, A_i, and B_i be positive constants such that the image of σ_{M_i} is δ_i–thick and

$$B_i \leq \operatorname{diam}(\sigma_{M_i}) \leq A_i,$$

as we may by Theorem 1 and the Bounded Image Theorem. Let $\varepsilon_i > 0$ be small enough so that the ε_i–neighborhood of the δ_i–thick part of $\mathcal{T}(\overline{\partial M_i})$ is $\delta_i/2$–thick,

$$\frac{B_i}{2} \leq B_i - \varepsilon_i\,,$$

and

$$A_i + \varepsilon_i \leq 2A_i.$$

For each i, let $\mathcal{E}_i$ be the set of filling slopes α for M_i such that the normalized length of *every* component of α is less than the constant $\hbar_{\varepsilon_i}(M_i)$ given by the Filling Theorem *and* such that there is a hyperbolic Dehn filling $M_i(\alpha)(\beta)$ of $M_i(\alpha)$ with totally geodesic boundary. Being the complement of a union of geodesics in $M_i(\alpha)(\beta)$, the manifold $M_i(\alpha)$ admits a hyperbolic structure with totally geodesic boundary—again, see [55].

For each α in $\mathcal{E}_i$, adjoin the manifold $M_i(\alpha)$ to the list $\mathfrak{M}_1$ to obtain a finite list of manifolds $\mathfrak{M}_2 = \{M_1, \ldots, M_{n_1}, M_{n_1+1}, \ldots, M_{n_2}\}$. Note that $\mathfrak{C}(\mathfrak{M}_2 - \mathfrak{M}_1) < \mathfrak{C}(\mathfrak{M}_1)$. For $n_1 + 1 \leq i \leq n_2$, we define ε_i, as before, to be small enough so that the ε_i–neighborhood of the δ_i–thick part of $\mathcal{T}(\overline{\partial M_i})$ is $\delta_i/2$–thick, $B_i/2 \leq B_i - \varepsilon_i$, and $A_i + \varepsilon_i \leq 2A_i$, where the image of σ_{M_i} is δ_i–thick and $B_i \leq \operatorname{diam}(\sigma_{M_i}) \leq A_i$.

We apply the argument again to the set $\mathfrak{M}_2$ to obtain a set $\mathfrak{M}_3$. Note that any element in $\mathfrak{M}_3 - \mathfrak{M}_2$ is obtained by filling a manifold in $\mathfrak{M}_2 - \mathfrak{M}_1$, and so

$$\mathfrak{C}(\mathfrak{M}_3 - \mathfrak{M}_2) < \mathfrak{C}(\mathfrak{M}_2 - \mathfrak{M}_1).$$

Continuing in this manner, we obtain a sequence of finite sets

$$\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{M}_i \subset \cdots$$

with

$$\mathfrak{C}(\mathfrak{M}_i - \mathfrak{M}_{i-1}) > \mathfrak{C}(\mathfrak{M}_{i+1} - \mathfrak{M}_i)$$

for all i. So this process terminates in a finite set $\mathfrak{M}_k = \{M_1, \ldots, M_{n_k}\}$.

Now, by construction, this set has the property that given a manifold M with totally geodesic boundary and volume less than V, there exists an i in $\{1, \ldots, n_k\}$ such that $M = M_i(\gamma)$ where the cores of the filling tori are geodesics in M and the normalized length of each component of γ is at least $\hbar_{\varepsilon_i}(M_i)$. So, letting

$$\delta = \min\left\{ \frac{\delta_i}{2} \,\middle|\, 1 \leq i \leq n_k \right\},$$

$$B = \min\left\{ \frac{B_i}{2} \;\middle|\; 1 \le i \le n_k \right\},$$

and

$$A = \max\{2A_i \mid 1 \le i \le n_k\},$$

we see that for any M with totally geodesic boundary and volume no more than V, the image of σ_M is δ–thick and $B \le \operatorname{diam}(\sigma_M) \le A$. □

5 Pinching

By Fujii and Soma's Density Theorem [39], there are hyperbolic 3–manifolds whose totally geodesic boundaries contain pants decompositions as short as desired, see also [53], and so the following theorem produces manifolds whose skinning maps have diameters descending to zero.

Theorem 39 (Bromberg–Kent)**.** *Let S be a closed hyperbolic surface. For each $\varepsilon > 0$ there is a $\delta > 0$ such that if M is a hyperbolic 3–manifold with totally geodesic boundary Σ homeomorphic to S containing a pants decomposition $\mathcal{P}$ each component of which has length less than δ, then the diameter of σ_M is less than ε.*

Proof. For simplicity we assume that S is connected of genus g. We let Γ denote the unifomizing Kleinian group for our 3–manifold, and as we work with the open manifold $\mathbb{H}^3/\Gamma$, we let $M = \mathbb{H}^3/\Gamma$ and consider Σ a totally geodesic surface inside M.

Let $\mathbf{T}_\Sigma$ be the ε_3–Margulis tube about $\mathcal{P}$ in M.

Now consider the double $D\mathcal{C}(M)$ of the convex core of M. The tube $\mathbf{T}_\Sigma$ isometrically embeds in the double and so we call its image by the same name.

