Skinning bounds along thick rays

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Abstract

We show that the diameter of the skinning map of an acylindrical hyperbolic 3–manifold M is bounded on ε–thick Teichmüller geodesics by a constant depending only on ε and the topological type of ∂M.

1 Introduction

Let M be a compact hyperbolic 3–manifold with totally geodesic boundary X_M. The space of convex cocompact hyperbolic metrics on the interior M° of M is naturally identified with the Teichmüller space $\mathscr{T}(\partial M)$. Given a convex cocompact hyperbolic metric M^X on M° associated to the marked Riemann surface X, the conformal boundary of M^X is X. The covering of M° corresponding to ∂M is a quasifuchsian manifold whose conformal boundary has two components: one conformally equivalent to X, the other the *skinning surface* $\sigma_M(X)$. This defines a map between Teichmüller spaces

$$\sigma_M : \mathscr{T}(\partial M) \to \mathscr{T}(\overline{\partial M})$$

called the *skinning map*. See [4] for more details.

Thurston's Bounded Image Theorem [16] states that

$$\operatorname{diam}(\sigma_M(\mathscr{T}(\partial M))) < \infty,$$

and is instrumental in his proof of hyperbolization for Haken 3–manifolds. Quantitative bounds on the diameter of this map would improve our understanding of the gluing of hyperbolic structures. One may conjecture that there is a bound

$$\operatorname{diam}(\sigma_M(\mathscr{T}(\partial M))) < \mathscr{D}$$

where $\mathscr{D}$ depends only on the topological type of ∂M and not on M itself. Our theorem supports this conjecture.

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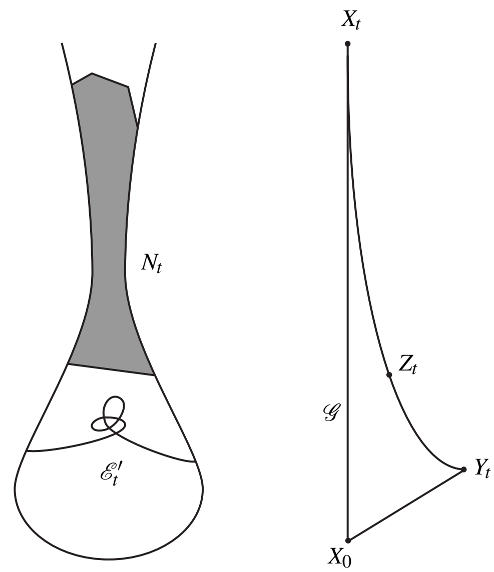

Figure 1: At left is the manifold M^{X_t}, with surface $\mathscr{E}_t'$ and collar N_t in the convex core about the convex core boundary. At right is the geodesic triangle $\triangle X_0 Y_t X_t$.

Theorem 1 (Bound along thick rays)**.** *Let S be a closed orientable surface of genus greater than 1 and let $\varepsilon > 0$. Then there is a $\mathscr{D}$ such that if M is any compact hyperbolic 3–manifold with totally geodesic boundary $X_M \cong S$ and $\mathscr{G}\colon [0,\infty) \to \mathscr{T}(\partial M)$ is any ε–thick Teichmüller geodesic ray with $\mathscr{G}(0) = X_M$, then*

$$\operatorname{diam}(\sigma_M(\mathscr{G}([0,\infty)))) \leqslant \mathscr{D}.$$

Specifically, there are constants A and B depending only on S and ε such that

$$\operatorname{diam}(\sigma_M(g([T,\infty)))) < Ae^{-BT} \text{ for all } T \geq 0.$$

Sketch of the proof

The idea of the proof is as follows, see Figure 1.

Let X_t be the surfaces along the geodesic ray, let Y_t be the mirror image of the skinning surface at X_t, and let $M_t = M^{X_t}$ be the interior of M equipped with the hyperbolic metric corresponding to X_t. McMullen proved [9] that the skinning map of an acylindrical manifold is uniformly contracting, and this means that the distance between X_t and Y_t is growing at a definite linear rate. The geodesic $[X_t, Y_t]$ from X_t to Y_t fellow travels our geodesic $\mathscr{G}$ along a thick segment $[X_t, Z_t]$ of linearly growing length, thanks to work of Rafi [14]. This implies, using work of Brock–Canary–Minsky [3], the existence of a linearly deep and uniformly thick collar about the convex hull boundary of M_t. We establish this in Theorem 3 in Section 3.

In Section 4, we use the Geometric Inflexibility Theorem of Brock–Bromberg [1]. This tells us that, in the complement of the thick collars of Theorem 3, the geometry of the manifold is changing, in a C^1–sense, at a rate exponentially small in t. (Here the metric distortion is measured in terms of the strain field of the family of metrics.)

We formulate two consequences of this. Theorem 11 gives the pointwise C^1 estimates in the form that we will use. Theorem 12 uses an additional estimate from [1] to show that every peripheral curve in M has an absolute lower bound on its geodesic length along the family M_t.

In Lemma 14 of Section 5 we show that, for sufficiently large t, there is a surface $\mathscr{E}_t'$ below the deep collar in M_t that serves as a proxy for the skinning surface Y_t. The surface $\mathscr{E}_t'$ is the immersion in M_t of a suitably smoothed neighborhood of the convex hull boundary facing the skinning end in the quasifuchsian cover of M_t.

In Lemma 15 and Proposition 16 we study the relation between $\mathscr{E}_t'$ and the skinning image Y_t, and use it to show that the speed of Y_t in Teichmüller space is controlled by the C^1 bounds on the strain field established in Theorem 11. The surface $\mathscr{E}_t'$ is uniformly thick by Theorem 12, and so we can apply Theorem 11 to see that Y_t moves exponentially slowly. It follows that the distance between Y_0 and Y_t is uniformly bounded for all t.

2 Constants, norms, and families of metrics

Throughout the paper we will want to keep track of the dependence of constants. To simplify our notation we will say constants are *nice* if they depend only on ε and the topological type of S.

