Atoroidal surface bundles

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Abstract

We show that there is a type–preserving homomorphism from the fundamental group of the figure–eight knot complement to the mapping class group of the thrice–punctured torus. As a corollary, we obtain infinitely many commensurability classes of purely pseudo-Anosov surface subgroups of mapping class groups of closed surfaces. This gives the first examples of compact atoroidal surface bundles over surfaces.

1 Introduction

Mapping class groups of surfaces are often viewed in analogy with Kleinian groups, with the Teichmüller space playing the role of $\mathbb{H}^3$ and the mapping class group playing the role of a finite covolume lattice in $\mathrm{PSL}_2(\mathbb{C})$. Finite covolume Kleinian groups contain swarms of quasifuchsian closed surface subgroups. In the noncocompact case, the existence of such subgroups is due to J. Masters and X. Zhang [66] and M. Baker and D. Cooper [6], while their ubiquity was proved by Cooper and D. Futer [18] and J. Kahn and A. Wright [43]. In the cocompact case, existence and ubiquity is due to Kahn and V. Marković [42]. Such quasifuchsian subgroups contain no parabolic elements, and so give examples of purely hyperbolic surface subgroups in every finite covolume Kleinian group.

At the turn of the century, B. Farb and L. Mosher [26] developed a theory of convex cocompact subgroups of mapping class groups that has been refined and

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explored over the years—see [27], [47], [33], [48], [50], [46], [55], [23], [24], [28], [64], [8], [60], and [80].

There are many equivalent definitions of convex cocompactness, see [47], [48], and [8] for a few, but the simplest to state is that sending the subgroup of $\mathrm{Mod}(S)$ to its orbit in the curve complex of S is a quasiisometric embedding. This was proven equivalent to convex cocompactness by the authors [47] and, independently, by U. Hamenstädt [33].

In [26], Farb and Mosher constructed convex cocompact free groups and asked whether or not there are examples of convex cocompact subgroups of $\mathrm{Mod}(S)$ that are not virtually free.

Question 1 (Questions 1.7 and 1.9 of [26])**.** Are there convex cocompact subgroups of $\mathrm{Mod}(S)$ that are not virtually free? Are there infinite convex cocompact subgroups that are isomorphic to the fundamental group of a closed surface? Are the surface group extensions of such subgroups δ–hyperbolic?

Any subgroup G of $\mathrm{Mod}(S)$ gives rise to a surface group extension

$$1 \to \pi_1(S) \to \Gamma_G \to G \to 1$$

by pulling G back to $\mathrm{Mod}(\mathring{S})$ in the Birman exact sequence [9]

$$1 \to \pi_1(S) \to \mathrm{Mod}(\mathring{S}) \to \mathrm{Mod}(S) \to 1.$$

Combining the work of Farb, Mosher, and Hamenstädt [26, 33] produces the following theorem characterizing the hyperbolic extensions.

Theorem (Farb–Mosher [26], Hamenstädt [33])**.** *Suppose that S is closed. Then the extension Γ_G is δ–hyperbolic if and only if G is a convex cocompact subgroup of* $\mathrm{Mod}(S)$. □

When G is a cyclic subgroup generated by a pseudo-Anosov mapping class g, the extension Γ_G is the fundamental group of a hyperbolic 3–manifold, by Thurston's Hyperbolization theorem for fibered 3–manifolds [79, 71]. It is then natural to wonder if there are closed surface bundles over surfaces that admit a hyperbolic structure as well.

Question 2. Is there a compact surface bundle over a surface that admits a hyperbolic metric?

Conjecturally, the answer to the first question is no, as it would violate certain conjectures about the Seiberg–Witten invariants of hyperbolic manifolds—see [72]. However, a negative answer would not forbid surface bundles over surfaces from having δ–hyperbolic fundamental groups, and so we focus our attention on Question 1.[◦] Note that convex cocompactness implies that the subgroup is virtually purely pseudo-Anosov [26], and so, implicit in Question 1 is the following question.

Question 3. Is there a purely pseudo-Anosov closed surface group in some $\mathrm{Mod}(S)$?

We give the first examples of such groups.

Theorem 1 (Purely pseudo-Anosov surface groups)**.** *There are infinitely many commensurability classes of purely pseudo-Anosov surface subgroups of* $\mathrm{Mod}(S_{g,0})$ *for all* $g \geq 4$.

We obtain Theorem 1 as a corollary of the following theorem, appealing to a branched cover construction.

Theorem 2 (Type preserving figure eight)**.** *There is a type–preserving representation of the fundamental group* $\pi_1(M_8)$ *of the figure–eight knot complement to the mapping class group of the thrice–punctured torus.*

Here a *type–preserving* representation is one that takes peripheral elements (that are parabolic in the associated Kleinian group) to reducible mapping classes and hyperbolic elements to pseudo-Anosov mapping classes. Viewing reducible elements of the mapping class groups in analogy with parabolic elements gives us a rather weak notion of type–preservation, as reducible elements may have positive infimal translation length. A stronger version of type–preservation might require that the parabolic elements pass to elements of zero translation length, which are, in a sense, the true parabolic elements in the isometry group of Teichmüller space. We note here that, while our representation does not have this stronger property, its restriction to the fiber subgroup does.

The abundance of purely hyperbolic surface subgroups of $\pi_1(M_8)$ then yields Theorem 1, which in turn yields the following theorem.

Theorem 3 (Atoroidal surface bundles)**.** *There is a closed* 4*–manifold* E *with no* $\mathbb{Z}^2$ *subgroups in* $\pi_1(E)$ *that fibers as a surface bundle over a surface* $S \to E \to B$.

[◦] It *is* known that such a bundle cannot be a complex hyperbolic surface [45].

We expect that our surface subgroups are convex cocompact, and plan to take up that topic in a subsequent paper.

In light of Theorem 2, it is natural to ask which Kleinian groups admit such type-preserving representations.

Question 4. Which hyperbolic n–manifold groups admit type-preserving representations into a mapping class group?

One of the original motivations for the study of convex cocompactness in mapping class groups was to approach the question of M. Gromov as to whether groups $\mathscr{G}$ with finite $K(\mathscr{G},1)$s are hyperbolic if and only if they contain no Baumslag–Solitar groups, as the extension Γ_G is hyperbolic if and only if G is convex cocompact, and has no Baumslag–Solitar subgroups if and only if G is purely pseudo-Anosov. This question has been remarkably shown to have a negative answer by G. Italiano, B. Martelli, and M. Migliorini [39], but whether or not there are counterexamples of the form Γ_G remains open. At any rate, we have the following corollary of our work here.

Corollary 4. *There is either a δ–hyperbolic surface-by-surface group or a non-hyperbolic surface-by-surface group that has no Baumslag–Solitar subgroups.*

Proof. See [46]. □

In the Kleinian setting, it is a theorem of Thurston [16, Theorem 5.2.18] that, given a finitely generated Kleinian group Γ, and a number $\chi < 0$, there are only finitely many conjugacy classes of quasifuchsian surface subgroups of Γ whose Euler characteristic is at least χ. B. Bowditch proved [13] the analogous statement for purely pseudo-Anosov surface subgroups of mapping class groups.

Theorem 5 (Bowditch [13])**.** *Given a number $\chi < 0$, there are only finitely many conjugacy classes of purely pseudo-Anosov subgroups isomorphic to $\pi_1(\Sigma)$, where Σ is a closed surface of Euler characteristic at least χ.*

Our main theorem provides a lower bound on the number of these conjugacy classes.

Theorem 6. *The number of commensurability classes of purely pseudo-Anosov subgroups of $\mathrm{Mod}(S_{g,0})$ and $\mathrm{Mod}(S_{1,3})$ that are isomorphic to the fundamental group of a surface of genus at most h is bounded below by a strictly increasing linear function of h.*

This estimate is likely a *lot* smaller than the number of all convex cocompact quasifuchsian subgroups of $\pi_1(M_8)$ of genus at most h, and one would expect the number to grow exponentially, if not super exponentially, in h, as is the case in the closed setting, where Kahn and Marković [41] show that there is a constant c depending only on the injectivity radius so that the number of commensurability classes of incompressible surfaces of genus h in a hyperbolic 3–manifold is at least $(ch)^{2h}$.

1.1 Historical notes and other approaches

1.1.1 Historical comments

As far as we are aware, the first mentions of Questions 2 and 3 in the literature occur in the 1990s, though we expect that Question 3 has been around longer than that. For example, Misha Kapovich [44] attributes Question 2 to Geoff Mess in 1991, and noted that Question 3 arose naturally at the same time.

Question 2 appears as Question 15 of [45], published 1998, and was put to W. J. Harvey in the 1990s by L. Potyagailo [31]—see also Question 4.1 of [72].

Question 3 was also put to Harvey by Potyagailo [31] sometime in the nineties, and variants were asked by Mosher [68] in 1997 and Kapovich [45] in 1998, who both asked if there are purely pseudo-Anosov groups that are not free. See also Questions 1.1 of [72], Problem 4.1 of [69], and Question 1.3 of [59]. Kapovich also asked [45] if there are δ–hyperbolic surface-by-surface groups, while Mosher [68] and M. Mj [67] asked if there are hyperbolic extensions $1 \to K \to \Gamma \to H \to 1$ where K is nonelementary and H is not free.

A natural question between Questions 1 and 2 (asked to us by D. Fisher) is whether there are surface bundles over surfaces that admit metrics of negative sectional curvature. While we could not find this exact question in the literature, Reznikov's 1993 paper [73], particularly his Corollary F.3, makes it clear that this question has also been around for a while.

A variant of Question 4 was posed by Reid (see Question 4.10 [72]). The type preserving assumption here is important, as M. Bridson [14] showed that many hyperbolic n–manifold groups embed into mapping class groups, including *all* hyperbolic 3–manifold groups, and, more generally, all virtually special groups.

1.1.2 Other surface groups

Representations of surface groups into mapping class groups arise naturally as monodromies of families of Riemann surfaces, which arise naturally in topology and algebraic geometry. See [5], [37], [56], [31], [15], and [75], for example.

The *universal* example of a family is the universal curve, which assigns to each point of the moduli space the corresponding algebraic curve. This family is a quotient of the Bers fibration over Teichmüller space [7], which is naturally identified with the Teichmüller space of the one punctured surface. Furthermore, the action of $\pi_1(S)$ on the fiber gives a faithful representation to the mapping class group of the once–punctured surface whose image is the Birman kernel [9]. By a theorem of I. Kra [57], an element of $\pi_1(S)$ is pseudo-Anosov under this representation if and only if it is a filling loop, and so these surface groups cannot be purely pseudo-Anosov.

Even more examples arise from the universal curve, for if you take a pseudo-Anosov $g\colon S \to S$, the fundamental group of the mapping torus injects into the mapping class group of the once-punctured surface. By [66], [6], [42], [43], and [18], these mapping tori contain many immersed incompressible surfaces, and we obtain many surface groups in mapping class groups. I. Agol has observed [2] that the only reducible elements arising from these representations must lie in the fiber subgroup, where being pseudo-Anosov is equivalent to being filling [57]. As shown in [23], any purely pseudo-Anosov, finitely generated subgroup arising in this way will be convex cocompact. A similar convex cocompactness statement holds when g is reducible [60], though all purely pseudo-Anosov subgroups are free in this case. See [49] for a discussion of this approach and a geometric criterion to be pseudo-Anosov that serves as an alternative to Kra's [49] in this setting.

In addition to the surfaces guaranteed by [66], [6], [42], [43], and [18], there are the "cut-and-cross–join" surfaces of Cooper, Long, and Reid [19], but it is worth noting that, while it is conceivable that there is a cut-and-cross–join surface that is purely pseudo-Anosov, one must be careful to distinguish between the geometrically finite and geometrically infinite cases, as the latter cannot be purely pseudo-Anosov—they are virtual fibers and must contain the commutator subgroup, which contains many reducible elements.

It seems likely that many of these quasi-Fuchsian surface subgroups pass over to purely pseudo-Anosov surface groups, but as of writing there are no known examples arising this way.

Prior to Theorem 1, the record for the "fewest" reducible elements of a surface subgroup of a mapping class group was given by the combination theorem of Leininger and A. Reid [59], where it is shown that one may combine Veech groups along parabolic subgroups to obtain closed surface groups in mapping class groups whose reducible elements are all conjugate into a single cyclic subgroup.

Another approach to construct purely pseudo-Anosov surface subgroups was given by Agol (see [72, Section 4.3]) who observed that certain small complexity compactified moduli spaces are in fact complex hyperbolic orbifolds, and that one may attempt to construct purely pseudo-Anosov surface subgroups by finding geodesic surfaces dodging the singularities.