Let $m = m_1 \sqcup \cdots \sqcup m_{3g-3}$ be a geodesic 1–manifold in $\partial\mathbf{T}_\Sigma$ (equipped with the induced path metric) such that each component m_i bounds a disk in $\mathbf{T}_\Sigma$. Each component m_i of m is a **meridian**. If the total length of $\mathcal{P}$ is small enough, then the depth of each component of $\mathbf{T}_\Sigma$ is large, by Brooks and Matelski's Theorem [27]. It follows that the *actual* lengths of the meridians m_i in $\partial\mathbf{T}_\Sigma$ are all large—this is true of the *meridian* of any deep Margulis tube about a geodesic.

The intersection of the surface $\partial\mathcal{C}(M)$ and $\mathbf{T}_\Sigma$ defines a slope in $\partial\mathbf{T}_\Sigma$ whose components we call the **longitudes**. Notice that the actual lengths of the longitudes are bounded above as ε tends to zero, as they are trapped in geodesic pairs of pants whose boundary components have length ε.

By Brock and Bromberg's Drilling Theorem, Theorem 18 here, we may drill $\mathcal{P}$ out from $D\mathcal{C}(M)$ to obtain a complete hyperbolic manifold M' and an L–bilipschitz map

$$f\colon\; D\mathcal{C}(M) - \mathbf{T}_\Sigma \longrightarrow M' - \mathbf{P}$$

where L is very close to one and $\mathbf{P}$ is a Margulis tube about the $3g-3$ new rank–two cusps in M'—here we are using the Drilling Theorem in the case when the conformal boundary is empty. Letting $m'_1, \ldots, m'_{3g-3}$ denote the geodesic representatives of the $f(m_1), \ldots, f(m_{3g-3})$, respectively, we see that the actual flat lengths of the m'_i are large in $\partial M'$. Also note that the longitudes are not much longer in $\partial\mathbf{P}$ than they were in $\mathbf{T}_\Sigma$.

The manifold M' is the double of a manifold whose interior is homeomorphic to M and whose boundary is a union $\mathcal{S} = S_1 \cup \cdots \cup S_{2g-2}$ of thrice–punctured spheres. The involution on M' implies that $\mathcal{S}$ may be isotoped to be totally geodesic in M'. So the surface $\mathcal{S}$ cuts the union of tori $\partial \mathbf{P}$ into a union $\mathbf{A}$ of annuli. It also cuts each meridian m'_i into two arcs of equal length. If ζ is one of these arcs and A its annulus in $\mathbf{A}$, then the involution implies that ζ intersects ∂A in right angles. Since the longitudes are bounded above in length, and ζ is long, it follows that the conformal modulus of A is large.

A separate application of the Drilling Theorem allows the drilling of $\mathcal{P}$ from M to obtain a complete hyperbolic manifold N with conformal boundary Σ and an L–bilipschitz map

$$f\colon\ M - \mathbf{T}_\Sigma \longrightarrow N - \mathbf{P}_\Sigma$$

where L is very close to one and $\mathbf{P}_\Sigma$ is a Margulis tube about the new rank–two cusps.

The manifold N contains $2g-2$ incompressible properly embedded thrice–punctured spheres $S_1, \ldots, S_{2g-2}$ whose ends lie in $\mathbf{P}_\Sigma$. A theorem of Adams [1] says that we may take the S_i to be totally geodesic in *any* complete hyperbolic structure on N—in [1], the theorem is stated for finite volume hyperbolic manifolds, but it is easily seen that the proof is valid as long as the ends of the punctured spheres lie in the cusps of the ambient manifold.

Given an Riemann surface X in $\mathcal{T}(S) \cong \mathrm{GF}(N)$, let $\mathcal{S}_X$ be the totally geodesic representative of $S_1 \cup \cdots \cup S_{2g-2}$ in N_X given by Adams' theorem. Let N'_X be the closure of the component of $N_X - \mathcal{S}_X$ homeomorphic to M, and let $N' = N'_\Sigma$. Since thrice–punctured spheres have no moduli, all of the N'_X are isometric to N', by Theorem 4, and so we have isometric embeddings

$$\iota_X\colon\ N' \to N_X.$$

Note that our manifold M' above is the double of N'.

For X in $\mathcal{T}(S)$, let $\mathbf{P}_X$ be a union of cuspidal ε_3–Margulis tubes in N_X. Since N' is isometrically embedded in N_X, the intersection $\partial \mathbf{P}_X \cap N'$ is a collection $\mathbf{B}$ of horospherical annuli perpendicular to $\mathcal{S}$. Moreover, each annulus in $\mathbf{B}$ is conformally equivalent to the corresponding annulus in $\mathbf{A}$.

Being trapped in finite volume geodesic pairs of pants, the longitudes in $\partial \mathbf{P}_X$ have lengths bounded above independent of X. Since the Margulis constant ε_3 is universal, their lengths are uniformly bounded below as well.

Now, if $n(X)$ is a geodesic meridian on $\partial \mathbf{P}_X$, the surface $\mathcal{S}$ cuts it into two arcs. Let ζ be the arc contained in N', and let B be the annulus in $\mathbf{B}$ containing it. Since B and the corresponding annulus A in $\mathbf{A}$ are conformally equivalent, they have the same conformal modulus. Since the modulus is large, and the actual length of the longitude is bounded *below*, the actual length of ζ must be large, and it follows that the actual length of $n(X)$ is large as well. Since the actual lengths of the longitudes are bounded above, we conclude that the normalized length of $n(X)$ is large for all X.

