A GEOMETRIC AND ALGEBRAIC DESCRIPTION OF ANNULAR BRAID GROUPS

Autumn Kent AND David Peifer

Department of Mathematics
University of North Carolina at Asheville
Asheville, NC 28804-3299

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Abstract. We provide a new presentation for the annular braid group. The annular braid group is known to be isomorphic to the finite type Artin group with Coxeter graph B_n. Using our presentation, we show that the annular braid group is a semidirect product of an infinite cyclic group and the affine Artin group with Coxeter graph $\tilde{A}_{n-1}$. This provides a new example of an infinite type Artin group which injects into a finite type Artin group. In fact, we show that the affine braid group with Coxeter graph $\tilde{A}_{n-1}$ injects into the braid group on $n+1$ stings. Recently it has been shown that the braid groups are linear, see [3]. Therefore, this shows that the affine braid groups are also linear.

§1: Introduction

In this paper we examine a variation on the classical braid groups known as the annular braid groups. We denote by CB_n the annular braid group on n strings. A precise definition is given below. In section two we will examine the geometry of CB_n and show that CB_n is isomorphic to D_{n+1}. The group D_{n+1}, first studied by W. Chow in 1948 [5], is a well known finite index subgroup of the braid group on $n+1$ strings. It is the subgroup of braids for which the string beginning in position one also ends in position one. The correspondence between CB_n and D_{n+1} has been examined by other authors. See for example Crisp [6], who provides a proof that CB_n is the finite type Artin group with Coxeter graph B_n. The geometry we construct for the annular braid group suggests a new presentation for this group that closely resembles the classical braid$_1$presentation. This is the presentation $\mathcal{P}$

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given below. In section three, we prove that this presentation is complete. In the final section we provide a few algebraic consequences of our presentation.

The braid groups were first defined by Emil Artin in 1925, [1]. Artin's presentation for the braid group on n strings is,

$$A_n = \langle \sigma_1, \sigma_2, ..., \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \text{ for } i = 1,..,n-2,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2\rangle.$$

There are several good references on the theory of braid groups, see for example [2], [4], or [9].

In order to construct Artin's presentation, picture a braid as n paths, "strings," in $\mathbb{R}^3$. These strings begin at equally spaced points along a line segment in the plane $z = a$, then run monotonically down with respect to the z-axis while twisting together, and finally end at analogous points in the plane $z = b$. Now, project the braid onto the plane that contains the initial and final line segments. For any braid this can be done so that one crossing is encountered at a time as one moves down the projection. The i^{th} generator represents the i^{th} string crossing over the $i+1^{st}$ string. The relations correspond to the equivalences of the braids pictured in Figure 1.

Figure 1: The relations of A_n.

Annular braids can be defined in analogy to classical braids. The strings are again continuous paths moving monotonically down the z-axis, in $\mathbb{R}^3$. The paths begin at equally spaced points along a unit circle in the plane $z = a$ and end along the analogous points of a unit circle in the plane $z = b$. For annular braids, we stipulate that the paths never intersect the z-axis. That is, they live in $\mathbb{R}^3$ minus the z-axis, denoted by L. Two annular braids are equivalent if one can be deformed within L by a braid isotopy into the second. Notice that the deformation is done without passing strings through each other, through the z-axis, nor moving any string outside the defining planes, $z = a$ and $z = b$. With the same stacking operation as braids, all annular braids on n strings form a group, denoted by CB_n.

Now, instead of projecting onto a plane, we project onto a cylinder as follows. First, thicken the z-axis to form a cylinder. We call this the core. Now, project the paths onto the core's surface and view the projection from the outside.

This projection suggests a presentation for CB_n similar to Artin's presentation of A_n. There is a generator σ_i corresponding to the i^{th} string passing over the $i+1^{st}$. Notice that CB_n will contain a generator σ_0, corresponding to the last string crossing over the first. Since we are projecting onto a cylinder, the twist that takes each string to the preceding position involves no crossings, and we have a new generator, τ. In section three, we will prove that the set $\{\tau, \sigma_0, ...\sigma_{n-1}\}$ is a generating set of CB_n.

Aside from our two new generators σ_0 and τ, the only difference between A_n and CB_n is that distance between generators is calculated mod n, due to our cylindrical projection. The generator τ introduces the nice property that conjugating σ_i by τ produces σ_{i+1}, see Figure 2. So, we arrive at the presentation

$$\mathcal{P} = \langle \tau, \sigma_0, \sigma_1, \sigma_2, ..., \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \text{for } i = 0, 1, .., n-1,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \neq 1, n-1, \quad \tau\sigma_i\bar{\tau} = \sigma_{i+1},\ \text{for } i = 0, 1, .., n-1\rangle.$$

Our main result is

Theorem One. *CB_n, the annular braid group on n strings, has presentation $\mathcal{P}$.*

Figure 2: Conjugation by τ.

In order to prove Theorem One, we prove the two following theorems.

Theorem Two. *CB_n is isomorphic to D_{n+1}.*

Theorem Three. *Chow's presentation $\mathcal{D}$, of D_{n+1}, is equivalent to $\mathcal{P}$.*

Theorem Two has been shown by others, for example see [6]. Most of the work in this paper is done to prove Theorem Three. The proof of Theorem Three is purely algebraic in nature. The argument is based on Tietze transformations. Together Theorem Two and Theorem Three prove Theorem One.