Embeddings of right-angled Artin groups into mapping class groups (considered in [20], [17], [54], [74], and [76]) provide another source of surface subgroups. Many of these groups (as well as more general Artin and Coxeter groups) contain surface subgroups. See, for example [78], [1], [32], [22], [21], [52], and [51]. In [22], J. Crisp and B. Wiest construct undistorted surface subgroups of right-angled Artin groups, which when applied to the embeddings constructed by M. Clay, Leininger, and J. Mangahas in [17] (and extended by I. Runnels in [74] and D. Seo in [76]) produce undistorted surface subgroups of the mapping class group. On the other hand, it was also observed in [17] that a surface subgroup that factors through an embedding of a right-angled Artin group is never purely pseudo-Anosov.

The authors's [47] and Hamenstädt's [33] characterization of convex cocompactness implies that a convex cocompact surface group gives rise to an equivariant quasiisometric embedding of the hyperbolic plane into the curve complex. Such a quasiisometric embedding extends continuously to an embedding of the circle at infinity into the Gromov boundary of the curve complex, which may be identified with the space of ending laminations [53]. This provides some potential obstructions to the existence of convex cocompact surface groups. Path connectivity of the boundary of the curve complex was proved for most surfaces by Gabai [29] (see also [58], [35], [30], and [84]), effectively ruling out this obstruction. Leininger and S. Schleimer [61] further proved the existence of quasiisometrically embedded hyperbolic planes (in fact, hyperbolic n–spaces for all n) in curve complexes, ruling out that obstruction as well.

At the 2007 Cornell Topology Festival, W. Thurston proposed yet another approach to the second author, similar in spirit to the constructions in [5], [56], and [31]. Thurston suggested finding a "sufficiently complicated" multisection of the trivial bundle $S \times S$, which produces a map of the base into the configuration space of several points on S, and hence a representation of the fundamental group into the mapping class group. While this is not too far from the ideas of the authors and Wright discussed below, the authors could never crystallize Thurston's ideas into a theorem.

Finally we note that one might hope to make the Kahn–Marković/Kahn–Wright strategy work in the moduli space, and that Kahn and Wright's work [43] was motivated by this.

1.2 Sketch of the proof of Theorem 2

Wright has posed [85] the following question, which arose in relation to conversations between him and Kahn.

Question 5 (Wright [85])**.** Does there exist a fixed–point-free homeomorphism $f: S \to S$ of a closed hyperbolic surface S with the property that every essential simple closed curve γ on S fills with its image $f(\gamma)$?

Wright's motivation for asking this question was his observation that, if there is such a homeomorphism, then we have a π_1–injective map $x \mapsto \{x, f(x)\}$ of S into the configuration space of two points on S, and, moreover, that the image of the fundamental group would be purely pseudo-Anosov in the fundamental group of the configuration space (viewed as a subgroup of the mapping class group of the twice–punctured S), via an analogue of Kra's Theorem [57] due to Imayoshi, Ito, and Yamamoto [38]. See Lemma 9 below.

In [5] and [56], M. Atiyah and K. Kodaira do something similar, though rather than requiring that curves fill with their image, the homeomorphism is taken to be holomorphic—see also G. González-Díez and Harvey [31]. Note that a mapping class represented by a holomorphic map is necessarily of finite order, and that, conversely, any finite order mapping class may be realized by a holomorphic map (in fact an isometry of some hyperbolic metric), by J. Nielsen's realization theorem [70]. It is conceivable that an irreducible periodic mapping class could have the second property in Wright's question, taking essential curves to mutually filling curves, and, by Nielsen realization, this would yield a *holomorphic* atoroidal surface bundle over a surface. Furthermore, since the moduli space of curves is a quasiprojective variety, the map from the base to the moduli space could be taken to be algebraic—by the GAGA principle [77]—and thus as in Atiyah's and Kodaira's construction, the bundle would be a complex algebraic family.

To illustrate the difficulty of Wright's question, note that T. Aougab, Futer, and S. Taylor [3] have recently shown that the number of fixed points of a pseudo-Anosov homeomorphism is coarsely bounded below by its curve complex translation length, which must be at least three if essential curves fill with their image.

One of our key innovations here is our observation that fixed–point free homeomorphisms of surfaces not only give us representations of the associated surface groups into mapping class groups, but representations of the fundamental groups of the mapping tori—see Corollary 13. This implies that if Wright's question has an affirmative answer for some pseudo-Anosov f, the resulting purely pseudo-Anosov surface subgroup would have infinite index in its normalizer, forbidding it from being convex cocompact.[†]

While we are unable to answer Wright's question, we do have available a fixed–point-free homeomorphism f of a *noncompact* surface S with the property that every essential loop fills with its image. This f is the homeomorphism of the once–punctured torus induced by the action of the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

on $\mathbb{R}^2$, which happens to be the monodromy of the fibration of the figure–eight knot complement. The argument works just as well for any affine homeomorphism of trace 3 and determinant 1 or trace 1 and determinant -1.

The point $z = 0.2[1,2]^T$ is fixed by f^2, and is a natural choice for the basepoint of $\pi_1(S)$. The representation Δ_f takes $\pi_1(M_f, z)$ into the mapping class group of the thrice punctured torus $T^2 - \{\mathbf{0}, z, f(z)\}$, and the image of $\pi_1(M_{f^2}, z)$ lies in the pure mapping class group.

A simple check shows that every essential loop in $S = T^2 - \mathbf{0}$ fills with its image under f. If γ is an essential *simple* closed curve, it represents a nontrivial homology class, and γ and $f\gamma$ fill since the matrix above is hyperbolic. This argument also shows that γ and $f^{-1}\gamma$ fill. In any case, if γ and $f\gamma$ don't fill, then, after some isotopies, there is an essential simple closed curve β that lies in their complement. But since γ *and* $f\gamma$ are disjoint from β, we have that γ is disjoint from β *and* $f^{-1}(\beta)$. Since γ is essential, this means that β and $f^{-1}(\beta)$ *don't* fill, contradicting the first case. It follows that every essential (nonperipheral) element of the fundamental group of S is sent to a pseudo-Anosov mapping class of the thrice–punctured torus.

Then, if we have a reducible element R in $\Delta_f(\pi_1(M_{f^2}))$ with nontrivial exponent sum in f, the reducing system in the thrice–punctured torus must contain a curve bounding a disk in T^2, since f^2 is Anosov on T^2, and this disk must contain at least two of our three punctures $\mathbf{0}$, z, and fz. This disk produces a solid torus in

[†] Wright was also aware of this normalizing behavior and its implication for non-convex cocompactness.

the mapping torus of the extension $\overline{f}$ of f to the closed torus T^2 (the Sol–manifold associated to the matrix). It follows that we may fill in one of the three punctures so that R descends to a reducible mapping class.

If we forget z or fz, the resulting representation of $\pi_1(M_{f^2})$ is the usual representation into the pure mapping class group of the twice–punctured torus given by the Birman exact sequence, and any reducible element here with nontrivial exponent sum in f must be peripheral.

If we forget $\mathbf{0}$, we obtain another representation of $\pi_1(M_{f^2})$ to the mapping class group of the twice–punctured torus, but this time it is less clear that we have an isomorphism onto a nice image. However, the additive group structure on T^2 implies that any two–strand pure braid on the torus is level isotopic to a one–strand braid on the punctured torus, and so this new representation *also* takes $\pi_1(M_{f^2})$ into the image of the usual representation. A calculation reveals that this representation is injective, and hence an isomorphism. Again we conclude that R must descend to a peripheral element, and since isomorphisms of hyperbolic 3–manifold groups are type preserving, we conclude that R was peripheral to begin with, and this completes the proof that Δ_f is type–preserving.

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2 Birman's Exact Sequence

Suppose S is a compact orientable surface punctured at a finite set of points. If $X \subset S$ is any finite subset, we write S_X to denote $S - X$, the surface S punctured at the additional set of points X. Any homeomorphism $h\colon S \to S$ with $h(X) = X$

restricts to a homeomorphism $h_X\colon S_X \to S_X$. Conversely, any homeomorphism $S_X \to S_X$ that preserves the punctures associated to X—the X*–punctures*—arises in this way. If $X = \{x\}$, we write $S_x = S_X$ and $h_x = h_X$. We write $[\![h_X]\!]$ for the isotopy class of h_X.

Given S and a (possibly empty) finite subset $X \subset S$ as above, we write $\mathrm{Mod}(S_X)$ for the mapping class group of S_X, the group of orientation preserving homeomorphisms of S_X up to isotopy, and

$$\mathrm{Mod}(S_X, X) = \{[\![h_X]\!] \mid \text{with } h(x) = x \text{ for all } x \in X\}$$

for the mapping classes fixing each of the X–punctures. The assignment $h_X \mapsto h$ descends to a homomorphism

$$\rho_X\colon \mathrm{Mod}(S_X, X) \to \mathrm{Mod}(S),$$

for if h_X and h'_X are isotopic, so are h and h'. The homomorphism ρ_X "forgets" (or "fills in") the X punctures.

If $h_X\colon S_X \to S_X$ represents an element of $\ker(\rho_X)$, then there is an isotopy $H\colon S \times [0,1] \to S$ from h to the identity $\mathbb{1}_S$. We write $h^t = H(\cdot, t)\colon S \to S$ for this 1–parameter family of homeomorphisms so that $h^0 = h$ and $h^1 = \mathbb{1}_S$. In this case, $t \mapsto h^t(X)$ is a loop in $\mathrm{Conf}_n(S)$, the configuration space of $n = |X|$ ordered distinct points on S. Recall that this is defined to be the open subset of the n–fold product

$$\mathrm{Conf}_n(S) = S \times S \times \cdots \times S - \left\{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j\right\}.$$

The loop $t \mapsto h^t(X)$ represents an element of $\mathrm{P}\mathscr{B}_n(S) = \pi_1(\mathrm{Conf}_n(S), X)$, the n–strand pure braid group on S (here we choose any ordering on X, and also use X to denote the n–tuple of its points that this ordering defines). Birman proved that for most surfaces S, this braid is uniquely determined by the element of the kernel—see Chapter 4 of [9], [7], and Theorem 9.1 of [25]

Theorem 7 (Birman Exact Sequence)**.** *If $\chi(S) < 0$ and $X \subset S$ is a set of n points, then there is an exact sequence*

$$1 \to \mathrm{P}\mathscr{B}_n(S) \to \mathrm{Mod}(S_X, X) \xrightarrow{\rho_X} \mathrm{Mod}(S) \to 1,$$

where the identification of an element of the kernel of ρ_X with $\mathrm{P}\mathscr{B}_n(S)$ is as described above.

Given a braid in $\mathrm{P}\mathscr{B}_n(S) = \pi_1(\mathrm{Conf}_n(S), X)$ represented by a loop γ in $\mathrm{Conf}_n(S)$ based at X, we may write $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, where $\gamma_1, \ldots, \gamma_n$ are paths in S with

$$(\gamma_1(0), \ldots, \gamma_n(0)) = X = (\gamma_1(1), \ldots, \gamma_n(1)).$$

The Isotopy Extension Theorem [36, Chapter 8, Theorem 1.3] implies that there is an isotopy $g^t \colon S \to S$ with $g^0 = \mathbb{1}_S$ and $g^t(X) = \gamma(t)$ for all $t \in [0,1]$. We think of g^t as "point pushing" X along γ. The reversed isotopy $h^t = g^{1-t} \colon S \to S$ is an isotopy from $h^0 = g^1 \colon S \to S$ to the identity $h^1 = \mathbb{1}_S$, so h^t point pushes X along $\overline{\gamma}$, the path with the reversed orientation. This identifies the braid $[\gamma]$ with the mapping class in $\mathrm{Mod}(S_X, X)$ obtained from $\mathbb{1}_S$ by point pushing *backwards* along γ and restricting to S_X.

The pure mapping class group $\mathrm{PMod}(S)$ of S is the subgroup of $\mathrm{Mod}(S)$ consisting of elements represented by homeomorphisms that fix each puncture. If S is closed and $Y \subset X \subset S$ then $\mathrm{PMod}(S_X) < \mathrm{Mod}(S_X, Y)$, and we can forget Y, defining a (sub) short exact sequence (when $\chi(S_{X-Y}) < 0$).

$$1 \to \mathrm{P}\mathscr{B}_n(S_{X-Y}) \to \mathrm{PMod}(S_X) \xrightarrow{\rho_Y} \mathrm{PMod}(S_{X-Y}) \to 1.$$

For $z \in S$ and $\chi(S) < 0$, another important instance of the exact sequence above is

$$1 \to \pi_1(S, z) \to \mathrm{PMod}(S_z) \to \mathrm{PMod}(S) \to 1,$$

since the 1–strand braid group is just the fundamental group of S.