If the normalized lengths of these meridians are large enough (which is guaranteed if the total hyperbolic length of $\mathcal{P}$ is small enough), we may perform hyperbolic Dehn filling at all of them to obtain M_X. Moreover, the Drilling Theorem again tells us that

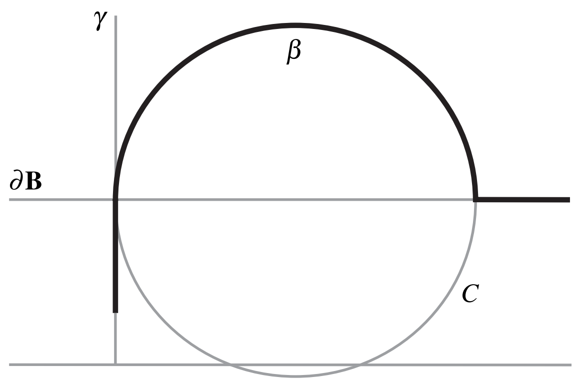


Figure 3: Building a slice of $\mathfrak{G}$ near $\mathbf{P}_\Sigma$.

for each X there is an L–bilipschitz map

$$g_X : N_X - \mathbf{P}_X \longrightarrow M_X - \mathbf{T}_X.$$

We now construct a surface $\mathfrak{G}$ in N' with diffeomorphic Gauss map that will allow us to compare skinning surfaces. Unlike the surfaces in the proofs of Theorems 29 and 36, this surface will not be convex. The idea is as follows. We would like to construct a surface by cutting the surface $\mathcal{S}$ along $\partial\mathbf{P}_\Sigma$, tubing the resulting boundary components together by annuli in $\partial\mathbf{P}_\Sigma$, and smoothing. Unfortunately, the hyperbolic Gauss map of such a surface will be a diffeomorphism, as it may have principal curvatures below -1. To correct this, we carefully extend the surface $\mathcal{S} - \mathbf{P}_\Sigma$ into the cusps $\mathbf{P}_\Sigma$ in a "rotationally" symmetric way *before* tubing to get a surface with diffeomorphic Gauss map after smoothing.

Outside a small neighborhood U of $\mathbf{P}_\Sigma$, the surface $\mathfrak{G}$ is simply $\mathcal{S} - U$. To describe $\mathfrak{G}$ in U, we begin by defining a meridional slice of $\mathfrak{G}$.

Pick a component $\mathbf{P}_0$ of $\mathbf{P}_\Sigma$ and consider one of its preimages in $\mathbb{H}^3$ normalized to be the horoball $\mathbf{B}$ centered at infinity that passes through i. Choose a sphere $S_i \subset \mathcal{S}$ that intersects $\mathbf{P}_0$ and further normalize so that a preimage $\widetilde{S}_i$ of S_i contains the geodesic γ passing through zero and infinity. There is a hyperbolic plane H that intersects both $\partial\mathbf{B}$ and $\widetilde{S}_i$ at right angles.

The plane H is pictured in Figure 3 with $\partial\mathbf{B}$ and γ in gray. Also in gray is a circle C tangent to γ at i and whose Euclidean radius is slightly larger than one—note that, choosing inward pointing normals, the hyperbolic curvature of C is strictly greater than -1. The black arc β is the concatenation of an arc in $\gamma = \widetilde{S}_i \cap H$, an arc in C, and an arc of $\partial\mathbf{B}$. Note that if the meridian in N is long enough, then β may be taken disjoint from all of the other preimages of components of $\mathcal{S}$. If the meridian is still longer, then two adjacent lifts of components of $\mathcal{S}$ that intersect $\mathbf{B}$ may be joined be an embedded arc formed by concatenating a copy of β and a copy of its reflection through a vertical line—this will guarantee that our surface $\mathfrak{G}$ is embedded—see Figure 4.

Now, the arc β is not smooth, but a small perturbation changes β into a smooth arc α that agrees with β near its endpoints and all of whose curvatures are strictly greater

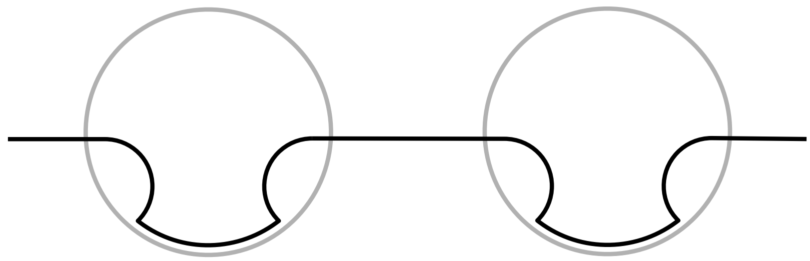


Figure 4: The surface $\mathfrak{G}$ intersecting two Margulis tubes. Here, the skinning surface lies "below" $\mathfrak{G}$.

than -1. Now, pushing α down into our neighborhood U of $\mathbf{P}_0$, revolving it through the product structure there, and taking the union with the surface $\mathbb{S}-U$, we obtain a smooth embedded surface $\mathfrak{G}$ all of whose normal curvatures are strictly greater than -1. See Figure 4 for a schematic of $\mathfrak{G}$ in M.