The significance of presentation $\mathcal{P}$ is that it implies that CB_n is the semidirect product of an infinite cyclic group and the affine (or euclidean) braid group with Coxeter graph $\tilde{A}_{n-1}$. For simplicity we will denote this affine Artin group by $\tilde{A}_{n-1}$. The group $\tilde{A}_{n-1}$ is an infinite type Artin group. That is, the associated Coxeter group is infinite. On the other hand, the group CB_n is known to be a finite type Artin group, see [6]. By the nature of the semidirect product, $\tilde{A}_{n-1}$ injects into CB_n. Likewise, CB_n injects into A_{n+1}. Therefore we have given an example of an infinite type Artin group $\tilde{A}_{n-1}$, that injects into both the finite type Artin group CB_n and the braid group A_{n+1}. Recently, Bigelow [3] has shown that the braid groups are linear. Therefore since $\tilde{A}_{n-1}$ injects into the braid group A_{n+1}, an immediate consequence is

Corollary One. *The affine braid group $\tilde{A}_n$ is linear.*

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§2: The geometry of D_{n+1} and CB_n.

The group D_{n+1} is a subgroup of B_{n+1} and has the geometry of the classical braids with the stipulation that the first string begins and ends at the same point in the initial and final planes. Chow's presentation of this group is

$$\mathcal{D} = \langle \gamma_2, \gamma_3, ..., \gamma_n, a_2, a_3, ..., a_{n+1} \mid \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1},$$

$$\gamma_i\gamma_j = \gamma_j\gamma_i \quad \text{for } |i-j| \geq 2, \quad \gamma_i a_k \bar{\gamma}_i = a_k \quad \text{for } k \neq i, i+1,$$

$$\gamma_i a_i \bar{\gamma}_i = a_{i+1}, \quad \gamma_i a_{i+1}\bar{\gamma}_i = \bar{a}_{i+1} a_i a_{i+1}\rangle.$$

In this presentation, γ_i corresponds to the i^{th} string passing over the $i+1^{st}$. The subscripts start at 2 so that none of these crossings involve the first string. The a_i correspond to braids involving the first string. Specifically, the generator a_i corresponds to the first string wrapping behind the 2^{nd} through $i-1^{st}$ strings, crossing over and then under the i^{th}, and finally returning back behind the other strings to it's original position. Examples of these generators are pictured in Figure 3.

Define the map ϕ from D_{n+1} to CB_n as follows. Let y be a braid in D_{n+1}. Notice that y is a classical braid, on $n+1$ strings, with the first string ending in the first position. Pull the first string tight and thicken it to form a cylinder. This cylinder will be the core of the corresponding circular braid. Carefully wrap the rest of the strings back around the core so that they reverse order. That is, the 2^{nd} braid string becomes the $n-1^{st}$ circular braid string, and the $n+1^{st}$ braid string becomes the 0^{th} circular braid string. While wrapping, no string is allowed to pass through the core or any of the other strings. This process takes the initial and final line segments of the braid and wraps them into circles, creating a circular braid on n strings. Figure 3 shows examples of this correspondence. The map, $\phi : D_{n+1} \to CB_n$, is given algebraically as follows,

$$\phi(\gamma_i) = \sigma_{n-i}, \quad \phi(a_2) = \bar{\tau}\bar{\sigma}_0 ... \bar{\sigma}_{n-2}, \quad \text{and}$$

$$\phi(a_i) = (\sigma_{(n+1)-i} ... \sigma_{n-2})(\bar{\tau}\bar{\sigma}_0 ... \bar{\sigma}_{n-2})(\bar{\sigma}_{n-2} ... \bar{\sigma}_{(n+1)-i}), \quad \text{for } i = 3, ..., n+1.$$

Figure 3: The isomorphism $\phi : D_{n+1} \to CB_n$.

This map is clearly a homomorphism since attaching two elements of D_{n+1} and then wrapping them around the first string results in the same c-braid as when the two $n+1$ braids are first wrapped around their first string and then attached. The generators, $\sigma_0, ..., \sigma_{n-2}$ are clearly in the image. It is easy to check that the generator τ is the image of $\bar{\gamma}_n ... \bar{\gamma}_2 \bar{a}_2$ and that the generator σ_{n-1} is the image of $(a_2\gamma_2 ... \gamma_n)\gamma_n(\bar{\gamma}_n ... \bar{\gamma}_2\bar{a}_2)$. So ϕ is onto. Suppose x is in the kernel of ϕ. Then since its image can be deformed into the trivial c-braid, an equivalent deformation of x will produce the trivial $n+1$ braid in D_{n+1}. Therefore, by the First Isomorphism Theorem, ϕ is an isomorphism. This proves Theorem Two.

Now, to prove that $\mathcal{P}$ is a presentation of CB_n, it will suffice to show that $\mathcal{P}$ is a presentation of D_{n+1}. This is the main result of the next section.

§3: The equivalence of the presentations $\mathcal{D}$ and $\mathcal{P}$.

The proof of Theorem Three is based on Tietze transformations, which are as follows. Given a group $G = \langle \mathcal{X} | \mathcal{R} \rangle$,

T1: If ω is a relation on $\mathcal{X}$ and ω can be derived from the set $\mathcal{R}$, then add ω to $\mathcal{R}$.

T2: If ω is a relation in $\mathcal{R}$ that can be derived from the other relations, then remove ν from $\mathcal{R}$.

T3: If ν is a word on $\mathcal{X}$ and t is a symbol not in $\mathcal{X}$, then add t to the generating set and $t = \nu$ to $\mathcal{R}$.

T4: If one relation takes the form $t = \nu$, where $t \in \mathcal{X}$ and ν is a word on $\mathcal{X} - \{t\}$, then remove t from the set $\mathcal{X}$, remove $t = \nu$ from $\mathcal{R}$, and substitute ν for all occurrences of t in the remaining relations.