Norms on tensors

Let V be a finite dimensional vector space with an inner product g. Since V is finite dimensional, all norms are equivalent, and we use the *operator norm*. Let x be a vector in V. Then $\|x\|^2 = g(x,x)$, and for a $(r,0)$–tensor τ on V, we define

$$\|\tau\| = \sup_{\|x_i\|=1} |\tau(x_1,\ldots,x_r)|.$$

If τ is an $(r,1)$–tensor, we define

$$\|\tau\| = \sup_{\|x_i\|=1} \|\tau(x_1,\ldots,x_r)\|.$$

If τ is an $(r,0)$– or $(r,1)$–tensor on a Riemannian manifold M then we have an operator norm $\|\tau_p\|$ at each point p and we define $\|\tau\| = \sup_{p\in M} \|\tau_p\|$.

Families

If we have a 1–parameter family of objects ob_t then we write $ob = ob_0$, and $\dot{ob}$ will denote the time–zero derivative.

Families of Riemannian metrics

Given a smooth 1–parameter family of Riemannian metrics g_t on a manifold M, there is, at each time t, a vector–valued 1–form η_t defined by

$$\frac{\partial}{\partial t} g_t(x,y) = 2g(\eta_t(x),y))$$

called the *strain field* at time t associated to the family g_t.

Families of conformal structures

A family of metrics on a surface determines a family of marked conformal structures. Such a family is a path in Teichmüller space, and we are interested in bounding the Teichmüller norm of its derivative in terms of the derivative of the metrics. Namely we will show

Lemma 2. *Let g_t be a smooth family of Riemannian metrics on a surface Σ and X_t the corresponding marked conformal structures in $\mathscr{T}(\Sigma)$. Then*

$$\|\dot{X}\|_{\mathscr{T}} \leq 2\|\dot{g}\|.$$

Proof. The proof is just a calculation. We begin with the case of a 2–dimensional vector space.

Let $\mathrm{Conf}(V)$ be the space of conformal structures on an oriented vector space V. If V is 2–dimensional then this can be identified with orientation preserving $\mathbb{R}$–linear maps from V to $\mathbb{C}$ where two such maps are equivalent if they differ by post-composition with a $\mathbb{C}$–linear map. Given two conformal structures ω_0, ω_1 in $\mathrm{Conf}(V)$, we define the *Teichmüller distance* between them as follows. Let $\lambda_0, \lambda_1 : V \to \mathbb{C}$ be $\mathbb{R}$–linear maps representing ω_0 and ω_1 and let $\mu = \frac{(\lambda_1 \circ \lambda_0^{-1}))_{\bar{z}}}{(\lambda_1 \circ \lambda_0^{-1})_z}$ be the *Beltrami differential*. Then $d_{\mathscr{T}}(\omega_0, \omega_1) = \frac{1}{2} \log \frac{1+|\mu_\lambda|}{1-|\mu_\lambda|}$. Note that while μ depends on the choice of λ_0 (but not λ_1), the absolute value $|\mu|$ only depends on ω_0 and ω_1 so $d_{\mathscr{T}}$ is well defined and one can check that it is a metric.

Let ω_t be a smooth family in $\mathrm{Conf}(V)$ with a smooth family of representatives λ_t and μ_t the Beltrami differentials between λ_0 and λ_t. A computation shows that the time zero derivative of the map $t \mapsto d_{\mathscr{T}}(\omega_0, \omega_t)$ is bounded by $|\dot{\mu}|$.

An inner product g determines a conformal structure by choosing an $\mathbb{R}$–linear map $\lambda : V \to \mathbb{C}$ to be an orientation preserving isometry from (V, g) to the usual Euclidean metric on $\mathbb{C}$. Note that if we multiply g by a scalar we get an equivalent conformal structure. Now take a smooth family g_t of inner products and isometries $\lambda_t : (V, g_t) \to \mathbb{C}$. If we choose an orthonormal basis for (V, g_0) taken by λ_0 to the standard basis of $\mathbb{C}$, then the traceless part $[\dot{g}]$ of $\dot{g}$ is represented by the matrix

$$[\dot{g}] = 2 \begin{pmatrix} \Re\dot{\mu} & \Im\dot{\mu} \\ \Im\dot{\mu} & -\Re\dot{\mu} \end{pmatrix}.$$

Another direct computation gives

$$|\dot{\mu}| = 2\|[\dot{g}]\| \leq 2\|\dot{g}\|. \tag{2.1}$$

A Riemannian metric g on a surface Σ defines a conformal structure on each tangent space and this defines a conformal structure on Σ, and hence a point in $\mathscr{T}(\Sigma)$. Given a diffeomorphism $f : (\Sigma, g_0) \to (\Sigma, g_1)$, the pointwise identification $df_p : T_p\Sigma \to T_{f(p)}\Sigma$ allows us to compare the conformal structures as above, and in particular to define a Beltrami differential μ_f whose absolute value $|\mu_f|$ is well-defined independently of coordinates. We can then write

$$d_{\mathscr{T}}(g_0, g_1) = \inf_{f \in \mathrm{Diff}_0(\Sigma)} \frac{1}{2} \log \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty} \tag{2.2}$$

where $\mathrm{Diff}_0(\Sigma)$ is the space of diffeomorphisms of Σ isotopic to the identity. This defines a pseudometric on the space of Riemannian metrics on Σ. The quotient metric space is the Teichmüller space $\mathscr{T}(\Sigma)$, which can be given a differentiable structure so that $d_{\mathscr{T}}$ is a Finsler metric, associated to a norm $\|\cdot\|_{\mathscr{T}}$ on the tangent space. If g_t is a smooth family of Riemannian metrics on Σ, and X_t are the corresponding marked conformal structures in $\mathscr{T}(\Sigma)$, then by differentiating (2.2) we obtain $\|\dot{X}\|_{\mathscr{T}} \leq \|\dot{\mu}\|_\infty$. Now using (2.1) we complete the proof of Lemma 2. □

