Linear Algebra - Fall 2018 - Dr. A. Kent

Course web page: www.math.wisc.edu/~kent/Math340.Fall.2018.html

MIDTERM 1: Tuesday, October 16, in class

MIDTERM 2: Thursday, November 15, in class

FINAL EXAM: Friday, December 14, 5:05-7:05 in a location TBD

LINEAR ALGEBRA.

- SOLVE SYSTEMS OF LINEAR EQUATIONS.
- IMAGE COMPRESSION (SVD CAN COMPRESS SIZE OF RGB IMAGE 768 × 1024 BY FACTOR OF 18.)
- CAN USE IT TO SHOW EVERY RATIONAL FUNCTION HAS PARTIAL FRAC. DECOMP.)
- GOOGLE PAGE RANK ALGORITHM.

CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS.

EX. UNKNOWNS x_1, x_2

SUBSCRIPTS WILL BE NEC. LATER WHEN WE HAVE HUNDREDS OF XS.

$$x_1 - 3x_2 = -7$$
$$2x_1 + x_2 = 7$$

THERE IS A UNIQUE SOLUTION HERE.

NAMELY $x_1 = 2$ AND $x_2 = 3$.

EX.

$$8x_1 - 3x_2 = 7$$
$$3x_1 - 2x_2 = 0$$
$$10x_1 - 2x_2 = 14$$

ALSO HAS A UNIQUE SOLUTION.

IN FACT THE SAME ONE.

$x_1 = 2$ AND $x_2 = 3$.

HOW TO SOLVE?

EX.

① $x + 2y + 3z = 6$

② $2x - 3y + 2z = 14$

③ $3x + y - z = -2$

NOTE ORDER OF VARIABLES

WE CAN ELIMINATE X.

WE COULD SOLVE FOR X IN ①

AND PLUG INTO ②.

AND PLUG INTO ③.

SLEEKER THING TO DO IS

MULT. ① BY 2 AND SUBTRACT

FROM ②.

AND

MULT. ① BY 3 AND SUBTRACT

FROM ③.

↓

2 TIMES ① IS

$$2x + 4y + 6z = 12$$

SUBTRACT FROM

② $2x - 3y + 2z = 14$

AND GET

$$\boxed{-7y - 4z = 2}$$

& SUBTRACTING 3 TIMES ① FROM 3 YIELDS

$$-5y - 10z = -20.$$

SO NEW SYSTEM:

$$-7y - 4z = 2$$
$$-5y - 10z = -20.$$

DIVIDE SECOND BY -5

$\rightsquigarrow$

$$-7y - 4z = 2$$
$$y + 2z = 4$$

$\rightsquigarrow$ ⭑ $y + 2z = 4$

⭑⭑ $-7y - 4z = 2$

ELIMINATE y.

ADD 7 TIMES ⭑ TO ⭑⭑

$\rightsquigarrow$ $10z = 30$

$\rightsquigarrow$ $z = 3.$

$z = 3$ AND ⭑

$\Rightarrow$ $y = -2.$

W/ ①, WE HAVE

$x = 1.$

THIS ENTIRE PROCESS PRODUCED
A LINEAR SYSTEM

$$x + 2y + 3z = 6$$
$$y + 2z = 4$$
$$z = 3$$

MUCH EASIER TO SOLVE.

CONSIDER (A) $x - 3y = -7$

(B) $2x - 6y = 7.$

ELIMINATE x.

SUBTRACT $2 \cdot$ (A) FROM (B).

$\leadsto$ $0 = 21.$ ABSURD.

$\boxed{\text{NO SOLUTION}}$.

EX Ⓒ $x + 2y - 3z = -4$

Ⓓ $2x + y - 3z = 4.$

ELIMINATE x. FROM Ⓒ

$\rightsquigarrow \quad -3y + 3z = 12$

$y = z - 4.$

SO z CAN BE ANYTHING

Ⓒ $\rightsquigarrow x + 2(z-4) - 3z = -4$

$x = z + 4$

INFINITELY MANY SOLUTIONS.

IN GENERAL

LINEAR EQUATION:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b.$$

x_i CALLED UNKNOWNS.

a_i CONSTANTS CALLED COEFFICIENTS.

A SOLUTION IS A SEQUENCE

$$s_1, \dots, s_n$$

OF #S S.T.

$$a_1 s_1 + \dots + a_n s_n = b.$$

IF WE HAVE m LINEAR EQUATIONS IN n UNKNOWNS, SAY WE HAVE A SYSTEM OF m LINEAR EQUATIONS IN n UNKNOWNS.

A SOLUTION OF THE LINEAR SYSTEM IS A SEQ. $s_1, \ldots, s_n$ THAT SOLVES ALL SIMULTANEOUSLY.

IF NO SOL^n, INCONSISTENT.

OTHERWISE CONSISTENT.

EASILY WRITTEN:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m$$

IF $b_1 = b_2 = \ldots = b_m = 0$

SYSTEM IS HOMOGENEOUS.

ALWAYS HAS TRIVIAL SOLUTION.

TWO SYSTEMS ARE EQUIV. IF
OF SAME TYPE

SAME SET OF SOLUTIONS.

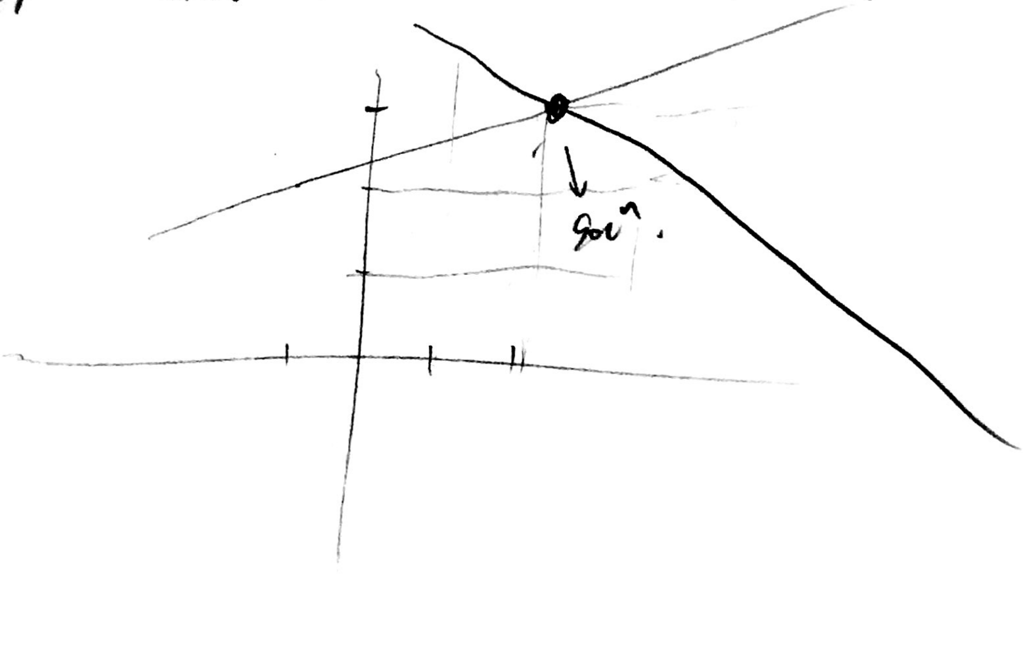
CAN USE ELIMINATION TO TRY AND SOLVE.
(ALWAYS WORKS.)

WE SAW ONE, NO, ∞ly MANY

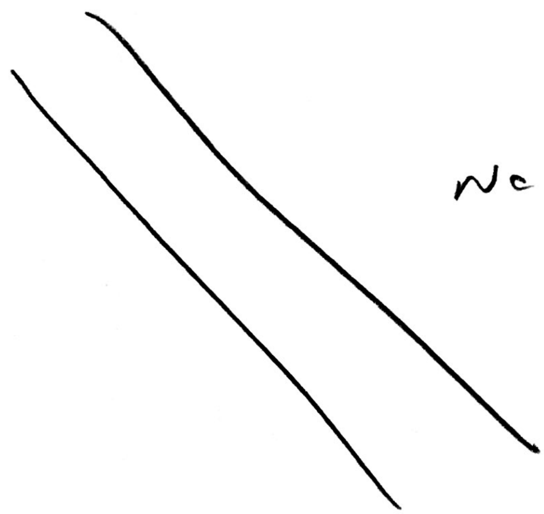
SOLUTIONS.

THIS IS ALWAYS CASE.

OUR VERY FIRST EXAMPLE WAS A PAIR
OF EQUATIONS FOR LINES.



IF TWO LINES PARALLEL, BUT **NOT** SAME:

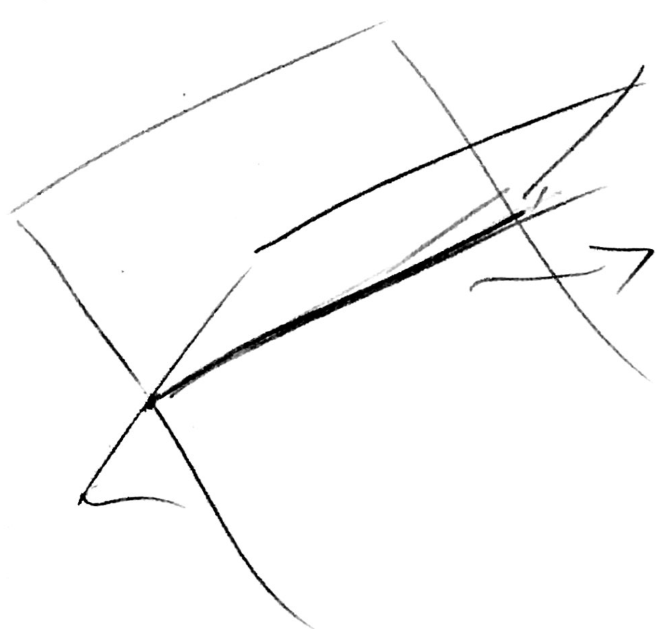


NO SOLUTION.

OUR INFINITE # SOL^{N} EX.

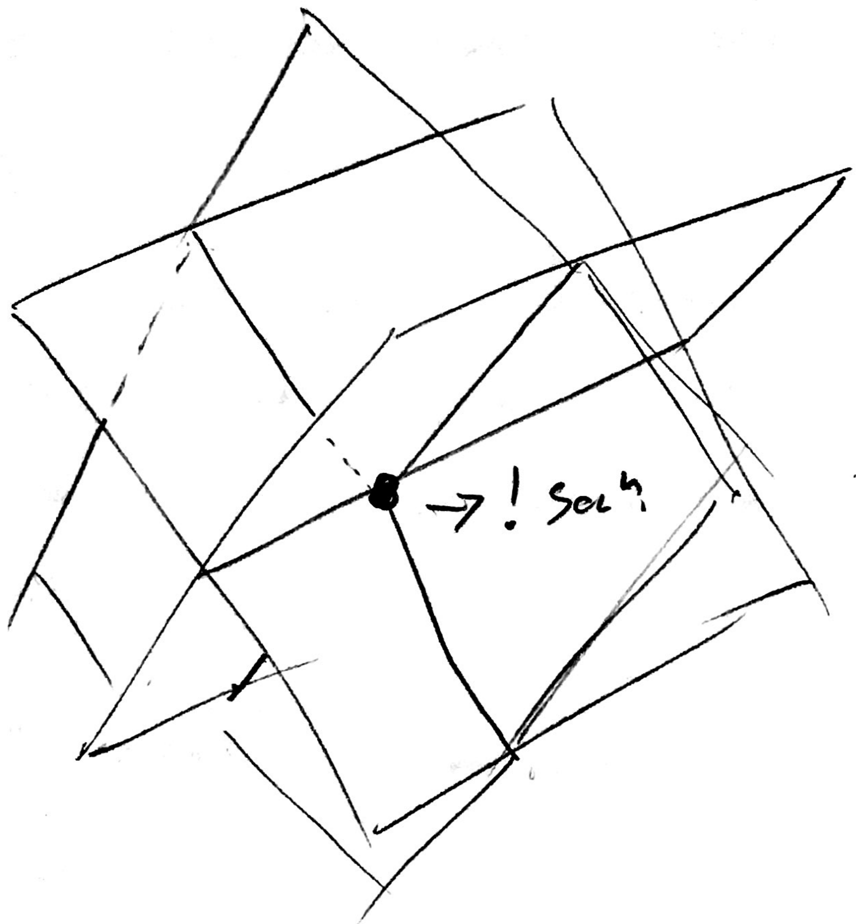
WE HAVE 2 EQ^{NS} FOR PLANES:

NOT PARALLEL



LINE OF SOLUTIONS.

THREE PLANES:



1.2 MATRICES.

KEEPING TRACK OF OUR OPERATIONS DURING ELIMINATION IS CUMBERSOME.

OBSERVE: WE ONLY MANIPULATE THE COEFFICIENTS a_{ij} AND THE b_i.

WE CAN MORE EASILY KEEP TRACK BY FOCUSING ONLY ON THE a_{ij} AND b_i.

DEF^n An $\underset{\text{ROWS}}{m} \times \overset{\text{COLUMNS}}{n}$ MATRIX A IS AN ARRAY OF $m \cdot n$ NUMBERS, m ROWS, n COLUMNS.

$$A = \begin{array}{cc} \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ & \vdots & & & \\ & & & a_{ij} & \\ & & & & \\ a_{m1} & a_{m2} & & & a_{mn} \end{bmatrix} & \begin{matrix} \\ \\ \\ i\text{th row} \\ \\ \\ \end{matrix} \\ \begin{matrix} \hphantom{a_{11}} & \hphantom{a_{12}} & & j\text{th col.} & \hphantom{a_{1n}} \end{matrix} & \end{array} = [a_{ij}]$$

"ijth" entry

ith ROW

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

jth COLUMN

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

IF $m=n$, A IS SQUARE.

THE TERMS

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \rightarrow \begin{matrix} \text{(MAIN)} \\ \text{DIAGONAL.} \end{matrix}$$

EX.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \qquad 2 \times 3$$

$$B = \begin{bmatrix} 7 & 5 & -\pi \\ 6 & 1 & 1 \\ \frac{1}{2} & 1 & 0 \end{bmatrix} \qquad 3 \times 3$$

$$C = \begin{bmatrix} 4 \end{bmatrix} \qquad 1 \times 1$$

$row_i(A)$ ith row

$col_i(A)$ ith col.

ROWS AND COLUMNS ARE VECTORS.

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \qquad \text{COLUMN VECTOR}$$

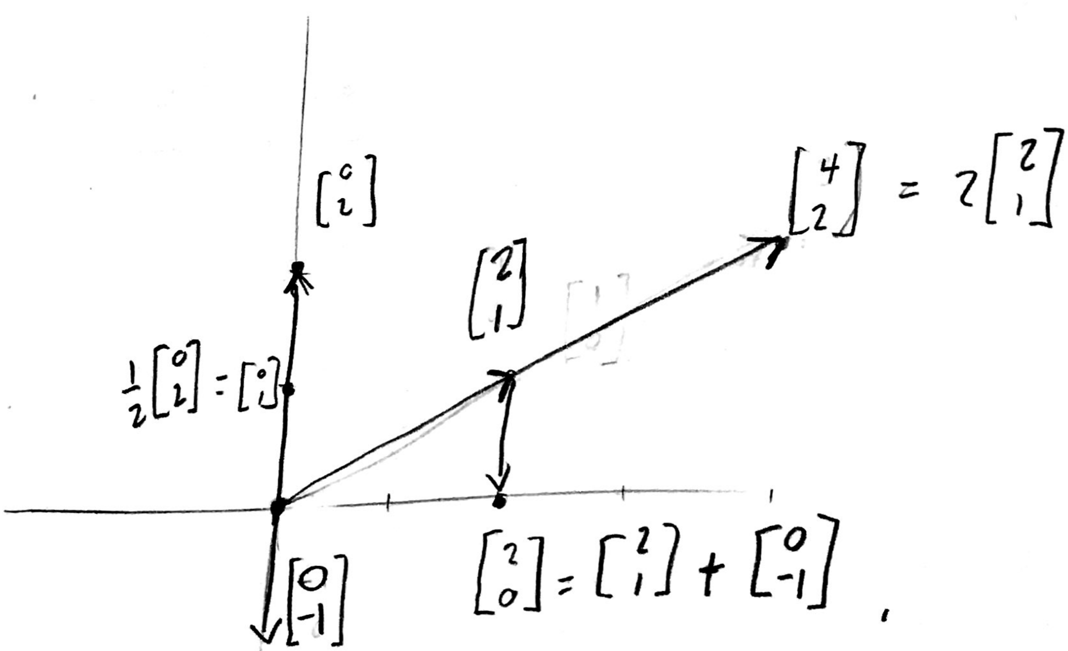
$$\underline{v} = \begin{bmatrix} 1 & 2 & 7 & -6 \end{bmatrix} \qquad \text{ROW VECTOR.}$$

MOSTLY USE COLUMN VECTORS.

WHILE THE NOTATION OF COLUMNS AND ROWS OF NUMBERS ARE CONVENIENT COMPUTATIONAL TOOLS, IT IS IMPORTANT TO HAVE A MORE GEOMETRIC PICTURE:

A VECTOR IS AN "ARROW" THAT MAY BE SCALED (MULTIPLIED BY A NUMBER) AND VECTORS MAY BE ADDED

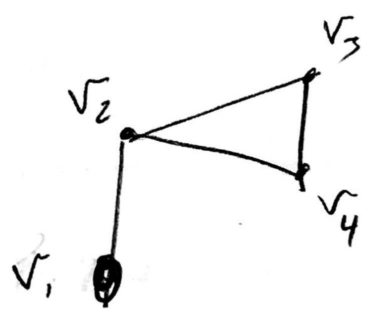
(BY ADDING THEIR COORDINATES.)