By construction, $\mathfrak{G}$ is contained in the ε'–thick part of N for some universal ε'. Note that the surfaces $f^{-1}(\mathfrak{G})$ and $g_X \circ \iota_X(\mathfrak{G})$ are homotopic to the convex core boundaries of M and M_X, respectively. Theorem 11 and Proposition 20 tell us that the normal curvatures of $f^{-1}(\mathfrak{G})$ and $g_X \circ \iota_X(\mathfrak{G})$ are very close to those of $\mathfrak{G}$, and, in particular, are greater than -1. This is all that is needed for the existence of diffeomorphic normal projections $\Pi\colon\ \overline{\Sigma} \to f^{-1}(\mathfrak{G})$ and $\Pi_X\colon\ \sigma_M(X) \to g_X \circ \iota_X(\mathfrak{G})$, as the hyperbolic Gauss map of a smooth embedded surface in $\mathbb{H}^3$ is a diffeomorphism provided all of its normal curvatures are strictly greater than -1.

As in the proof of Theorem 36, as we let the normal curvatures of $g_X \circ \iota_X(\mathfrak{G})$ and $f^{-1}(\mathfrak{G})$ tend to those of $\mathfrak{G}$, letting our bilipschitz constant L tend to one, we see that the composition

$$\Pi_X^{-1} \circ g_X \circ \iota_X \circ f \circ \Pi\colon\ \overline{\Sigma} \to \sigma_M(X)$$

is very close to conformal in a manner independent of X. □

6 No universal bound

Theorem 40 (Flexible Manifolds)**.** *For any $R > 0$, there is a small d such that if M is a hyperbolic 3–manifold with connected totally geodesic boundary Σ such that every metric collar about Σ has depth at most d, then the skinning map of M has diameter greater than R.*

Proof. Let N_n be any sequence of hyperbolic 3–manifolds with connected totally geodesic boundary Σ_n such that every collar about Σ_n has depth at most d_n with $\lim d_n = 0$. Let Γ_n be a uniformizing Kleinian group for N_n, and let $\sigma_n = \sigma_{N_n}$. We will show that the diameters of the σ_n tend to infinity.

After normalizing, we may assume that $\mathcal{U}$ is the component of the domain of discontinuity of Γ_n that uniformizes Γ_{Σ_n}. Since $\lim d_n = 0$, we may further normalize so that there are loxodromic elements g_n in the Γ_n with the property that

$$\lim_{n\to\infty} \operatorname{meas}(\mathcal{L} - g_n \mathcal{U}) = 0. \tag{6.1}$$

This normalization produces the picture in Figure 2. There is a fundamental domain for $\Gamma_{\overline{\Sigma}_n}$ in $\mathcal{L}$ that contains $g_n\mathcal{U}$, and so, by (6.1), the Γ_{Σ_n} have fundamental domains ω_n in $\mathcal{U}$ such that

$$\lim_{n\to\infty} \operatorname{meas}(g_n\mathcal{U} - g_n\omega_n) = 0.$$

With (6.1), we have

$$\lim_{n\to\infty} \operatorname{meas}(\mathcal{L} - g_n\omega_n) = 0. \tag{6.2}$$

Let k lie in the interval $(0,1)$ and let $K = (1+k)/(1-k)$. Define a Beltrami differential μ_n for Γ_{Σ_n} by declaring that

$$\mu_n(z) = k\,\frac{\overline{g_n'(z)}}{g_n'(z)}$$

on ω_n and extending to $\mathcal{U}$ equivariantly.

The constant function $1/\pi$ is a quadratic differential on $\mathcal{L}$. So

$$\begin{aligned}
\frac{1}{\mathrm{K}\big[\sigma_n([\mu_n])\big]_{\mathcal{L}}} &\le \iint_{\mathcal{L}} \frac{1}{\pi}\,\frac{|1-\sigma_n(\mu_n)|^2}{1-|\sigma_n(\mu_n)|^2}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{by (1.2)}\\
&= \iint_{g_n\omega_n} \frac{1}{\pi}\,\frac{|1-\sigma_n(\mu_n)|^2}{1-|\sigma_n(\mu_n)|^2}\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\qquad + \iint_{\mathcal{L}-g_n\omega_n} \frac{1}{\pi}\,\frac{|1-\sigma_n(\mu_n)|^2}{1-|\sigma_n(\mu_n)|^2}\,\mathrm{d}x\,\mathrm{d}y\\
&\le \iint_{\omega_n} \frac{1}{\pi}\,\frac{\left|1-k\frac{\overline{g_n'}\,\overline{g_n'}}{g_n'\,g_n'}\right|^2}{1-k^2}\,|g_n'|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\qquad + \frac{1+k^2}{\pi(1-k^2)}\operatorname{meas}(\mathcal{L}-g_n\omega_n)\\
&= \frac{1}{K}\iint_{\omega_n} \frac{1}{\pi}\,|g_n'|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\qquad + \frac{1+k^2}{\pi(1-k^2)}\operatorname{meas}(\mathcal{L}-g_n\omega_n) \qquad (6.3)
\end{aligned}$$

By (6.2), the term (6.3) is less than ε for all sufficiently large n, and so

$$\frac{1}{\mathrm{K}\big[\sigma_n([\mu_n])\big]_{\mathcal{L}}} \le \frac{1}{K} + \varepsilon,$$

since

$$\iint_{\mathcal{U}} \frac{1}{\pi}\,|g_n'|^2\,\mathrm{d}x\,\mathrm{d}y \le 1.$$

Putting k close to one—so that K is large—we see that we may take

$$\mathrm{K}\big[\sigma_n([\mu_n])\big]_{\mathcal{L}} \leq \mathrm{K}\big[\sigma_n([\mu_n])\big]_{\overline{\Sigma}_n}$$

as large as we like. This means that the distance $\mathrm{d}_{\mathcal{T}(\overline{\Sigma}_n)}\big([0], \sigma_n([\mu_n])\big)$ is large, and so the skinning map σ_n has large diameter. □

7 The Bounded Image Theorem

In the following M will denote a compact oriented irreducible atoroidal 3–manifold with incompressible boundary of negative Euler characteristic. Let $\partial_0 M$ be the part of the boundary that contains no tori.