In 1908, Tietze proved

Tietze's Theorem. *Given two finite presentations, they present the same group iff there is a finite sequence of transformations, T1-T4, that transforms the first presentation into the second.*

For further information on Tietze transformations see either [7] or [8].

As a first step towards Theorem Three, we provide two simplified versions of the presentation $\mathcal{P}$. First note that the relations involving τ have an immediate consequence. Take a relation of the first type in $\mathcal{P}$, for example $\sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1$. Conjugating both sides by τ, results in the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

$$\sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1 \implies \tau(\sigma_0\sigma_1\sigma_0)\bar{\tau} = \tau(\sigma_1\sigma_0\sigma_1)\bar{\tau}$$

$$\implies (\tau\sigma_0\bar{\tau})(\tau\sigma_1\bar{\tau})(\tau\sigma_0\bar{\tau}) = (\tau\sigma_1\bar{\tau})(\tau\sigma_0\bar{\tau})(\tau\sigma_1\bar{\tau}) \implies \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

Likewise, each relation of this type is obtained by repeated conjugation. Therefore, by repeated applications of $T2$, only one of these relations is necessary in the presentation. Conjugation by τ also effects the second type of relation. Take one of the relations stating that two generators commute if their indices are two apart, for example $\sigma_0\sigma_2 = \sigma_2\sigma_0$. Successive conjugation by τ generates all the commuting relations whose indices are two apart mod n. So, again by repeated applications of $T2$, only one relation of this type is needed. A similar argument shows that $\sigma_0\sigma_3 = \sigma_3\sigma_0$ generates all the commuting relations whose indices are three apart mod n. Thus the presentation can be simplified to include only one relation of the first type and of the second only one for each integer $2 \leq |i-j| \leq \frac{n}{2}$. This proves the following lemma.

Lemma A. *The presentation $\mathcal{P}$ is equivalent to the presentation*

$$\mathcal{P}_1 = \langle \tau, \sigma_0, \sigma_1, \sigma_2, ..., \sigma_{n-1} \mid \sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1,$$

$$\sigma_0\sigma_i = \sigma_i\sigma_0 \textit{ for } 2 \leq i \leq \frac{n}{2},$$

$$\tau\sigma_i\bar{\tau} = \sigma_{(i+1)(\text{mod n})} \textit{ for } i = 0, ..., n-1\rangle.$$

Continuing to simplify, we arrive at a presentation with two generators. Again, τ can be thought of as the twist that moves each string back one position, and σ represents a generic crossing. All other crossings are conjugates of σ by powers of τ.

Lemma B. *The presentation $\mathcal{P}$ is equivalent to the presentation*

$$\mathcal{P}_2 = \langle \tau, \sigma \mid \sigma(\tau\sigma\bar{\tau})\sigma = (\tau\sigma\bar{\tau})\sigma(\tau\sigma\bar{\tau}), \quad \sigma = \tau^n\sigma\bar{\tau}^n,$$

$$\sigma(\tau^k\sigma\bar{\tau}^k) = (\tau^k\sigma\bar{\tau}^k)\sigma \quad \textit{for } 2 \leq k \leq \frac{n}{2}\rangle.$$

Proof. First, by using T4, the generator σ_1 and the relation $\sigma_1 = \tau\sigma_0\bar{\tau}$ may be removed while changing all occurrences of σ_1 in the other relations to $\tau\sigma_0\bar{\tau}$. After this, perform another application of T4. This time the generator σ_2 and the relation $\sigma_2 = \tau^2\sigma_0\bar{\tau}^2$ (note $\sigma_2 = \tau\sigma_1\bar{\tau}$ has been converted to $\sigma_2 = \tau^2\sigma_0\bar{\tau}^2$ by the first application of T4) may be removed and every occurrence of σ_2 replaced by $\tau^2\sigma_0\bar{\tau}^2$.

Continue iterative applications of T4 to remove all σ_i except σ_0. Now rename σ_0, σ. The result is the desired presentation.

■

Lemmas A and B result in a simple two generator presentation for CB_n, with generators τ and σ. We now add to this presentation new generators that represent the generators of Chow's group, D_{n+1}. The generators in the set $\mathcal{Y} = \{\gamma_2, ..., \gamma_n, a_2, ..., a_{n+1}\}$ are added by applying a sequence of T3 transformations. To do this each element of $\mathcal{Y}$ must be assigned a word on $\mathcal{X}$. The intuition for choosing these words comes from the geometric correspondence discussed in section two. In essence, we have simply wrapped the classical braid around its first string to form a circular braid. The next lemma adds the generators in the set $\mathcal{Y}$ to the presentation.

Lemma C. *The presentation $\mathcal{P}$ is equivalent to the presentation*

$$\mathcal{P}_3 = \langle \tau, \sigma, \gamma_2, ..., \gamma_n, a_2, ..., a_{n+1} \mid \sigma(\tau\sigma\bar{\tau})\sigma = (\tau\sigma\bar{\tau})\sigma(\tau\sigma\bar{\tau}), \quad \sigma = \tau^n\sigma\bar{\tau}^n,$$

$$\sigma(\tau^k\sigma\bar{\tau}^k) = (\tau^k\sigma\bar{\tau}^k)\sigma \quad \textit{for } 2 \leq k \leq \frac{n}{2}, \quad \gamma_i = \bar{\tau}^i\sigma\tau^i \quad \textit{for } i = 2, ..., n,$$

$$a_2 = \bar{\tau}\bar{\gamma}_n...\bar{\gamma}_2, \quad a_{i+1} = \gamma_i a_i \bar{\gamma}_i \quad \textit{for } i = 2, ..., n\rangle.$$

At this point we could remove both the generators τ and σ, using two T4 transformations. In the next lemma we will remove σ, however it will be convenient to leave τ. Notice that, from this point on, τ could be thought of as shorthand notation for the word $\bar{\gamma}_n...\bar{\gamma}_2\bar{a}_2$. The rest of the proof of Theorem Three will require using T1 and T2 moves to add relations of the form found in $\mathcal{D}$ and to remove any other relations. This will require several lemmas. The next lemma will begin this process by converting the first three types of relations in $\mathcal{P}_3$ to relations found in $\mathcal{D}$.