3 Thick collar

In this section we let X_t and M_t be as in the introduction. Let $\mathrm{core}(M_t)$ denote the convex core of M_t. Our goal is the following statement:

Theorem 3 (Thick collar in $\mathrm{core}(M_t)$)**.** *There exists $t_1 > 0$ and $\delta > 0$ depending on S, ε such that for $t > t_1$ there is a collar neighborhood of $\partial\mathrm{core}(M_t)$ in $\mathrm{core}(M_t)$ which is ε_2–thick and contains a δt–neighborhood of the boundary.*

Uniform contraction

McMullen showed [9] that if M is a compact hyperbolic 3–manifold with totally geodesic boundary, then σ_M is uniformly contracting. Namely, if $\mathrm{d}\sigma_M$ is the derivative of σ_M and $\|\,\mathrm{d}\sigma_M\,\|_{\mathscr{T}}$ its Teichmüller norm as in Section 2, then

$$\|\,\mathrm{d}\sigma_M\,\|_{\mathscr{T}} < c_M < 1$$

over the entire Teichmüller space for some constant c_M depending on M. Remarkably, the proof provides uniform contraction independent of M.

Theorem 4 (McMullen [9])**.** *There is a constant c_S such that if M is any hyperbolic 3–manifold with totally geodesic boundary $\partial M \cong S$, then*

$$\|\,\mathrm{d}\sigma_M\,\|_{\mathscr{T}} < c_S < 1$$

at every point in $\mathscr{T}(S)$.

Remark on the proof. The proof of uniform contraction in [9] makes very little use of the topology of the 3–manifold M, and relies only on the facts that M is compact, irreducible, acylindrical, atoroidal, and boundary incompressible. The main argument, in the proof of Theorem 6.1 of [9], obtains uniform contraction by considering the possible geometric limits of a sequence of potential counterexamples. This argument works just as well if one allows the underlying 3–manifolds to vary over the sequence while fixing the topological type of the boundary. □

Thick segments

Let $X_t = \mathscr{G}(t)$ be as in the hypothesis of the main theorem, and let $Y_t = \overline{\sigma_M(X_t)}$. Note that $Y_0 = \overline{\sigma_M(X_0)} = X_0$.

Theorem 4 bounds the speed

$$\|Y_t'\|_{\mathscr{T}} < c < 1,$$

where $c = c_S$ depends only on S. We conclude that

$$\mathrm{d}(Y_0, Y_t) \leq ct \tag{3.1}$$

and so

$$\mathrm{d}(X_t, Y_t) \geq (1-c)t$$

by the triangle inequality.

For X, Y in $\mathscr{T}(S)$, let $[X,Y]$ denote the Teichmüller geodesic between them.

To produce our thick collar, we begin by showing that the geodesic $[X_t, Y_t]$ has an initial segment $[X_t, Z_t]$ that is ε_1–thick for a nice ε_1. To do this, we use the coarse hyperbolicity that Teichmüller space exhibits in its thick part. Theorem 5 below, due to Minsky, says that thick geodesics have coarsely Lipschitz closest points projections. Theorem 6 below, due to Rafi, says that geodesic triangles in $\mathscr{T}(S)$ try to be thin triangles when they are in the thick part. That is, a point in a long thick segment in the side of triangle is close to the union of the other two sides. Together, these theorems tell us that, since Y_t is far from X_t, the geodesic $[X_t, Y_t]$ must fellow travel $[X_t, X_0]$ for a long time.

We now make this precise.

Theorem 5 (Minsky, Second part of Corollary 4.1 of [12])**.** *Let $\varepsilon > 0$ and let S be a closed orientable surface. There is a constant b such that if $\mathscr{G}$ is an ε–thick geodesic in $\mathscr{T}(S)$ with closest points projection map $\pi_{\mathscr{G}}$, then*

$$\mathrm{diam}(\pi_{\mathscr{G}}(X) \cup \pi_{\mathscr{G}}(Y)) \leq \mathrm{d}(X,Y) + b$$

for any points X and Y in $\mathscr{T}(S)$. □

Theorem 6 (Rafi, Theorem 8.1 of [14])**.** *Let $\varepsilon > 0$ and let S be a closed hyperbolic surface. There are constants A and B such that the following holds. Let X, Y, and Z be three points in $\mathscr{T}(S)$. If $[C,D] \subset [X,Y]$ such that $d(C,D) > A$ and every t in $[C,D]$ is ε–thick, then there is a w in $[C,D]$ with*

$$\min\{\mathrm{d}(w,[X,Z]), \mathrm{d}(w,[Y,Z])\} \leq B.$$ □

Theorem 7 (Minsky, Theorem 4.2 of [12])**.** *Let $K, C, \varepsilon > 0$. Then there is a $D > 0$ such that the following holds. Let $\mathscr{P}$ be a (K,C)–quasigeodesic path in $\mathscr{T}(S)$ whose endpoints are connected by an ε–thick Teichmüller geodesic $\mathscr{G}$. Then $\mathscr{P}$ remains in a D–neighborhood of $\mathscr{G}$.* □

Lemma 8 (Big thick segment)**.** *There exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ depending only on S and ε such that the segment $[X_t, Y_t]$ contains a point Z_t such that*

$$\mathrm{d}(X_t, Z_t) > \delta_1 t$$

and $[X_t, Z_t]$ is ε_1–thick.