EX

A VERY USEFUL APPLICATION OF MATRICES COMES UP IN THE STUDY OF GRAPHS.

A **GRAPH** IS A COLLECTION OF VERTICES (NODES) CONNECTED BY **EDGES**.



THIS KIND OF THING IS IMPORTANT IN GOOGLE'S PAGE-RANK ALGORITHM.

INCIDENCE MATRIX

$$
\begin{array}{c c}
 & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

ENTRY IS 1 IF JOINED BY EDGE, 0 OTHERWISE.

CAN EASILY COMPUTE # PATHS BETWEEN EDGES USING THIS MATRIX. (LATER).

EX ANOTHER USE OF MATRICES COMES UP IN IMAGE COMPRESSION.

IMAGE IS A (BIG) MATRIX OF COLOR VALUES RGB OR INTENSITY VALUE.

LINEAR ALGEBRA CAN STORE THIS INFO EFFICIENTLY.

MATRIX ADDITION

$A = [a_{ij}]$ $B = [b_{ij}]$ BOTH $m \times n$.

$$A + B = [a_{ij} + b_{ij}]$$

So $A + B = C = [c_{ij}]$ WHERE $c_{ij} = a_{ij} + b_{ij}$.

WE ONLY WRITE $A+B$ IF A AND B SAME SIZE.

ALSO HAVE $A - B$.

SCALE MATRICES:

$A = [a_{ij}]$

λ IS A REAL #,

$\lambda A = [c_{ij}]$ WHERE $c_{ij} = \lambda a_{ij}$.

EX So $7\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 14 & 7 \end{bmatrix}$.

EX CAN THINK OF A STORE'S INVENTORY AS A VECTOR. (EACH COORD. CORRESPONDS TO AN ITEM AND ENTRY IS THE # OF THAT ITEM.

IF $\underline{u}$ IS INVENTORY VECTOR AND $\underline{v}$ IS VECTOR FOR AMOUNT SOLD, RESULTING INVENTORY IS

$\underline{u} - \underline{v}$

RECALL SUMMATION NOTATION:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n .$$

(SEE PAGE 17 FOR A REVIEW.)

A LINEAR COMBINATION OF $m \times n$ MATRICES

IS A SUM

$$\sum_{i=1}^{\ell} c_i A_i \quad \text{WHERE} \quad c_1, \dots, c_\ell \ \#\text{s}$$

$$A_1, \dots, A_\ell \quad m \times n \text{ MATRICES.}$$

MOST COMMON SITUATION IS WHEN THE

A_i ARE VECTORS:

$$7\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + \pi\begin{bmatrix}1\\1\\7\end{bmatrix}$$

SOME MORE NOTATION.

DEF

$A = [a_{ij}]$ $m \times n$ - MATRIX,

$A^T = [a_{ji}]$ IS THE TRANSPOSE OF A.

$n \times m$ INTERCHANGE ROWS & COLUMNS.

EX.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 5 \end{bmatrix}$$

1.3 MATRIX MULTIPLICATION.

DOT PRODUCT (INNER PRODUCT)

$$\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\underline{a} \cdot \underline{b} = a_1 b_1 + \dots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

EX

$$\underline{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{AND} \quad \underline{v} = \begin{bmatrix} 3 \\ 5 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \underline{u} \cdot \underline{v} &= 2 \cdot 3 + (-1) \cdot 5 + 0 \cdot (-1) + 1 \cdot 2 \\ &= 6 - 5 + 0 + 2 \\ &= 3. \end{aligned}$$

MATRIX MULTIPLICATION

DEFINITION

$B = [b_{ij}]$ $p \times n$

$A = [a_{ij}]$ $m \times p$

AB IS $m \times n$ MATRIX $C = [c_{ij}]$

WHERE c_{ij} IS THE DOT PRODUCT OF THE ith ROW OF A AND THE jth

COLUMN OF B.

EX.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1\cdot 2+1\cdot 5 & 1\cdot 3+1\cdot 7 \\ 0\cdot 2+1\cdot 5 & 0\cdot 3+1\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}.$$

So $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ip}b_{pj}$

$$= \sum_{k=1}^{p} a_{ik} b_{kj} .$$

Non SQ. EX.

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6-6 & 4+2+6 \\ 4+6-2 & 16-2+2 \\ 0+3+4 & 0-1-4 \end{bmatrix} = \begin{bmatrix} -11 & 12 \\ 8 & 16 \\ 7 & -5 \end{bmatrix} .$$

IMPORTANT EXAMPLE IS

MATRIX TIMES A VECTOR

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}.$$

WE WERE MOTIVATED BY LINEAR SYSTEMS TO THINK ABOUT MATRICES, WHICH ARE MORE IMPORTANT. "TRANSFORMATIONS" ARE EVEN MORE IMPORTANT.

2x2 MATRIX IS A TRANSFORMATION OF PLANE:

E.G. $\begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$ STRETCHES X-AXIS SHRINKS Y-AXIS.

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ROTATES PLANE.

3x3 MATRIX TRANSFORMATION OF $\mathbb{R}^3$

ETC.

CAVEATS:

- BA MAY NOT BE DEFINED. (EVEN IF AB IS.)
- IF BA IS DEFINED, THEN $m=q$.
  AB IS $m \times m$ AND BA IS $p \times p$.
- IF A AND B ARE SQUARE, THEN BOTH AB AND BA ARE DEFINED BUT COULD BE DIFFERENT!

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

RETURN TO LINEAR SYSTEMS.

$\underline{m}$ EQS IN $\underline{n}$ UNKNOWNS.

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}}$$

CAN ENCODE THIS IN

$$A\underline{x} = \underline{b} \quad \text{WHERE}$$

$$\underbrace{A = [a_{ij}]}_{\text{"COEFFICIENT MATRIX"}}, \quad \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

(A HOMOGENEOUS SYSTEM IS $A\underline{x} = \underline{0}$.)

WE WILL ALSO USE WHAT'S CALLED THE <u>AUGMENTED MATRIX</u> OF THE SYSTEM,

$$[A \,\vdots\, \underline{b}]$$

$$= \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

EX

$$\begin{aligned} -4x + y &= 2 \\ 2x + 3y - z &= 3 \\ x + y + z &= 4 \end{aligned}$$

$$\underbrace{\begin{bmatrix} -4 & 1 & 0 \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}_{\underline{b}}.$$

USEFUL OBSERVATION: (IN THIS EX

$$A\underline{x} = x\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

SO $A\underline{x} = \underline{b}$ IS CONSISTENT

IFF $\underline{b}$ IS A LINEAR COMBO

OF COLUMNS OF A.

PROPERTIES OF MATRIX ADD & MULT.

PAGE 3

ADD

a) $A+B=B+A$

b) $A+(B+C)=(A+B)+C$.

c) $A+O=A$ where O zero MX.

$(A+-A=O)$.

MULT.

a) $A(BC)=(AB)C$

b) $(A+B)C=AC+BC$

c) $C(A+B)=CA+CB$.

SCALAR MULT.

a) $r(sA)=(rs)A$

b) $(r+s)A=rA+sA$

c) $r(A+B)=rA+rB$

d) $A(rB)=rAB=(rA)B$).

TRANSPOSE

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(AB)^T = B^T A^T$

d) $(rA)^T = rA^T$

EX. $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} 12 & 5 \\ 7 & -3 \end{bmatrix}$ $(AB)^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix}$ $B^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix} = (AB)^T$

WARNINGS

$AB = 0$ DOES NOT MEAN ONE OF THE MATRICES IS ZERO.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

ALSO $AB = AC$ DOESN'T MEAN $B = C$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$$

SOME SPECIAL MATRICES.

$n \times n$ $A = [a_{ij}]$ IS DIAGONAL IF $a_{ij} = 0$ WHEN $i \neq j$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ DIAGONAL.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$ NOT DIAGONAL.

A IS SCALAR IF DIAGONAL AND ALL DIAGONAL ENTRIES EQUAL.

EG. $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$n \times n$ $I_n = [\delta_{ij}]$ (ALSO JUST I)

$\delta_{ij} = 1$ IF $i = j$

$\delta_{ij} = 0$ IF $i \neq j$

IS $n \times n$ IDENTITY MATRIX.

IF A IS $m \times n$

$$I_m A = A$$

$$A I_n = A.$$

POWERS. A $n \times n$.

$$A^p = \underbrace{A \cdot A \cdots A}_{p \text{ TIMES}}.$$

$$A^0 = I_n.$$

$$A^p A^q = A^{p+q} \text{ AND } (A^p)^q = A^{pq}$$

WARNING $(AB)^p = A^p B^p$ ONLY WHEN $AB = BA$

UPPER TRIANGULAR

5.6. $\begin{bmatrix} 1 & 5 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$A = [a_{ij}]$
upper triang.
if $a_{ij} = 0$ if $i > j$.

LOWER TRIANGULAR

$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 7 & 2 & 1 \end{bmatrix}$

$a_{ij} = 0$ if $i < j$.

A is **SYMMETRIC** if $A = A^T$

$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

(SKEW SYMMETRIC IF $A^T = -A$.)

[NOTE TO ~~SELF~~.
SKIP OVER PARTITIONED MATRICES.]

NONSINGULARITY.

WE SAW ABOVE THAT $AB = AC$

DOES <u>NOT</u> MEAN THAT $B = C$.

IF WE HAVE NUMBERS $a \neq 0$ AND $b \neq 0$ AND $c \neq 0$

THEN $ab = ac$ IMPLIES

THAT $b = c$.

CUZ $$\frac{1}{a}(ab) = \frac{1}{a}(ac)$$

$$\Rightarrow b = c.$$

BUT WE CAN'T DIVIDE BY A MATRIX!

SO WE CAN'T DO

$$\frac{1}{A} AB = \frac{1}{A} AC.$$

BUT SOMETIMES WE SORTA CAN!

WATCH!

CONSIDER $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$.

AND $B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$

$$BA = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & -3+3 \\ 2-2 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

SO IF WE HAD C, D WITH

$AC = AD$, THEN $BAC = BAD \Rightarrow C = D$.

$\Rightarrow I_2C = I_2D$

WE CALL B AN INVERSE OF A.

AND WRITE $A^{-1} = B$.

YOU CAN CHECK THAT $AA^{-1} = I_2$.

<u>DEF.</u> A $n \times n$ IS INVERTIBLE (NONSINGULAR)

IF THERE IS AN $n \times n$ MATRIX B

s.t. $AB = BA = I_n$.

B IS CALLED AN <u>INVERSE</u> OF A

AND IS WRITTEN A^{-1}.

(IT IS UNIQUE)

NOT ALL MATRICES ARE INVERTIBLE

BUT "MOST" ARE.

FACTS. (SEE BOOK FOR MOST

- IF A AND B ARE NONSINGULAR,
  THEN $(AB)^{-1} = B^{-1}A^{-1}$.

PROOF:

$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1}$$
$$= AIA^{-1}$$
$$= AA^{-1}$$
$$= I.$$

AND

$$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB$$
$$= B^{-1}IB$$
$$= B^{-1}B$$
$$= I.$$

$\square$

ALSO, $$(A^{-1})^{-1} = A.$$

$$(A^{-1})^T = (A^T)^{-1}.$$

CHECK ...

THIS IDEA HAS SURPRISING POWER.

BACK TO LINEAR SYSTEMS:

SUPPOSE WE HAVE SYSTEM

$$A\underline{x} = \underline{b}$$

IF A IS INVERTIBLE:

$$A^{-1}A\underline{x} = A^{-1}\underline{b}$$

$$I\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1}\underline{b}$$

IS A FORMULA FOR A SOLUTION!

NOTE THAT THIS GIVES A UNIQUE SOL'N SO WILL ONLY WORK IN THAT CASE.

EX $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

IF $\underline{b} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$,

THEN $A\underline{x} = \underline{b}$ HAS UNIQUE

SOLUTION

$$\underline{x} = A^{-1}\underline{b}$$

$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

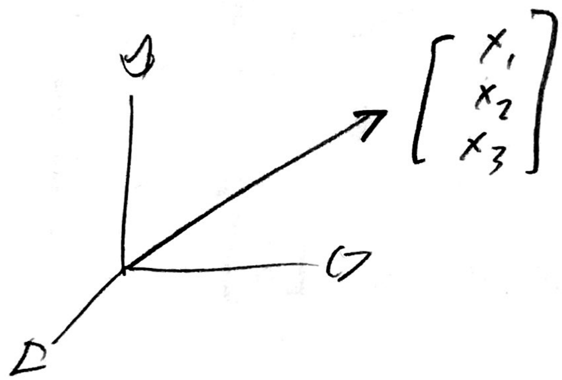
$$= \begin{bmatrix} -10 \\ 9 \end{bmatrix}.$$

RETURN TO THE IDEA OF TRANSFORMATIONS.

LET $\mathbb{R}^n$ BE THE COLLECTION OF ALL n-VECTORS $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. WRITTEN AS COLUMNS.

AGAIN, THINK GEOMETRICALLY

$\boxed{\text{VECTOR IS AN ARROW.}}$

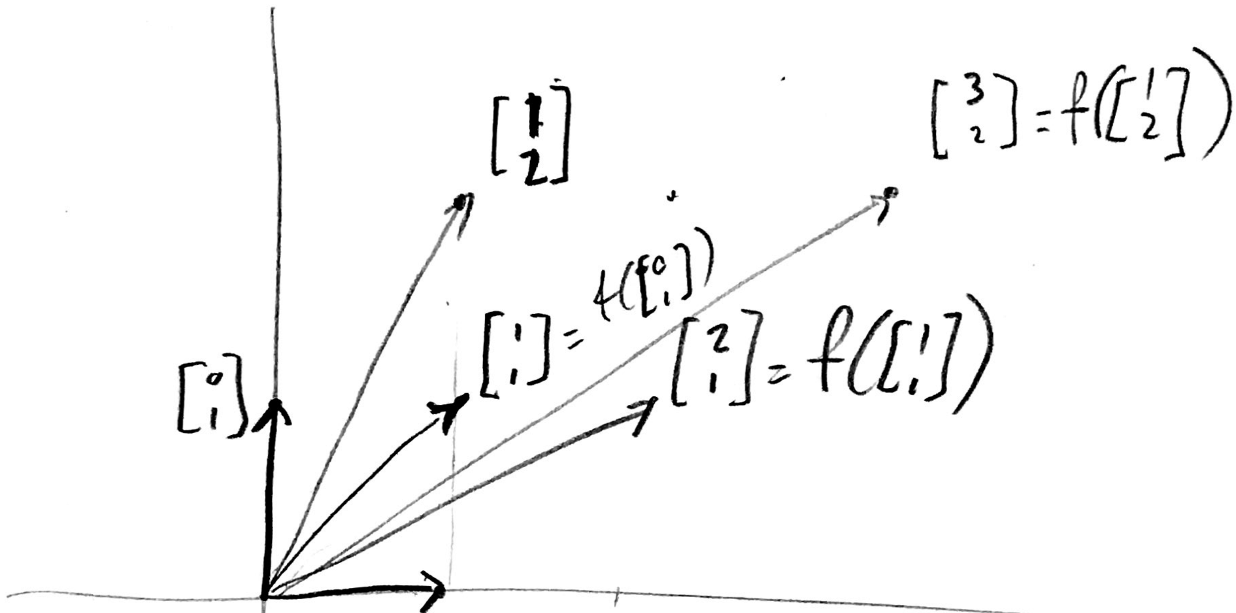


AN $n \times n$ MATRIX A WILL DEFINE A TRANSFORMATION $f: \mathbb{R}^n \to \mathbb{R}^n$ DEFINED BY $f(\underline{v}) = A\underline{v}$.

THIS IS A FUNCTION WHOSE INPUT IS A VECTOR $\underline{v}$ AND WHOSE OUTPUT IS A VECTOR $A\underline{v}$.

E.G.

$$\begin{array}{ccccc} A & \underline{v} & & f(\underline{v}) \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} x+y \\ y \end{bmatrix} \end{array}$$



$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ IS FIXED BY THE TRANSFORM.

MORE GENERALLY, YOU CAN HAVE LINEAR TRANSFORMATIONS

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

GIVEN BY AN $m \times n$ MATRIX A.

$$f(\underline{v}) = A \cdot \underline{v} \to m\text{-VECTOR}.$$

$\underline{v}$ ↓ n-VECTOR

EX. SCALAR MATRIX LIKE

DILATION $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ JUST STRETCHES EVERY VECTOR (EXCEPT $\underline{0}$).

ONE LIKE

CONTRACTION $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ SHRINKS EVERYTHING.

EX THE ZERO MATRIX JUST ZAPS EVERYTHING TO $\underline{0}$.

EX $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ IS A REFLECTION.

EX AND AS MENTIONED

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ROTATION.

SEE PAGE 61 FOR DERIVATION.

EX: TO REFLECT IN A LINE.