Lemma D. *The presentation $\mathcal{P}$ is equivalent to the presentation*

$$\mathcal{P}_4 = \langle \tau, \gamma_2, ..., \gamma_n, a_2, ..., a_{n+1} \mid \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1} \textit{ for } i = 2, ..., n-1$$

$$\gamma_i\gamma_j = \gamma_j\gamma_i \quad \textit{for } |i-j| \geq 2, \quad \gamma_i = \bar{\tau}^i\gamma_n\tau^i \quad \textit{for } i = 2, ..., n,$$

$$a_2 = \bar{\tau}\bar{\gamma}_n...\bar{\gamma}_2, \quad a_{i+1} = \gamma_i a_i \bar{\gamma}_i \quad \textit{for } i = 2, ..., n\rangle.$$

Proof. Notice that the second and fourth relations of $\mathcal{P}_3$ imply that $\gamma_n = \sigma$. Thus we may remove σ from the generators and replace all occurrences of σ in the relation by γ_n. Therefore, the first four relation types of $\mathcal{P}_3$ become

$$\gamma_n(\tau\gamma_n\bar{\tau})\gamma_n = (\tau\gamma_n\bar{\tau})\gamma_n(\tau\gamma_n\bar{\tau}), \quad \gamma_n = (\tau^n\gamma_n\bar{\tau}^n),$$

$$\gamma_n(\tau^k\gamma_n\bar{\tau}^k) = (\tau^k\gamma_n\bar{\tau}^k)\gamma_n \;\text{ for } 2 \leq k \leq \frac{n}{2}, \quad \gamma_i = \bar{\tau}^i\gamma_n\tau^i \;\text{ for } i = 2, ..., n.$$

The second of these relations is the last case of the fourth type, thus it may be removed.

To convert the first type, use the fourth type relation as follows,

$$\gamma_n(\tau\gamma_n\bar{\tau})\gamma_n = (\tau\gamma_n\bar{\tau})\gamma_n(\tau\gamma_n\bar{\tau})$$

$$\Leftrightarrow \gamma_n\tau^{i+1}\bar{\tau}^i\gamma_n\tau^i\bar{\tau}^{i+1}\gamma_n = \tau^{i+1}\bar{\tau}^i\gamma_n\tau^i\bar{\tau}^{i+1}\gamma_n\tau^{i+1}\bar{\tau}^i\gamma_n\tau^i\bar{\tau}^{i+1}$$

$$\Leftrightarrow (\bar{\tau}^{i+1}\gamma_n\tau^{i+1})(\bar{\tau}^i\gamma_n\tau^i)(\bar{\tau}^{i+1}\gamma_n\tau^{i+1}) = (\bar{\tau}^i\gamma_n\tau^i)(\bar{\tau}^{i+1}\gamma_n\tau^{i+1})(\bar{\tau}^i\gamma_n\tau^i)$$

$$\Leftrightarrow \gamma_{i+1}\gamma_i\gamma_{i+1} = \gamma_i\gamma_{i+1}\gamma_i \text{ for } i = 2, ..., n-1.$$

Now an application of T1 and T2 convert the first type of relation in $\mathcal{P}_3$ to the first type in $\mathcal{P}_4$. Similarly, the third type of relation can be converted. First notice that the third type relation in $\mathcal{P}_3$ is actually true for $2 \leq k \leq n-2$. This is a consequence of the fourth type relation. Now use T1, T2, and the following equivalence.

$$\gamma_n(\tau^k\gamma_n\bar{\tau}^k) = (\tau^k\gamma_n\bar{\tau}^k)\gamma_n \;\text{ for } 2 \leq k \leq n-2$$

$$\Leftrightarrow \gamma_n\tau^i\bar{\tau}^j\gamma_n\tau^j\bar{\tau}^i = \tau^i\bar{\tau}^j\gamma_n\tau^j\bar{\tau}^i\gamma_n \;\text{ for } i-j = k$$

$$\Leftrightarrow (\bar{\tau}^i\gamma_n\tau^i)(\bar{\tau}^j\gamma_n\tau^j) = (\bar{\tau}^j\gamma_n\tau^j)(\bar{\tau}^i\gamma_n\tau^i) \Leftrightarrow \gamma_i\gamma_j = \gamma_j\gamma_i \text{ for } i-j = 2, ..., n-2.$$

This proves the lemma.

■

In the following proofs, frequently an equivalent relation will be a direct result of one of the five types of relations found in Lemma D. We use Di to signify an application of the i^{th} type of relation in $\mathcal{P}_4$. The next lemma states a few helpful consequences of the relations in $\mathcal{P}_4$.