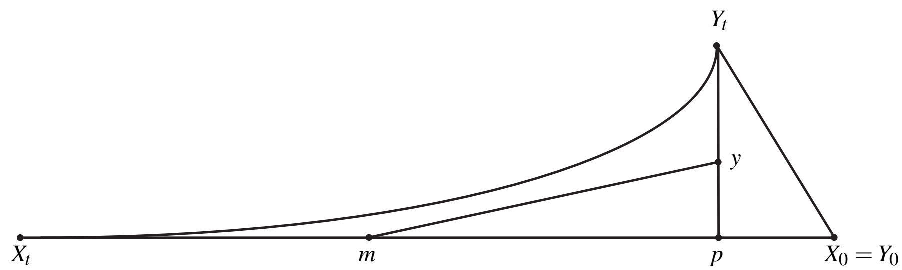

Figure 2: Geodesics and points of interest in the proof of Lemma 8.

Proof. See Figure 2 for geodesics and points of interest throughout the proof.

Let p be a nearest point to Y_t on the geodesic $\mathscr{G}$, and let m be the midpoint of the geodesic joining p to X_t.

Applying Theorem 5 to the nearest–points projection $\pi_{\mathscr{G}}$, we find that $d(X_0, p) \leq d(X_0, Y_t) + b < ct + b$. Thus $d(p, X_t) > (1-c)t - b$, and so $d(m, p) > ((1-c)t - b)/2$.

Let y in $[p, Y_t]$ minimize distance from $[p, Y_t]$ to m and note that $d(m, y) \geq d(p, y)$. The triangle inequality gives us $d(m, p) \leq d(m, y) + d(y, p) \leq 2d(m, y)$.

Putting these two paragraphs together we obtain $d(m, [p, Y_t]) > ((1-c)t - b)/4$. Applying Theorem 6 to the triangle $\triangle pX_tY_t$ we can conclude that the point m is a bounded distance B from a point Z_t on the geodesic $[X_t, Y_t]$, where B is a nice constant.

By the triangle equality, this point Z_t is at a distance at least $(1-c)t/2 - B'$ from X_t, where $B' = B + b/2$. This gives $d(Z_t, X_t) > \delta_1 t$ for suitable δ_1 and t larger than a nice constant. For small t we take $Z_t = Y_t$.

By Theorem 7, the geodesic $[Z_t, X_t]$ lies in a D–neighborhood of the geodesic $[m, X_t]$ for some nice D (the path $[m, Z_t] \cup [Z_t, X_t]$ is a $(1, 2B)$–quasigeodesic). Since $\mathscr{H}$ (and hence $[m, X_t]$) is ε–thick, there is then a nice ε_1 such that $[Z_t, X_t]$ is ε_1–thick. □

Thick collar

Let $\operatorname{core}(X_t, Y_t)$ be the convex core of the quasifuchsian manifold $\operatorname{qf}(X_t, Y_t)$ and let $\mathscr{X}_t$ and $\mathscr{Y}_t$ be the components of $\partial\operatorname{core}(X_t, Y_t)$ facing X_t and Y_t, respectively.

We say a subset $\mathscr{A}$ of a hyperbolic manifold is *ℓ–thick* if it is contained in the ℓ-thick part.

Lemma 9 (Thick collar in $\operatorname{qf}(X_t, Y_t)$)**.** *There are constants t_0, ε_2, and δ_2 depending only on S and ε*

such that $\operatorname{core}(X_t, Y_t)$ *contains an ε_2–thick submanifold $B_t \cong S \times [0,1]$ such that $S \times \{0\} = \mathscr{X}_t$ and* $\operatorname{d}(\mathscr{X}_t, S \times \{1\}) \geq \delta_2 t$.

Proof. The key point is that the thick subsegment $[X_t, Z_t]$ from Lemma 8 is reflected in the structure of the model manifold of [3]. To explain this, let $s : \mathscr{T}(S) \to \mathscr{C}(S)$

be the systole map from Teichmüller space to the complex of curves. Then s takes a Teichmüller geodesic to an unparameterized quasigeodesic, as in [7], with the quality of the quasigeodesic depending only on S. Let $\mathscr{H}$ be a $\mathscr{C}(S)$–geodesic connecting $s(X_t)$ to $s(Y_t)$. By hyperbolicity of $\mathscr{C}(S)$ there is $K = K_S$ so that $\mathscr{H}$ is at Hausdorff distance at most K from $s([X_t, Y_t])$, and we can find an initial segment $\mathscr{H}_1$ of $\mathscr{H}$ which lies Hausdorff distance at most K from $s([X_t, Z_t])$.

Because $[X_t, Z_t]$ is ε_1–thick, by [13] there is a bound $d_W(X_t, Z_t) < B$ for $B = B(S, \varepsilon_1)$ and any subsurface W of S. (Here, $d_W(X,Y)$ is the "subsurface projection distance" discussed in [8, 11] and [3]. Namely it is the distance in the arc/curve complex of W between the intersections with W of the shortest filling curve systems in X and Y respectively.) The Bounded Geodesic Image Theorem [8] provides constants b, k such that if the $\mathscr{C}(S)$–distance between $[\partial W]$ and $s([Z_t, Y_t])$ is at least k then $d_W(Z_t, Y_t) \leq b$. It follows, after trimming the end of $\mathscr{H}_1$ by a bounded amount, that for any W with $d(\partial W, \mathscr{H}_1) \leq 1$ we have $d_W(X_t, Y_t) \leq c'$.

The Bilipschitz Model Theorem of [3, 11] provides a manifold $\mathbb{M}$ depending on (X_t, Y_t) and a bilipschitz homeomorphism $f : \mathbb{M} \to \mathrm{qf}(X_t, Y_t)$. The structure of $\mathbb{M}$ is determined by $\mathscr{H}$, and in particular any vertex v of $\mathscr{H}$ can be associated to a "cut surface" τ_v in $\mathbb{M}$, whose inclusion is a homotopy equivalence and whose geometry is determined by v and the projections $d_W(X_t, Y_t)$ for subsurfaces W with $d(\partial W, v) \leq 1$. When two vertices are sufficiently far apart their cut surfaces cobound a product region. The bounds on $d_W(X_t, Y_t)$ from the previous paragraph imply that product regions determined by cut surfaces based on vertices of $\mathscr{H}_1$ have bounded geometry, and in particular are ε'–thick for some nice ε'. See Sections 4 and 5, and particularly Lemma 5.7, of [3] for the construction of these regions in the general setting. Theorem 7.1 of [3] indicates how bounds on d_W give rise to bounded–geometry regions.