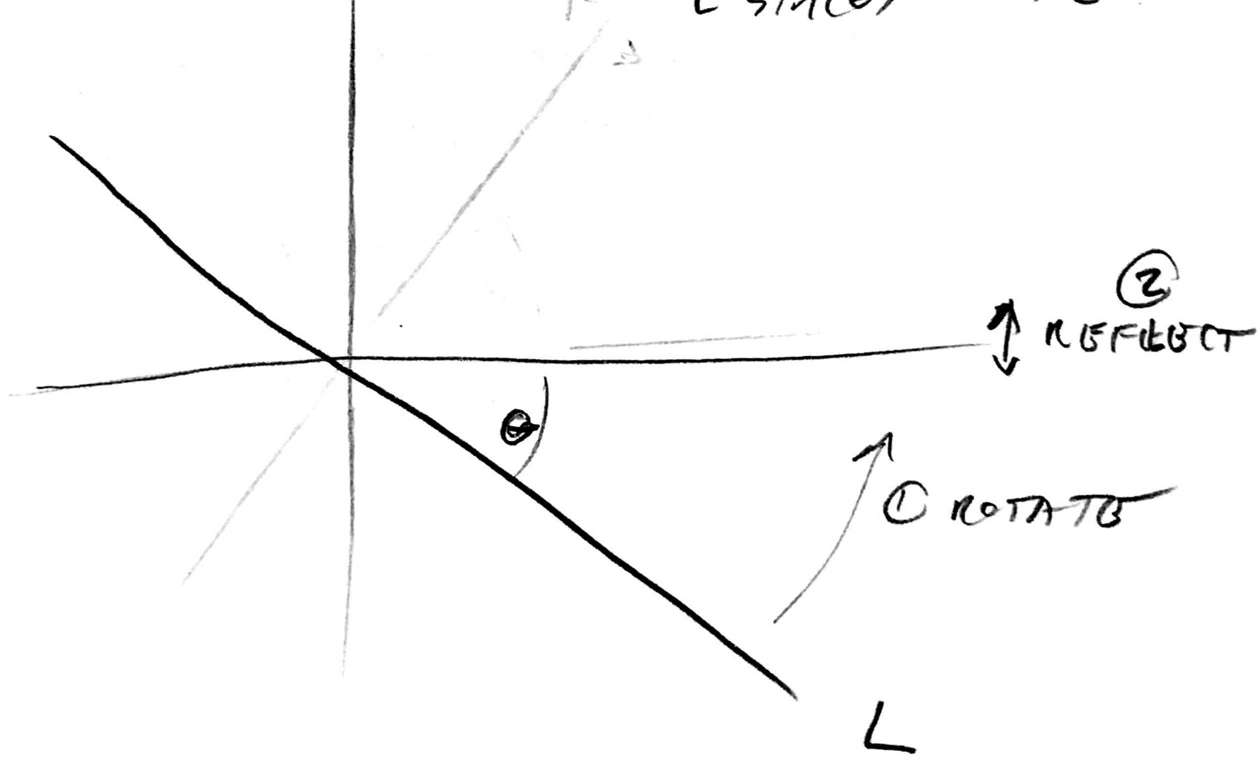
IF YOU WANT TO REFLECT IN A LINE L THAT ISN'T THE X-AXIS:

① FIRST ROTATE L TO X-AXIS USING SOME ROTATION MATRIX $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

② REFLECT IN X-AXIS USING $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

③ ROTATE BACK W/ $\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix}^{-1}$$



SO REFLECTION IN L IS TRANSFORM $R_\theta^{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}R_\theta$.

CHAPTER 2. SOLVING LINEAR SYSTEMS

2.1 ECHELON FORM OF A MATRIX.

IN THIS SECTION WE STUDY MOVES THAT PUT A MATRIX IN A SPECIFIC FORM.

THIS PROCEDURE WILL STREAMLINE THE PROCESS OF ELIMINATION WE SAW BEFORE.

AT BEGINNING OF SEMESTER, WE USED ELIMINATION TO CHANGE A LINEAR SYSTEM INTO A SYSTEM OF THE FORM

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3 \end{aligned}$$

THIS IS EASY TO SOLVE. (EACH EQ. HAS 1 VARIABLE WITH COEFFICIENT 1.

THE COEFFICIENT MATRIX LOOKS NICE:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

THIS MATRIX IS IN ROW ECHELON FORM.

Def AN $m \times n$ MATRIX IS IN ROW ECHELON FORM IF

a) ALL ZERO ROWS (IF ANY), AT BOTTOM.

b) IF A ROW IS NONZERO, ITS FIRST NONZERO ENTRY IS 1. (THE "LEADING 1")

c) THE LEADING 1 OF A NONZERO ROW IS TO THE RIGHT OF ANY LEADING IN THE PREVIOUS ROW.

IF A MATRIX SATISFIES a) — c) AND

d) THE LEADING ENTRY OF A NONZERO ROW IS THE ONLY NONZERO ENTRY IN ITS COLUMN.

THEN MATRIX IS IN REDUCED ROW ECHELON FORM.

EX.

$$B = \begin{bmatrix} 1 & 0 & 2 & 5 & 6 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ROW ECHELON

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REDUCED ROW ECHELON.

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

NEITHER

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

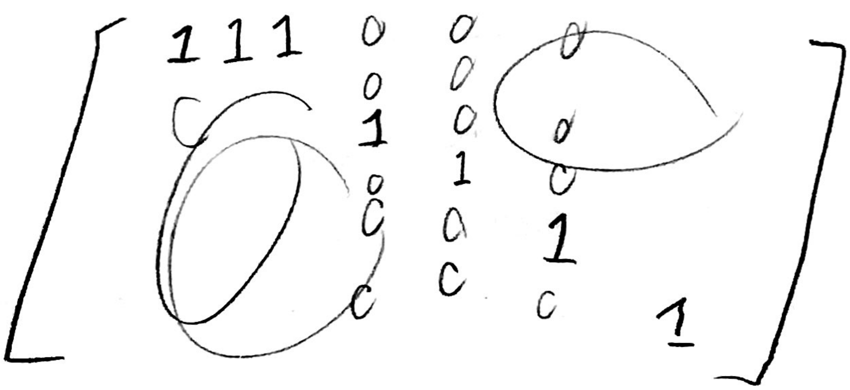
NEITHER.

GO TO RREF:

ECHELON:

$$\begin{bmatrix} \text{— zeros . 1 ——— STUFF ———} \\ \text{— zeros —— 1 ——— STUFF ——} \\ \text{— zeros —— 1 STUFF} \\ \vdots \\ \text{— zeros —————————} \end{bmatrix}$$

REDUCED ECHELON



OPERATIONS.

ELEMENTARY ROW OPERATIONS:

TYPE I. INTERCHANGE TWO ROWS

TYPE II. MULTIPLY A ROW BY A NONZERO NUMBER

TYPE III. ADD A MULTIPLE OF ONE ROW TO ANOTHER.

(ELEMENTARY COLUMN OPERATIONS ARE SAME BUT W/ COLUMNS.)

NOTE: IF WE PERFORM THESE OPERATIONS ON THE AUGMENTED MATRIX OF A LINEAR SYSTEM, WE GET THE AUGMENTED MATRIX OF AN EQUIV. SYSTEM.

NOTATION TYPE I: INTERCHANGE ROWS (COLS) i AND j

$$\underline{r}_i \leftrightarrow \underline{r}_j \qquad (c_i \leftrightarrow c_j)$$

II, REPLACE ROW i BY k TIMES ROW i

$k\underline{r}_i \to \underline{r}_i$

NEW ROW   OLD ROW

($k\underline{c}_i \to \underline{c}_i$ FOR COLS)

III REPLACE ROW j WITH k TIMES ROW i + ROW j

$k\underline{r}_i + \underline{r}_j \to \underline{r}_j$

NEW ROW   OLD ROW.

WE USE THESE AS SUBSCRIPTS

SO GIVEN A, $A_{r_1 \leftrightarrow r_3}$

MEANS THE MATRIX AFTER INTERCHANGING ROWS 1 AND 3.

EX. $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}$

$$B = A_{r_1 \leftrightarrow r_3} = \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$C = A_{\frac{1}{3}r_3 \to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

$$D = A_{-2r_2 + r_3 \to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$ → ROW 2 SAME.

WE SAY B IS ROW EQUIVALENT TO A IF B CAN BE PRODUCED BY APPLYING A FTE SEQUENCE OF ROW OPERATIONS.

EX $\quad A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -2 & 2 & 3 \end{bmatrix}$

$$B = A_{2r_3 + r_2 \to r_2} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & -3 & 7 & 8 \\ 1 & -2 & 2 & 3 \end{bmatrix}$$

$$C = B_{r_2 \leftrightarrow r_3} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & -2 & 2 & 3 \\ 4 & -3 & 7 & 8 \end{bmatrix}$$

$$D = C_{2r_1 \to r_1} = \begin{bmatrix} 2 & 4 & 8 & 6 \\ 1 & -2 & 2 & 3 \\ 4 & -3 & 7 & 8 \end{bmatrix}$$

SAY $\quad D = A_{\substack{2r_3 + r_2 \to r_2 \\ r_2 \leftrightarrow r_3 \\ 2r_1 \to r_1}}$

EQ. REL. $\quad$ A ROW EQ. A.

IF A ROW EQ. B THEN B ROW EQUIVALENT TO A.

IF A REQ TO B ROW EQ. C, THEN A ROW EQ C.

THM. EVERY MATRIX IS ROW EQUIVALENT TO A MATRIX IN ROW ECHELON FORM.

PROOF EXAMPLE FIRST.

① FIND FIRST NONZERO COLUMN, COL j SAY

$$A = \begin{array}{cc} & \begin{array}{ccc} & & \text{col } j \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} & \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & p & \\ 0 & 0 & \vdots & \\ 0 & 0 & \vdots & \\ 0 & 0 & \vdots & \end{bmatrix} \end{array}$$

② CALL THIS COLUMN'S FIRST NONZERO ENTRY p. (THE "PIVOT")

IT IS IN SOME ROW, i SAY.

③ SWAP ROW 1 AND i.

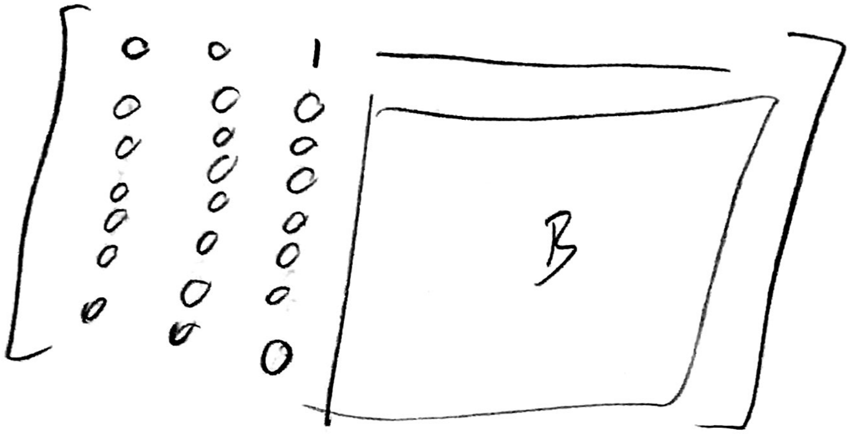
$$\rightsquigarrow \begin{bmatrix} 0 & 0 & p & \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & 0 & \\ \vdots & \vdots & 0 & \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \vdots & \end{bmatrix}$$

④ DIVIDE ROW 1 by p

$$\rightsquigarrow \begin{bmatrix} 0 & 0 & 1 & \\ \vdots & \vdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & * & \end{bmatrix}$$

(5) IF THERE IS A NONZERO ENTRY IN COL j BELOW ROW 1, WE CAN CLEAR IT TO ZERO BY SUBTRACTING A MULTIPLE OF ROW 1.

RESULT:



(6) REPEAT WITH B.

THIS ENDS EVENTUALLY. $\square$ ✓

EX.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \checkmark$$

BACK TO PROOF.

THEN

THM. A IS ROW EQUIV. TO A MATRIX IN REDUCED ECHELON FORM.

PROOF. FIRST USE FORM FROM PREVIOUS THM AND THEN CLEAR ABOVE PIVOTS. ∎

IN EX.
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

2.2 LINEAR SYSTEMS.

THM $A\underline{x} = \underline{b}$ $C\underline{x} = \underline{d}$

LINEAR SYSTEMS.

THESE ARE EQUIVALENT IF THE AUGMENTED MATRICES ARE ROW EQUIVALENT.

PROOF. ELEM. MOVES ON MATRICES CORRESPOND TO MOVES ON SYSTEMS.

ALGORITHM TO SOLVE $A\underline{x} = \underline{b}$.

PUT THE AUGMENTED MATRIX INTO

ECHELON OR RED. ECHELON FORM
(GAUSS ELIM.) (GAUSS-JORDAN ELIM.)

AND THEN SOLVE USING BACK. SUBST. FOR NEW SYSTEM.

<u>ex</u>

$$
\begin{aligned}
x + 2y + 3z &= 9 \\
2x - y + z &= 8 \\
3x \qquad - z &= 3
\end{aligned}
$$

$$
[A \mid \underline{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]
$$

$$
\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]
$$

$$
\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right]
$$

$$
\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right]
$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\rightsquigarrow \quad x + 2y + 3z = 9$$

$$y + z = 2$$

$$z = 3$$

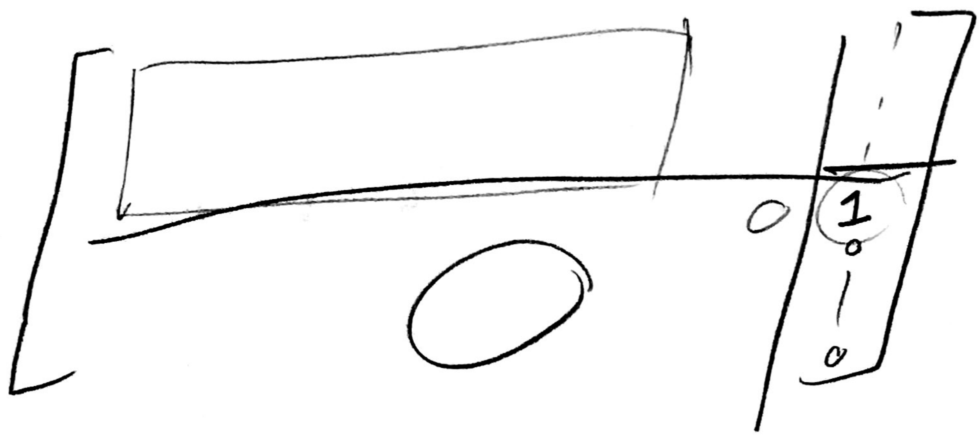
$$\rightsquigarrow \quad y = -1$$

$$\rightsquigarrow \quad x = 2$$

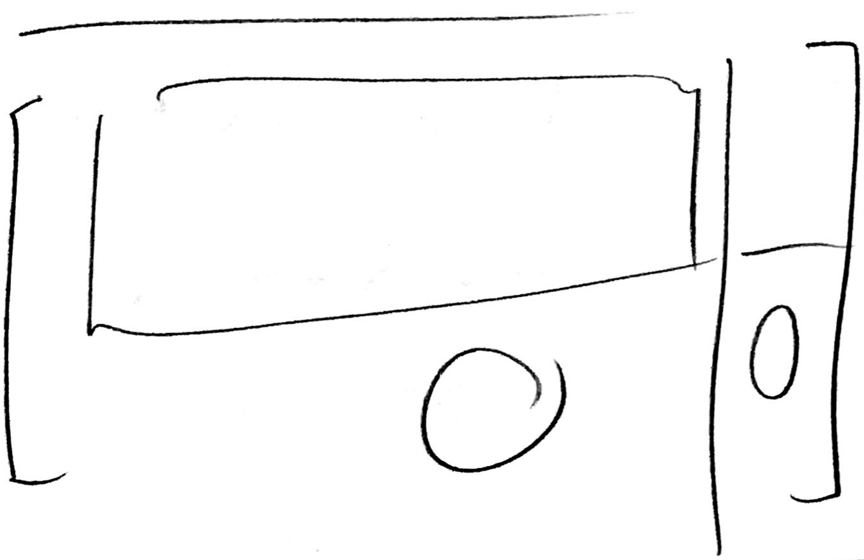
UNIQUE SOLUTION.

WHEN THERE ARE NO SOLUTIONS,

YOU SEE A MATRIX LIKE



INFINITELY MANY SOLUTIONS:



BACK SUBSTITUTE AND SOME VARIABLES
COULD BE ANYTHING.

EG.

X ANYTHING.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$w \quad x \quad y \quad z$

$z = 1$

$y = 1$

$w + x + z = 1$

$\Rightarrow \quad w + x = 0$

$w = -x$

HOMOGENEOUS SYSTEMS

$A\underline{x} = \underline{0}$

THM IF THERE ARE MORE UNKNOWNS THAN EQ^S, THERE IS A NONTRIVIAL SOL^N.

Proof.

i.e. THERE IS A COLUMN W/ NO PIVOT.

THAT VARIABLE IS UNDETERMINED. $\square$

HOMOG. AND NONHOMOG.

$A\underline{x} = \underline{b} \neq \underline{0}$ $\rightsquigarrow$ A HOMOG. SYSTEM. $A\underline{x} = \underline{0}$.

IF $\underline{x}_p$ IS A SOL^N TO $A\underline{x} = \underline{b}$

AND $\underline{x}_h$ A " TO $A\underline{x} = \underline{0}$

then $\underline{x}_h + \underline{x}_p$ A SOL^N TO $A\underline{x} = \underline{b}$.

$A(\underline{x}_h + \underline{x}_p) = A\underline{x}_h + A\underline{x}_p = \underline{0} + \underline{b}$)

ALL SOL'NS OF $A\underline{x} = \underline{b}$ ARE LIKE THIS.

PROOF ; EXERCISE 29. IN 2.2.

2.3 ELEMENTARY MATRICES.

FINDING THE INVERSE OF A MATRIX.