Theorem 41 (Thurston)**.** *If M is acylindrical, the skinning map σ_M admits a continuous extension*

$$\sigma_M\colon \mathrm{AH}(M) \to \mathcal{T}(\overline{\partial_0 M}).$$

Since $\mathrm{AH}(M)$ is compact, this implies the Bounded Image Theorem.

To complete the final gluing step in the Geometrization of Haken Manifolds, Thurston originally suggested the following strong form of the Bounded Image Theorem—see statement (II′) on page 85 of [61].

Statement. *Let M be an orientable irreducible 3–manifold with incompressible boundary that is not an interval bundle over a surface. Let $\tau\colon \overline{\partial_0 M} \to \partial_0 M$ be a homeomorphism such that the 3–manifold M/τ is atoroidal. Then there is a number n such that the map $(\tau\circ\sigma_M)^n\colon \mathcal{T}(\partial_0 M) \to \mathcal{T}(\partial_0 M)$ has bounded image.*

We know of no proof of this global statement. In all accounts of Thurston's argument, a local statement is used at the final gluing step, namely that for any X in $\mathcal{T}(\partial_0 M)$, the sequence $(\tau\circ\sigma_M)^n(X)$ is bounded. See [61, 58, 67, 52].

We continue to the proof of Theorem 41.

Let N be a manifold. A **compact core** of N is a compact submanifold with the property that the inclusion is a homotopy equivalence. It is a theorem due to G. P. Scott and P. Shalen that aspherical 3–manifolds with finitely generated fundamental groups have compact cores [76].

If N is a hyperbolic 3–manifold, let N^0 denote the complement of its ε_3–cuspidal thin part. An end E of N^0 with a neighborhood U homeomorphic to $S\times[0,\infty)$ is **simply degenerate** if there is a sequence of pleated surfaces $f_n\colon \Sigma_n \to U$ homotopic in U to the inclusion $S\times\{0\}\to U$ whose images leave every compact set in U. As is common, we will often refer to a neighborhood of an end *as* the end.

To show that the image of our map lies in the Teichmüller space, we will need Thurston's Covering Theorem, Theorem 9.2.2 of [83], see [30]:

Covering Theorem. *Let $f\colon N\to M$ be a locally isometric covering of hyperbolic 3–manifolds such that $\pi_1(N)$ is finitely generated. Let E be a simply degenerate end of N^0 corresponding to an incompressible surface in N^0. Then either*

1. *E has a neighborhood U such that f is finite-to-one on U, or*

2. *M has finite volume and has a finite cover M' that fibers over the circle and N is finitely covered by the cover of M' corresponding to the fiber subgroup.* □

Canary's strong version of this theorem removes the requirement that the end correspond to an incompressible surface, provided the manifolds in question are tame. Tameness is provided in Thurston's theorem by a theorem of Bonahon [15], and is now known for all hyperbolic manifolds with finitely generated fundamental group by the recent solution to Marden's Tameness Conjecture by I. Agol [2] and D. Calegari and D. Gabai [28].

For the continuity we will need the following embedding theorem of Canary and J. Anderson, which is implicit in [9].

Theorem 42 (Anderson–Canary [9])**.** *Let M be a hyperbolic 3–manifold with finitely generated fundamental group and totally geodesic boundary. Let $M_n \in \mathrm{AH}(M)$ be a sequence of manifolds converging algebraically to M^a_∞ and geometrically to M^g_∞. Then there is a compact core $\mathfrak{K}$ of M^a_∞ such that the restriction of the covering $M^a_\infty \to M^g_\infty$ to $\mathfrak{K}$ is an embedding.*

Proof. Let Γ^a and Γ^g be the Kleinian groups for M^a_∞ and M^g_∞, respectively. By tameness and Proposition 2.7 of [10], for any element γ of $\Gamma^g - \Gamma^a$, the intersection $\Lambda_{\Gamma^a} \cap \gamma\Lambda_{\Gamma^a}$ has cardinality at most one. So either $\Gamma^g = \Gamma^a$ or Γ^a has a nonempty domain of discontinuity. In the first case, the theorem is obvious. In the second, the theorem follows from Corollary B of [9]. □

Proof of Theorem 41 (Brock–Kent–Minsky). For simplicity we assume that only one component $\partial_0 M$ of ∂M has negative Euler characteristic.

The map. Let M_ρ be a hyperbolic manifold in $\mathrm{AH}(M)$, with end invariant λ, corresponding to the character of a representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2\mathbb{C}$. Let $f\colon N_\rho \to M_\rho$ be the cover of M_ρ corresponding to $\partial_0 M$. This manifold has two end invariants, one of them λ corresponding to the lift of the end of M_ρ, and a skinning invariant $\sigma_M(\rho)$ corresponding to the new end E (the end E is tame by Bonahon's theorem).