2.1 Closed braids in mapping tori

Given a homeomorphism $h \colon S \to S$, the mapping torus M_h is the quotient

$$M_h = S \times [0,1]/(x,0) \sim (h(x),1).$$

We write $\mathsf{q} \colon S \times [0,1] \to M_h$ for the quotient map. The fundamental group $\pi_1(M_h)$ is a semidirect product

$$\pi_1(M_h) \cong \pi_1(S) \rtimes \mathbb{Z},$$

where the stable letter acts by h_* on $\pi_1(S)$. We describe this explicitly when h fixes a basepoint z in S, as we will make use of this throughout. With such an

h, the induced map $h_*\colon \pi_1(S,z) \to \pi_1(S,z)$ is a well-defined automorphism (not just an outer automorphism). We then let $\mathfrak{q}\colon S\times[0,1] \to M_h$ denote the quotient map and consider the inclusion $\iota\colon S \to M_h$ by $\iota(x) = \mathfrak{q}(x,\frac{1}{2})$. We consider z as a basepoint for both S and M_h by identifying z with $\iota(z)$. We further write $\gamma = \iota\circ\gamma$, and identify $\pi_1(S,z)$ with its image in $\pi_1(M_h,z)$ by ι_*. The loop $\tau\colon [0,1] \to M_h$ defined by $\tau(t) = \mathfrak{q}(z,t+\frac{1}{2})$ (modulo 1) then represents the stable letter in the semidirect product, and, for all γ in $\pi_1(S,z)$, we have

$$\tau\gamma\tau^{-1} = h_*(\gamma).$$

Puncturing S at z, we may then view $M_{h_{\{z\}}} \subset M_h$, where $M_{h_{\{z\}}}$ is obtained from M_h by deleting the image of τ.

If a finite set $X \subset S$ is preserved by a homeomorphism $h\colon S\to S$, then we can view $M_{h_X} = M_h - \mathscr{L}$, where $\mathscr{L} = \mathfrak{p}(X\times[0,1])$.

When h_X represents an element of the kernel of ρ_X, then the isotopy of $h = h^0$ to the identity $h^1 = \mathbb{1}_S$ defines a homeomorphism $H\colon S\times[0,1] \to S\times[0,1]$ given by $H(y,t) = (h^t(y),t)$. This descends to a homeomorphism

$$\hat{H}\colon M_h \to M_{\mathbb{1}_S} \cong S\times S^1.$$

To see this, observe that points $(y,0)$ and $(h(y),1)$ identified in the domain are mapped by H to $H(y,0) = (h(y),0)$ and $H(h(y),1) = (h(y),1)$, which are identified in the range. The image $\mathscr{L}_0 = H(\mathscr{L}) \subset S\times S^1$ of $\mathscr{L}$ is a link in "closed braid form" in the product, being transverse to the fibers $S\times\{*\}$ of the product structure. This allows us to view $M_{h_X} = S\times S^1 - \mathscr{L}_0$.

More generally, if $f\colon S\to S$ is *any* homeomorphism, an isotopic homeomorphism can be expressed as $fh\colon S\to S$, where $h\colon S\to S$ is isotopic to the identity. The isotopy h^t of $h = h^0$ to the identity similarly defines a homeomorphism $\hat{H}_f\colon M_{fh}\to M_f$. In fact, $\hat{H}_f$ is the descent of the exact same homeomorphism $H\colon S\times[0,1]\to S\times[0,1]$ above, since $H(x,0) = (h(x),0)$ and $H(fh(x),1) = (fh(x),1)$. If $X\subset S$ is a finite set preserved by both homeomorphisms h and f, then we have $(fh)_X = f_X h_X\colon S_X\to S_X$, and the mapping torus of $M_{(fh)_X}$ can be viewed as the complement of a link $\mathscr{L}\subset M_f$, depending on h and X, which is transverse to the S–fibers.

We return to the case of a homeomorphism $f\colon S\to S$ fixing a point z. When the mapping class of f has infinite order in $\mathrm{Mod}(S)$, the short exact sequence from the semidirect product embeds into the Birman Exact Sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S,z) & \longrightarrow & \pi_1(M_f,z) & \longrightarrow & \langle\tau\rangle & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(S,z) & \longrightarrow & \mathrm{Mod}(S_z,z) & \longrightarrow & \mathrm{Mod}(S) & \longrightarrow & 1 \end{array} \tag{2.1}$$

The vertical homomorphism $\langle\tau\rangle \to \mathrm{Mod}(S)$ sends τ to the mapping class of f and the image of $\pi_1(M_f,z)$ is precisely the preimage $\rho_z^{-1}(\langle f\rangle)$, with the loop τ in M_f sent to the mapping class f_z. We think of f_z as the first return map to $S_z = \iota(S_z)$ of the "downward flow" on M_f, that is, the flow generated by the vector field $-\frac{\partial}{\partial t}$ on $S\times[0,1]$ projected to M_f: When S_z reaches $S_z\times\{0\}$, it is identified with $S_z\times\{1\}$ via f_z on the first factor, and then continues to flow down until it reaches $S_z = \iota(S_z)$.

If we change f by an isotopy to fh, where h also fixes z and is isotopic to the identity, then we have $(fh)_* = f_*h_*$. On the other hand, the isotopy h^t traces out the loop $\beta(t) = h^t(z)$, and so h_* is the inner automorphism i_β obtained by conjugating by β. Flowing downward in M_{fh}, the first return map to S_z is fh. We can see the loop in M_f by applying the homeomorphism $\hat{H}_f$ above, but because the basepoint z is not preserved, it is more convenient to adjust it as follows. Define $G\colon S\times[0,1]\to S\times[0,1]$ by

$$G(x,t) = \begin{cases} (h^0(x),t) & \text{for } 0\le t\le \frac{1}{4} \\ (h^{4t-1}(x),t) & \text{for } \frac{1}{4}\le t\le \frac{1}{2} \\ (x,t) & \text{for } \frac{1}{2}\le t\le 1 \end{cases}$$

This homeomorphism G agrees with H on $S\times\{0\}$ and $S\times\{1\}$, so it still descends to a homomorphism $\hat{G}_f\colon M_{fh}\to M_f$, but we have "combed the interesting part" of the map into the subspace $S\times[\frac{1}{4},\frac{1}{2}]$ in the range. Conjugating the downward flow on M_{fh} to a flow on M_f via $\hat{G}_f$, we see that the map pushes $z=\iota(z)$ backwards (and down), first along β, and then backwards along (the rest of) τ. See Figure 2.1. Thus we see the loop $t\mapsto \hat{G}_f(\mathfrak{q}(z,t+\frac{1}{2}))$ (modulo 1), represents the same element of $\pi_1(M_f)$ as $\tau\beta$. We record this in the following lemma.

Lemma 8. *Suppose that z is a point in S fixed by $f\colon S\to S$ and $h\colon S\to S$, and that there is an isotopy h^t from $h=h^0$ to the identity $h^1=\mathbb{1}_S$. Let τ and β be the paths defined by $\tau(t)=\mathfrak{q}(z,t+\frac{1}{2})$ (modulo 1) and $\beta(t)=h^t(z)$, for $t\in[0,1]$. Then*

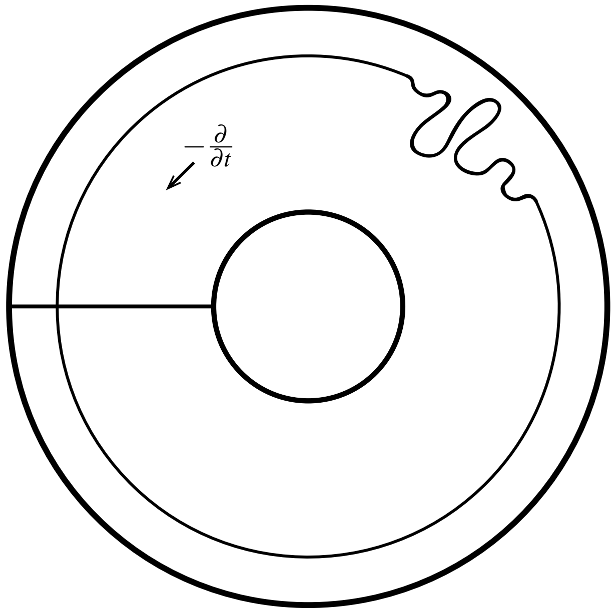


Figure 1: The flow.

$t \mapsto \mathfrak{q}(h^t(z),t)$ is a loop in M_f in the conjugacy class of $[\tau\beta]$ in $\pi_1(M_f,z)$, up to basepoint change isomorphism. Furthermore, this element represents $[\![(fh)_z]\!] = [\![f_z h_z]\!]$ in $\mathrm{Mod}(S_z,z)$, up to conjugacy.

Proof. The loop in the lemma is homotopic to the one described in the paragraph preceding the statement of the lemma: it is straight forward to construct an isotopy from H to G from their formulas. This is not a basepoint preserving homotopy (indeed, the loop from the lemma need not be based at z), but the homotopy ensures that the two loops necessarily represent conjugate elements via a basepoint change isomorphism. The loop defined by G was already noted to represent $[\tau\beta]$ in $\pi_1(M_f,z)$ and $[\![(fh)_z]\!]$ in $\mathrm{Mod}(S_z,z)$, and this completes the proof. □

3 The dancing representation

We continue to assume that S is the compact orientable surface punctured at a finite set of points. Let f be any fixed–point-free homeomorphism of S. Then there is a map $D\colon S \to \mathrm{Conf}_2 S$ given by

$$D(x) = (x, f(x))$$

and so a representation

$$D_*\colon \pi_1(S,z) \to \pi_1(\mathrm{Conf}_2 S, (z, f(z))) = \mathrm{P}\mathscr{B}_2(S).$$

The inclusion from the Birman Exact Sequence allows us to view this as a homomorphism

$$\Delta_f\colon\ \pi_1(S,z) \to \mathrm{Mod}(S_{\{z,f(z)\}},\{z,f(z)\}).$$

Given a loop β, there is an isotopy $h^t\colon S \to S$ from $h^0\colon S \to S$ to the identity $h^1 = \mathbb{1}_S$ with $h^t(z) = \beta(t)$ and $h^t(f(z))) = f(\beta(t))$. More concretely, we can view h^t as pushing z and $f(z)$ along β and $f(\beta)$ simultaneously, and then $\Delta_f(\beta) = h^0_{\{z,f(z)\}}$. The next lemma was Wright's motivation for asking Question 5.

Lemma 9 (A. Wright)**.** *Suppose $f\colon S \to S$ is a fixed point free homeomorphism. Then if $\beta \in \pi_1(S,z)$ and β and $f(\beta)$ are distinct homotopy classes of curves that fill S, then $\Delta_f(\beta)$ is pseudo-Anosov.*

Proof. The main result of [38] completely describes the Nielsen–Thurston type of a two–strand braid on any orientable finite–type surface, and may be applied in our setting to prove the lemma. As the full statement is somewhat technical, we give a self–contained proof for the reader's convenience.

Suppose to the contrary that $\Delta_f(\beta)$ is reducible. Since a pure braid is pure as a mapping class (in the sense of [40]), we may assume that a homeomorphism h representing $\Delta_f(\beta)$ fixes the isotopy class of a simple closed curve γ in $S_{\{z,f(z)\}}$. As explained in §2.1, there is a link $\mathscr{L}_\beta \subset S \times S^1$, transverse to the S–fibers so that $S \times S^1 - \mathscr{L}_\beta \cong M_h$. Projecting $S \times S^1$ onto the first factor, $\mathscr{L}_\beta$ projects to $\beta \cup f\beta$.