DEF. AN $n \times n$ ELEMENTARY MATRIX OF TYPE I, II, III

IS A MATRIX OBTAINED FROM I_n

BY AN ELEM. MOVE OF TYPE I, II, III.

EX

$$E_1 \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{I}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{II}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>THM</u>

A $m \times n$ MATRIX.

B MATRIX OBTAINED BY AN ELEM. MOVE.

E ELEM. MX " FROM I_m BY PERFORMING

SAME MOVE.

THEN $B = EA$.

<u>PROOF:</u> EXERCISE 1.

<u>EX.</u>

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

BY USING MULT. BY ELEM. MATRICES INSTEAD OF MOVES, WE CAN WRITE OUR ROW REDUCTION USING MX MULT.

THM A AND B $m \times n$ MXS.

A ROW EQ. B IFF THERE ARE ELEMENTARY MATRICES $E_1, \ldots E_k$ S.T.

$$B = E_k E_{k-1} \cdots E_2 E_1 A.$$

THM ELEMENTARY MATRICES ARE INVERTIBLE AND INVERSES ARE ELEM. OF SAME TYPE.

PROOF EXER. 6

THM A invertible iff A is prod. of elementary matrices.

LEMMA. If A $n \times n$ and $A\underline{x} = \underline{0}$ has only trivial sol^n, then A is row equivalent to I.

A is row equivalent to such a B in row echelon form, which will have the same number of solutions as A.

Proof. If B is in reduced row echelon form and $B\underline{x} = \underline{0}$ has only triv. sol^n, then $B = I$. since it can't have any zero rows. $\square$

PROOF OF THM.

If A is prod $E_1 \dots E_k$ E_i elem, then $A^{-1} = E_k^{-1} \cdot E_{k-1}^{-1} \cdots E_1^{-1}$.

If A is invertible, then

$$A\underline{x} = \underline{0} \Rightarrow A^{-1}(A\underline{x}) = A^{-1}\underline{0} = 0$$

so $I_n \underline{x} = 0 \Rightarrow \underline{x} = 0$.

So $A\underline{x}$ has only triv. sol^n.

By lemma, done. $\square$

<u>COR.</u> A INVERTIBLE IFF
A ROW EQUIV. TO I_n.

<u>THM</u> $A\underline{x} = \underline{0}$ (A $n \times n$) IFF A SINGULAR.

THIS GIVES US AN ALGORITHM TO FIND A^{-1}.

WRITE $E_k E_{k-1} \cdots E_1 A = I_n$

SO $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.

THEN $A^{-1} = E_k E_{k-1} \cdots E_1$.

<u>EASY METHOD.</u>

$$\underbrace{E_k \cdots E_1}\left[A \mid I_n\right] \rightsquigarrow \left[I \mid A^{-1}\right]$$

ELEMENTARY MOVES ON LEFT TURNING A INTO I GIVES A^{-1} ON RIGHT.

ex. (simple?) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

- Don't mult.
- Do row operations directly.

$$\begin{array}{c} \sim \\ -5r_1 + r_3 \\ \to r_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

EASY EX

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

---

check!

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

THM A SINGULAR IFF EQ. B w/ ROW OF ZEROES.

I'VE BEEN SAYING $B = A^{-1}$ EVEN IF WE ONLY HAVE $AB = I$ OR $BA = I$.

THM, IF A, B $n \times n$ AND

$AB = I$, THEN $BA = I$.

PROOF: SEE TEXT.

IDEA. IF $AB = I$ THEN A IS NONSING.

IF NOT, $A \sim C$ ROW OF ZEROES.

SO $C = E_k \cdots E_1 A$

ROW OF ZEROES.

THEN $E_k \cdots E_1 AB$ HAS

ROW OF ZEROES.

SO AB ROW EQ. TO CB

SO AB IS SINGULAR, BY LEMMA.

BUT $AB = I$. ※.

SO A HAS AN INVERSE A^{-1}.

$$A^{-1}AB = A^{-1} \Rightarrow B = A^{-1}.$$

2.3

DEF.

A, B EQUIVALENT

(NOT JUST "ROW EQ.")

IF. B OBTAINED FROM A BY FINITE

SEQ. OF ELEM. ROW or COL. OPERATIONS.

THM. A NONZERO.

A EQ TO

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$\square$

THM A EQUIV. B

IFF $B = PAQ$

FOR NONSINGULAR P, Q.

THM A INVERTIBLE IFF A EQUIV. I.

CHAPTER 3.

DETERMINANTS

CAN ASSOC. #S TO MATRIX.

E.G.

TRACE OF nxn MATRIX $A = [a_{ij}]$

IS

$$TR(A) = \sum_{i=1}^{n} a_{ii}$$

ANOTHER VERY IMPORTANT. #

IS THE DETERMINANT.

LET A BE nxn MATRIX.

START WITH n=2.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

LET'S THINK ABOUT A. AS A TRANSFORMATION.

LET'S USE PARTICULAR #S.

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$C\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

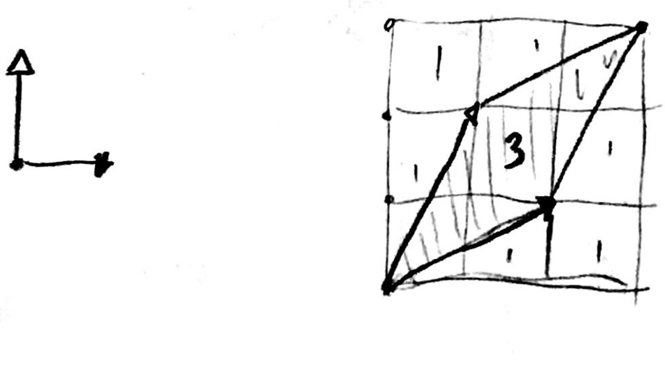
$$C\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

AREA IS 2.

$$D = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$D\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

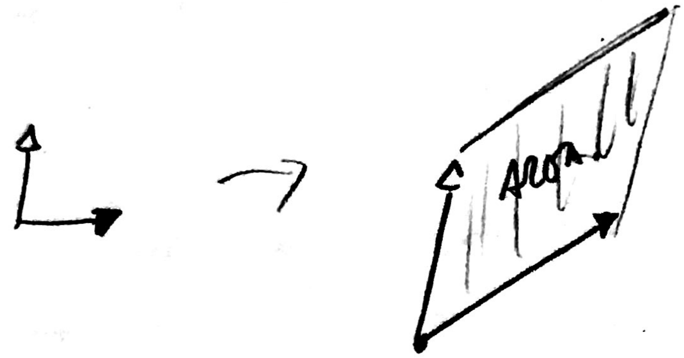
$$D\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$



DEFINE

$$\mathrm{DET}(A) = ad - bc$$

NOTICE!

$\mathrm{DET}(B) = 2$

$\mathrm{DET}(C) = 2$

$\mathrm{DET}(D) = 3$

WARNING.

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathrm{DET}\ E = -1$$

AREA IS ONE BUT BACKWARDS.

IF COLS PARALLEL DET: $A = 0$.

WE WILL DEFINE $A \mapsto \det(A)$

FOR ARBITRARY $n \times n$ MX A.

$|\det(A)|$ IS VOLUME OF PARALLELEPIPED.

NOTICE THAT A WILL BE INVERTIBLE

IFF $\det A \neq 0$.

JUST STRAIGHTEN OUT THE VECTORS.

TO DEFINE IT WE NEED TO WORK.

PERMUTATIONS.

IN BOOK

$S = \{1, \ldots, n\}$ $\qquad$ $j_1 \cdots j_n$

A PERMUTATION IS A REARRANGEMENT OF TERMS.

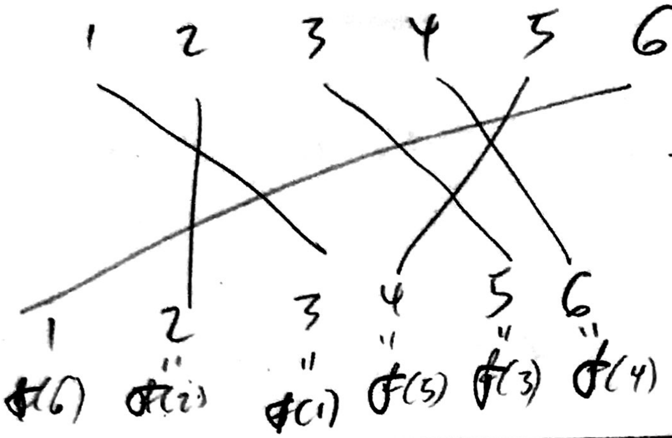
PRECISELY, A BIJECTION $\sigma: S \to S$.

$j_k = \sigma(k)$.

EX.

$\sigma(1) = 4$
$\sigma(2) = 2$
$\sigma(3) = 3$
$\sigma(4) = 1$

THINK THIS WAY · 3, 2, 5, 6, 4, 1.



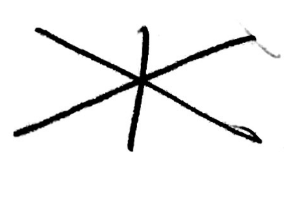
→ ALWAYS DRAW SO THAT IN GENERAL POSITION.

HOW MANY PERMUTATIONS ARE THERE WHEN $S = \{1, 2, 3\}$.

HOW MANY LISTS OF LENGTH 3 FROM S?

3 choices 2 1

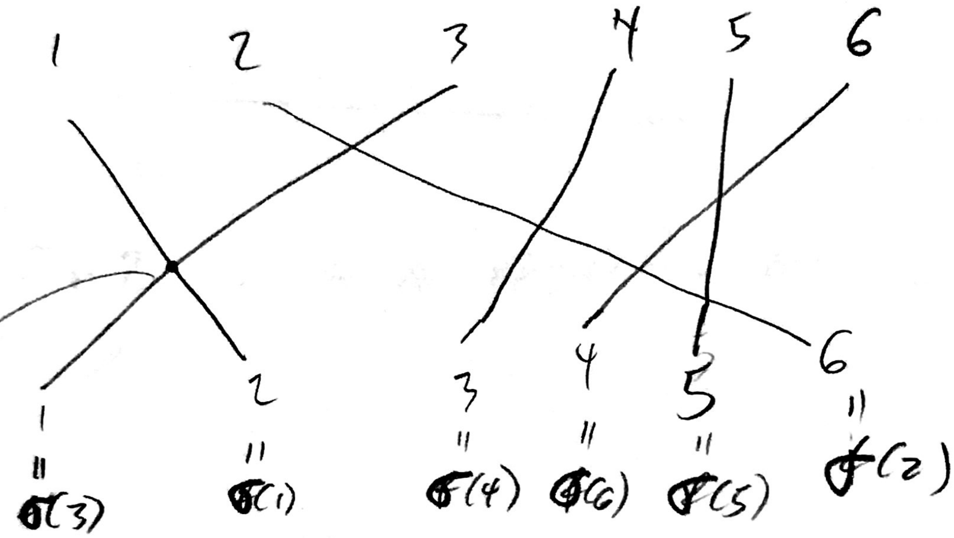
NOT LIKE



<u>TYPES OF PERMUTATIONS.</u>

LOOK AT A BRAID PICTURE.

2, 6, 1, 3, 5, 4



WHAT DOES A <u>CROSSING</u> MEAN?

IT MEANS THAT A NUMBER SHOWS UP <u>BEFORE</u> A SMALLER ONE.

CALLED AN <u>INVERSION</u>.

MEANS THAT 2 SHOWS UP <u>BEFORE</u> 1.

THEIR ORDER IS <u>FLIPPED</u>.

CROSSINGS IS # OF INVERSIONS.

f IS <u>EVEN</u> IF <u>EVEN</u>.

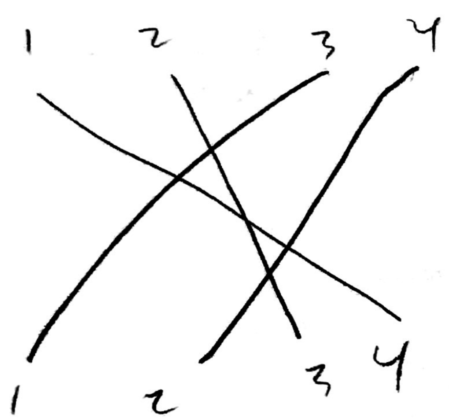
f IS <u>ODD</u> IF <u>ODD</u>.

$n \geq 2$

THERE ARE $\frac{n!}{2}$ EVEN PERMS

AND $\frac{n!}{2}$ ODD. PERMS

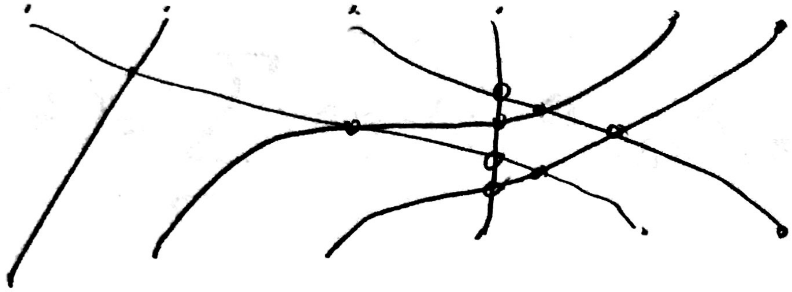
4312 IS A PERMUTATION.



ODD.

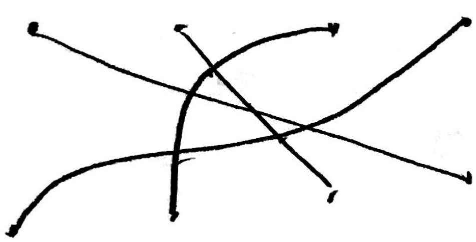
516423 9

ODD



4321

6



$\mathrm{sign}(\sigma) = 1$ IF σ EVEN

$= -1$ IF σ ODD.

DEF. $A = [a_{ij}]$

$$|A| = \det(A) = \sum_{\sigma} \mathrm{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

EX $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (BEFORE WE SAID $a_{11}a_{22} - a_{12}a_{21}$)

WHAT ARE THE PERMUTATIONS OF $\{1,2\}$?

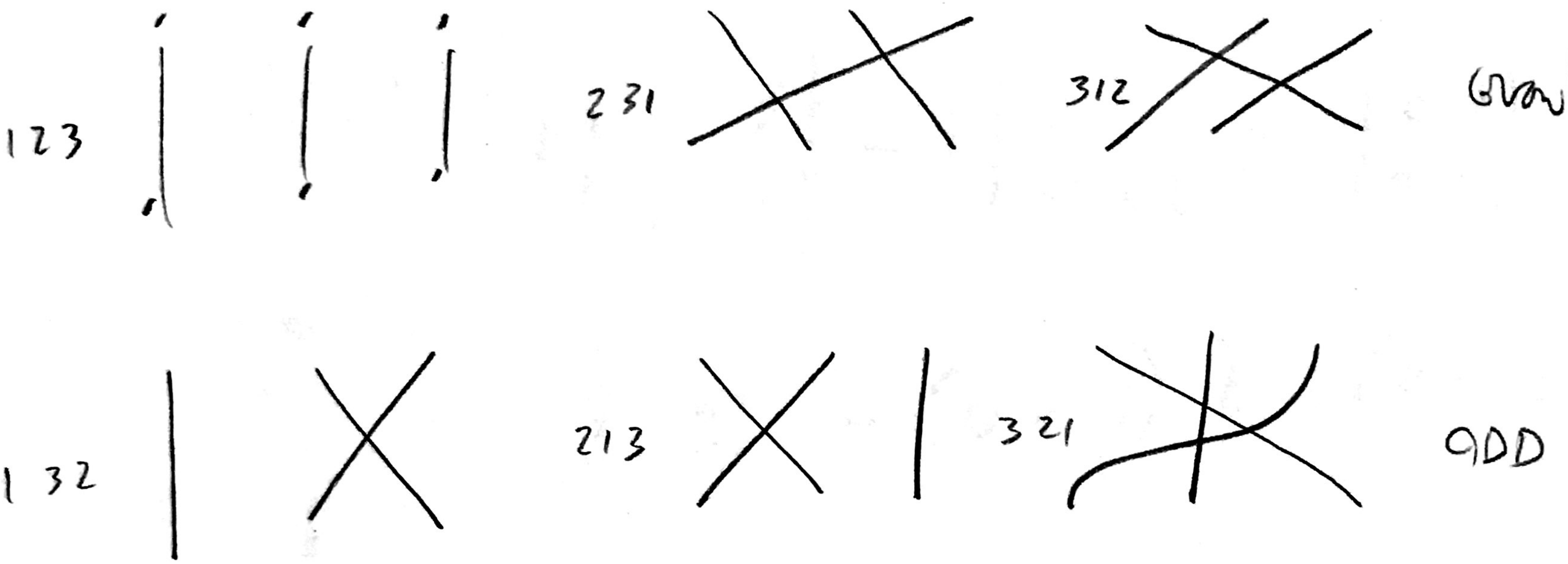
EVEN 12 AND ODD 21.

$\sigma(1)=1 \ \sigma(2)=2$ $\quad$ $\tau(1)=2 \ \tau(2)=1$.

$$\det(A) = +a_{1\sigma(1)}a_{2\sigma(2)} - a_{1\tau(1)}a_{2\tau(2)}$$

$$= a_{11}a_{22} - a_{12}a_{21}.$$

ex. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$



$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

<u>ex</u>

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\det(A) = 1\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} + 5\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= (2-3) - (0-9) + 5(0-3)$$

$$= -1 + 9 - 15$$

$$= -7.$$

3.2 PROPERTIES.

Thm $\det(A) = \det(A^T)$

Pf SKIP. USEFUL!