Lemma E. *The following relations hold in $\mathcal{P}_4$.*

1) $\gamma_j \bar{\tau} \bar{\gamma}_{j-1} = \bar{\tau}$ *for* $j = 3, ..., n$
2) $\gamma_j(\gamma_{k-1}...\gamma_2) = (\gamma_{k-1}...\gamma_2)\gamma_j, \quad \gamma_j(\gamma_2...\gamma_{k-1}) = (\gamma_2...\gamma_{k-1})\gamma_j$ *for* $j \geq k+1$ *and*
   $\gamma_j(\gamma_{k-1}...\gamma_2) = (\gamma_{k-1}...\gamma_2)\gamma_{j+1}, \quad \gamma_{j+1}(\gamma_2...\gamma_{k-1}) = (\gamma_2...\gamma_{k-1})\gamma_j$ *for* $j \leq k$
3) $\gamma_j a_2 \bar{\gamma}_j = a_2$ *for* $j = 3, ..., n$
4) $\gamma_{j-1}(\gamma_i...\gamma_2 a_j \bar{\gamma}_2...\bar{\gamma}_i)\bar{\gamma}_{j-1} = \gamma_i...\gamma_2 a_{j+1} \bar{\gamma}_2...\bar{\gamma}_i$ *for* $j = 3, ..., i-1$
5) $a_k = \gamma_{k-1}...\gamma_2 a_2 \bar{\gamma}_2...\bar{\gamma}_{k-1}$ *for* $k = 3, ..., n+1$
6) $\gamma_i \bar{\tau} = \bar{\tau}\gamma_{i-1}$ *for* $i = 3, ..., n$.

Proof. (1) Using $D3$, we have $\bar{\tau}\gamma_j\tau = \bar{\tau}(\bar{\tau}^j\gamma_n\tau^j)\tau = \bar{\tau}^{j+1}\gamma_n\tau^{j+1} = \gamma_{j+1}$ for $j = 2, ..., n-1$. This implies that $\gamma_{j+1}\bar{\tau}\bar{\gamma}_j = \bar{\tau}$.

(2) The first half is clear because when $j \geq k+1$, γ_j commutes with all the other γ's by $D2$. Suppose $j \leq k-2$. Both equalities follow from similar proofs. The first is done by using $D1$, the fact that γ_j commutes with the first γ's, and γ_{j+1} commutes with the rest to get

$$\gamma_j(\gamma_{k-1}...\gamma_2) = \gamma_{k-1}...\gamma_{j+2}(\gamma_j\gamma_{j+1}\gamma_j)\gamma_{j-1}...\gamma_2$$

$$= \gamma_{k-1}...\gamma_{j+2}(\gamma_{j+1}\gamma_j\gamma_{j+1})\gamma_{j-1}...\gamma_2 = (\gamma_{k-1}...\gamma_2)\gamma_{j+1}.$$

(3) Noting that $j \leq n$, use part 2 and then part 1 from above to get the following.

$$\gamma_j a_2 \bar{\gamma}_j = \gamma_j(\bar{\tau}\bar{\gamma}_n...\bar{\gamma}_2)\bar{\gamma}_j = \gamma_j\bar{\tau}\bar{\gamma}_{j-1}(\bar{\gamma}_n...\bar{\gamma}_2) = \bar{\tau}(\bar{\gamma}_n...\bar{\gamma}_2) = a_2.$$

(4) This follows from part 2,

$$\gamma_{j-1}(\gamma_i...\gamma_2 a_j\bar{\gamma}_2...\bar{\gamma}_i)\bar{\gamma}_{j-1} = \gamma_i...\gamma_2(\gamma_j a_j \bar{\gamma}_j)\bar{\gamma}_2...\bar{\gamma}_i = \gamma_i...\gamma_2(a_{j+1})\bar{\gamma}_2...\bar{\gamma}_i.$$

(5) and (6) follow easily from $D5$.

■

In the following proofs equivalences based on Lemma E will be denoted by E1 - E6.

Lemma F. *The relations $\gamma_i a_k \bar{\gamma}_i = a_k$ for $k \neq i, i+1$, are consequences of the relations in $\mathcal{P}_4$, and therefore may be added to the presentation.*

Proof. Consider a fixed value of k. Note that $k = 2$ is part 3 of the previous lemma, so we may assume $k \geq 3$. There are two cases to consider, $i > k$ and $i < k$.

Case 1: $i > k$. $\gamma_i a_k \bar{\gamma}_i =_{E5} \gamma_i(\gamma_{k-1}...\gamma_2 a_2 \bar{\gamma}_2...\bar{\gamma}_{k-1})\bar{\gamma}_i =_{E2} (\gamma_{k-1}...\gamma_2)\gamma_i a_2 \bar{\gamma}_i(\bar{\gamma}_2...\bar{\gamma}_{k-1}) =_{E3} (\gamma_{k-1}...\gamma_2)a_2(\bar{\gamma}_2...\bar{\gamma}_{k-1}) = a_k$.

Case 2: $i < k$. We need only consider $i < k-1$, because $k \neq i, i+1$. $\gamma_i a_k \bar{\gamma}_i =_{E5} \gamma_i(\gamma_{k-1}...\gamma_2 a_2 \bar{\gamma}_2...\bar{\gamma}_{k-1})\bar{\gamma}_i =_{E2} (\gamma_{k-1}...\gamma_2)\gamma_{i+1} a_2 \bar{\gamma}_{i+1}(\bar{\gamma}_2...\bar{\gamma}_{k-1}) =_{E3} (\gamma_{k-1}...\gamma_2)a_2(\bar{\gamma}_2...\bar{\gamma}_{k-1}) = a_k$.

■

There is one more type of relation that must be added to the presentation $\mathcal{P}_4$. We do this inductively: the first case in Lemma G, the generalization follows in Lemma H.