The fixed curve γ defines an essential torus $T \subset S \times S^1 - \mathscr{L}_\beta$ meeting each S–fiber in a simple closed curve. Assume first that γ is homotopically nontrivial in S. Then work of Waldhausen [82] implies that, after an isotopy preserving the S–fibers, we may assume $T = \gamma \times S^1$. The isotopy replaces $\mathscr{L}_\beta$ with an isotopic link $\mathscr{L}'_\beta$. Projecting $\mathscr{L}'_\beta$ onto the first factor of $S \times S^1$ produces the union of two curves $\beta' \cup \beta''$ homotopic to $\beta \cup f(\beta)$. In particular, $\beta' \cup \beta''$ is disjoint from the projection of T, which is γ. However, $\beta \cup f(\beta)$ is assumed to fill, hence so does $\beta' \cup \beta''$, and so the homotopically nontrivial curve γ on S must be peripheral. Therefore, γ bounds a once-punctured disk $B \subset S$, and T bounds $B \times S^1$ in $S \times S^1$. Since T is essential in its complement, the link $\mathscr{L}'_\beta$ must nontrivially intersect $B \times S^1$, and hence at least one of β' or β'' must project into B. If β' projects into B, then β', and hence β, is peripheral, which then implies that $f\beta$ is peripheral as well. No such pair of curves can fill S, and from this contradiction we conclude that either γ is homotopically trivial in S, or else Δ_f is pseudo-Anosov (and in the latter case, we're done, so we assume the former). Similarly, if β'' projects into

B, we have that $f\beta$, and hence also β, is peripheral, again concluding that γ is homotopically trivial.

Now, since γ is homotopically trivial, T bounds a solid torus $V \subset S \times S^1$ containing $\mathscr{L}_\beta$. Since $\Delta_f(\beta)$ is a pure braid, both components of $\mathscr{L}_\beta$ are homotopic to a core of V. In particular, they are homotopic to each other. Projecting this homotopy to S determines a homotopy from β to $f\beta$, another contradiction. This exhausts all possibilities assuming $\Delta_f(\beta)$ were reducible. Therefore, $\Delta_f(\beta)$ is pseudo-Anosov, as required. $\square$

When convenient, we will write $\Delta_f(\beta) = (\beta, f\beta)$ when given a loop β in $\pi_1(S,z)$, which makes sense as an element of $\mathrm{P}\mathscr{B}_2(S) = \pi_1(\mathrm{Conf}_2(S),(z,f(z)))$.

3.1 The configuration space bundle

Continue to assume that $f\colon S \to S$ is a fixed point free homeomorphism. Letting $\hat{f}\colon \mathrm{Conf}_2(S) \to \mathrm{Conf}_2(S)$ denote the homeomorphism $\hat{f}(x,y) = (f(x), f(y))$, we construct the mapping torus $M_{\hat{f}}$ of $\hat{f}$, which is a $\mathrm{Conf}_2(S)$–bundle over the circle. The embedding $D(x) = (x, f(x))$ above defines a embedding

$$D \times \mathbb{1}_{[0,1]}\colon S \times [0,1] \to \mathrm{Conf}_2(S) \times [0,1]$$

that descends to an embedding

$$\overline{D}\colon M_f \to M_{\hat{f}},$$

since $D \circ f = \hat{f} \circ D$. Write $\Pi\colon \mathrm{Conf}_2(S) \to S$ for the projection $\Pi(x,y) = x$. Since $\Pi \circ \hat{f} = f \circ \Pi$, we have that $\Pi \times \mathbb{1}_{[0,1]}$ descends to a map

$$\overline{\Pi}\colon M_{\hat{f}} \to M_f.$$

Lemma 10. *The composition $D \circ \Pi\colon \mathrm{Conf}_2(S) \to \mathrm{Conf}_2(S)$ is a retraction onto $D(S)$. Similarly, $\overline{D} \circ \overline{\Pi}\colon M_{\hat{f}} \to M_{\hat{f}}$ is a retraction onto the image $\overline{D}(M_f)$.*

Proof. The first claim is just the observation $D \circ \Pi(x,y) = (x, f(x))$. The second follows from the first, and the definition of $\overline{D}$ and $\overline{\Pi}$. $\square$

Corollary 11. *The induced map $\overline{D}_*\colon \pi_1(M_f) \to \pi_1(M_{\hat{f}})$ is injective.*

Picking a basepoint z, we have the isomorphism $f_*\colon \pi_1(S,z) \to \pi_1(S,f(z))$. Choosing a path δ from z to $f(z)$ in S, we write $\delta_*\colon \pi_1(S,f(z)) \to \pi_1(S,z)$ for the basepoint change isomorphism given by $\delta_*([\gamma]) = [\delta\gamma\overline{\delta}]$, and we denote the composition of these two isomorphisms

$$f_*^\delta = \delta_* f_*\colon \pi_1(S,z) \to \pi_1(S,z).$$

Viewing $z \in S \subset M_f$ as in §2.1, we have $\pi_1(M_f,z) \cong \pi_1(S,z) \rtimes \langle\sigma\rangle$ where the stable letter σ acts as f_*^δ. The image path $D(\delta) = (\delta, f\delta)$ in $\mathrm{Conf}_2(S)$, together with $\hat{f}$, similarly defines an isomorphism on $\mathrm{P}\mathscr{B}_2(S) = \pi_1(\mathrm{Conf}_2(S),(z,f(z)))$,

$$\hat{f}_*^\delta\colon \mathrm{P}\mathscr{B}_2(S) \to \mathrm{P}\mathscr{B}_2(S),$$

and we can thus write

$$\pi_1(M_{\hat{f}},(z,f(z))) \cong \mathrm{P}\mathscr{B}_2(S) \rtimes \left\langle \overline{D}_*(\sigma) \right\rangle,$$

so that $\overline{D}_*(\sigma)$ acts as $\hat{f}_*^\delta$, and the isomorphism $\overline{D}_*$ preserves the semidirect product structure, restricting to $D_*\colon \pi_1(S,z) \to \pi_1(\mathrm{Conf}_2(S),(z,f(z))) = \mathrm{P}\mathscr{B}_2(S)$ on the normal subgroup.

Proposition 12. *Suppose f is fixed point free and $[\![f]\!] \in \mathrm{Mod}(S)$ has infinite order. Then the short exact sequence coming from the semidirect product structure on $\pi_1(M_{\hat{f}})$ embeds into the Birman Exact Sequence*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{P}\mathscr{B}_2(S) & \longrightarrow & \pi_1(M_{\hat{f}},(z,f(z))) & \longrightarrow & \left\langle \overline{D}_*(\sigma) \right\rangle & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{P}\mathscr{B}_2(S) & \longrightarrow & \mathrm{Mod}(S_{\{z,f(z)\}},\{z,f(z)\}) & \xrightarrow{\rho_{\{z,f(z)\}}} & \mathrm{Mod}(S) & \longrightarrow & 1.
\end{array}$$

Proof. Let $g\colon S \to S$ be a homeomorphism so that $g(f(z)) = z$ and $g(f^2(z)) = f(z)$, and so that g is isotopic to the identity by an isotopy g^t with $g^0 = g$, $g^1 = \mathbb{1}_S$, and so that $g^t(f(z)) = \delta(t)$ and $g^t(f^2(z)) = f\delta(t)$. Then

$$(g^t f, g^t f)\colon \mathrm{Conf}_2(S) \to \mathrm{Conf}_2(S),$$

defines an isotopy from (gf,gf) to (f,f) and $(g^t f(z), g^t f(f(z)) = (\delta(t), f\delta(t))$. Consequently, for any loop (α,β) in $\mathrm{Conf}_2(S)$ based at $(z,f(z))$, we have

$$(gf\alpha, gf\beta) \simeq \Big(\delta(f\alpha)\overline{\delta}, (f\delta)(f\beta)(f\overline{\delta})\Big),$$

as loops based at $(z, f(z))$. See e.g. [34, Lemma 1.19].

We now construct the embedding of short exact sequences. For this, we start by defining the homomorphism on the stable letter $\overline{D}_*(\sigma)$ in $\pi_1(M_{\hat{f}})$, sending it to the element $[\![(gf)_{\{z,f(z)\}}]\!]$. Any element of the kernel of $\rho_{\{z,f(z)\}}$ is represented by $h_{\{z,f(z)\}} \colon S_{\{z,f(z)\}} \to S_{\{z,f(z)\}}$, and we let h^t be an isotopy with $h^0 = h$ and $h^1 = \mathbb{1}_S$, so that $[\![h]\!]$ corresponds to the pure braid represented by the loop

$$t \mapsto (\alpha(t), \beta(t)) = (h^t(z), h^t(f(z)).$$

Conjugating $[\![h_{\{z,f(z)\}}]\!]$ by $[\![(gf)_{\{z,f(z)\}}]\!]$ defines another element of the kernel of $\rho_{\{z,f(z)\}}$ and the associated braid is represented by the loop

$$\begin{aligned} t \mapsto\ & (gf)h^t(gf)^{-1}(z, f(z)) \\ &= gfh^t(z, f(z)) \\ &= (gf(\alpha(t)), gf(\beta(t))) \\ &\simeq \Big(\big(\delta(f\alpha)\overline{\delta}\big)(t), \big((f\delta)(f\beta)(f(\overline{\delta}))\big)(t) \Big), \end{aligned}$$

as explained above. It follows that the image of $\overline{D}_*(\sigma)$ conjugates the image of (α, β) to the image of the $\overline{D}_*(\sigma)$ conjugate of (α, β), and thus we have a well-defined homomorphism from $\pi_1(M_{\hat{f}}, (z, f(z)))$ to $\mathrm{Mod}(S_{\{z,f(z)\}}, \{z, f(z)\})$ that is the “identity" on $\mathrm{P}\mathscr{B}_2(S)$. Sending the quotient $\langle \overline{D}_*(\sigma) \rangle$ to $\langle [\![f]\!] \rangle$ then completes the embedding of the short exact sequence. $\square$

We write $\Delta_f \colon \pi_1(M_f) \to \mathrm{Mod}(S_{\{z,f(z)\}})$ for the composition of $\overline{D}_*$ with the embedding from the Proposition.

Corollary 13. *If f is fixed point free and $[\![f]\!] \in \mathrm{Mod}(S)$ has infinite order, then*

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(S, z) \arrow[r] \arrow[d, "D_*"] & \pi_1(M_f, z) \arrow[r] \arrow[d, "\Delta_f"] & \langle \sigma \rangle \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{P}\mathscr{B}_2(S) \arrow[r] & \mathrm{Mod}\big(S_{\{z,f(z)\}}, \{z, f(z)\}\big) \arrow[r, "\rho_{\{z,f(z)\}}"] & \mathrm{Mod}(S) \arrow[r] & 1, \end{tikzcd}$$

is an embedding of short exact sequences.

If $f^2(z) = z$, we can pass to a 2–fold cover $M_{f^2} \to M_f$, and the issues with basepoints in the proof of Proposition 12 disappear. Write τ for the loop in M_{f^2}

based at $z \in S \subset M_{f^2}$ as in §2.1, representing the stable letter. Then τ acts like f_*^2 on $\pi_1(S,z)$, and restricting Δ_f to $\pi_1(M_{f^2})$ we get

$$\Delta_f \colon \pi_1(M_{f^2}, z) \cong \pi_1(S,z) \rtimes \langle \tau \rangle \to \mathrm{Mod}(S_{\{z,f(z)\}}, \{z, f(z)\}),$$

where τ is sent to $f^2_{\{z,f(z)\}}$. We note that Δ_f on $\pi_1(M_f)$ is type preserving if and only if this restriction is, so it will suffice to work with this restriction.

4 Figure eight

We now focus on the figure–eight knot group.

4.1 A linear map

Let $L \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$L = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and let $\overline{L}$ be the induced self-map of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The restriction

$$f = \overline{L}_{\mathbf{0}} \colon T^2_{\mathbf{0}} \to T^2_{\mathbf{0}}$$

is the monodromy of the fibration of the figure–eight knot complement.

Lemma 14. *The homeomorphism $\overline{L}$ fixes exactly one point, $\mathbf{0}$, and its square fixes exactly five points $\mathbf{0}$, $z = 0.2[1,2]^T$, $w = fz = 0.2[4,3]^T$, $0.2[2,4]^T$, and $0.2[3,1]^T$.*

Proof. Let m and n be integers and consider the linear system

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}.$$

The matrix on the left is invertible, and so we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}.$$

The only solution to this is $\mathbf{0}$ in T^2.

Similarly, let m and n be integers and consider the linear system

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$

The matrix on the left is invertible, and we have that our fixed points are those points in the unit square $[0,1] \times [0,1]$ of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix},$$

which yields the list above. $\square$

Lemma 15. *For every essential loop β in $T^2_{\mathbf{0}}$, β and $f\beta$ are distinct homotopy classes, and their union is filling.*

Proof. Let β be an essential loop in $T^2_{\mathbf{0}}$. Since f is pseudo-Anosov, it cannot fix the homotopy class of an essential curve, and β and $f\beta$ are not homotopic.