OBV. FOR 2×2. COME BACK LATER.

Thm IF WE OBTAIN B FROM A BY SWITCHING ROWS, THEN $\det(B) = -\det(A)$.

Ex. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \qquad \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1.$

GEOMETRIC IDEA. IF WE SWITCH TWO COLUMNS SIGN CHANGES. (MIRROR IMAGE).

CHANGING TWO ROWS = CHANGES TWO COLUMNS OF TRANSPOSE, SO SAME EFFECT.

[THINK OF THE EFFECT OF THE ELEMENTARY MATRICES OF TYPE I.]
E.G.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Proof that $\det(A^T) = \det(A)$.

Given a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$.

There is an inverse permutation σ^{-1} that undoes σ.

Example.

Inverse of 312 is 231.

Easier to see with bijections

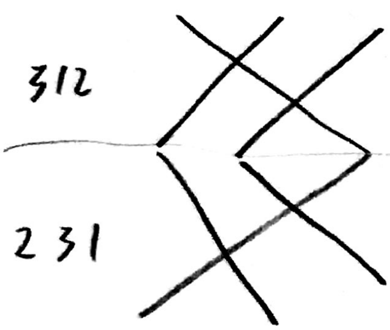
$$\sigma(1) = 3 \qquad \sigma(2) = 1 \qquad \sigma(3) = 2$$

$$1 = \sigma^{-1}(3) \qquad 2 = \sigma^{-1}(1) \qquad 3 = \sigma^{-1}(2)$$

So $\sigma^{-1}(1) = 2$, $\sigma^{-1}(2) = 3$, $\sigma^{-1}(3) = 1$.

Picture:

312



231

Notice σ^{-1} has same sign as σ.

Since # of crossings same.

Consider $B = A^T$. $B = [b_{ij}] = [a_{ji}]$

$b_{13} b_{21} b_{32}$ in def. of $\det(A^T)$

$$= a_{31} a_{12} a_{23}$$

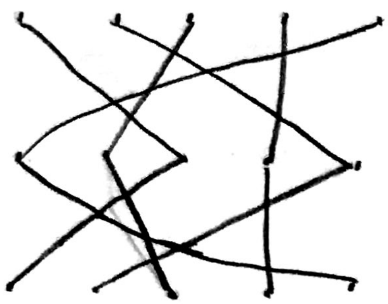
— reorder

$$= a_{12} a_{23} a_{31}$$

$$= a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} a_{3\sigma^{-1}(3)}$$

35241

53142



INVERSE IS SAME SIGN.

$[b_{ij}] = B = A^T \qquad b_{ij} = a_{ji}$

$$b_{1\sigma(1)}\, b_{2\sigma(2)}\, b_{3\sigma(3)}\, b_{4\sigma(4)}\, b_{5\sigma(5)}$$

$$= b_{13}\, b_{25}\, b_{32}\, b_{44}\, b_{51}$$

$$= a_{31}\, a_{52}\, a_{23}\, a_{44}\, a_{15}$$

REORDER

$$= a_{15}\, a_{23}\, a_{31}\, a_{44}\, a_{52}$$

$$= a_{1\sigma^{-1}(1)}\, a_{2\sigma^{-1}(2)}\, a_{3\sigma^{-1}(3)}\, a_{4\sigma^{-1}(4)}\, a_{5\sigma^{-1}(5)}.$$

$B = [b_{ij}] = [a_{ji}] = A^T$

$$\det(B) = \sum_{\sigma} \operatorname{sign}(\sigma)\, b_{1\sigma(1)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma^{-1}} \operatorname{sign}(\sigma^{-1})\, a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}$$

$$= \sum_{\tau} \operatorname{sign}(\tau)\, a_{1\tau(1)} \cdots a_{n\tau(n)} = \det(A)$$

$\square$

<u>THM</u> IF TWO ROWS (OR COLUMNS) ARE EQUAL, THEN $\det(A) = 0$.

<u>PF.</u> INTERCHANGE ROWS TO GET B.

SO $$\det(B) = -\det(A).$$

BUT $A = B$, SO
$$\det(A) = -\det(A).$$

SO $$\det(A) = 0.$$

$\square$

<u>THM</u> IF A HAS A ZERO ROW,
$$\det(A) = 0.$$

<u>PROOF</u> SUPPOSE ROW i IS ZERO. EVERY TERM OF THE SUM DEFINING $\det(A)$ INVOLVES A TERM FROM ROW i:

$$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots \underbrace{a_{i\sigma(i)}}_{=\,0} \cdots a_{n\sigma(n)}$$

$\square$

ex

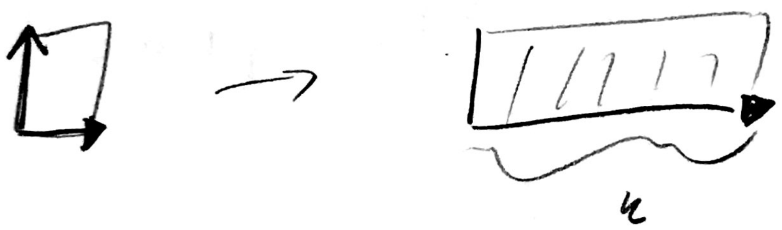
$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

THM. IF OBTAIN B FROM A BY MULTIPLYING A ROW BY k, (OR COLUMN) THEN $\det(B) = k\det(A)$.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ ✓

(THINK COLUMNS. THEN PARALLELEPIPED STRETCHED BY k:



)

Pf

$$\sum \operatorname{sign}(\sigma)\, b_{1\sigma(1)} \cdots b_{r\sigma(r)} \cdots b_{n\sigma(n)}$$

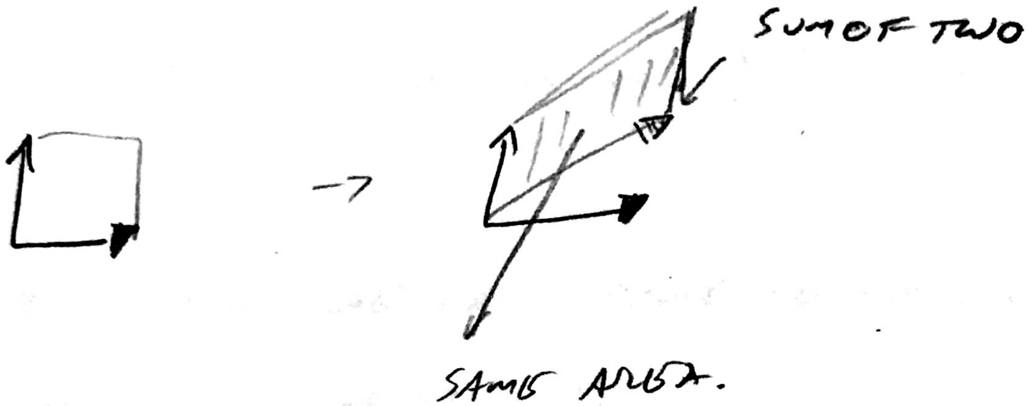
$$= \sum \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots (k a_{r\sigma(r)}) \cdots a_{n\sigma(n)}$$

$$= k \sum \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots\cdots a_{n\sigma(n)}$$

$$= k \det(A).$$

$\square$

THM TYPE III ROW OPERATIONS DON'T CHANGE DETERMINANT. $\square$



EX

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

or

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

or

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

<u>THM</u> IF $A = [a_{ij}]$ IS <u>UPPER</u> (LOWER)

TRIANGULAR,

THEN $\det A = a_{11} a_{22} \cdots a_{nn}$.

(FOLLOWS FROM PREVIOUS THM.)

USING THESE WE CAN COMPUTE DETERMINANT

USING ROW OPERATIONS.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \det = 1$$

$\Uparrow$ TYPE I MULT. ROW 1 BY 2.

$$\sim \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix} \qquad \det = 1/2$$

$\Uparrow$ TYPE III

$$\sim \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} \qquad \det = \frac{1}{2}$$

(PRODUCT OF DIAGONALS)

THM IF E IS ELEMENTARY

$$\det(EA) = \det(E)\det(A).$$

PF. USE THMS ABOUT CHANGE IN DETERMINANT UNDER ROW OPERATIONS. $\square$

THM A INVERTIBLE IFF $\det(A) \neq 0$.

PF. IF A IS INVERTIBLE THEN

$A = E_1 \cdots E_n$ WHERE E_i ELEM.

So $\det(A) = \det(E_1) \cdots \det(E_n) \neq 0.$ $\square$

IF A SINGULAR, A ROW EQUIV. TO B WITH ROW OF ZEROES.

$A = E_1 \cdots E_n B$

AND $\det(A) = \det(E_1)\det(E_2) \cdots \det(E_k)\det(B)$

$= 0.$ $\square$

COR

$\underline{A}\underline{x} = \underline{0}$ HAS NONZERO SOLUTION IFF A SINGULAR.

THM IF A AND B ARE $n \times n$ THEN $\det(AB) = \det(A)\det(B)$.

PROOF: SUPPOSE A IS SINGULAR, THEN RHS IS ZERO.

ALSO A IS ROW EQUIV. TO C WITH ROW OF ZEROES.

$C = E_k \cdots E_1 A$.

SO $\underbrace{CB}_{\substack{\text{HAS}\\ \text{ROW OF}\\ \text{ZEROES}}} = \underbrace{E_k \cdots E_1 AB} \Rightarrow AB$ IS SINGULAR.

SO $\det(AB) = 0$.

IF A INVERTIBLE, THEN $A = E_1 \cdots E_k$

SO $\det(AB) = \det(E_1 \cdots E_k)\det(B)$ ✓.

Cor. $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof:

$$1 = \det(I) \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

$\square$

3.3 COFACTOR EXPANSION.

DEF. LET $A = [a_{ij}]$.

LET M_{ij} BE THE $(n-1) \times (n-1)$ SUBMATRIX OBTAINED BY DELETING THE i^{th} ROW AND THE j^{TH} COLUMN.

$\det(M_{ij})$ IS THE MINOR OF a_{ij}.

THE COFACTOR A_{ij} OF a_{ij} IS

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

EX

$$A = \begin{bmatrix} 5 & 7 & 1 \\ 2 & 6 & 4 \\ 2 & -1 & 0 \end{bmatrix} \qquad M_{12} = \begin{bmatrix} 2 & 4 \\ 2 & 0 \end{bmatrix}$$

$$\det(M_{12}) = -8$$

$$A_{12} = (-1)(-8)$$

$$M_{22} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix} \qquad \det(M_{22}) = -2$$

$$A_{22} = (+1)(-2) = -2$$

THE SIGN IN FRONT OF $\det(M_{ij})$ IN COFACTOR CAN BE REMEMBERED BY PUTTING +S AND −S IN A CHECKERBOARD PATTERN IN A MATRIX:

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix}$$

THM $A = [a_{ij}]$ $n \times n$ MX.

"EXPANSION ALONG ith ROW"

THEN $\det(A) = a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}$

(just like 3×3 case!)

AND

$$\det(A) = a_{1j} A_{1j} + \dots \; a_{nj} A_{nj}$$

"EXPANSION ALONG jth COLUMN.

$\square$

SHOULD PICK ROW OR COLUMN WITH LOTS OF ZEROS.

Ex.

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$\begin{vmatrix} 2 & 5 & 7 & 1 \\ 6 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 10 & 1 \end{vmatrix}$$

$$= 0\begin{vmatrix} 5 & 7 & 1 \\ 3 & 0 & 0 \\ 0 & 10 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 7 & 1 \\ 6 & 0 & 0 \\ 0 & 10 & 1 \end{vmatrix}$$

$$+ 2\begin{vmatrix} 2 & 5 & 1 \\ 6 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 5 & 7 \\ 6 & 3 & 0 \\ 0 & 0 & 10 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 5 & 1 \\ 6 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= 2\left(0\cdot\begin{vmatrix} 5 & 1 \\ 3 & 0 \end{vmatrix} - 0\cdot\begin{vmatrix} 2 & 1 \\ 6 & 0 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix}\right)$$

$$= 2\cdot(6-30) = -48$$

CAN USE ROW OPERATIONS
TO INTRODUCE ZEROES:

$$\begin{vmatrix} 1 & 2 & 5 & 6 \\ 7 & 1 & 1 & 1 \\ 22 & 1 & 1 & 1 \\ -1 & 5 & 5 & 6 \end{vmatrix}$$

SUBTRACT ROW 2 FROM ROW 3.

$$\xrightarrow{-r_2+r_3\to r_3} \;=\; \begin{vmatrix} 1 & 2 & 5 & 6 \\ 7 & 1 & 1 & 1 \\ 15 & 0 & 0 & 0 \\ -1 & 5 & 5 & 6 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$= +15 \begin{vmatrix} 2 & 5 & 6 \\ 1 & 1 & 1 \\ 5 & 5 & 6 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\xrightarrow{-r_1+r_3\to r_3} \;=\; 15 \begin{vmatrix} 2 & 5 & 6 \\ 1 & 1 & 1 \\ 3 & 0 & 0 \end{vmatrix} = 15 \left(3 \begin{vmatrix} 5 & 6 \\ 1 & 1 \end{vmatrix} \right)$$

$$= 45 \begin{vmatrix} 5 & 6 \\ 1 & 1 \end{vmatrix} =$$ [redacted]

WHY IS DETERMINANT THE SIGNED AREA FOR 2×2 MATRICES?

BOOK USES COFACTOR EXPANSION TO SHOW THIS.

THINK GEOMETRICALLY:

SUPPOSE A IS A 2×2 MATRIX.

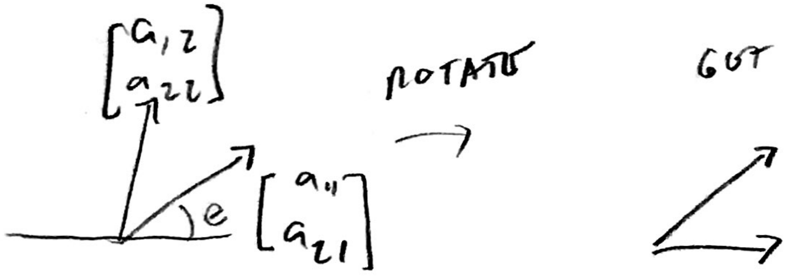
LET $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ BE A ROTATION MATRIX. NOTE $\det R_\theta = 1$ BY PYTHAGOREAN THM,

SO $\det(R_\theta A) = \det(R_\theta)\det(A) = 1 \cdot \det(A)$
$= \det(A)$.

FOR SOME θ, $R_\theta A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

THINK:

IF $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$



GET TWO NEW VECTORS AND 1ST ONE HAS 0 SECOND COORD.

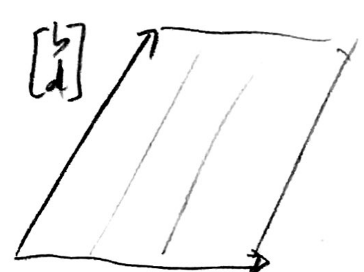
ROTATION ALSO DOESN'T CHANGE AREA.

SO WE CAN PROVE THIS FOR MATRICES

LIKE

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$$\det(A) = ad - 0$$
$$= ad \neq 0$$



AREA

BASE IS a
HEIGHT IS d

ad

3.4. INVERSES AGAIN

DEF^{n} $A = [a_{ij}]$ $n \times n$.

A_{ij} IS THE ijTH COFACTOR.

THEN THE **ADJOINT** OF A IS

$$\operatorname{adj} A = [A_{ij}]^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

EX

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

EX

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

NOTE $\det A$

$$= 1 \cdot \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= 3 - 4 + 3 = 2$$

$$A_{11} = + \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3$$

$$A_{12} = - \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = -2$$

$$A_{13} = + \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} = 3$$

$$A_{21} = - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2$$

$$A_{22} = + \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

$$A_{23} = - \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = -2$$

$$A_{31} = + \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$$

$$A_{32} = - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2$$

$$A_{33} = + \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

So $$[A_{ij}] = \begin{bmatrix} 3 & -2 & 3 \\ -2 & 2 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\operatorname{adj} A = [A_{ij}]^T$$

$$= \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$

MAGIC:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \det A \cdot I$$

!!!

THM.

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A) I$$

$$\Rightarrow \quad A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

WHY DOES THIS work?

$A \operatorname{adj} A$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{j1} & \cdots & A_{n1} \\ & & \vdots & & \\ A_{1n} & \cdots & A_{jn} & \cdots & A_{nn} \end{bmatrix}$$

ijth ENTRY IS

$$a_{i1} A_{j1} + a_{i2} A_{j2} + \cdots + a_{in} A_{jn}.$$

IF $i = j$ THIS IS $\det A$.

IF NOT,

THEN

WHAT IS

$$a_{i1} A_{j1} + \cdots + a_{in} A_{jn} ?$$

CLAIM THIS IS 0.

WHY? FORM B BY REPLACING row j with row i,

$\det B = 0$ (TWO OF ITS ROWS ARE EQUAL!)

BUT $B_{jk} = A_{jk}$ i.e. COFACTORS OF A AND B ARE THE SAME AT ROW j.

SO

$$\begin{aligned} 0 &= \det B \\ &= b_{j1} B_{j1} + \cdots + b_{jn} B_{jn} \\ &= a_{i1} B_{j1} + \cdots + a_{in} B_{jn} \\ &= a_{i1} A_{j1} + \cdots a_{in} A_{jn}. \end{aligned}$$

$\square$

3.5. CRAMER'S RULE.

$\det A \neq 0.$

USE ADJOINT TO FIND A^{-1}!