Lemma G. *In $\mathcal{P}_4$, $\bar{a}_3 a_2 a_3 = \gamma_2 a_3 \bar{\gamma}_2$.*

Proof. $\bar{a}_3 a_2 a_3 =_{D5} (\gamma_2 \bar{a}_2 \bar{\gamma}_2) a_2 (\gamma_2 a_2 \bar{\gamma}_2)$

$$=_{D4} \gamma_2(\gamma_2...\gamma_n\tau)\bar{\gamma}_2(\bar{\tau}\bar{\gamma}_n...\bar{\gamma}_2)\gamma_2(\bar{\tau}\bar{\gamma}_n...\bar{\gamma}_2)\bar{\gamma}_2$$

$$= \gamma_2(\gamma_2...\gamma_n\tau)\bar{\gamma}_2\bar{\tau}(\bar{\gamma}_n...\bar{\gamma}_3)\bar{\tau}\bar{\gamma}_n...\bar{\gamma}_2\bar{\gamma}_2$$

$$=_{E6} \gamma_2\gamma_2...\gamma_n(\tau\bar{\gamma}_2\bar{\tau}\bar{\tau})(\bar{\gamma}_{n-1}...\bar{\gamma}_2)(\bar{\gamma}_n...\bar{\gamma}_2)\bar{\gamma}_2$$

$$=_{D3} \gamma_2\gamma_2...\gamma_n(\bar{\tau}\bar{\gamma}_n)(\bar{\gamma}_{n-1}...\bar{\gamma}_2)\bar{\gamma}_n...\bar{\gamma}_2\bar{\gamma}_2$$

$$=_{D4} \gamma_2\gamma_2...\gamma_n(a_2)\bar{\gamma}_n...\bar{\gamma}_2\bar{\gamma}_2 =_F \gamma_2\gamma_2 a_2\bar{\gamma}_2\bar{\gamma}_2 =_{D5} \gamma_2 a_3 \bar{\gamma}_2.$$

■

Lemma H. *The relations* $\bar{a}_{i+1}a_ia_{i+1} = \gamma_i a_{i+1}\bar{\gamma}_i$ *for* $i = 2, ..., n$ *are consequences of the relations in* $\mathcal{P}_4$*, and therefore may be added to the presentation.*

Proof. By Lemma G, we may assume $i \geq 3$. Consider the following relation.

$$\bar{a}_{i+1}a_ia_{i+1} =_{E5} \bar{a}_{i+1}(\gamma_{i-1}...\gamma_2 a_2\bar{\gamma}_2...\bar{\gamma}_{i-1})a_{i+1}$$

$$=_F \gamma_{i-1}...\gamma_2(\bar{a}_{i+1}a_2a_{i+1})\bar{\gamma}_2...\bar{\gamma}_{i-1}$$

$$=_{E5} \gamma_{i-1}...\gamma_2(\gamma_i...\gamma_3\bar{a}_3\bar{\gamma}_3...\bar{\gamma}_i)a_2(\gamma_i...\gamma_3a_3\bar{\gamma}_3...\bar{\gamma}_i)\bar{\gamma}_2...\bar{\gamma}_{i-1}$$

$$=_F \gamma_{i-1}...\gamma_2(\gamma_i...\gamma_3\bar{a}_3a_2a_3\bar{\gamma}_3...\bar{\gamma}_i)\bar{\gamma}_2...\bar{\gamma}_{i-1}$$

$$=_G \gamma_{i-1}...\gamma_2(\gamma_i...\gamma_3)(\gamma_2a_3\bar{\gamma}_2)(\bar{\gamma}_3...\bar{\gamma}_i)\bar{\gamma}_2...\bar{\gamma}_{i-1}$$

Now by repeated applications of E4, the conjugation by $\gamma_{i-1}...\gamma_2$ may be eliminated starting with γ_2. Each application of E4 pushes the index of a up by one. The result is $\gamma_i...\gamma_2a_{i+1}\bar{\gamma}_2...\bar{\gamma}_i$. Now noticing that a_{i+1} commutes with all the γ's except for γ_i we get that,

$$\bar{a}_{i+1}a_ia_{i+1} = \gamma_i...\gamma_2a_{i+1}\bar{\gamma}_2...\bar{\gamma}_i = \gamma_ia_{i+1}\bar{\gamma}_i.$$

Using T3 and this relation proves the lemma.

■

Now that all the relations of $\mathcal{D}$ are added to the presentation $\mathcal{P}_4$, we must show that the other two types of relations, $D3$ and $D4$, can be removed. Lemma I removes from the presentation the non-$\mathcal{D}$ type relations, $\gamma_i = \bar{\tau}^i\gamma_n\tau^i$. The proof has two steps. The first, and harder step, involves showing that the lowest index case, $\gamma_2 = \bar{\tau}^2\gamma_n\tau^2$, is a consequence of the other relators. After this it is fairly easy to see that the remaining relations of this type are also consequences of the other relators.