The initial cut surface of $\mathscr{H}_1$ is the surface in the model that maps to the convex hull boundary $\mathscr{X}_t$. If we build a product region $B(\mathscr{H}_1)$ bounded by the initial and final vertices of $\mathscr{H}_1$, the distance between its two boundary components is at least a uniform multiple of the length $|\mathscr{H}_1|$ of $\mathscr{H}_1$. One can see this by dividing it up using cut surfaces for equally–spaced vertices of $\mathscr{H}_1$. Since the length $|\mathscr{H}_1|$ is a uniform multiple of t, the image of $B(\mathscr{H}_1)$ in $\mathrm{qf}(X_t, Y_t)$ under the bilipschitz model map f is the desired product region. $\square$

It will be convenient to talk about product subregions of B_t. From the fact that B_t has bounded geometry (or the construction itself) one has for each x in B_t a surface F_x isotopic to $\mathscr{X}$ which contains x and has diameter bounded by D depending only on ε. It follows that for each s in $[0, \delta_2 t]$ there exists a region $B_t[s] \subset B_t$ such that

1. $\mathscr{N}_s(\mathscr{X}) \subset B_t[s] \subset \mathscr{N}_{s+2D}(\mathscr{X})$,
2. $B_t[s]$ is homeomorphic to $S \times [0,1]$

(here $\mathscr{N}_s$ denotes an s–neighborhood within $\mathrm{core}(X_t, Y_t)$). Simply pick the region between F_x and $\mathscr{X}$, where $\mathrm{dist}(x, \mathscr{X}) = s + D$.

Let $\pi : \mathrm{qf}(X_t, Y_t) \to M_t$ be the covering map and let $\mathscr{X}_t' = \pi(\mathscr{X}_t)$.

Lemma 10 (Embedding of collar)**.** *There exists $t_1 > t_0$ depending on S, ε, such that for $t > t_1$ the covering map π embeds $B_t[\delta_2 t/3]$ in* $\mathrm{core}(M_t)$*, and the image is in the ε_2–thick part of M_t.*

Proof. Note first that $\mathrm{core}(X_t, Y_t)$ is contained in the pullback $\pi^{-1}(\mathrm{core}(M_t))$, and that $\mathscr{X}_t$ is a boundary component of both $\mathrm{core}(X_t, Y_t)$ and $\pi^{-1}(\mathrm{core}(M_t))$. Therefore any component $\mathscr{Z}$ of $\pi^{-1}(\pi(\mathscr{X}_t))$ cannot meet $int(\mathrm{core}(X_t, Y_t))$, and if $\mathscr{Z} \neq \mathscr{X}_t$ then $\mathscr{Z}$ is disjoint from B_t, which then separates it from $\mathscr{X}_t$. It follows that the distance from $\mathscr{Z}$ to $\mathscr{X}_t$ is at least $\delta_2 t$.

Thus for $s < \delta_2 t/2$, the s–neighborhood C_s of $\mathscr{X}_t$ in $\mathrm{core}(X_t, Y_t)$ is disjoint from the s–neighborhoods of the other components of $\pi^{-1}(\pi(\mathscr{X}_t))$. We conclude that $\pi|_{C_s}$ is an embedding into M_t. For suitable t_1 we have that the product region $B_t[\delta_2 t/3]$ is in such a neighborhood, so π embeds it.

Now as soon as $\delta_2 t/6 > \varepsilon_2 + 2D$ we find that any loop of length ε_2 based at a point in $\pi(B_t[\delta_2 t/3])$ lifts to a loop in $B_t[\delta_2 t/2]$, so since B_t is in the ε_2–thick part of $\mathrm{qf}(X_t, Y_t)$, we conclude that $\pi(B_t[\delta_2 t/3])$ is in the ε_2–thick part of M_t. □

Theorem 3 is now just a rewording of Lemma 10.

4 Geometric inflexibility

The goals of this section are Theorem 11, which uses Geometric Inflexibility to give exponentially shrinking bounds on the time and space derivatives of our family of metrics; and Proposition 16, which uses these bounds and the proxy surfaces of Lemma 13 to control the speed of the skinning image.

It is a classical fact that two hyperbolic 3–manifolds are K–quasiconformally conjugate if and only if there is an L–bi-Lipschitz map between them and each of the constants K and L can be effectively controlled in terms of the other, see, for example, Theorem 2.5 and Corollary B.23 of [10]. McMullen [10] showed that if the injectivity radius is bounded away from zero, the bi-Lipschitz map may be chosen so that the pointwise bi-Lipschitz constant decays exponentially to 1 as the point moves deeper into the convex core. McMullen called this *geometric inflexibility*. In Brock–Bromberg [1], an alternative approach to geometric inflexibility removes the global restriction on the injectivity radius and shows that the bi-Lipschitz constant decays exponentially away from the thin part. We use this version here.

Theorem 11. *Let M be a compact, smooth hyperbolizable 3–manifold with boundary and X_t a smooth 1–parameter family of conformal structures on ∂M such that $\|\dot{X}_t\|_{\mathscr{T}} \leq 1$. Then there exists a smooth family of complete hyperbolic metrics g_t on the interior of M that extend continuously to the conformal structures X_t on ∂M and such that for x in the ε-thick part of M_t,*

$$\|\eta_t(x)\| \leq A e^{-Bd(x, M_t - \mathrm{core}(M_t))}$$

and

$$\|\nabla^t \eta_t(x)\| \leq A e^{-Bd(x, M_t - \mathrm{core}(M_t))}.$$

The constants A and B depend only on the topological type of ∂M and on ε.