THEN ! SOLN TO $A\underline{x} = \underline{b}$

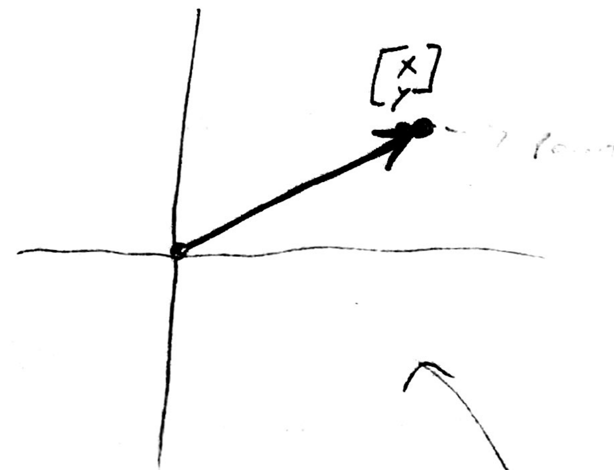
IS $A^{-1}\underline{b}$.

CHAPTER 4.

VECTOR SPACES.

CHAPTER 4.

VECTOR SPACES.



AS WE'VE BEEN DOING, WE THINK OF TWO DIMENSIONAL SPACE $\mathbb{R}^2$ AS THE SET OF ALL 2×1 COLUMN VECTORS $\begin{bmatrix} x \\ y \end{bmatrix}$.

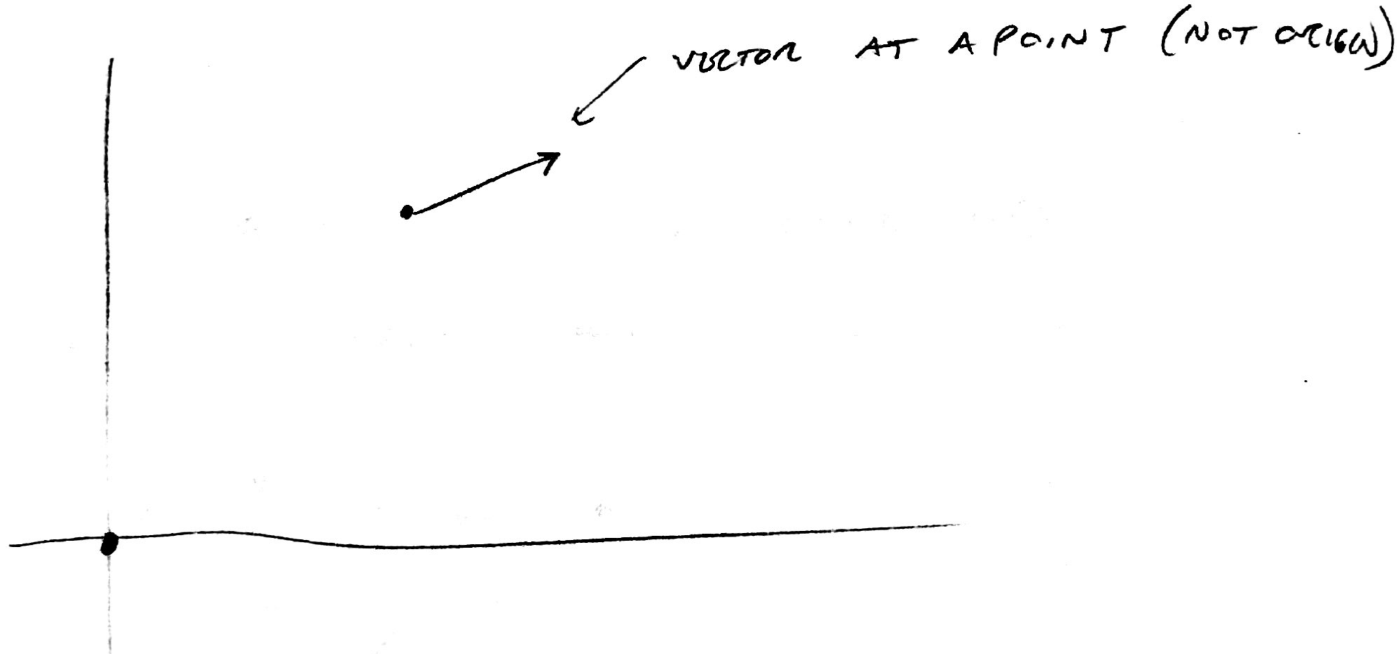
WE WILL USE <u>UNDERLINES</u> TO REPRESENT VECTORS:

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

THE BOOK USES **BOLD** TYPE, AND A LOT OF PEOPLE WRITE $\vec{x}$.

IT IS USEFUL TO DISTINGUISH BETWEEN VECTORS AND POINTS.

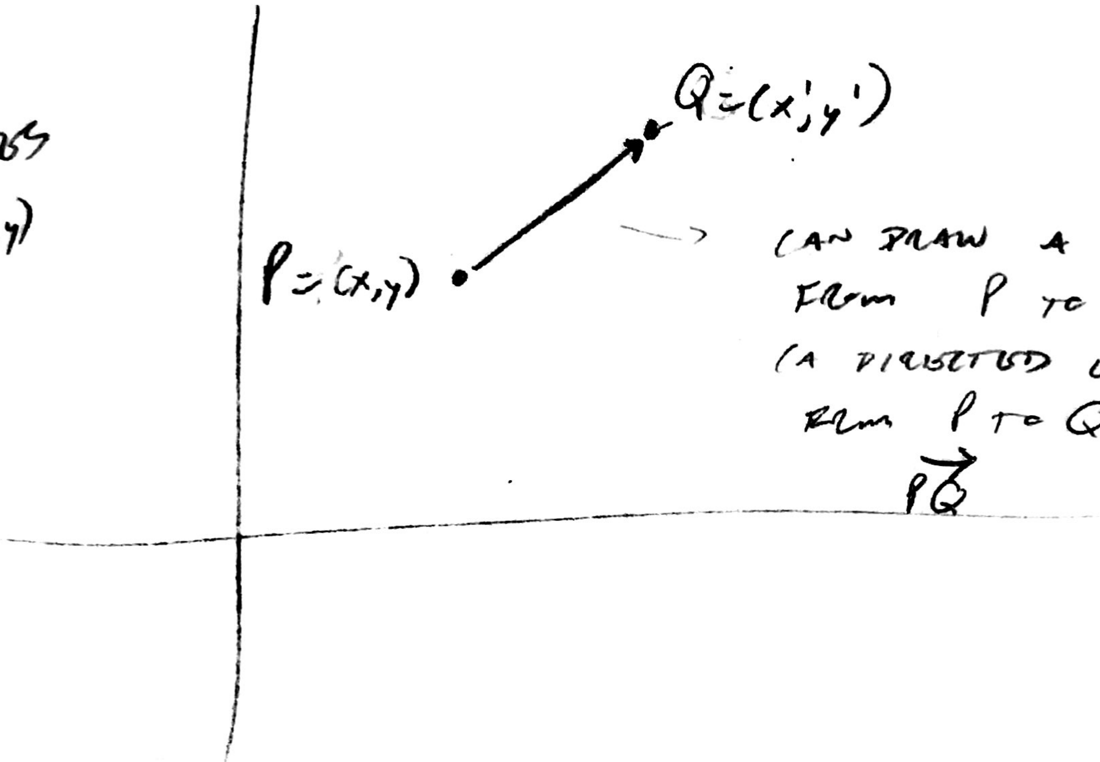
(WE OFTEN WOULD LIKE TO TALK ABOUT A VELOCITY VECTOR AT A POINT, FOR EXAMPLE.)



WE WILL WRITE (x,y) FOR POINTS IN THE PLANE

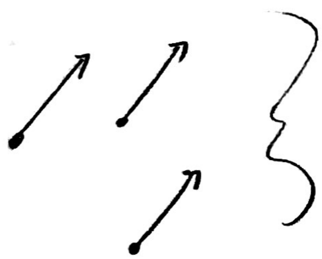
AND $\begin{bmatrix} x \\ y \end{bmatrix}$ FOR VECTORS (START AT ORIGIN).

BOOK WRITES $P(x,y)$

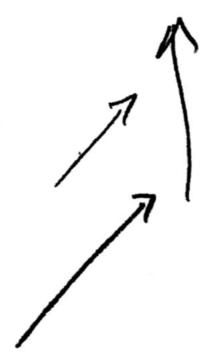


CAN DRAW A VECTOR FROM P TO Q.
(A DIRECTED LINE SEGMENT FROM P TO Q DENOTED $\overrightarrow{PQ}$

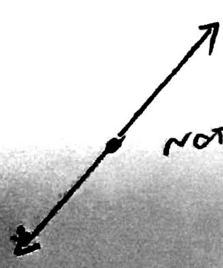
WE THINK OF TWO VECTORS BEING EQUAL IF THEY HAVE SAME LENGTH AND DIRECTION.



ALL SAME

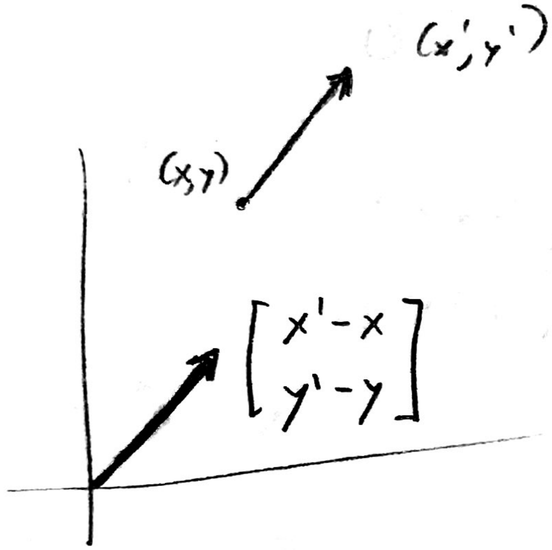


NOT SAME.

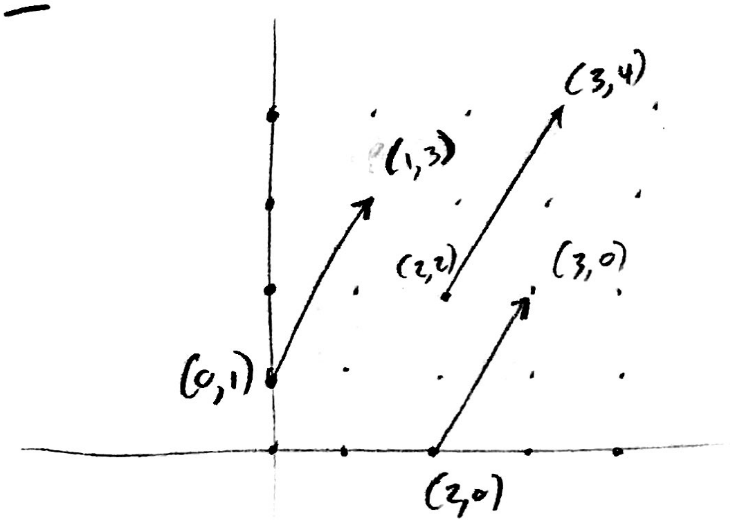


NOT SAME.

CAN TRANSLATE SUCH A VECTOR TO ORIGIN WITHOUT CHANGING ITS DIRECTION TO GET A VECTOR WITH SAME DIRECTION AND MAGNITUDE AT ORIGIN.



EX

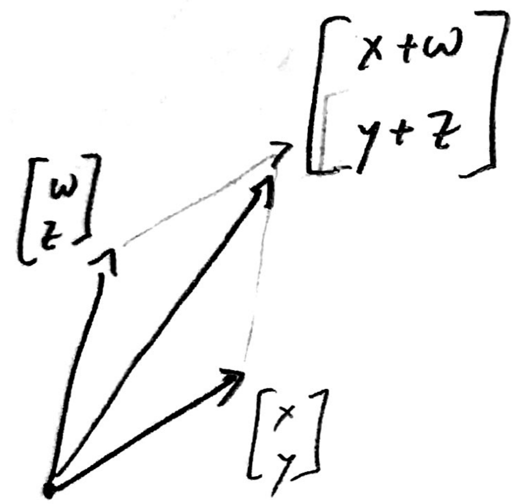


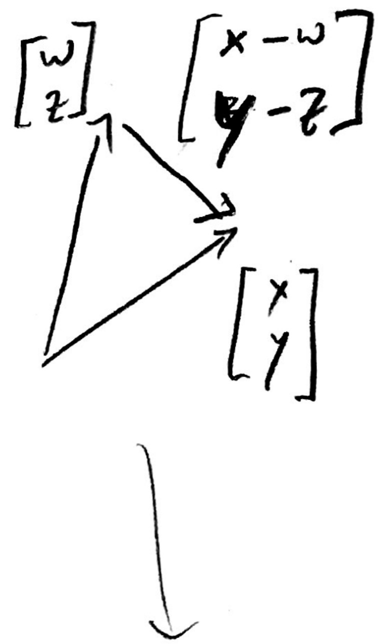
ALL EQUAL.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

RECALL VECTOR ADDITION HAS A NICE PICTURE: $\underline{u} = [$

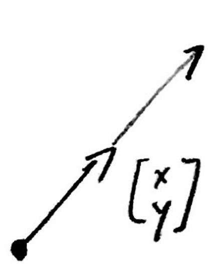
$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}$$



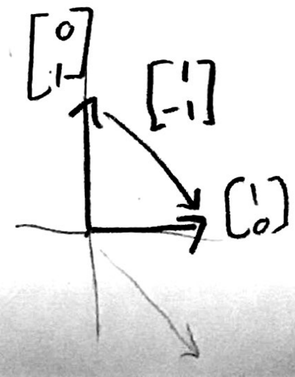


SCALAR MULTIPLICATION

$$2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

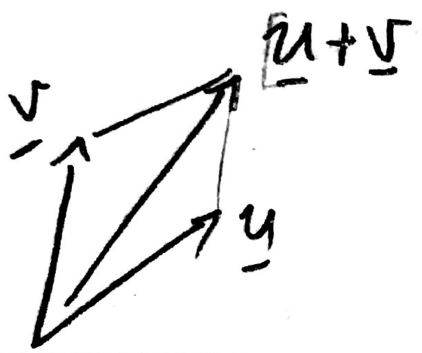


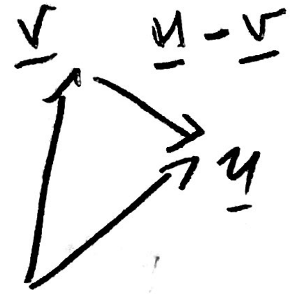
OR



WRITING

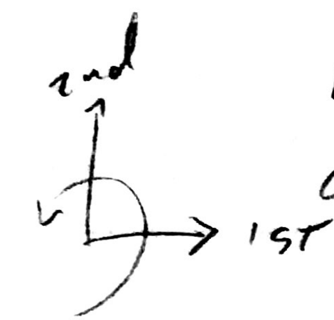
$$\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} w \\ z \end{bmatrix}$$





IN PLANE, IT'S IMPORTANT TO REMEMBER

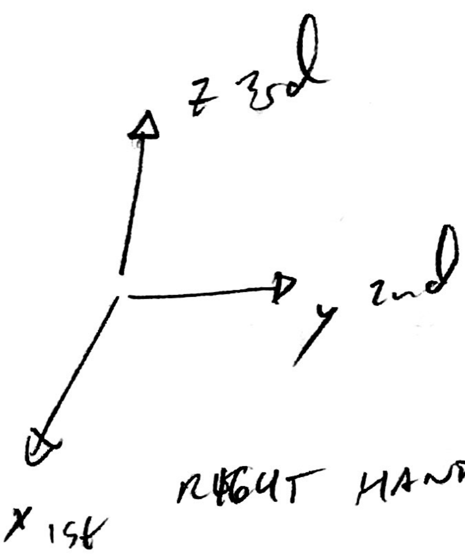
"X COMES FIRST"



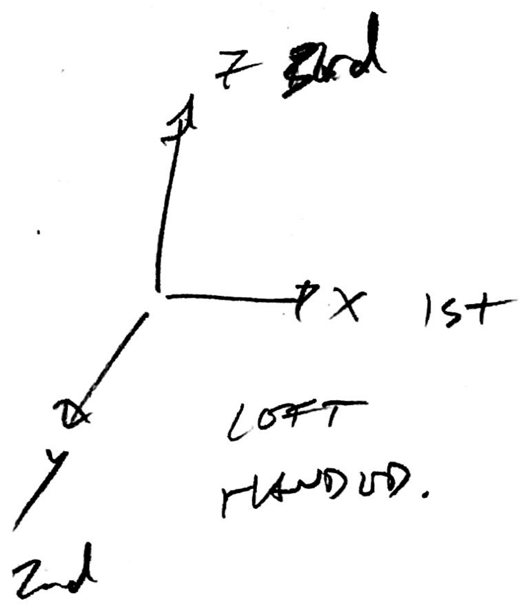
PLANE ORIENTED

COUNTER-CLOCKWISE.

IN SPACE, WE HAVE RIGHT HAND RULE:



RIGHT HANDED



LEFT HANDED.

HAVE SAME DISCUSSION OF VECTORS IN SPACE, AND FOR $\mathbb{R}^n$ = SPACE OF $n \times 1$ COLUMN VECTORS.

VECTOR SPACES.