We claim that E has no rank–one cusp. To see this, suppose to the contrary that there is a rank–one cusp $\mathbf{P}$ in E. If $\mathbf{P}$ covers a rank–two cusp in M_ρ, we conclude that a closed curve in $\partial_0 M$ is homotopic into a torus in ∂M, contrary to the fact that M is acylindrical. So $\mathbf{P}$ must cover a rank–one cusp in M_ρ. But then, by considering the corresponding annulus in N_ρ and its image in M_ρ, we conclude that there is a nontrivial conjugacy in $\pi_1(M)$ between two elements of $\pi_1(\partial_0 M)$, again contradicting the fact that M is acylindrical. So E is an end of N^0_ρ. Furthermore, it is either simply degenerate or conformally compact.

Now, the restriction of f to any neighborhood of E is infinite-to-one, and so, by the Covering Theorem, E cannot be simply degenerate, as M_ρ has infinite volume. This means that $\sigma_M(\rho)$ is a Riemann surface homeomorphic to $\partial_0 M$. Now, it may happen that the orientation of M_ρ naturally places its skinning invariant in $\mathcal{T}(\partial_0 M)$ rather than

$\mathcal{T}(\overline{\partial_0 M})$, in which case we let $\sigma_M(\rho)$ be the mirror image of this invariant. We thus obtain a map

$$\sigma_M\colon\ \mathrm{AH}(M) \to \mathcal{T}(\overline{\partial_0 M})$$

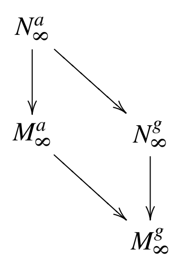
extending the skinning map, defined by $\sigma_M(M_\rho) = \sigma_M(\rho)$.

Continuity. To see that σ_M is continuous, let M_n be a sequence in $\mathrm{AH}(M)$ converging algebraically to M_∞^a. Let N_n be the cover of M_n corresponding to $\partial_0 M$. Since the map

$$\mathrm{AH}(M) \to \mathrm{AH}(\partial_0 M)$$

induced by inclusion is continuous (being the restriction of the regular function induced on $\mathrm{SL}_2\mathbb{C}$–character varieties), the N_n converge in $\mathrm{AH}(\partial_0 M)$ to a manifold N_∞^a.

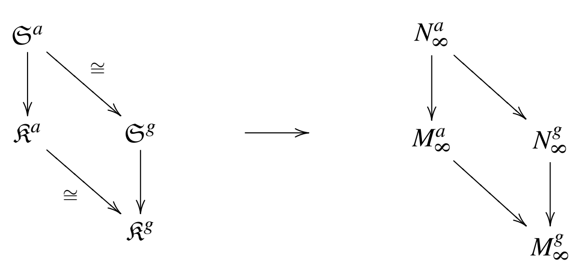
We pass to a subsequence so that the M_n converge geometrically—see Corollary 9.1.8 of [83] and Proposition 3.8 of [50]. After passing to a deeper subsequence, this produces geometric convergence of the N_n, and considering that this is convergence of the uniformizing Kleinian groups in the Chabauty topology, we have the following commutative diagram of covering spaces:



Let Γ_n, Γ_∞^a, and Γ_∞^g be the uniformizing Kleinian groups for N_n, N_∞^a, and N_∞^g, respectively.

By Theorem 42, there is a core $\mathfrak{K}^a$ of M_∞^a that embeds in M_∞^g. We call its image $\mathfrak{K}^g$. We let $\mathfrak{K}_n$ be the image of the core $\mathfrak{K}^g$ in the approximate M_n.

Now, a peripheral π_1–injective surface $\mathcal{S} \to M$ in an acylindrical manifold M has a unique non-simply connected lift $\widetilde{\mathcal{S}} \to \widetilde{M}$ to the cover corresponding to the boundary, and we let $\mathfrak{S}^a$ denote this *bona fide* lift of $\partial\mathfrak{K}^a$ to N_∞^a. We let $\mathfrak{S}^g$ denote the image of $\mathfrak{S}^a$ in N_∞^g. Diagrammatically:



We let $\mathfrak{S}_n$ be the image of $\mathfrak{S}^g$ in N_n.

The algebraic limit N^a_∞ has two ends: one the isometric lift of the end of M^a_∞, the "left" side; and another we call E^a, the "right" side—for psychological reasons, we are choosing a homeomorphism $N^a_\infty \cong \mathbb{R} \times \partial_0 M$. Since $\sigma_M(M^a_\infty)$ lies in $\mathcal{T}(\overline{\partial_0 M})$, the end E^a is conformally compact.

Each $\mathfrak{S}_n$ cuts each N_n into two pieces A_n and E_n: the first facing the end lifted from M_n; the second facing the skinning surface of M_n. The geometric convergence implies that $\mathfrak{S}^g$ separates N^g_∞ into two components A and E^g: the first the geometric limit of the A_n; the second the geometric limit of the E_n.

Lemma. *The subgroup Γ^a_∞ carries the fundamental group of E^g, and so E^g is an end of N^g_∞ that lifts isometrically to N^a_∞.*

We establish some notation before proving the lemma.

Let $\mathcal{C}^a$ be the component of the boundary of the convex core of N^a_∞ facing the end E^a, and let $\mathcal{C}^g$ be its image in N^g_∞. The surfaces $\mathcal{C}^g$ and $\mathfrak{S}^g$ are homotopic in N^g_∞, and we fix a homotopy.