Proof. The proof is a straightforward combination of several results. Work of Reimann [15] supplies a family of hyperbolic metrics g_t that extend continuously to X_t and such that the associated strain fields η_t are *harmonic*. One obtains bounds on the L^2–norm of η_t in $\mathrm{core}(M_t)$ that are linear functions of the genus of ∂M (Lemma 5.2 in [1]). By Theorem 3.6 of [1], the L^2–norm of η_t on the submanifold of points in $\mathrm{core}(M_t)$ a distance $> r$ from $\partial\mathrm{core}(M_t)$ decays exponentially in r. From this one obtains bounds on the L^2–norm of η_t in an ε–ball centered at x. The pointwise norm bounds on η and $\nabla\eta$ then follow from standard estimates in partial differential equations, see [2]. $\square$

Peripheral curves do not get short

By assumption the length of any closed curve on X_t is at least ε. However, a lower bound on the length of curve on the conformal boundary does not, in general, imply lower bounds on length in the hyperbolic 3–manifold. We now combine Theorem 3 and geometric inflexiblity to show that such a bound does hold for the manifolds in our family.

Theorem 12. *There exists $\varepsilon' > 0$ depending on S and ε such that for all $t > 0$ every curve γ in S has $\ell_{M_t}(\gamma) > \varepsilon'$.*

Proof. Let t_1 be the constant from Lemma 10. Then for all $t < t_1$ there is an ε_3, depending only on ε and t_1, such that $\ell_{M_t}(\gamma) \geq \varepsilon_3$.

Let ε_2 also be the constant from Lemma 10 and choose ε_4 to be the minimum of ε_2, ε_3 and the 3–dimensional Margulis constant. If $\ell_{M_t}(\gamma) < \varepsilon_4$, let $\mathbb{T}_t(\gamma)$ be the ε_4–Margulis tube for γ in M_t. Then by Lemma 10, $d(\mathbb{T}_t(\gamma), M_t - \mathrm{core}(M_t)) \geq \delta_2 t/3 = \delta_3 t$ where, again, δ_2 is from Lemma 10.

By Theorem 5.8 in Brock–Bromberg [1], there exist constants C_1 and C_2, depending only on ∂M, such that if $\ell_{M_t} < \varepsilon_4$ then

$$\left|\log \frac{\ell_{M_{t+s}}(\gamma)}{\ell_{M_t}(\gamma)}\right| \leq C_1 e^{-C_2 d(\mathbb{T}_t(\gamma), M_t - \mathrm{core}(M_t))} \tag{4.1}$$

for $|s| \leq 1$. (This is a consequence of their geometric inflexibility theorem, applied to the boundary of $\mathbb{T}_t(\gamma)$.) Choose $\varepsilon' < \varepsilon_4$ such that

$$-\log \frac{\varepsilon'}{\varepsilon_4} = \frac{C_1 e^{-C_2 \delta_3 t_1}}{1 - e^{-C_2 \delta_3}}.$$

We will show that $\ell_{M_t}(\gamma) \geq \varepsilon'$.

If $\ell_{M_t}(\gamma) \geq \varepsilon_4$ we are done. So assume to the contrary that $\ell_{M_t}(\gamma) < \varepsilon_4$. Choose $t_\gamma < t$ such that $\ell_{M_{t_\gamma}}(\gamma) = \varepsilon_4$ and $\ell_{M_s}(\gamma) \leq \varepsilon_4$ for all s in $[t_\gamma, t]$. Since $\ell_{M_s}(\gamma) \geq \varepsilon_4$ when $s \leq t_1$ and $\ell_{M_s}(\gamma)$ is continuous in s, such a t_γ exists and is bigger than t_1 .

Using the fact that for s in $[t_\gamma, t]$ we have $d(\mathbb{T}_s(\gamma), M_s - \mathrm{core}(M_s)) \geq \delta_3 s$, we can repeatedly apply (4.1) to see that

$$\left|\log \frac{\ell_{M_t}(\gamma)}{\ell_{M_{t_\gamma}}(\gamma)}\right| \leq \sum_{k=0}^{n} C_1 e^{-C_2 \delta_3 (t_\gamma + k)} < \sum_{k=0}^{\infty} C_1 e^{-C_2 \delta_3 (t_\gamma + k)}$$

where n is the least integer greater than $t - t_\gamma$. Summing this geometric series gives the desired bound. □

5 Proxy surfaces

We now introduce the smooth locally convex surfaces in M_t whose geometry will give us good control of the conformal geometry of the skinning surface Y_t. In Lemma 14, we locate these surfaces with respect to our thick collars.

Lemma 13. *There is a smooth surface $\mathscr{E}_t$ in the Y_t–end of* $\mathrm{qf}(X_t, Y_t)$ *in the* 3*–neighborhood of $\mathscr{Y}_t$ whose principal curvatures are within $\frac{1}{4}$ of* 1.

Proof. Observe that there is a smooth convex surface arbitrarily close to $\mathscr{Y}_t$. One can construct such a surface in several ways. For example, one can smoothly approximate the distance function from $\mathscr{Y}_t$ by convex functions and take a level set. A more concrete construction is due to Labourie ([6]) who showed that, for any κ in $(0,-1)$, there is a surface $\mathscr{L}_\kappa$ of constant Gaussian curvature κ in the Y_t–end of $\mathrm{qf}(X_t, Y_t)$. As $\kappa \to -1$ the surfaces $\mathscr{L}_\kappa$ will converge uniformly to $\mathscr{Y}_t$. If we flow any convex surface a distance r in the normal direction then the curvatures are bounded between $\tanh r$ and $\coth r$. We then obtain $\mathscr{E}_t$ by flowing the smooth convex surface near $\mathscr{Y}_t$ a distance 2. □

Lemma 14. *There is a time $t_2 > t_1$ depending only on S and ε such that, for all $t \geq t_2$, the surface $\mathscr{E}_t' = \pi(\mathscr{E}_t)$ lies in* $\mathrm{core}(M_t) \smallsetminus \pi(B_t[\delta_2 t/4])$.