Def A REAL VECTOR SPACE IS A SET V WITH TWO OPERATIONS $+$ AND $\cdot$, (BOOK CIRCLES THESE, BUT I DON'T CARE, UNLESS IT WOULD BE CONFUSING.) SATISFYING THE FOLLOWING RULES:

A. IF $\underline{u}, \underline{v} \in V$ THEN $\underline{u} + \underline{v} \in V$.

(V IS "CLOSED UNDER ADDITION.")

1. $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
2. $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$
3. THERE IS AN ELEMENT $\underline{0}$ OF V SUCH THAT $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$ FOR ALL $\underline{u}$ IN V
4. FOR EACH $\underline{u}$ THERE IS $-\underline{u}$ IN V SUCH THAT $\underline{u} + -\underline{u} = -\underline{u} + \underline{u} = \underline{0}$.

B. IF $\underline{u}$ IN V AND $c \in \mathbb{R}$,

$c \cdot \underline{u}$ IN V (V IS "CLOSED UNDER MULTIPLICATION.")

5. $c \cdot (\underline{u} + \underline{v}) = c \cdot \underline{u} + c \cdot \underline{v}$

6. $(c+d) \cdot \underline{u} = c\underline{u} + d \cdot \underline{u}$

7. $c \cdot (d \cdot \underline{u}) = (cd) \cdot \underline{u}$

8. $1 \cdot \underline{u} = \underline{u}$

ELEMENTS OF V ARE <u>VECTORS</u>

ELEMENTS OF $\mathbb{R}$ ARE <u>SCALARS</u>.

EXAMPLES.

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \,\middle|\, x_i \in \mathbb{R} \right\}$$

EX $\{ A \mid A$ IS AN $m \times n$ MATRIX $\}$

WITH MATRIX ADDITION.

EX. $\mathbb{R}$.

EX. $\mathbb{R}_n$ ROW VECTORS

$[x_1, \ldots, x_n]$.

~~EX~~

6X

$V = \{ A \mid A$ 2x2 MATRIX OF TRACE $0 \}$

WITH MATRIX ADDITION.

RECALL $tr(A)$ = SUM OF DIAGONAL ENTRIES,

SO

$$tr\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+d.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} r & s \\ t & p \end{bmatrix} = \begin{bmatrix} a+r & b+s \\ c+t & d+p \end{bmatrix}$$

$$\begin{aligned} tr(RHS) &= a+r+d+p \\ &= a+d+r+p \\ &= tr\begin{bmatrix} a & b \\ c & d \end{bmatrix} + tr\begin{bmatrix} r & s \\ t & p \end{bmatrix}. \end{aligned}$$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ IS ZERO VECTOR.

EX POLYNOMIALS.

IF

$$P(t) = a_n t^n + \cdots a_0$$

IF $a_n \neq 0$, DEGREE OF P IS n.

$P_n = \{ \text{POLYS} \mid \text{DEGREE OF } P \leq n \}$

WITH ADDITION OF FUNCTIONS

AND USUAL SCALAR MULT. IS VECTOR SPACE.

~~AS~~ AS A VECTOR SPACE,

THERE ISN'T MUCH DIFFERENCE

BETWEEN P_n AND $\mathbb{R}^{n+1}$.

EX V SET OF ALL CONTINUOUS REAL VALUED FUNCTIONS ON $\mathbb{R}$.

NONEX V SET OF REAL MULTIPLES OF FUNCTIONS OF THE FORM

e^{kx} WHERE $k \in \mathbb{R}$.

DEFINE

$$ce^{kx} \oplus de^{\ell x} = cd\, e^{(k+\ell)x}$$

$$r \odot ce^{kx} = (rc)e^{kx}$$

CLOSED UNDER BOTH OPERATIONS.

NOTICE THAT THE CONSTANT FUNCTION

$\mathbb{1} = e^{0x}$ IS IN V

AND $ce^{kx} \oplus \mathbb{1} = \mathbb{1} \oplus ce^{kx} = ce^{kx}$

AND SO $\mathbb{1}$ IS THE ZERO VECTOR $\underline{0}$.

ALSO NOTICE THAT THE CONSTANT FUNCTION $0 = 0e^{kx}$ IS IN V,

BUT 0 HAS NO ADDITIVE INVERSE!

$$0 \oplus ce^{kx} = 1$$

$$\Rightarrow 0e^{lx} \oplus ce^{kx} = 1$$

$$\Rightarrow (0 \cdot c)e^{(l+k)x} = 1$$

$$\Rightarrow 0 = 1. \quad \text{FALSE.}$$

SO NOT A VECTOR SPACE.

Thm V v.s.

a) $0 \cdot \underline{u} = \underline{0}$

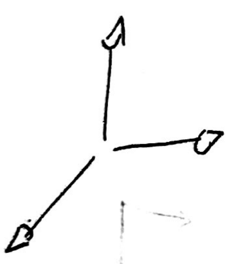
b) $c \cdot \underline{0} = \underline{0}$

c) IF $c \cdot \underline{u} = \underline{0}$ THEN

EITHER $c = 0$ OR $\underline{u} = \underline{0}$.

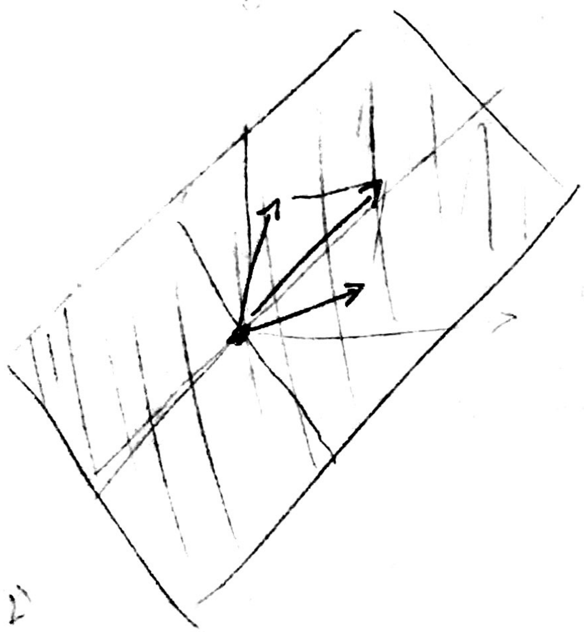
d) $(-1) \cdot \underline{u} = -\underline{u}$.

SUBSPACES.



NOTICE THAT IF WE ADD OR SCALE VECTORS IN XY-PLANE YOU STAY IN THE XY PLANE.

LET W BE A PLANE THROUGH ORIGIN.



IF WE HAVE TWO VECTORS $\underline{v}$ AND $\underline{w}$ IN W, THEIR SUM IS STILL IN W.

(IMAGINE ROTATING SPACE SO THAT $\underline{v}$ AND $\underline{w}$ ARE LYING ON THE "FLOOR.")

<u>DEFINITION</u>. LET V BE A VECTOR SPACE AND LET W BE A NONEMPTY SUBSET OF V. IF W IS A VECTOR SPACE WITH THE OPERATIONS OF V, THEN W IS A <u>SUBSPACE</u>.

<u>THM</u> $W \subset V$ WITH V A VECTOR SPACE.

THEN W IS A SUBSPACE IFF

THE FOLLOWING HOLD:

a) IF $\underline{u}, \underline{v} \in W$, THEN $\underline{u} + \underline{v} \in W$

b) IF $\underline{u} \in W$, THEN $c \cdot \underline{u} \in W$.

<u>EX.</u> V IS A SUBSPACE OF ITSELF.

$\{\underline{0}\}$ IS A SUBSPACE.

<u>EX.</u> LET V BE THE VECTOR SPACE OF ALL 2×2 MATRICES WITH MATRIX ADD. AND SCALAR MULT.

LET W BE THE SET OF 2×2 MATRICES OF TRACE 0.

W IS A SUBSPACE.

<u>EX</u> P_2 BE THE SET OF ALL POLYNOMIALS OF DEGREE ≤ 2.

P_2 IS A SUBSPACE OF THE VECTOR SPACE P OF ALL POLYNOMIALS.

NON EX

LET W BE THE SUBSET OF $\mathcal{P}$ CONSISTING OF ALL POLYS OF DEGREE EXACTLY 3.

W IS NOT A SUBSPACE, SINCE

$$(x^3 + x^2 + 2) - (x^3 + x^2 + 1)$$

HAS DEGREE 1.

HOW CAN WE BUILD SUBSPACES?

LET V BE A VECTOR SPACE.

LET $\underline{u} \in V$.

① THEN $W = \{ c\underline{u} \mid c \in \mathbb{R} \}$ IS A SUBSPACE.

② LET $\underline{u}, \underline{v} \in V$.

THEN $W = \{ a\underline{u} + b\underline{v} \mid a, b \in \mathbb{R} \}$

IS A SUBSPACE.

$$(a\underline{u} + b\underline{v}) + (c\underline{u} + d\underline{v})$$
$$= (a+c)\underline{u} + (b+d)\underline{v} \in W.$$

AND $c(a\underline{u} + b\underline{v}) = ca\underline{u} + cb\underline{v} \in W$ ✓

DEF. LET $\underline{v}_1, \dots, \underline{v}_n$ BE VECTORS IN A VECTOR SPACE V.

THEN

$\underline{v}$ IS A LINEAR COMBINATION OF $\underline{v}_1, \dots, \underline{v}_k$ IF

$$\underline{v} = a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_k\underline{v}_k$$
$$= \sum_{i=1}^{n} a_i\underline{v}_i .$$

EX. EVERY VECTOR IN $\mathbb{R}^n$ IS A LINEAR COMBINATION OF

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

LET $\underline{v} \in \mathbb{R}^n$.

THEN $\underline{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \underline{e}_i$.

EX. EVERY VECTOR IN P_2 IS A LINEAR COMBINATION OF $t^2, t, 1$.

EX.

LET A BE AN $m \times n$ MATRIX.

CONSIDER THE HOMOGENEOUS SYSTEM

$$A\underline{x} = \underline{0}. \qquad (\star)$$

LET W BE THE SET OF SOLUTIONS TO $(\star)$.

W IS A SUBSPACE:

IF $A\underline{u} = \underline{0}$ AND $A\underline{v} = \underline{0}$

THEN $A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0}$

$= \underline{0}$

AND $A c\underline{u} = cA\underline{u}$

SO IF $A\underline{u} = \underline{0}$ SO DOES $Ac\underline{u}$.

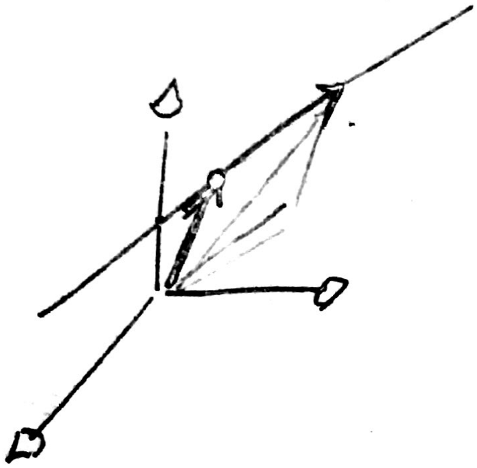
W IS CALLED THE <u>NULL SPACE</u> OF A.

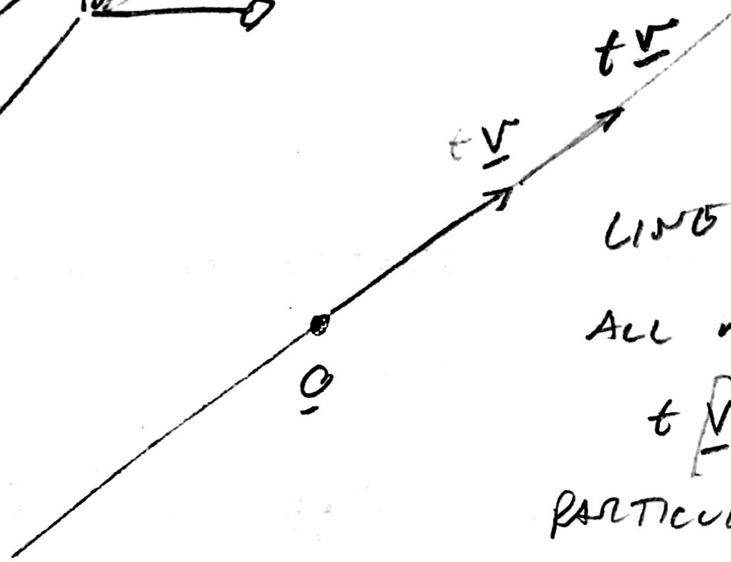
(OR THE "KERNEL" OF A.)

[IF $\underline{b} \neq \underline{0}$, THE SET OF SOL'NS TO $A\underline{x} = \underline{b}$ IS <u>NOT</u> A SUBSPACE.)

PARAMETRIC EQUATIONS FOR LINES.

WHAT IS A LINE?

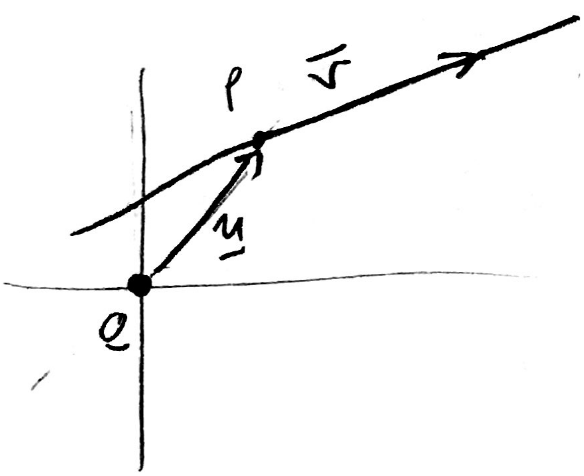


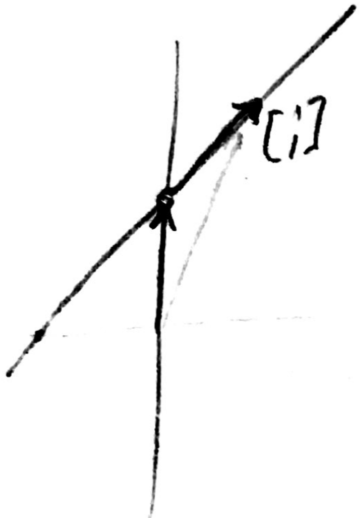


LINE AT $\underline{0}$ IS ALL MULTIPLES $t\underline{v}$ OF A PARTICULAR VECTOR.

LINE THROUGH ANOTHER POINT LOOKS LIKE ALL VECTORS

$\underline{u} + t\underline{v}$ FOR FIXED $\underline{u}$ AND $\underline{v}$.





$$\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix}$$

SPAN

A NONZERO VECTOR $\underline{v}$ DETERMINES A LINE $\{t\underline{v} \mid t \in \mathbb{R}\}$.

TWO NONZERO VECTORS $\underline{0} \neq \underline{u} \neq \underline{v} \neq \underline{0}$ GIVE A PLANE

$$\{t\underline{u} + s\underline{v} \mid t, s \in \mathbb{R}\}.$$

Def. LET $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\} \subset V$.

THEN $\text{SPAN } S = \text{span}\{\underline{v}_1, \ldots, \underline{v}_k\}$

$$= \left\{ \sum_{i=1}^{k} a_i \underline{v}_i \;\middle|\; a_i \in \mathbb{R} \right\}.$$

(ALSO WORKS FOR INFINITE S, WHERE WE TAKE ALL FINITE LINEAR COMBOS.)

EX. $S = \{t^2, t, 1\} \subset P_2$

SPAN $S = P_2$.

EX $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

SPAN $S = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \right\}$.

THM SPAN S IS A SUBSPACE.

PROOF. COMBINE LIKE TERMS. ✓

EX. $S = \{t^2, t\}$

SPAN $= \{at^2 + bt \mid a, b \in \mathbb{R}\}$.

Def IF SPAN $S = V$ WE SAY

S SPANS V OR V IS SPANNED BY S,

AND CALL S A SPANNING SET.

EX. LET $\mathcal{P}$ BE V.S. OF ALL POLYNOMIALS.

$$S = \{1, t, t^2, t^3, \dots\}$$

IS A SPANNING SET.

EVERY SPANNING SET OF $\mathcal{P}$ IS INFINITE!

(BECAUSE $\mathcal{P}$ CONTAINS POLYS OF ARBITRARILY LARGE DEGREES.)

HOW DO WE TELL IF A GIVEN $\underline{v}$ IS IN SPAN S?

EX: LET $S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$

IS $\underline{v} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix} \in$ SPAN S?

i.e. ARE THERE a_1, a_2 S.t.

$$a_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix} ?$$

THIS IS A LINEAR SYSTEM

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 3 & -7 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

RED. ECHELON FORM

$a_1 = 2 \qquad a_2 = -3.$

SO $\underline{v} \in \operatorname{SPAN} S.$

EX. IN $\mathcal{P}$.

$$S = \{ \underline{v}_1 = 2t^2 + t + 2$$
$$\underline{v}_2 = t^2 - 2t$$
$$\underline{v}_3 = 5t^2 - 5t + 2$$
$$\underline{v}_4 = -t^2 - 3t - 2 \}$$

IS $\underline{v} = t^2 + t + 2 \in$ SPAN S ?

$$a_1(2t^2 + t + 2) + a_2(t^2 - 2t) + a_3(5t^2 - 5t + 2) + a_4(-t^2 - 3t - 2) = t^2 + t + 2.$$

$$2a_1 + a_2 + 5a_3 - a_4 = 1$$
$$a_1 - 2a_2 - 5a_3 - 3a_4 = 1$$
$$2a_1 + 0a_2 + 2a_3 - 2a_4 = 2.$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

RED. ROW ECH. FORM

NO SOLUTION

$\underline{v}$ NOT IN SPAN.