If β is a simple closed curve, then $f(\beta)$ and β are a pair of distinct essential simple closed curves on $T^2_{\mathbf{0}}$, hence their union fills.

If β is *not* a simple closed curve, and β and $f\beta$ don't fill, then there is an essential simple closed curve γ in the complement of $\beta \cup f\beta$. Applying f^{-1}, we see that $f^{-1}\gamma$ and β are also disjoint. This means that β is disjoint from both γ and $f^{-1}(\gamma)$. By the simple closed curve case, we know that γ and $f^{-1}(\gamma)$ fill, and so β must be inessential, a contradiction. $\square$

4.2 Notation, conventions, and the main theorem

Write $F = \overline{L}^2 \colon T^2 \to T^2$ and let $X = \{\mathbf{0}, z, w\}$ be the first three fixed points in Lemma 14. We continue to use the notation that, for $Z \subset X$, the map $F_Z \colon T^2_Z \to T^2_Z$ is the restriction of F to $T^2_Z = T^2 - Z$.

The homeomorphism $f = \overline{L}_{\mathbf{0}}$ is fixed point free and its square $f^2 = F_{\mathbf{0}}$ fixes z and w, by Lemma 14. We let $\Gamma = \pi_1(M_{F_{\mathbf{0}}}, z) = \pi_1(M_{f^2}, z)$, and let $K = \pi_1(T^2_{\mathbf{0}}, z)$ be the fiber subgroup, which is free of rank two. We will also write Γ as $\Gamma = K \rtimes \langle \tau \rangle$

so that $\tau \in \Gamma$ is the stable letter as described in §2.1. Corollary 13 now gives us a representation

$$\Delta_f \colon \Gamma \to \mathrm{PMod}(T^2_X).$$

With this set up we have $\Delta_f(\tau) = F_X$, as described after Corollary 13. Given an element β of K, we may write $\Delta_f(\beta) = (\beta, f\beta)$ when convenient, (harmlessly) blurring the distinction between a loop in $\mathrm{Conf}_2(T^2_{\mathbf{0}}))$ and its corresponding mapping class.

From Lemmas 9 and 15, we see that Δ_f sends nonperipheral loops in K to pseudo-Anosov elements of $\mathrm{PMod}(T^2_X)$. Our main theorem states that this is also the case on all of Γ, yielding the precise version of Theorem 2.

Theorem 16. *The representation Δ_f is type–preserving. In other words, the mapping class $\Delta_f(\gamma)$ is reducible if and only if γ is a peripheral element of Γ.*

We will make use of the additive group structure on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with identity $\mathbf{0}$. Given x in T^2, we let $\mu_x \colon T^2 \to T^2$ be the translation $\mu_x(y) = y + x$, whose inverse is $\mu_x^{-1}(y) = \mu_{-x}(y) = y - x$. Observe that for each fixed point x of F, we have $F_x = \mu_x F_{\mathbf{0}} \mu_x^{-1}$, since F is linear. In particular, μ_x determines a canonical homeomorphism

$$\hat{\mu}_x \colon M_{F_{\mathbf{0}}} \to M_{F_x}.$$

Explicitly, M_{F_x} is an open submanifold of M_F obtained by deleting the image of $\{x\} \times [0,1] \subset T^2 \times [0,1]$ in the quotient M_F, and $\hat{\mu}_x$ is the descent of the homeomorphism $\mu_x \times \mathbb{1}_{T^2} \colon T^2 \times [0,1] \to T^2 \times [0,1]$ to M_F restricted to $M_{F_{\mathbf{0}}}$ on the domain and M_{F_x} on the range.

4.3 Forgetting punctures

For each $x \in X = \{\mathbf{0}, z, w\}$, we consider the Birman Exact Sequence

$$1 \to \pi_1\left(T^2_{X-\{x\}}, x\right) \to \mathrm{PMod}\left(T^2_X\right) \xrightarrow{\rho_x} \mathrm{PMod}\left(T^2_{X-\{x\}}\right) \to 1,$$

where ρ_x forgets x.

We also consider two element subsets $Y = \{x, y\} \subset X$, and the Birman Exact Sequence

$$1 \to \pi_1\left(T^2_y, x\right) \to \mathrm{PMod}\left(T^2_Y\right) \xrightarrow{\pi_x} \mathrm{Mod}\left(T^2_y\right) \to 1$$

where we write π_x for the homomorphism that forgets x to distinguish it from the map in the previous sequence. The domain of π_x also depends on the choice of two point set Y containing x, which we will make clear in context. Of course, we can interchange the roles of x and y.

For $Y = \{x, y\} \subset X$ a two point set as above, there is an associated isomorphism

$$\eta_{x,y} \colon \pi_1\left(M_{F_x}, y\right) \to \pi_y^{-1}\langle F_x\rangle < \mathrm{PMod}\left(T_Y^2\right)$$

coming from the embedding of short exact sequences in (2.1).

The stable letter in the semidirect product $\pi_1(M_{F_x}, y)$ is mapped to F_Y by this isomorphism. Since $M_{F_x} \cong M_{F_{\mathbf{0}}}$, we can view the domain of $\eta_{x,y}$ as Γ, when convenient, after choosing an appropriate basepoint change isomorphism. The next lemma tells us that the image depends only on Y.

Lemma 17. *For $Y = \{x, y\} \subset X$, we have $\pi_y^{-1}\langle F_x\rangle = \pi_x^{-1}\langle F_y\rangle$.*

Proof. Observe that $\pi_x(F_Y) = F_y$ and $\pi_y(F_Y) = F_x$. Furthermore, the inclusions of $\pi_1(T_x^2, y)$ and $\pi_1(T_y^2, x)$ into $\mathrm{PMod}(T_Y^2)$ via the Birman Exact Sequence have the same image: namely they are both equal to the kernel of the forgetful map $\mathrm{PMod}(T_Y^2) \to \mathrm{Mod}(T^2)$ since $\mathrm{Mod}(T_x^2) \cong \mathrm{Mod}(T^2)$ with the isomorphism obtained by forgetting the puncture x. Therefore, we have

$$\pi_y^{-1}\langle F_x\rangle = \left\langle \pi_1\left(T_x^2, y\right), F_Y\right\rangle = \left\langle \pi_1\left(T_y^2, x\right), F_Y\right\rangle = \pi_x^{-1}\langle F_y\rangle. \qquad \square$$

For any two element subset $Y = \{x, y\} \subset X$, we write $\Gamma_Y = \pi_y^{-1}\langle F_x\rangle = \pi_x^{-1}\langle F_y\rangle$. We also write $K_Y \lhd \Gamma_Y$ for the fiber subgroup (which, as the proof shows, is independent of the forgotten point).

We now come to the key lemma.

Lemma 18. *For each x in $X = \{\mathbf{0}, z, w\}$, the composition $\rho_x \circ \Delta_f$ is an isomorphism onto $\Gamma_{X-\{x\}}$, sending K to the fiber subgroup $K_{X-\{x\}}$.*

Remark. As we will see, the case when $x = z$ or $x = w$ is straightforward. When $x = \mathbf{0}$, this is surprising, for the following reasons. Filling in $\mathbf{0}$ has the effect of abelianizing $K = \pi_1(T_{\mathbf{0}}^2)$. This map to $\mathbb{Z}^2$ has a very large kernel, and makes injectivity of the composition $\rho_{\mathbf{0}} \circ \Delta_f$ seem unlikely. This is an illusory problem, as we are filling in $\mathbf{0}$ *after* mapping K into the mapping class group, and furthermore, the homomorphism Δ_f does not descend to a representation of the solvable group $\mathbb{Z}^2 \rtimes \mathbb{Z}$.

We note that although the proof below shows directly that the isomorphism sends K to $K_{X-\{x\}}$ in each case, this actually follows immediately since Γ (and hence each Γ_Y) has a unique homomorphism to $\mathbb{Z}$ since $\overline{L}^2$ is Anosov.

Proof of Lemma 18. First consider the case $x = w$ and let $Y = \{\mathbf{0}, z\}$ with associated forgetful homomorphism π_z. Then $\rho_w \circ \Delta_f(\tau) = \rho_w(F_X) = F_Y$, and for any γ in $\pi_1(T^2_{\mathbf{0}}, z)$, we have

$$\rho_w \circ \Delta_f(\gamma) = \rho_w(\gamma, f\gamma) = \gamma,$$

viewing $(\gamma, f\gamma)$ as an element of $\pi_1(\mathrm{Conf}(T^2_{\mathbf{0}}), \{z,w\}) < \mathrm{PMod}(T^2_X)$ and γ as an element of $\pi_1(T^2_{\mathbf{0}}, z) < \mathrm{PMod}(T^2_Y)$. It follows that $\rho_w \circ \Delta_f$ is an isomorphism onto Γ_Y. A similar argument holds for $x = z$ (where $Y = \{\mathbf{0}, w\}$ with associated forgetful homomorphism π_w). The only exception is that the displayed equation becomes

$$\rho_z \circ \Delta_f(\gamma) = \rho_z(\gamma, f\gamma) = f\gamma.$$

We now consider the case that $x = \mathbf{0}$ and let $Y = \{z, w\}$. Let us first restrict our attention to the fiber subgroup K.

Let γ be an element of K, and let $h\colon T^2 \to T^2$ be a homeomorphism so that $h_X\colon T^2_X \to T^2_X$ represents $\Delta_f(\gamma)$. The mapping class $\rho_{\mathbf{0}} \circ \Delta_f(\gamma)$ is then represented by the homeomorphism

$$h_Y\colon T^2_Y \to T^2_Y.$$

Since $\Delta_f(K)$ is in the kernel of the homomorphism obtained by forgetting *both* z and w, the restriction $h_{\mathbf{0}}\colon T^2_{\mathbf{0}} \to T^2_{\mathbf{0}}$ is isotopic to the identity on $T^2_{\mathbf{0}}$. We let $h^t\colon T^2 \to T^2$ be the extension of that isotopy to an isotopy from $h = h^0$ to the identity $h^1 = \mathbb{1}_{T^2}$.

We define a new one–parameter family of maps by

$$\psi^t = h^t - h^t(z) + z.$$

For each t, the map ψ^t is a homeomorphism (namely, the homeomorphism h^t composed with the translation $\mu_{z-h^t(z)}$) and so ψ^t defines an isotopy from ψ^0 to ψ^1 on T^2. Now, since $h^0(z) = z = h^1(z)$, we have

$$\psi^0 = h^0 - h^0(z) + z = h^0$$

and

$$\psi^1 = h^1 - h^1(z) + z = h^1 = \mathbb{1}_{T^2}$$

Also note that, for all t, we have

$$\psi^t(z) = h^t(z) - h^t(z) + z = z.$$

Since ψ^t fixes z for all t, the isotopy $\psi_z^t \colon T_z^2 \to T_z^2$ is well–defined.

Since ψ_z^t is an isotopy from $\psi_z^0 = h_z^0$ to $\psi_z^1 = h_z^1 = \mathbb{1}_{T_z^2}$, the homeomorphism

$$h_Y \colon T_Y^2 \to T_Y^2$$

representing $\rho_{\mathbf{0}} \circ \Delta_f(\gamma)$ lies in the kernel of the map

$$\pi_w \colon \mathrm{PMod}\big(T_Y^2\big) \to \mathrm{Mod}\big(T_z^2\big)$$

forgetting w. Moreover, this mapping class corresponds to the element of $\pi_1(T_z^2, w)$ (via the Birman sequence) represented by the loop

$$\begin{aligned}\psi_z^t(w) &= h^t(w) - h^t(z) + z \\ &= f\gamma(t) - \gamma(t) + z.\end{aligned}$$

At this point, we have shown that, for every γ in K, the mapping class $\rho_{\mathbf{0}} \circ \Delta_f(\gamma)$ corresponds to the element of $K_Y = \pi_1(T_z^2, w)$ represented by the loop $f\gamma(t) - \gamma(t) + z$. Now, consider the generators of $K = \pi_1(T_{\mathbf{0}}^2, z)$ given by the loops $A(t) = z + [t,0]^T$ and $B(t) = z + [0,t]^T$. For these loops, we have

$$\begin{aligned}\rho_{\mathbf{0}} \circ \Delta_f(A) \approx fA(t) - A(t) + z & \\ &= f\left(z + \begin{bmatrix}1\\0\end{bmatrix} t\right) - \begin{bmatrix}1\\0\end{bmatrix} t - z + z \\ &= \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\left(z + \begin{bmatrix}1\\0\end{bmatrix} t\right) - \begin{bmatrix}1\\0\end{bmatrix} t \\ &= fz + \begin{bmatrix}1\\1\end{bmatrix} t \\ &= w + \begin{bmatrix}1\\1\end{bmatrix} t\end{aligned}$$

and

$$\begin{aligned}
\rho_{\mathbf{0}} \circ \Delta_f(B) &\approx fB(t) - B(t) + z \\
&= f\left(z + \begin{bmatrix}0\\1\end{bmatrix} t\right) - \begin{bmatrix}0\\1\end{bmatrix} t - z + z \\
&= \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} \left(z + \begin{bmatrix}0\\1\end{bmatrix} t\right) - \begin{bmatrix}0\\1\end{bmatrix} t \\
&= fz + \begin{bmatrix}1\\0\end{bmatrix} t \\
&= w + \begin{bmatrix}1\\0\end{bmatrix} t.
\end{aligned}$$

These two loops form a basis for the free group $K_Y = \pi_1(T^2_z, w)$, and so $\rho_{\mathbf{0}} \circ \Delta_f|_K$ is a homomorphism taking a basis to a basis and therefore must be an isomorphism onto its image K_Y.