Proof. The surface $\mathscr{Y}_t$ is ε'–thick by Theorem 12 and so the diameter of $\mathscr{Y}_t$ is bounded by a nice constant.

Since the covering map π maps $\mathrm{core}(X_t, Y_t)$ into $\mathrm{core}(M_t)$, the image $\mathscr{Y}_t'$ of $\mathscr{Y}_t$ lies in $\mathrm{core}(M_t)$. If $\mathscr{Y}_t'$ lies entirely in $\pi(B_t[\delta_2 t/3])$ then $\mathscr{Y}_t$ lies in a component of $\pi^{-1}(\pi(B_t[\delta_2 t/3]))$, and as in Lemma 10 these components are retracts of the lifts of $\mathscr{X}_t'$. All of them except $B_t[\delta_2 t/3]$ itself are simply connected (since M is acylindrical) and thus cannot contain $\mathscr{Y}_t$. Furthermore, $B_t[\delta_2 t/3]$ cannot contain $\mathscr{Y}_t$ since it is inside $\mathrm{core}(X_t, Y_t)$.

We conclude that $\mathscr{Y}_t'$ cannot lie in $B_t[\delta_2 t/3]$. So, if t is sufficiently large (depending on the diameter bound for $\mathscr{Y}_t$), then $\mathscr{Y}_t'$ will be disjoint from $\pi(B_t[\delta_2 t/4])$.

Since $\mathscr{E}_t'$ is in a 3–neighborhood of $\mathscr{Y}_t'$, it is also disjoint from $\pi(B_t[\delta_2 t/4])$ when t is large enough. □

Horocylically convex surfaces and their conformal structures

Let Σ be a transversally oriented surface immersed in a hyperbolic 3–manifold (M, g). This gives a normal vector field $\mathbf{n}$ to Σ, and a shape operator $B : T\Sigma \to T\Sigma$ given by $B(x) = \nabla_x \mathbf{n}$. If the eigenvalues of B lie in $(-1, \infty)$ (i.e. the principal curvatures are bigger than -1), then Σ is *horocyclically convex* and the geodesic flow to infinity along $\mathbf{n}$ gives a complex structure ω on Σ (in fact a complex projective structure). More precisely let Σ_r be the surface obtained by flowing Σ in the direction of $\mathbf{n}$ a distance r. The condition that Σ is horocylically convex is equivalent to this normal flow being non-singular for all $r \geq 0$. If we pull the metrics on Σ_r back to Σ by the normal flow, the metrics diverge, but the conformal structures converge to a conformal structure Y_Σ.

In our setting, Σ is the locally convex surface $\mathscr{E}_t'$ and the conformal structure is the skinning surface Y_t.

A 1–parameter family of hyperbolic metrics on M determines a 1–parameter family of conformal structures on Σ. We want to convert bounds on the derivative of the metric to bounds on the derivative of the conformal structures in Teichmüller space. The key to this is the following formula which gives the conformal structure Y_Σ in terms of the geometry of Σ.

Lemma 15 (Krasnov–Schlenker [5])**.** *Let Σ be a horocyclically convex surface in a hyperbolic 3–manifold (M,g) with first fundamental form $I = g|_\Sigma$ and shape operator B. Then $I^*(x,y) = I(x+Bx, y+By)$ is a Riemannian metric on Σ in the conformal class Y_Σ.*

If g_t is a smooth family of complete hyperbolic metrics on M we obtain a family of shape operators B_t and conformal structures ω_t. At each t we have a strain field η_t defined as before. We wish to control the speed of ω_t in $\mathscr{T}(\Sigma)$ in terms of the behavior of B_t and η_t.

Proposition 16. *Let (M, g_t) be a manifold with a smooth family g_t of complete hyperbolic metrics. Let Σ be a closed immersed transversally oriented surface in M and let ω_t, B_t and η_t be the conformal structure, shape operator and strain field for g_t, respectively. Given k there exists C such that, if the eigenvalues of B_0 lie in $[-1+1/k, k]$, then*

$$\|\dot{\omega}\|_{\mathscr{T}} < C \max\left(\|\eta\|_g, \|\nabla\eta\|_g\right).$$

Proof. Let x, y and z be tangent vector fields on Σ. Differentiating the formula from Lemma 15 we have

$$\dot{I^*}(x,y) = 2I(\eta(x+Bx), y+By) + I(\dot{B}x, y+By) + I(x+Bx, \dot{B}y). \tag{5.1}$$

From Lemma 2 we see that a bound on $\|\dot{I^*}\|$ for all points in Σ gives a bound on $\|\dot{\omega}\|_{\mathscr{T}}$. From (5.1) we have that given a bound on $\|B\|$, $\|\dot{I^*}\|$ is bounded by a linear function of $\|\eta\|$ and $\|\dot{B}\|$. If ∇^t is the Riemannian connection for g_t and $\mathbf{n}_t$ is the unit normal outward vector field for (Σ, g_t), then $B_t x = \nabla^t_x \mathbf{n}_t$. Therefore $\dot{B}x = \nabla_x \dot{\mathbf{n}} + \dot{\nabla}_x \mathbf{n}$, and so we need to control $\nabla\dot{\mathbf{n}}$ and $\dot{\nabla}$.

Given a vector $\mathbf{v}$ at a point in Σ, we let $\mathbf{v}^\top$ be the component of $\mathbf{v}$ tangent to Σ.

First consider $\nabla\dot{\mathbf{n}}$. We only need to bound $\nabla\dot{\mathbf{n}}^\top$ as we are taking the inner product against tangent vectors.