Lemma I. *The relations $\gamma_i = \bar{\tau}^i \gamma_n \tau^i$ for $i = 2, ..., n$ are consequences of the relations in Lemma F, Lemma H, and the other relations in $\mathcal{P}_4$.*

Proof. The proof is by induction on i. The base case is $\gamma_2 = \bar{\tau}^2 \gamma_n \tau^2$. Begin with the word $\bar{\tau}^2 \gamma_n \tau^2$. This is equivalent to the word $a_2\gamma_2 a_2 \gamma_2 \bar{a}_2 \bar{\gamma}_2 \bar{a}_2$ by the following,

$$\bar{\tau}^2 \gamma_n \tau^2 =_{D4} (a_2\gamma_2 ... \gamma_n)(a_2\gamma_2 ... \gamma_n)\gamma_n(\bar{\gamma}_n ... \bar{\gamma}_2\bar{a}_2)(\bar{\gamma}_n ... \bar{\gamma}_2\bar{a}_2)$$

$$= a_2\gamma_2 ... \gamma_n a_2 \gamma_2 ... \gamma_n \bar{\gamma}_{n-1} ... \bar{\gamma}_2 \bar{a}_2 \bar{\gamma}_n ... \bar{\gamma}_2 \bar{a}_2$$

$$=_{E2} \ a_2\gamma_2 ... \gamma_n a_2(\bar{\gamma}_n ... \bar{\gamma}_3)(\gamma_2 ... \gamma_n)\bar{a}_2\bar{\gamma}_n ... \bar{\gamma}_2\bar{a}_2 =_F \ a_2\gamma_2 a_2\gamma_2\bar{a}_2\bar{\gamma}_2\bar{a}_2$$

Notice that E2 was used $n-2$ times, each time increasing the index of the $\bar{\gamma}$ as it passed left across $(\gamma_2 ... \gamma_n)$. Then Lemma F was used to cancel all the γ's that commute with a_2. Now by the proper insertion of inverse pairs we get

$$a_2\gamma_2 a_2\gamma_2\bar{a}_2\bar{\gamma}_2\bar{a}_2 = a_2(\gamma_2 a_2\bar{\gamma}_2)\gamma_2(\gamma_2\bar{a}_2\bar{\gamma}_2)\bar{a}_2 =_{D5} \ a_2 a_3\gamma_2\bar{a}_3\bar{a}_2$$

$$= a_2 a_3(\gamma_2\bar{a}_3\bar{\gamma}_2)\gamma_2\bar{a}_2 =_H \ a_2 a_3(\bar{a}_3\bar{a}_2 a_3)\gamma_2\bar{a}_2 = a_3\gamma_2\bar{a}_2$$

$$= a_3(\gamma_2\bar{a}_2\bar{\gamma}_2)\gamma_2 =_{D5} a_3\bar{a}_3\gamma_2 = \gamma_2.$$

Now assume that $i = 3, ..., n$ and that $\gamma_j = \bar{\tau}^j\gamma_n\tau^j$, for all $j < i$. Then

$$\bar{\tau}^i\gamma_n\tau^i = \bar{\tau}(\bar{\tau}^{i-1}\gamma_n\tau^{i-1})\tau = \bar{\tau}\gamma_{i-1}\tau =_{D4} \ a_2(\gamma_2 ... \gamma_n)\gamma_{i-1}\bar{\gamma}_n ... \bar{\gamma}_2\bar{a}_2$$

$$=_{E2} \ a_2\gamma_i(\gamma_2 ... \gamma_n)\bar{\gamma}_n ... \bar{\gamma}_2\bar{a}_2 = a_2\gamma_i\bar{a}_2 =_F \ a_2\bar{a}_2\gamma_i = \gamma_i.$$

Therefore the result is true for $i = 2, ..., n$.

■

Proof of Theorem Three. By Lemma D, $\mathcal{P}$ is equivalent to $\mathcal{P}_4$. By Lemma F and Lemma H, the remaining $\mathcal{D}$ relations can be added to the presentation by T3. By Lemma I the relations of the form $\gamma_i = \bar{\tau}^i\gamma_n\tau^i$ may be removed from the presentation by T4. Finally, τ may be removed along with the relation $a_2 = \bar{\tau}\bar{\gamma}_n ... \bar{\gamma}_2$ by another application of T4. This successfully transforms the presentation $\mathcal{P}$ to the presentation $\mathcal{D}$. ■

§4: Further Results

In this section we give a few algebraic consequences that follow easily from the presentation $\mathcal{P}$ for CB_n. By the structure of the presentation $\mathcal{P}$, it is clear that conjugation by the generator t induces an automorphism of the subgroup generated by the σ_i's. It is also clear that the subgroup generated by the σ_i's is the affine braid group $\tilde{A}_{n-1}$, with presentation,

$$\langle \sigma_0, \sigma_1, \sigma_2, ..., \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \text{ for } i = 0, 1, .., n-1,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \neq 1, n-1\rangle.$$

This shows that the group CB_n is a semidirect product of the infinite cyclic group generated by t, and the affine braid group $\tilde{A}_{n-1}$. Therefore, $\tilde{A}_{n-1}$ injects into CB_n. It is known that Chow's group D_{n+1}, and thus CB_n, is a finite type Artin group, see [6]. On the other hand, $\tilde{A}_{n-1}$ is an infinite type Artin group. Therefore this gives a new example of an infinite type Artin group that injects into a finite type Artin group. Furthermore, CB_n injects into A_{n+1}, so in fact, $\tilde{A}_{n-1}$ also injects into the braid group A_{n+1}. Bigelow has shown that the braid groups are linear, [3]. Therefore we have

Corollary One. *The affine braid group* $\tilde{A}_n$ *is linear.*

To end with, we will show how the presentation $\mathcal{P}$ can be used to construct a two generator presentation for the braid groups. Consider Chow's presentation Δ of D_{n+1}. Notice that the γ_i generate a copy of A_n. So setting the generators a_i equal to the identity leaves a copy of A_n. In D_{n+1}, this is equivalent to assuming that there are no crossings involving the first string. In the geometry of CB_n this is equivalent to removing the core, or filling in the z-axis. This suggests a method for finding a two generator presentation of the braid group A_n. In Δ, setting $a_2 = 1$ results in $a_i = 1$ for all i, since $\gamma_i a_i \bar{\gamma} = a_{i+1}$. Thus if the word that is equivalent to a_2 in $\mathcal{P}_2$ is set equal to the identity, the result will be a two generator presentation of A_n. We prove this algebraically in Corollary Two.