The mapping class $\rho_{\mathbf{0}} \circ \Delta_f(\tau)$ is represented by the homeomorphism F_Y. Therefore, $\rho_{\mathbf{0}} \circ \Delta_f$ restricts to a homomorphism

$$\Gamma \cong K \rtimes \langle \tau \rangle \to K_Y \rtimes \langle F_Y \rangle \cong \Gamma_Y,$$

taking K isomorphically to K_Y and τ to F_Y. This implies that $\rho_{\mathbf{0}} \circ \Delta_f$ is an isomorphism from Γ to Γ_Y, as required. □

Proof of the Theorem 16. We have already noted that for γ in K, Lemmas 9 and 15 imply that $\Delta_f(\gamma)$ is reducible if and only if γ is peripheral. Thus, we consider $\tau^m\gamma$ for some $m \neq 0$. Then $\Delta_f(\tau^m\gamma)$ is represented by a homeomorphism $F^m_X h_X$ for h_X in the kernel of the homomorphism $\pi_w\rho_z$ that forgets both z and w. Suppose $F^m_X h_X$ is reducible. After passing to a power, we may assume that $F^m_X h_X$ fixes an essential simple closed curve $\alpha \subset T^2_X$. The mapping torus $M_{F^m_X h_X}$ then contains an essential torus $\mathscr{T}$ meeting each T^2_X–fiber transversely in an essential simple closed curve.

We consider $M_{F^m_X h_X}$ an open submanifold of $M_{F^m h}$. As described in §2.1, there is a T^2–fiber-preserving homeomorphism $M_{F^m h} \cong M_{F^m}$ and $M_{F^m_X h_X}$ is the complement of a link $\mathscr{L} \subset M_{F^m}$ transverse to the T^2 fibers. Moreover, $\mathscr{L}$ must have three components since each point of X is fixed by $F^m h$. The homeomorphism sends $\mathscr{T}$ to a torus meeting each fiber T^2 in a simple closed curve, and the torus defines a homotopy from one such curve to its F^m–image. Since F^m is Anosov, it preserves no homotopy class of essential simple closed curve on T^2, and hence each curve of intersection of $\mathscr{T}$ with the fiber T^2 is null homotopic in T^2. In particular, $\mathscr{T}$ bounds a solid torus V in M_{F^m}.

Since $\mathscr{T}$ is essential in $M_{F_X^m h_X} = M_{F^m} - \mathscr{L}$, the solid torus V must contain at least two components of $\mathscr{L}$, and each component must be a core of V. Let $Y \subset X$ be the set of two points determining this two component link $\mathscr{L}_0 \subset \mathscr{L}$ contained in V, and let $x \in X - Y$ be the third point, whose corresponding component of $\mathscr{L}$ may or may not lie in V. Observe that $\rho_x \Delta_f(\tau^m \gamma)$ is represented by $F_Y^m h_Y \in \Gamma_Y$, and $M_{F_Y^m h_Y} = M_{F^m} - \mathscr{L}_0$. In particular, since $\mathscr{L}_0 \subset V$, it follows that the torus $\mathscr{T}$ is still essential in $M_{F_Y^m h_Y}$, and hence $F_Y^m h_Y$ is reducible.

Now write $Y = \{y, u\}$ and note that $\rho_x \Delta_f(\tau^m \gamma) \in \Gamma_Y$ is represented by $F_Y^m h_Y$. Forgetting u, we have that h_y must in fact be isotopic to the identity in T_y^2, and we let h_y^t be the isotopy from h_y^0 to the identity. This traces out a loop $\beta(t) = h_y^t(u)$ that represents the element of $\pi_1(T_y^2, u)$ corresponding to mapping class h_Y. The mapping torus $M_{F_Y^m h_Y}$ is homeomorphic to $M_{F_y^m} - \mathscr{K}$, where $\mathscr{K}$ is the knot traced out in $T_y^2 \times [0,1]$ by $(h_y^t(u), t)$ and projected to $M_{F_y^m}$. The knot $\mathscr{K}$ is a loop based at u which maps to $\rho_x \Delta_f(\tau_m \gamma)$ up to conjugacy, by Lemma 8. Since $\mathscr{K}$ is contained in V, it follows that this loop is peripheral. Since peripheral elements are precisely those whose centralizers are isomorphic to $\mathbb{Z}^2$, the isomorphism from $\Gamma \to \Gamma_Y$ maps peripheral elements precisely to the peripheral elements, and hence $\tau^m \gamma$ is peripheral in Γ, as required. $\square$

4.4 Surface subgroups

We have the following consequence of Theorem 16.

Corollary 19. *There are infinitely many commensurability classes of purely pseudo-Anosov closed surface subgroups of* $\mathrm{Mod}(T_X^2)$.

Proof. The figure–eight knot complement contains infinitely many commensurability classes of totally geodesic closed immersed surfaces [62], and the Δ_f–image of these are purely pseudo-Anosov, by Theorem 16. We claim that infinitely many distinct commensurability classes remain distinct commensurability classes in $\mathrm{Mod}(T_X^2)$.

Suppose $G_1, G_2 < \Gamma$ are two closed surface subgroups such that $\Delta_f(G_1)$ and $\Delta_f(G_2)$ are conjugate by an element $g \in \mathrm{PMod}(T_X^2)$. Since $K = \pi_1(T_{\mathbf{0}}^2, z)$ is free, G_1 is not contained in K, and so there is an element $\tau^m \gamma \in G_1$ with $m > 0$ and γ in K. Then

$$g\Delta_f(\tau^m \gamma) g^{-1} = \left(g F_X^m g^{-1}\right)\left(g \Delta_f(\gamma) g^{-1}\right) \in \Delta_f(G_2).$$

Applying $\rho_{\{z,w\}}$ we have

$$\rho_{\{z,w\}}(g\Delta_f(\tau^m\gamma)g^{-1}) = \rho_{\{z,w\}}(g)F_{\mathbf{0}}^m\rho_{\{z,w\}}(g)^{-1}. \tag{4.1}$$

Since $g\Delta_f(\tau^m\gamma)g^{-1}$ is in $\Delta_f(G_2) < \Delta_f(\Gamma)$ and $\rho_{\{z,w\}}\colon \Delta_f(\Gamma) \to \mathrm{Mod}(T^2_{\mathbf{0}})$ has image $\langle F_{\mathbf{0}}\rangle$, the element (4.1) above is in $\langle F_{\mathbf{0}}\rangle$. Consequently, $\rho_{\{z,w\}}(g)$ is in the normalizer N of $\langle F_{\mathbf{0}}\rangle$. Since $F_{\mathbf{0}}$ is pseudo-Anosov, N contains $\langle F_{\mathbf{0}}\rangle$ with finite index, and hence $g \in \mathscr{N} = \rho^{-1}_{\{z,w\}}(N)$, which contains $\rho^{-1}_{\{z,w\}}(\langle F_{\mathbf{0}}\rangle)$ with finite index. We note that $\mathscr{G} = \rho^{-1}_{\{z,w\}}(\langle F_{\mathbf{0}}\rangle)$ is the image of $\pi_1(M_{\hat{f}^2},(z,f(z)) < \pi_1(M_{\hat{f}},(z,f(z)))$ under the embedding from Proposition 12.

Now, suppose there are infinitely many surface subgroups $G_1, G_2, \ldots < \Gamma$ that are pairwise non-conjugate in Γ, but whose Δ_f–images are conjugate in $\mathrm{PMod}(T^2_X)$. For each $i \geq 2$, let $g_i \in \mathrm{PMod}(T^2_X)$ be an element that conjugates $\Delta_f(G_1)$ to $\Delta_f(G_i)$. After passing to a subsequence, we can assume that $g_i\mathscr{G} = g_j\mathscr{G}$ for all i,j, since g_i and g_j lie in $\mathscr{N}$ and the index $[\mathscr{N} : \mathscr{G}]$ is finite. But then $g_ig_2^{-1}$ conjugates $\Delta_f(G_2)$ to $\Delta_f(G_i)$, for all i, and $g_ig_2^{-1}$ lies in $\mathscr{G}$ for all i. By Lemma 10, there is a retraction $r\colon \mathscr{G} \to \Delta_f(\Gamma)$ induced by the retraction $M_{\hat{f}^2} \to M_{f^2}$. It follows that $r(g_ig_2^{-1})$ also conjugates $\Delta_f(G_2)$ to $\Delta_f(G_i)$ for all i, but $r(g_ig_2^{-1}) = \Delta_f(\gamma_i)$ for some γ_i in Γ. Since Δ_f is an isomorphism, γ_i conjugates G_2 to G_i for all i, contradicting the fact that the subgroups G_i were all non-conjugate in Γ.

Therefore, there are infinitely many $\mathrm{PMod}(T^2_X)$–commensurability classes of purely pseudo-Anosov surface subgroups. Since $\mathrm{PMod}(T^2_X)$ has finite index in $\mathrm{Mod}(T^2_X)$, there are also infinitely many $\mathrm{Mod}(T^2_X)$–commensurability classes of purely pseudo-Anosov surface subgroups. □

5 Mapping class groups of closed surfaces

Theorem 1 follows using a well–known branched covering trick following the ideas of J. Birman and H. Hilden [10, 11, 12] (see also [63, 83, 65]). The precise fact we need is the following.

Proposition 20. *Suppose $p\colon \widetilde{S} \to T^2$ is a branched cover, branched over $X = \{\mathbf{0}, z, w\}$, so that the local degree at each point of $p^{-1}(X)$ is greater than 1. Then there is a finite index subgroup $\mathrm{Mod}_p(T^2_X) < \mathrm{Mod}(T^2_X)$ and a type–preserving, injective homomorphism $p^*\colon \mathrm{Mod}_p(T^2_X) \to \mathrm{Mod}(\widetilde{S})$.*

A homomorphism between torsion–free subgroups of mapping class groups is *type–preserving* if an element is pseudo-Anosov if and only if its image is.

Proof. Let $Y = p^{-1}(X)$ and $p_0\colon \widetilde{S}_Y \to T^2_X$ the associated unbranched covering. From standard covering space theory, a homeomorphism h_X of T^2_X lifts to a homeomorphism of $\widetilde{S}_Y$ if and only if the conjugacy class of $(p_0)_*(\pi_1(\widetilde{S}_Y))$ is preserved by $(h_X)_*$. Since this subgroup has finite index, there is a finite index subgroup of $\mathrm{Homeo}^+(T^2_X)$ consisting of homeomorphisms that lift to homeomorphisms of $\widetilde{S}_Y$, and this defines a finite index subgroup $\mathrm{Mod}_0(T^2_X)$ of elements that lift. The set of lifts $\widetilde{\mathrm{Mod}}_0(T^2_X) < \mathrm{Mod}(\widetilde{S}_Y)$ forms a subgroup that fits into a short exact sequence

$$1 \to G \to \widetilde{\mathrm{Mod}}_0(T^2_X) \to \mathrm{Mod}_0(T^2_X) \to 1,$$

where G is the covering group of p_0. See, e.g. [4].