We begin by differentiating the formula $g_t(\mathbf{n}_t, y) = 0$ to see that

$$2g(\eta\mathbf{n}, y) + g(\dot{\mathbf{n}}, y) = 0.$$

Note that this implies that $g(2\eta\mathbf{n} + \dot{\mathbf{n}}, y) = 0$ and so $2\eta\mathbf{n} + \dot{\mathbf{n}}$ is orthogonal to Σ. We will use this later. Differentiating in the x–direction we have

$$\begin{aligned} 0 &= x(2g(\eta\mathbf{n},y)+g(\dot{\mathbf{n}},y)) \\ &= 2g(\nabla_x(\eta\mathbf{n}),y)+2g(\eta\mathbf{n},\nabla_x y)+g(\nabla_x\dot{\mathbf{n}},y)+g(\dot{\mathbf{n}},\nabla_x y) \\ &= 2g((\nabla_x\eta)\mathbf{n},y)+2g(\eta(Bx),y)+g(2\eta\mathbf{n}+\dot{\mathbf{n}},\nabla_x y)+g(\nabla_x\dot{\mathbf{n}},y). \end{aligned}$$

As $2\eta\mathbf{n}+\dot{\mathbf{n}}$ is normal we only need to know the normal component of $\nabla_x y$. Since $g(\nabla_x y,\mathbf{n})+g(y,\nabla_x\mathbf{n}) = xg(y,\mathbf{n}) = 0$, we have $g(\nabla_x y,\mathbf{n}) = -g(y,Bx)$ and so

$$|g(2\eta\mathbf{n}+\dot{\mathbf{n}},\nabla_x y)| = \|2\eta\mathbf{n}+\dot{\mathbf{n}}\||g(y,Bx)|.$$

Combining we have

$$\|\nabla_x\dot{\mathbf{n}}^\top\| \leq 2\|\nabla\eta\|+4\|\eta\|\|B\|+\|\dot{\mathbf{n}}\|\|B\|.$$

We now bound $\dot{\mathbf{n}}$. Let x be a unit vector in the direction $\dot{\mathbf{n}}^\top$. Differentiating the formula $g_t(x,\mathbf{n}_t) = 0$, we have

$$2g(\eta x,\mathbf{n})+g(x,\dot{\mathbf{n}}) = 0$$

and so $|g(x,\dot{\mathbf{n}})| \leq 2\|\eta\|$. Differentiating $g_t(\mathbf{n}_t,\mathbf{n}_t) = 1$, we see that

$$2g(\eta\mathbf{n},\mathbf{n})+2g(\dot{\mathbf{n}},\mathbf{n}) = 0$$

and so $|g(\dot{\mathbf{n}},\mathbf{n})| \leq \|\eta\|$. Therefore $\|\dot{\mathbf{n}}\| \leq 3\|\eta\|$.

To bound $\dot{\nabla}$ we differentiate the formula $xg_t(y,z) = g_t(\nabla^t_x y,z)+g_t(y,\nabla^t_x z)$. The left hand side is

$$2xg(\eta y,z) = 2\left(g(\nabla_x(\eta y),z)+g\left(\eta y,\nabla_x z\right)\right)$$

and the right hand side is

$$2g\left(\eta(\nabla_x y),z\right)+g(\dot{\nabla}_x y,z)+2g(\eta y,\nabla_x z)+g(y,\dot{\nabla}_x z).$$

Rearranging and applying the Leibnitz rule to $\nabla_x(\eta y)$, this becomes

$$2g((\nabla_x\eta)y,z) = g(\dot{\nabla}_x y,z)+g(y,\dot{\nabla}_x z). \tag{5.2}$$

As the Riemannian connections are torsion free we have

$$\nabla^t_x y-\nabla^t_y x = [x,y]$$

and differentiating we see that $\dot{\nabla}_x y = \dot{\nabla}_y x$. Taking the three permutations of (5.2), the symmetry of $\dot{\nabla}$ gives

$$g(\dot{\nabla}_x y,z) = g((\nabla_x\eta)y,z)+g((\nabla_y\eta)z,x)-g((\nabla_z\eta)x,y),$$

and so $\|\dot{\nabla}\| \leq 3\|\nabla\eta\|$.

Combing the bounds on $\|\nabla\dot{\mathbf{n}}\|$ and $\|\dot{\nabla}\|$ we have

$$\begin{aligned} \|\dot{B}\| &\leq 2\|\nabla\eta\|+4\|\eta\|\|B\|+3\|\eta\|\|B\|+3\|\nabla\eta\| \\ &\leq 5\|\nabla\eta\|+7\|\eta\|\|B\|. \end{aligned}$$

□

6 Finishing the proof

Let M be a hyperbolizable acylindrical 3–manifold and assume that X is the conformal boundary of the unique hyperbolic structure on M whose convex core boundary is totally geodesic. Let X_t be an ε–thick Teichmüller geodesic ray in $\mathscr{T}(\partial M)$ with $X = X_0$. Let $M_t = (M^\circ, g_t)$ be the hyperbolic metrics given by Theorem 11 with conformal boundary X_t and let $Y_t = \overline{\sigma_M(X_t)}$ be the skinning surface with its orientation reversed. By Lemma 13, there are convex surfaces $\mathscr{E}_t$ in $\mathrm{qf}(X_t, Y_t)$ with curvatures within $\frac{1}{4}$ of 1 and whose conformal structures at infinity are Y_t. By Lemma 14, the image $\mathscr{E}_t'$ of $\mathscr{E}_t$ in M_t is contained in $\mathrm{core}(M_t)$ and $d(\mathscr{E}_t', M_t - \mathrm{core}(M_t)) \geq \delta_2 t$. By Theorem 11 we have

$$\|\eta_t(x)\| \leq Ae^{-B\delta_2 t} \quad \text{and} \quad \|\nabla^t \eta_t(x)\| \leq Ae^{-B\delta_2 t}$$

for any x in $\mathscr{E}_t'$. By Proposition 16 we have

$$\|\dot{Y}_t\|_{\mathscr{T}} < ACe^{-B\delta_2 t}. \tag{6.1}$$

All of these constants are nice, and Theorem 1 follows by integrating (6.1).

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