We consider the component of the convex core boundary of N_n facing $\sigma_M(M_n)$ as a pleated surface $\mathcal{C}_n \to N_n$. When no confusion can arise, we blur the distinction between $\mathcal{C}_n \to N_n$ and its image. Each $\mathcal{C}_n$ is compact since $\sigma_M(M_n)$ lies in $\mathcal{T}(\overline{\partial_0 M})$.

After identifying the $\mathfrak{K}_n$ with M, the manifolds N_n are each marked by the unique *bona fide* lift $\partial_0 M \to N_n$, which, in turn, marks each $\mathcal{C}_n$.

The pleated surfaces $\mathcal{C}_n \to N_n$ descend to pleated surfaces $\mathcal{D}_n \to M_n$.

Proof of lemma. We show that every essential closed curve in N^g_∞ is homotopic into A.

CASE I. Suppose that, after passing to a subsequence, there is a compact set K in N^g_∞ whose preimage K_n in the approximate N_n intersects $\mathcal{C}_n$ for all n. By Theorem 6 (Pleated Surfaces Compact) and geometric convergence of the N_n, we see, after choosing base frames and taking a subsequence, that the $\mathcal{C}_n \to N_n$ are converging to a pleated surface $\mathcal{C} \to N^g_\infty$.

If $\mathcal{C}$ is noncompact, then there is a fixed curve γ in $\partial_0 M$ whose length in $\mathcal{C}_n$ is tending to zero. This implies that the end E^a has a rank–one cusp, which we have excluded. To see this, let γ_n be the geodesic representative of γ in $\mathcal{C}_n$, and let γ^*_n be its geodesic representative in N_n. The cores $\mathfrak{K}_n$ have uniformly bounded diameters, so we may choose an ε so that the ε–Margulis tubes $\mathbf{T}_n$ about the γ^*_n miss the $\mathfrak{K}_n$, by Brooks and Matelski's theorem [27]. Since the lengths of the γ_n are tending to zero, the depths at which they lie in the $\mathbf{T}_n$ must be tending to infinity, again by [27]. It follows that $\partial \mathbf{T}_n$ intersects the boundary of a large neighborhood of $\mathcal{C}_n$ in the end of N_n facing $\sigma_M(M_n)$. Since $\mathfrak{K}_n$ misses $\mathbf{T}_n$, the distance from $\mathfrak{K}_n$ to $\mathcal{C}_n$ is uniformly bounded, and the $\mathfrak{K}_n$ have uniformly bounded diameters, the $\mathfrak{K}_n$ must eventually lie to the left of the $\mathbf{T}_n$. So the γ^*_n eventually lie to the right of $\mathfrak{K}_n$, and we discover a rank–one cusp in E^a.

So $\mathcal{C}$ is compact. It follows that the $\mathcal{C}_n$ have uniformly bounded diameter, and so, for sufficiently large n, we may push the $\mathcal{C}_n$ into N^g_∞ to obtain surfaces $\mathcal{C}^g_n$ converging to $\mathcal{C}$.

Moreover, since the Γ_n converge algebraically, the surface $\mathcal{C}$ is in the same homotopy class as $\mathcal{C}^g$, and we fix a homotopy between them. Since the $\mathcal{C}^g_n$ are converging to $\mathcal{C}$, the surfaces $\mathcal{C}^g_n$ and $\mathcal{C}$ are eventually homotopic in the 1–neighborhood of $\mathcal{C}$.

It follows that the $\mathcal{C}_n$ admit homotopies to the $\mathfrak{S}_n$ of uniformly bounded diameter: the $\mathcal{C}_n^g$ are uniformly homotopic to $\mathcal{C}$ in N_∞^g, which is homotopic to $\mathcal{C}^g$, which is homotopic to $\mathfrak{S}^g$; and we may push the resulting homotopy from $\mathcal{C}_n^g$ to $\mathfrak{S}^g$ back into the approximates.

Now, let γ be a closed geodesic in N_∞^g, and push it back to curves γ_n in the approximates. The geometric convergence implies that the geodesic curvatures of the γ_n are all uniformly very nearly zero in the N_n for all large n. It follows that each γ_n is homotopic to its geodesic representative γ_n^* in the ε–neighborhood of γ_n^*. Now, each γ_n^* lies in the convex core of N_n, to the left of the pleated surface $\mathcal{C}_n$. Since the $\mathcal{C}_n$ admit homotopies to the $\mathfrak{S}_n$ of uniformly bounded diameter, there are ambient homotopies in the N_n with supports of uniformly bounded diameter that carry the γ_n^* to the left of $\mathfrak{S}_n$. Since the ε–neighborhoods of the γ_n^* eventually map to the geometric limit, we conclude that γ is homotopic into A.

So any closed geodesic in N_∞^g is homotopic into A.

Now let $\mathbf{P}$ be a cusp in N_∞^g, let p be a parabolic element in the corresponding subgroup of Γ_∞^g, and let η be a geodesic representing p in the flat metric on $\partial\mathbf{P}$.

Suppose that there is a sequence $\{p_n \,|\, p_n \in \Gamma_n\}$ of loxodromic elements converging to p. Then the cusp $\mathbf{P}$ is the geometric limit of Margulis tubes $\mathbf{T}_n$ about the corresponding geodesics γ_n^* in N_n. Pushing η into the approximates yields a sequence of closed curves η_n whose geodesic curvatures are close to one, and which are hence uniformly close to the geodesic η_n^* in $\partial\mathbf{T}_n$ in the same homotopy class.