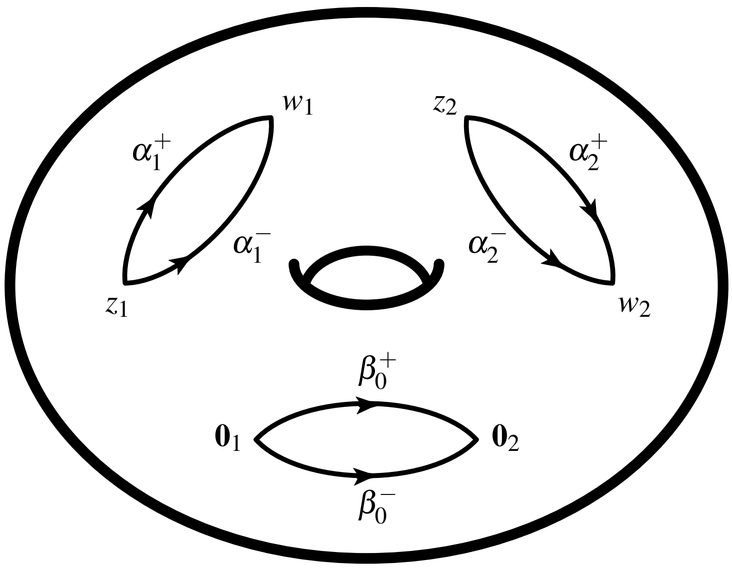
Any finite index torsion free subgroup of $\widetilde{\mathrm{Mod}}_0(T^2_X)$ will map isomorphically onto a finite index subgroup of $\mathrm{Mod}_0(T^2_X)$. Denoting such a subgroup $\mathrm{Mod}_p(T^2_X) < \mathrm{Mod}_0(T^2_X)$, we obtain an (inverse) isomorphism back to the finite index subgroup of $\widetilde{\mathrm{Mod}}_0(T^2_X)$, and hence to $\mathrm{Mod}(\widetilde{S}_Y)$. This homomorphism is clearly type preserving: for any pseudo-Anosov element $\mathrm{Mod}_p(T^2_X)$, its stable and unstable foliations lift to stable and unstable foliations on $\widetilde{S}_Y$ for the lifted mapping class. Now we compose this homomorphism with the forgetting homomorphism $\rho_Y\colon \mathrm{Mod}(\widetilde{S}_Y) \to \mathrm{Mod}(\widetilde{S})$ to define $p^*\colon \mathrm{Mod}_p(T^2_X) \to \mathrm{Mod}(\widetilde{S})$. If $g \in \mathrm{Mod}_p(T^2_X)$ is any pseudo-Anosov element, then $p^*(g)$ is pseudo-Anosov since the stable and unstable foliations for the lift to $\widetilde{S}_Y$ have at least 2 prongs at each puncture, because of the local degree assumption at every point of Y. Since no pseudo-Anosov element is in the kernel (as its p^*–image is pseudo-Anosov), it also follows that p^* is injective. $\square$

Remark. Pulling back complex structures, we actually get an isometric embedding of the Teichmüller space of T^2_X into that of $\widetilde{S}$ (with the Teichmüller metric) since every meromorphic quadratic differential on T^2 with only simple poles, and only at points of X, pulls back to a holomorphic quadratic differential (without poles). This gives an alternate proof that p^* is type preserving since pseudo-Anosov elements are the only ones with geodesic axes.

5.1 Explicit branched covers

It is not difficult to produce some closed surfaces that admit branched covers to T^2 branched over X and satisfying the hypotheses of Proposition 20. To see that one can do this for all closed surfaces of genus at least 4, we proceed as follows.

Consider a two–fold unbranched cover $S \to T^2$ and let $\{\mathbf{0}_1, \mathbf{0}_2, z_1, z_2, w_1, w_2\} \subset S$ be such that the covering map sends each point in our set to the point with the subscript erased (in the set $\{\mathbf{0}, z, w\}$). Let α be an arc in T^2 connecting z and w and β a disjoint essential loop based at $\mathbf{0}$. Next, let α_1 and α_2 be the components of the preimage, and choose the labeling so that they connect z_1, w_1 and z_2, w_2, respectively. Assume that β does not lift to a loop, and let β_0 be a component of the preimage of β. Finally, cut S open along these three arcs and label the exposed arcs by $\alpha_1^+, \alpha_1^-, \alpha_2^+, \alpha_2^-, \beta_0^+, \beta_0^-$, as illustrated. Call this cut open surface-with-boundary Σ.



Now consider any surface $\widetilde{S}$ constructed from finitely many copies $\Sigma_1, \ldots, \Sigma_k$ of Σ by gluing the copies along arcs in pairs subject to the following constraints:

1. Any α_1^+ arc in any Σ_i is glued to any α_2^- arc in any Σ_j, or to any α_1^- in any Σ_j *except* Σ_i. Similarly for α_2^+.

2. Any α_1^- arc in any Σ_i is glued to any α_2^+ arc in any Σ_j, or to any α_1^+ in any Σ_j *except* Σ_i. Similarly for α_2^-.

3. Any β_0^+ arc in any Σ_i may be glued to any β_0^- arc in any Σ_j, except Σ_i, and vice versa for β_0^- arcs in any Σ_i.

Lemma 21. *If $\widetilde{S}$ is constructed as above, then $\widetilde{S}$ admits a branched cover $p\colon \widetilde{S} \to T^2$, branching precisely over $X = \{\mathbf{0}, z, w\}$ with local degree at least 2 at every point of $p^{-1}(X)$.*

Proof. Extend the arcs to a cellular triangulation of T^2 with vertices at $\{\mathbf{0}, z, w\}$ and label the triangles. This triangulation defines a triangulation of Σ with the same labels (so that the map to T^2 sends triangles with a given label to a triangle with the same label). There is thus a triangulation of $\widetilde{S}$ with triangles labeled with the same set of labels, and mapping each triangle in $\widetilde{S}$ to the triangle of the same label is a branched cover, branched over $\{\mathbf{0}, z, w\}$. The restrictions on the gluings ensure that every point branches nontrivially at every vertex, as required. □

Proposition 22. *Every surface of genus at least* 4 *arises from the construction above.*

Proof. Note that for a single copy of Σ, we can glue α_1^+ to α_2^-, α_2^+ to α_1^-, and leaving the $\beta_0^\pm$ unidentified, we get a surface with boundary Ω which has genus 2 and one boundary component divided into the two arcs $\beta_0^\pm$. We can string $k \geq 2$ of these together joining β_0^+ on one to β_0^- on the next, to produce a surface of genus $2k$, thus producing all surfaces of even genus at least 4.

Taking two copies of Σ and gluing so that all z_i vertices get identified and all w_i vertices get identified, and so that both $\mathbf{0}_1$ vertices get identified and both $\mathbf{0}_2$ vertices get identified, we get a surface of genus 5. To see this, note that we have a four fold cover of $T^2_{\{\mathbf{0},z,w\}}$ with one puncture mapping to each of z and w and two mapping to $\mathbf{0}$, and we have capped off all four punctures with a disk. So χ is $4(-3)+4 = -8 = 2-2(5)$, so the genus is 5.

Cutting this genus 5 surface open along one of the β arcs and inserting k genus 2 pieces from the even genus construction, we get all odd genus at least 7, hence all odd genus at least 5. □

5.2 Proof of Theorem 1

We now give the proof of the main theorem.

Theorem 1 *There are infinitely many commensurability classes of purely pseudo-Anosov surface subgroups of* $\mathrm{Mod}(S_{g,0})$ *for all* $g \geq 4$.

Proof of Theorem 1. For any closed surface S of genus at least 4, Proposition 22 and Proposition 20 produce an embedding of finite index subgroups $\mathrm{Mod}_0(T^2_X) < \mathrm{Mod}(T^2_X)$ into $\mathrm{Mod}(S)$. Together with Corollary 19, this produces infinitely many surface subgroups in $\mathrm{Mod}(S)$. To explain why there are infinitely many commensurability classes, identify $\mathrm{Mod}_0(T^2_X)$ with its image in $\mathrm{Mod}(S)$, and observe that it suffices to prove that there are only finitely many $\mathrm{Mod}(T^2_X)$–conjugacy classes of intersections of $\mathrm{Mod}_0(T^2_X)$ with a $\mathrm{Mod}(S)$–conjugate, that contains a pseudo-Anosov.

This required finiteness statement can be seen by considering the equivariant isometric embedding of Teichmüller spaces $\mathscr{T}(T^2_X) \to \mathscr{T}(S)$. Identifying $\mathscr{T}(T^2_X)$ with its image, we note that $\mathrm{Mod}_0(T^2_X)$ has finite index in the stabilizer of the image of $\mathscr{T}(T^2_X)$ in $\mathscr{T}(S)$, and *up to the action of this stabilizer*, $\mathscr{T}(T^2_X)$ intersects only finitely many $\mathrm{Mod}(S)$–translates of $\mathscr{T}(T^2_X)$. These both follow from proper discontinuity of the action on Teichmüller space, and the cocompactness of this action on any thick part. Then we observe that any intersection of $\mathrm{Mod}_0(T^2_X)$ with a $\mathrm{Mod}(S)$–conjugate of itself that contains a pseudo-Anosov element has an associated intersection of $\mathscr{T}(T^2_X)$ and a $\mathrm{Mod}(S)$–translate, and moreover a $\mathrm{Mod}_0(T^2_X)$–conjugate of such intersection of groups corresponds to $\mathrm{Mod}_0(T^2_X)$–translates of the intersection with $\mathscr{T}(T^2_X)$. □

5.3 Sketch of the proof of Theorem 6

Closer examination of the proofs of Corollary 19 and Theorem 1 establishes Theorem 6.

Theorem 6 *The number of commensurability classes of purely pseudo-Anosov subgroups of* $\mathrm{Mod}(S_{g,0})$ *and* $\mathrm{Mod}(S_{1,3})$ *that are isomorphic to the fundamental group of a surface of genus at most h is bounded below by a strictly increasing linear function of h.*

As mentioned in the introduction, we expect the actual number of commensurability classes to be much much larger, and so we only sketch the proof.

Sketch of proof. Let $\mathscr{S}_\Gamma$ be the set of conjugacy classes of convex cocompact surface subgroups of Γ and let $\mathscr{S}_{\mathrm{Mod}(S_{g,n})}$ be the set of conjugacy classes of purely pseudo-Anosov subgroups of $\mathrm{Mod}(S_{g,n})$.

The indices $[\mathscr{N} : \mathscr{G}]$ and $[\mathrm{Mod}(T^2_X) : \mathrm{PMod}(T^2_X)]$ in the proof of Corollary 19 are bounded by a universal constant, and tracing through the proof with this in mind reveals that there is a universal constant m such that the the natural map $\mathscr{S}_\Gamma \to \mathscr{S}_{\mathrm{Mod}(T^2_X)}$ induced by the representation Δ_f is m-to-1. We conclude that the number of commensurability classes of purely pseudo-Anosov surface groups in $\mathrm{Mod}(S_{1,3})$ of genus at most h is comparable to the number of commensurability classes of such convex cocompact surface groups in $\pi_1(M_8)$.

Let $S_{g,0}$ be the branched cover constructed in the proof of Theorem 1, let Ξ be the finite index subgroup of Γ such that $\Delta_f(\Xi)$ lifts to a subgroup Ξ_g of

$S_{g,0}$, and note that the index of Ξ in Γ is bounded by a constant depending only on g. Close examination of the proof of Theorem 1 reveals that the number of Ξ_g–commensurability classes of purely pseudo-Anosov surface groups in Ξ_g is comparable to the number of $\mathrm{Mod}(S_{g,0})$–commensurability classes, where the constants of comparison depend only on Ξ and g. It follows that the number of $\mathrm{Mod}(S_{g,0})$–commensurability classes of purely pseudo-Anosov surface groups of genus at most h is comparable to the number of $\pi_1(M_8)$–commensurability classes of convex cocompact surface groups of genus at most h.

The proof is completed by noting that the number of commensurability classes of cocompact *fuchsian* subgroups of $\pi_1(M_8)$ of genus at most h is comparable to a linear function of h, by the discussion at the end of [81].• $\square$

References

[1] A. D. Abrams. *Configuration spaces and braid groups of graphs*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–University of California, Berkeley.

[2] I. Agol. Private communication.

[3] T. Aougab, D. Futer, and S. Taylor. On fixed points of pseudo-anosov maps. In preparation.

[4] J. Aramayona, C. J. Leininger, and J. Souto. Injections of mapping class groups. *Geom. Topol.*, 13(5):2523–2541, 2009.

[5] M. F. Atiyah. The signature of fibre-bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 73–84. Univ. Tokyo Press, Tokyo, 1969.

[6] M. Baker and D. Cooper. Immersed, virtually-embedded, boundary slopes. *Topology Appl.*, 102(3):239–252, 2000.