Recall that ϕ maps a_2 to $\bar{\tau}\bar{\sigma}_0...\bar{\sigma}_{n-2}$ in $\mathcal{P}$. Using the fact that $\sigma_i = \tau^i\sigma\bar{\tau}^i$, for $i = 3, ..., n-1$ we can show that a_2 maps to $\bar{\tau}(\bar{\sigma}_0\tau)^{n-1}\bar{\tau}^{n-1}$. To pass from the presentation $\mathcal{P}$ to the presentation $\mathcal{P}_2$ we set σ_0 equal to σ. Therefore setting $a_2 = 1$ in Δ is equivalent to setting $\bar{\tau}(\bar{\sigma}\tau)^{n-1}\bar{\tau}^{n-1} = 1$ in $\mathcal{P}_2$. This relation can also be stated as $\tau^n = (\bar{\sigma}\tau)^{n-1}$. The next theorem shows that adding the relation $\tau^n = (\bar{\sigma}\tau)^{n-1}$ to the presentation $\mathcal{P}_2$, which geometrically corresponds to removing the core, gives a presentation of A_n.

Corollary Two. *A_n has presentation*

$$\mathcal{B} = \langle \tau, \sigma \mid \sigma(\tau\sigma\bar{\tau})\sigma = (\tau\sigma\bar{\tau})\sigma(\tau\sigma\bar{\tau}), \quad \sigma = \tau^n\sigma\bar{\tau}^n,$$

$$\sigma(\tau^i\sigma\bar{\tau}^i) = (\tau^i\sigma\bar{\tau}^i)\sigma \quad \textit{for } 2 \le i \le \frac{n}{2}, \quad \tau^n = (\bar{\sigma}\tau)^{n-1}\rangle.$$

Proof. In his 1925 paper [1], Artin gave a simple two generator presentation of the braid groups. Artin's presentation is

$$\mathcal{A} = \langle \tau, \delta \mid \tau^n = (\tau\delta)^{n-1}, \quad \delta(\tau^i\delta\bar{\tau}^i) = (\tau^i\delta\bar{\tau}^i)\delta, \quad \text{for } 2 \le i \le \frac{n}{2}\rangle.$$

To prove Corollary Two, it will suffice to show that $\mathcal{B}$ is equivalent to $\mathcal{A}$.

First notice that $\tau^n = (\bar{\sigma}\tau)^{n-1} \Leftrightarrow \tau\tau^n\bar{\tau} = \tau(\bar{\sigma}\tau)^{n-1}\bar{\tau} \Leftrightarrow \tau^n = (\tau\bar{\sigma})^{n-1}$. Therefore the last relation of $\mathcal{B}$ may be exchanged for the relation $\tau^n = (\tau\bar{\sigma})^{n-1}$. The above equivalence also implies that $\bar{\sigma}\tau^n = \bar{\sigma}(\tau\bar{\sigma})^{n-1} = (\bar{\sigma}\tau)^{n-1}\bar{\sigma} = \tau^n\bar{\sigma}$. Since $\bar{\sigma}$ commutes with τ, σ must also commute with τ. Thus the second relation in $\mathcal{B}$ is a consequence of the other relations and may be removed.

To remove the first relation in $\mathcal{B}$ it will help to revert back to the notation from the presentation $\mathcal{P}$. That is, we will use σ_i as short hand notation for $\tau^i\sigma_0\bar{\tau}^i$. This part of the proof follows the same line of argument used by Artin. Recall, adding the relation $\tau^n = (\bar{\sigma}\tau)^{n-1}$ to $\mathcal{P}_2$ is equivalent to adding the relation $\tau = \bar{\sigma}_0...\bar{\sigma}_{n-2}$ to $\mathcal{P}$. Now, σ_0 commutes with $\bar{\sigma}_2...\bar{\sigma}_{n-2}$ and so σ_0 commutes with $\sigma_1\sigma_0\tau$. That is, $\sigma_0\sigma_1\sigma_0\tau = \sigma_1\sigma_0\tau\sigma_0$. Since $\tau\sigma_0 = \sigma_1\tau$, we have that $\sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1$. Translating this back into the generators of $\mathcal{P}_2$ we get, $\sigma(\tau\sigma\bar{\tau})\sigma = (\tau\sigma\bar{\tau})\sigma(\tau\sigma\bar{\tau})$ as desired. Therefore the first relation of $\mathcal{B}$ may also be removed.

Finally, substitute $\delta = \bar{\sigma}$. Note that the relation $\tau^n = (\tau\bar{\sigma})^{n-1}$ becomes $\tau^n = (\tau\delta)^{n-1}$ as desired. The remaining relations in $\mathcal{B}$ become $\bar{\delta}(\tau^i\bar{\delta}\bar{\tau}^i) = (\tau^i\bar{\delta}\bar{\tau}^i)\bar{\delta}$ for $2 \leq i \leq \frac{n}{2}$. By taking the inverse of each of these, we get the desired relation in $\mathcal{A}$. This completes the proof.

■

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Department of Mathematics, University of North Carolina at Asheville, Asheville, NC 28804-3299
Current address: Department of Mathematics University of Texas at Austin, Austin, TX 78712
E-mail address: rkent@math.utexas.edu

Department of Mathematics, University of North Carolina at Asheville, Asheville, NC 28804-3299
E-mail address: dpeifer@unca.edu