The depths of the tubes $\mathbf{T}_n$ are tending to infinity. The geodesic γ_n^* lies in the convex core of N_n, to the left of $\mathcal{C}_n$, and since the $\mathcal{C}_n$ have uniformly bounded diameter, and hence cannot penetrate too deeply into $\mathbf{T}_n$, there is a smaller Margulis tube about γ_n^* that lies to the left of $\mathcal{C}_n$ and whose boundary is a uniformly bounded distance from η_n.

If p is the limit of parabolic elements p_n, then $\mathbf{P}$ is the geometric limit of Margulis tubes $\mathbf{T}_n$, and, as above, we may take these to lie in the convex cores of the approximates, perhaps after changing $\mathbf{P}$ slightly.

In either case, we obtain bounded diameter homotopies carrying the η_n to the left of the $\mathcal{C}_n$, and then ambient homotopies with supports of uniformly bounded diameter carrying the result to the left of $\mathfrak{S}_n$. We conclude that η is homotopic into A.

So, in CASE I, every essential closed curve in N_∞^g is homotopic into A.

CASE II. Suppose that for every compact set K in N_∞^g, there are only finitely many n such that the preimage K_n of K in N_n intersects $\mathcal{C}_n$.

It follows that the images $\mathcal{D}_n$ of the $\mathcal{C}_n$ in M_n must eventually lie outside of the compact cores $\mathfrak{K}_n$.

To see this, suppose to the contrary that, after passing to a subsequence, each $\mathcal{D}_n$ intersects $\mathfrak{K}_n$. Then, after choosing base frames and passing to another subsequence, the $\mathcal{D}_n \to M_n$ converge to a finite area pleated surface $\mathcal{D} \to M_\infty^g$ freely homotopic into $\partial\mathfrak{K}^g$ in M_∞^g. So, if $\mathcal{G}$ is a closed geodesic in $\mathcal{D}$, and $\mathcal{G}_n$ a sequence of geodesics in $\mathcal{D}_n$ converging to $\mathcal{G}$, then there is a uniform $R > 0$ such that the $\mathcal{G}_n \to M_n$ are eventually homotopic into $\partial\mathfrak{K}_n$ in the R–neighborhood of the latter.

Now, since $\mathcal{D}_n$ is peripheral and M is acylindrical, the surface $\mathcal{C}_n \to N_n$ is the unique lift of $\mathcal{D}_n \to M_n$ that is not simply connected. So, lifting the homotopies of $\mathcal{G}_n \to M_n$ into $\partial\mathfrak{K}_n$ to N_n, we find that each $\mathcal{C}_n$ intersects the R–neighborhood of $\mathfrak{S}_n$. But the $\mathfrak{S}_n$

are the preimages of the compact set $\mathfrak{S}^g$ in N^g_∞.

We conclude that the $\mathcal{D}_n \to M_n$ eventually miss the cores $\mathfrak{K}_n$. Since they cannot lie in the rank–two cusps of the M_n, they eventually lie to the left of the $\partial\mathfrak{K}_n$. It follows that the $\mathcal{C}_n$ eventually lie to the left of the $\mathfrak{S}_n$. (In the case where $\partial_0 M$ is disconnected, the $\mathcal{D}_n$ are all homotopic to a fixed component of $\partial_0 M$, and as distinct components of $\partial_0 M$ are not homotopic, one concludes that the $\mathcal{D}_n$ lie to the left of the chosen one.)

Again, let γ be a closed geodesic in N^g_∞, and push it back to curves γ_n in the N_n. As before, the γ_n are homotopic to their geodesic representatives γ^*_n in the ε–neighborhood of γ^*_n. But now, γ^*_n lies in the convex core of N_n which lies to the left of $\mathfrak{S}_n$, and, going back to the limit, we conclude that γ is homotopic into A.

Similarly, the argument that the parabolic elements are homotopic into A proceeds as in CASE I, the argument simplified by $\mathcal{C}_n$ lying to the left of $\mathfrak{S}_n$.

We have now shown that any essential closed curve in N^g_∞ is homotopic into A. It follows that $\pi_1(\mathfrak{S}^g) \cong \Gamma^a_\infty < \Gamma^g_\infty$ carries the fundamental group of E^g, and so E^g is an end of N^g_∞ that lifts isometrically to E^a. □

To complete the proof, choose an Epstein surface $\mathfrak{F}$ in E^g (see Section 1.10), and push it into the approximates to obtain surfaces $\mathfrak{F}_n$. By the geometric convergence, for all large n the surfaces $\mathfrak{F}_n$ are strictly convex and the normal curvatures of the $\mathfrak{F}_n$ converge to those of $\mathfrak{F}$. Paired with the isometry $E^g \to E^a$, convexity provides normal projections

$$\sigma_M(M_n) \to \mathfrak{F}_n \quad \text{and} \quad \sigma_M(M^a_\infty) \to \mathfrak{F}$$

whose derivatives depend only on the normal curvatures of the $\mathfrak{F}_n$ and $\mathfrak{F}$. Composing with the approximating bilipschitz maps, we see that the derivatives of the normal projections will "cancel" in the limit, as the limiting bilipschitz map is the restriction of an isometry (which will carry principal directions to principal directions)—as we are only concerned with continuity here, we do pass to the limit (compare the proofs of Theorems 29 and 36). By C^∞–convergence, we conclude that for large n the composition is very close to conformal and so the $\sigma_M(M_n)$ are converging to $\sigma_M(M^a_\infty)$. □

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