[7] L. Bers. Fiber spaces over Teichmüller spaces. *Acta Math.*, 130:89–126, 1973.

[8] M. Bestvina, K. Bromberg, A. E. Kent, and C. J. Leininger. Undistorted purely pseudo-Anosov groups. *J. Reine Angew. Math.*, 760:213–227, 2020.

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[9] J. S. Birman. Mapping class groups and their relationship to braid groups. *Comm. Pure Appl. Math.*, 22:213–238, 1969.

[10] J. S. Birman and H. M. Hilden. On the mapping class groups of closed surfaces as covering spaces. In *Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, volume No. 66 of *Ann. of Math. Stud.*, pages 81–115. Princeton Univ. Press, Princeton, NJ, 1971.

[11] J. S. Birman and H. M. Hilden. On isotopies of homeomorphisms of Riemann surfaces. *Ann. of Math. (2)*, 97:424–439, 1973.

[12] J. S. Birman and H. M. Hilden. Erratum to "Isotopies of homeomorphisms of Riemann surfaces" [MR0325959]. *Ann. of Math. (2)*, 185(1):345, 2017.

[13] B. H. Bowditch. Atoroidal surface bundles over surfaces. *Geom. Funct. Anal.*, 19(4):943–988, 2009.

[14] M. R. Bridson. On the subgroups of right-angled Artin groups and mapping class groups. *Math. Res. Lett.*, 20(2):203–212, 2013.

[15] J. Bryan and R. Donagi. Surface bundles over surfaces of small genus. *Geom. Topol.*, 6:59–67, 2002.

[16] R. D. Canary, D. B. A. Epstein, and P. L. Green. Notes on notes of Thurston. In *Fundamentals of hyperbolic geometry: selected expositions*, volume 328 of *London Math. Soc. Lecture Note Ser.*, pages 1–115. Cambridge Univ. Press, Cambridge, 2006.

[17] M. T. Clay, C. J. Leininger, and J. Mangahas. The geometry of right-angled Artin subgroups of mapping class groups. *Groups Geom. Dyn.*, 6(2):249–278, 2012.

[18] D. Cooper and D. Futer. Ubiquitous quasi-Fuchsian surfaces in cusped hyperbolic 3-manifolds. *Geom. Topol.*, 23(1):241–298, 2019.

[19] D. Cooper, D. D. Long, and A. W. Reid. Bundles and finite foliations. *Invent. Math.*, 118(2):255–283, 1994.

[20] J. Crisp and L. Paris. The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group. *Invent. Math.*, 145(1):19–36, 2001.

[21] J. Crisp, M. Sageev, and M. Sapir. Surface subgroups of right-angled Artin groups. *Internat. J. Algebra Comput.*, 18(3):443–491, 2008.

[22] J. Crisp and B. Wiest. Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups. *Algebr. Geom. Topol.*, 4:439–472, 2004.

[23] S. Dowdall, A. E. Kent, and C. J. Leininger. Pseudo-Anosov subgroups of fibered 3-manifold groups. *Groups Geom. Dyn.*, 8(4):1247–1282, 2014.

[24] M. G. Durham and S. J. Taylor. Convex cocompactness and stability in mapping class groups. *Algebr. Geom. Topol.*, 15(5):2839–2859, 2015.

[25] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[26] B. Farb and L. Mosher. Convex cocompact subgroups of mapping class groups. *Geom. Topol.*, 6:91–152 (electronic), 2002.

[27] B. Farb and L. Mosher. The geometry of surface-by-free groups. *Geom. Funct. Anal.*, 12(5):915–963, 2002.

[28] K. Fujiwara. Subgroups generated by two pseudo-Anosov elements in a mapping class group. II. Uniform bound on exponents. *Trans. Amer. Math. Soc.*, 367(6):4377–4405, 2015.

[29] D. Gabai. Almost filling laminations and the connectivity of ending lamination space. *Geom. Topol.*, 13(2):1017–1041, 2009.

[30] D. Gabai. On the topology of ending lamination space. *Geom. Topol.*, 18(5):2683–2745, 2014.

[31] G. González-Díez and W. J. Harvey. Surface groups inside mapping class groups. *Topology*, 38(1):57–69, 1999.

[32] C. McA. Gordon, D. D. Long, and A. W. Reid. Surface subgroups of Coxeter and Artin groups. *J. Pure Appl. Algebra*, 189(1-3):135–148, 2004.

[33] U. Hamenstädt. Word hyperbolic extensions of surface groups. Preprint, `arXiv:math.GT/0505244`.

[34] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[35] S. Hensel and P. Przytycki. The ending lamination space of the five-punctured sphere is the Nöbeling curve. *J. Lond. Math. Soc. (2)*, 84(1):103–119, 2011.

[36] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[37] F. Hirzebruch. The signature of ramified coverings. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 253–265. Univ. Tokyo Press, Tokyo, 1969.

[38] Y. Imayoshi, M. Ito, and H. Yamamoto. A remark on the Bers type of some self-maps of Riemann surfaces with two specified points. In *Proceedings of the Second ISAAC Congress, Vol. 2 (Fukuoka, 1999)*, volume 8 of *Int. Soc. Anal. Appl. Comput.*, pages 871–875. Kluwer Acad. Publ., Dordrecht, 2000.

[39] G. Italiano, B. Martelli, and M. Migliorini. Hyperbolic 5-manifolds that fiber over S^1. *Invent. Math.*, 231(1):1–38, 2023.

[40] N. V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.

[41] J. Kahn and V. Marković. Counting essential surfaces in a closed hyperbolic three-manifold. *Geom. Topol.*, 16(1):601–624, 2012.

[42] J. Kahn and V. Markovic. Immersing almost geodesic surfaces in a closed hyperbolic three manifold. *Ann. of Math. (2)*, 175(3):1127–1190, 2012.

[43] J. Kahn and A. Wright. Nearly Fuchsian surface subgroups of finite covolume Kleinian groups. *Duke Math. J.*, 170(3):503–573, 2021.

[44] M. Kapovich. Private communication.

[45] M. Kapovich. On normal subgroups in the fundamental groups of complex surfaces, 1998. `arXiv:math/9808085`.

[46] A. E. Kent and C. J. Leininger. Subgroups of mapping class groups from the geometrical viewpoint. In *In the tradition of Ahlfors-Bers. IV*, volume 432 of *Contemp. Math.*, pages 119–141. Amer. Math. Soc., Providence, RI, 2007.

[47] A. E. Kent and C. J. Leininger. Shadows of mapping class groups: capturing convex cocompactness. *Geom. Funct. Anal.*, 18(4):1270–1325, 2008.

[48] A. E. Kent and C. J. Leininger. Uniform convergence in the mapping class group. *Ergodic Theory Dynam. Systems*, 28(4):1177–1195, 2008.

[49] A. E. Kent and C. J. Leininger. A geometric criterion to be pseudo-Anosov. *Michigan Math. J.*, 63(2):227–251, 2014.

[50] A. E. Kent, C. J. Leininger, and Saul Schleimer. Trees and mapping class groups. *J. Reine Angew. Math.*, 637:1–21, 2009.

[51] S.-h. Kim. *Hyperbolic surface subgroups of right-angled Artin groups and graph products of groups*. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–Yale University.

[52] S.-h. Kim. Co-contractions of graphs and right-angled Artin groups. *Algebr. Geom. Topol.*, 8(2):849–868, 2008.

[53] E. Klarreich. The boundary at infinity of the curve complex and the relative Teichmüller space. *Groups Geom. Dyn.*, 16(2):705–723, 2022.

[54] T. Koberda. Right-angled Artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups. *Geom. Funct. Anal.*, 22(6):1541–1590, 2012.

[55] T. Koberda, J. Mangahas, and S. J. Taylor. The geometry of purely loxodromic subgroups of right-angled Artin groups. *Trans. Amer. Math. Soc.*, 369(11):8179–8208, 2017.

[56] K. Kodaira. A certain type of irregular algebraic surfaces. *J. Analyse Math.*, 19:207–215, 1967.

[57] I. Kra. On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces. *Acta Math.*, 146(3-4):231–270, 1981.

[58] C. J. Leininger, M. Mj, and S. Schleimer. The universal Cannon-Thurston map and the boundary of the curve complex. *Comment. Math. Helv.*, 86(4):769–816, 2011.

[59] C. J. Leininger and A. W. Reid. A combination theorem for Veech subgroups of the mapping class group. *Geom. Funct. Anal.*, 16(2):403–436, 2006.

[60] C. J. Leininger and J. Russell. Pseudo-Anosov subgroups of general fibered 3-manifold groups. *Trans. Amer. Math. Soc. Ser. B*, 10:1141–1172, 2023.

[61] C. J. Leininger and S. Schleimer. Hyperbolic spaces in Teichmüller spaces. *J. Eur. Math. Soc. (JEMS)*, 16(12):2669–2692, 2014.

[62] C. Maclachlan. Fuchsian subgroups of the groups $\mathrm{PSL}_2(O_d)$. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 305–311. Cambridge Univ. Press, Cambridge, 1986.

[63] C. Maclachlan and W. J. Harvey. On mapping-class groups and Teichmüller spaces. *Proc. London Math. Soc. (3)*, 30(part):496–512, 1975.

[64] J. Mangahas and S. J. Taylor. Convex cocompactness in mapping class groups via quasiconvexity in right-angled Artin groups. *Proc. Lond. Math. Soc. (3)*, 112(5):855–881, 2016.

[65] D. Margalit and R. R. Winarski. Braids groups and mapping class groups: the Birman-Hilden theory. *Bull. Lond. Math. Soc.*, 53(3):643–659, 2021.

[66] J. D. Masters and X. Zhang. Closed quasi-Fuchsian surfaces in hyperbolic knot complements. *Geom. Topol.*, 12(4):2095–2171, 2008.

[67] M. Mitra. Cannon-Thurston maps for hyperbolic group extensions. *Topology*, 37(3):527–538, 1998.

[68] L. Mosher. A hyperbolic-by-hyperbolic hyperbolic group. *Proc. Amer. Math. Soc.*, 125(12):3447–3455, 1997.

[69] L. Mosher. Problems in the geometry of surface group extensions. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 245–256. Amer. Math. Soc., Providence, RI, 2006.

[70] J. Nielsen. Abbildungsklassen endlicher Ordnung. *Acta Math.*, 75:23–115, 1943.

[71] J.-P. Otal. *The hyperbolization theorem for fibered 3-manifolds*, volume 7 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1996 French original by Leslie D. Kay.

[72] A. W. Reid. Surface subgroups of mapping class groups. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 257–268. Amer. Math. Soc., Providence, RI, 2006.

[73] A. G. Reznikov. Harmonic maps, hyperbolic cohomology and higher Milnor inequalities. *Topology*, 32(4):899–907, 1993.

[74] I. Runnels. Effective generation of right-angled Artin groups in mapping class groups. *Geom. Dedicata*, 214:277–294, 2021.

[75] N. Salter and B. Tshishiku. Arithmeticity of the monodromy of some Kodaira fibrations. *Compos. Math.*, 156(1):114–157, 2020.

[76] D. Seo. Powers of Dehn twists generating right-angled Artin groups. *Algebr. Geom. Topol.*, 21(3):1511–1533, 2021.

[77] J.-P. Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier (Grenoble)*, 6:1–42, 1955/56.

[78] H. Servatius, C. Droms, and B. Servatius. Surface subgroups of graph groups. *Proc. Amer. Math. Soc.*, 106(3):573–578, 1989.

[79] W. P. Thurston. Hyperbolic structures on 3-manifolds, II: surface groups and 3-manifolds which fiber over the circle. In *Collected works of William P. Thurston with commentary. Vol. II. 3-manifolds, complexity and geometric group theory*, pages 79–110. Amer. Math. Soc., Providence, RI, [2022] ©2022. August 1986 preprint, January 1998 eprint.

[80] B. Tshishiku. Convex-compact subgroups of the Goeritz group. *Trans. Amer. Math. Soc.*, 377(1):271–322, 2024.

[81] L. Ya. Vulakh. Classification of maximal Fuchsian subgroups of some Bianchi groups. *Canad. Math. Bull.*, 34(3):417–422, 1991.

[82] F. Waldhausen. Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II. *Invent. Math.*, 3:308–333; ibid. 4 (1967), 87–117, 1967.

[83] R. R. Winarski. Symmetry, isotopy, and irregular covers. *Geom. Dedicata*, 177:213–227, 2015.

[84] A. Wright. Spheres in the curve graph and linear connectivity of the Gromov boundary. Preprint, `arXiv:2304.03004`.

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