EX $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

DO THESE SPAN $\mathbb{R}^3$?

ARBITRARY VECTOR.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 2 & 0 & 1 & b \\ 1 & 2 & 0 & c \end{array}\right]$$

REDUCE MATRIX AND SOLUTION OF

$$a_1 = \frac{-2a + 2b + c}{3}$$

$$a_2 = \frac{a - b + c}{3}$$

$$a_3 = \frac{4a - b - 2c}{3}.$$

SO, YES, $\mathbb{R}^3 = \text{SPAN}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$

LINEAR INDEPENDENCE.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\text{SPAN } S = \mathbb{R}^2 = \text{SPAN} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

SO WE DON'T NEED $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ TO GET $\mathbb{R}^2$.

ALSO NOTICE

$$\text{SPAN} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$$

SO WE DON'T NEED $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

AND $\text{SPAN} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$.

SINCE $\begin{bmatrix}1\\1\end{bmatrix}$ IS IN $\text{SPAN}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$,

WE CAN WRITE $\begin{bmatrix}1\\1\end{bmatrix}$ AS A LINEAR COMBINATION OF $\begin{bmatrix}1\\0\end{bmatrix}$ AND $\begin{bmatrix}0\\1\end{bmatrix}$.

NAMELY

$$\begin{bmatrix}1\\1\end{bmatrix} = 1 \cdot \begin{bmatrix}1\\0\end{bmatrix} + 1 \cdot \begin{bmatrix}0\\1\end{bmatrix}.$$

__DEFN__ A COLLECTION $\{\underline{v}_1, \dots, \underline{v}_k\}$ OF VECTORS IS __LINEARLY DEPENDENT__ IF WE CAN WRITE ONE OF THEM AS LINEAR COMBINATION OF THE OTHERS:

$$\underline{v}_i = a_1 \underline{v}_1 + \dots + a_{i-1} \underline{v}_{i-1} + a_{i+1} \underline{v}_{i+1} + \dots + a_k \underline{v}_k$$

WE SAY $\{\underline{v}_1, \ldots, \underline{v}_k\}$ IS <u>LINEARLY</u> <u>INDEPENDENT</u> IF IT IS <u>NOT</u> LINEARLY DEPENDENT.

HOW TO THINK

SO, $S = \{\underline{v}_1, \ldots \underline{v}_k\}$ IS <u>LINEARLY DEPENDENT</u> IFF IT IS RE<u>DUNDANT</u>, i.e. WE CAN REMOVE ONE VECTOR AND NOT CHANGE THE SPAN.

<u>ALT DEF.:</u>

$S = \{\underline{v}_1, \ldots, \underline{v}_k\}$ <u>LINEARLY DEPENDENT</u> IF THERE IS A <u>NONTRIVIAL</u> LINEAR COMBINATION OF THE $\underline{v}_i$ THAT EQUALS $\underline{0}$, i.e.

$$\underline{0} = \sum_{i=1}^{k} a_i \underline{v}_i$$

FOR SOME a_i NOT ALL ZERO.

USING THIS DEFINITION,

S IS <u>LINEARLY INDEPENDENT</u>

IF $\underline{0} = \sum a_i v_i$ IMPLIES THAT

$a_1 = a_2 = \dots = a_k = 0.$

<u>EX</u>. ARE $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 17 \\ 6 \\ 11 \end{bmatrix}$ LINEARLY DEPENDENT?

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 17 \\ 6 \\ 11 \end{bmatrix} = \underline{0}$$

$$\left[\begin{array}{ccc|c} 1 & 7 & 17 & 0 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 11 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 7 & 17 & 0 \\ 0 & -14 & -28 & 0 \\ 0 & -20 & -40 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & 17 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 7 & 17 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$a_1 = -7a_2 - 17a_3 = 14k - 17k = -3k$

$a_2 = -2k$

$a_3 = k$

So $k = 1$

$$-3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 17 \\ 6 \\ 11 \end{bmatrix} = 0$$

Ex

$\underline{v}_1 = t^2 + t + 2$

$\underline{v}_2 = 2t^2 + t$

$\underline{v}_3 = 3t^2 + 2t + 2$

LINEARLY DEPENDENT.

$$a_1 \underline{v}_1 + a_2 \underline{v}_2 + a_3 \underline{v}_3 = \underline{0} \ ?$$

$$a_1(t^2 + t + 2) + a_2(2t^2 + t) + a_3(3t^2 + 2t + 2) = \underline{0} \ ?$$

$$a_1 + 2a_2 + 3a_3 = 0$$
$$a_1 + a_2 + 2a_3 = 0$$
$$2a_1 + 0a_2 + 2a_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right]$$

∞ly many.

THM Let $\underline{v}_1, \ldots, \underline{v}_n \in \mathbb{R}^n$.

$\underline{v}_1, \ldots, \underline{v}_n$ ARE LINEARLY INDEPENDENT

IFF $\det \underbrace{\begin{bmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{bmatrix}}_{\text{MATRIX}} \neq 0.$

WHOSE COLS ARE THE $\underline{v}_i$'s.

Proof: IF THEY ARE LINEARLY INDEPENDENT, THEN $\begin{bmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{bmatrix}$ IS ROW EQUIVALENT TO I_n. (BECAUSE THE HOMOGENEOUS SYSTEM HAS ONLY THE TRIVIAL SOLUTION)

So det IS $\neq 0$.

OTOH: IF $\det \neq 0$ THEN

$\begin{bmatrix} \underline{v}_1 & \cdots & \underline{v}_n \end{bmatrix}$ IS ROW EQ. TO I,

AND SO THE COLLECTION IS LINEARLY INDEPENDENT.

<u>THM</u> LET $R \subseteq S \subseteq V$.

IF S IS LINEARLY INDEPENDENT THEN SO IS R.

IF R IS LINEARLY DEPENDENT SO IS S.

<u>PROOF:</u> THINK ABOUT IT IN TERMS OF REDUNDANCY:

IF S IS <u>NOT</u> <u>REDUNDANT</u>, HOW COULD R BE?

IF R <u>IS</u> REDUNDANT THEN SO IS S!

$\square$

NOTE:

IF $\underline{0} \in S$, THEN S IS LINEARLY DEPENDENT.

$$7\underline{0} = \sum_{\substack{\underline{v} \in S \\ \underline{v} \neq \underline{0}}} 0\underline{v} \quad 1\underline{0} + 0v_2 + \cdots + 0v_k \, .$$

THM IF $v_1, \ldots, v_k$ ARE LINEARLY DEPENDENT, THEN THERE IS AN i s.t. $v_i \in \operatorname{SPAN}\{v_1, \ldots, v_{i-1}\}$, AND $v_1, \ldots, v_{i-1}$ ARE LINEARLY INDEPENDENT.

PROOF: LET i BE THE FIRST i SUCH THAT $\{v_1, \ldots, v_i\}$ IS L.D.

$\triangledown$

BASES AND DIMENSION.

DEF.

$S \subset V$ is a **basis** of V

IF 1) $\operatorname{span} S = V$

2) every finite subset of S is linearly independent.

EX

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

is a basis for $\mathbb{R}^n$.

EX $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

IS ALSO A BASIS.

STD BASIS.

EX. $S = \{1, t, t^2, t^3\}$ IS A BASIS FOR P_3.

SO IS $\{1, 2+t, t^2, 1+t^2+t^3\}$

EX LET $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a+d=0 \right\}$

CLAIM: $S = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

IS A BASIS.

PROOF:

A MATRIX IN W LOOKS LIKE

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

SO SPAN $S = W$.

ALSO: IF

$$\underline{0} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

THEN

$$\underline{0} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \Rightarrow a = 0,\ b = 0,\ c = 0.$$

✓

DEF:

V IS FINITE DIMENSIONAL

IF IT HAS A FINITE BASIS, AND THE DIMENSION OF V IS # OF VECTORS IN A BASIS. INFINITE DIMENSIONAL OTHERWISE.

WAIT. DO TWO DIFFERENT BASES HAVE SAME SIZE?

WE'LL COME BACK TO THIS SOON.

THM IF $\{v_1, \dots, v_k\}$ IS A BASIS OF V, THEN EVERY v MAY BE WRITTEN AS A LINEAR COMBO OF THE v_i IN A UNIQUE WAY.

<u>Proof</u> The v_i span so

$$v = \sum a_i v_i$$

IF $v = \sum b_i v_i$

THEN $\underline{0} = v - v = \sum a_i v_i - \sum b_i v_i$

$$= \sum (a_i - b_i) v_i$$

$\Rightarrow$ $\underline{0}$ IS A LINEAR COMBO OF THE v_i $\Rightarrow$ COEFFICIENTS $(a_i - b_i)$ ARE ALL <u>Zero</u>, SINCE THE v_i ARE LINEARLY INDEPENDENT.

SO $a_i = b_i$ FOR ALL i.

$\square$

THM V V.S.

$S = \{v_1, \ldots, v_m\} \subset V$

WITH. SPAN $S = V$.

THEN SOME SUBSET OF S IS A BASIS.

PROOF:

BY EARLIER THM,

THERE IS A FIRST i SUCH THAT

$v_1, \ldots, v_{i-1}$ IS LINEARLY INDEPENDENT

AND $v_1, \ldots, v_i$ ISN'T.

SO SPAN $\{v_1, \ldots, v_{i-1}\}$ = SPAN $\{v_1, \ldots, v_i\}$.

DELETE v_i FROM S TO GET

A SUBSET $S_1 \subset S$

$$S_1 = \{u_1, \ldots, u_{m-1}\}.$$

CONTINUE THIS WITH S_1 TO GET

$S_2 = \{w_1, \ldots, w_{m-2}\} \subset S$ WITH SAME

SPAN.

THIS EVENTUALLY STOPS AT A BASIS.

$\square$

THM LET $S = \{v_1, \ldots v_n\}$ BE A BASIS.

IF $T = \{w_1, \ldots, w_m\}$ IS LINEARLY

INDEPENDENT, THEN $m \leq n$.

PROOF:

$T_1 = \{w_1, v_1, \ldots, v_n\}$ LINEARLY DEPENDENT

$S_1 = \{w_1, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$

WHERE v_i IS A LINEAR COMBO

OF PREVIOUS VECTORS IN T_1.

S_1 SPANS.

LET

$T_2 = \{w_2, w_1, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$

SOME VECTOR IN T_2 IS A LIN. COMBO OF PREVIOUS ONES.

NOT w_1. (CUZ T IS LIN. INDEP.)

SO WE CAN DELETE ANOTHER v_j AND CONTINUE.

EACH TIME WE ADD A w, WE CAN DELETE A v, UNTIL THE vs ARE GONE.

SO THERE MUST BE AS MANY vs AS ws. SO $n \geq m$.

$\square$

COR

IF $S = \{v_1, \ldots, v_m\}$

AND $T = \{w_1, \ldots, w_n\}$ ARE BASES OF V.

THEN $m = n$.

SO DIMENSION MAKES SENSE!

PROOF: BY PREVIOUS THM,

$m \le n$ AND $n \le m$.

$\square$

DIMENSION.

DEF. LET $S \subset V$.

$T \subset S$ IS A MAXIMAL INDEPENDENT SUBSET OF S IF IT IS LINEARLY INDEPENDENT AND IS NOT PROPERLY CONTAINED IN ANOTHER LINEARLY IND. SUBSET OF S.

(i.e. IF $T \subset T' \subset S$ WITH T' LINEARLY IND. THEN $T' = T$.)

OFTEN WE TAKE $S = V$. WE WILL SEE THAT IF $S = V$, THEN A MAX. LIN. IND. SET IS A BASIS.

THM V HAS DIMENSION n,

THEN A MAXIMALLY IND. SUBSET OF V

CONTAINS n VECTORS.

PROOF: LET $S = \{v_1, \dots, v_k\}$ BE M.I. SUBSET

OF V.

IF SPAN $S = V$ THEN S IS A BASIS AND $k = n$.

IF SPAN $S \neq V$ THEN THERE

IS A v THAT ISN'T IN SPAN S,

BUT THEN $\{v, v_1, \dots, v_k\}$

IS LINEARLY INDEPENDENT.

THIS CONTRADICTS MAXIMALITY. $\square$.

THM IF S IS A MINIMAL SPANNING SET,

THEN S HAS n ELEMENTS.

PROOF. EX. $\square$

THM. IF V HAS DIMENSION n AND $S \subset V$ HAS ~~m~~ $> n$ ELEMENTS, THEN S IS LINEARLY DEPENDENT.

THM IF S HAS $< n$ ELEMENTS AND $\dim V = n$, THEN $\operatorname{SPAN} S \neq V$.

THM IF S IS LINEARLY INDEPENDENT SUBSET OF A FINITE DIM^{l} VECTOR SPACE, THEN S IS CONTAINED IN A BASIS. (MAY BE "FILLED OUT" TO A BASIS.)

PROOF: IF S SPANS, IT IS A BASIS.

IF IT $\{v_1, \ldots, v_k\}$ DOESN'T, THEN ADD A VECTOR w_1 THAT ISN'T IN SPAN S.

IF $\{v_1, \ldots, v_k, w_1\}$ IS A BASIS, DONE.

CONTINUE UNTIL $\{v_1, \ldots v_k, w_1, \ldots, w_{n-k}\}$ IS A ~~BASIS~~ SPANS. ✓

$\square$

THM ① IF V HAS DIMENSION n

AND $\{v_1, \dots, v_n\}$ IS LINEARLY IND,

THEN $\{v_1, \dots, v_n\}$ IS A BASIS.

② IF $\operatorname{SPAN}\{v_1, \dots, v_n\} = V$,

THEN $\{v_1, \dots, v_n\}$ IS A BASIS.

THM MAXIMALLY INDEP. SUBSET OF V

IS A BASIS. $\square$

SKIP 4.7. FOR NOW.

4.8. COORDINATES.

AN ORDERED BASIS IS A BASIS

$\{v_1, v_2, \dots, v_n\}$ WHERE THE ORDER MATTERS.

SO $\{v_2, v_1, \dots v_n\}$ IS A DIFFERENT

ORDERED BASIS.

IF $S = \{v_1, \dots, v_n\}$ IS AN ORDERED BASIS

WE CAN WRITE EACH v AS

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

UNIQUELY AND THE ORDER ALLOWS

US TO KEEP JUST THE COEFFICIENTS.

i.e. WE WRITE

$$[\underline{v}]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

COORD. VECTOR OF $\underline{v}$ WITH RESPECT TO

THE ORDERED BASIS S.

SO WE CAN PRETEND LIKE V IS

$\mathbb{R}^n$.

EX. $S = \{1, t, t^2\}$ ORDERED BASIS

OF $\mathcal{P}_2$.

COORDINATES

$$[1 + 2t + 3t^2]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

EX $T = \{t, 1, t^2\}$

$$[1 + 2t + 3t^2]_T = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

EX $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

WHAT IS $[v]_S$?

WANT $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

SOLVE $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\boxed{c = 3}$

$b + c = 2$

$\Rightarrow b = -1$

$a + b + c = 1$

$a - 1 + 3 = 1$

$a = -1$

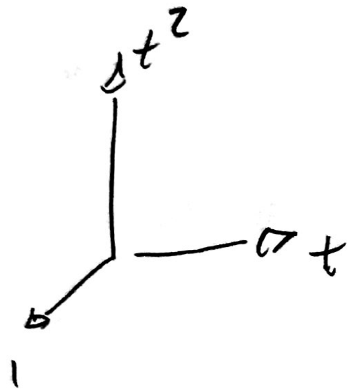
SO

$[v]_S = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$

THIS ALLOWS US TO

PICTURE THE VECTOR SPACE:

$\mathcal{P}_2$ IS JUST LIKE $\mathbb{R}^3$:



ISOMORPHISMS.

LET $S = \{v_1, \ldots, v_n\}$ BE AN ORDERED BASIS

FOR V.

$v = a_1 v_1 + \ldots + a_n v_n \qquad w = b_1 v_1 + \ldots + b_n v_n$

IN $\mathbb{R}^n$, WE HAVE

$$[v]_S \text{ AND } [w]_S.$$

NOTICE THAT $[v+w]_S = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} = [v]_S + [w]_S$

AND

$$[cv]_S = c[v]_S$$

DEFINE A FUNCTION $L : V \to \mathbb{R}^n$

BY $L(v) = [v]_S$.

L IS <u>SURJECTIVE</u> ("ONTO")

i.e. EVERY VECTOR IN $\mathbb{R}^n$

IS THE IMAGE OF SOME v IN V,

L IS <u>INJECTIVE</u> ("1-1")

i.e. IF $L(v) = L(w)$, THEN $v = w$.

L IS <u>BIJECTIVE</u> IF BOTH.

<u>DEFINITION</u> LET V AND W BE VECTOR SPACES.

A BIJECTION $L : V \to W$

s.t. a) $L(v+w) = L(v) + L(w)$

AND b) $L(cv) = cL(v)$

IS AN <u>ISOMORPHISM</u>.

AND SAY V AND W ARE ISOMORPHIC.

THM. IF V HAS DIMENSION n,

THEN V IS ISOMORPHIC TO $\mathbb{R}^n$.

PF. $L: V \to \mathbb{R}^n$ GIVEN BY

$L(v) = [v]_S$ IS AN ISO. $\square$

THM a) V ISOMORPHIC TO V

b) IF V ISO. TO W, THEN W ISO TO V.

c) IF U ISO TO V AND V ISO TO W

THEN U ISO TO W.

THM. TWO FTE DIM^l V.S.s ARE

ISOMORPHIC IFF THEY HAVE SAME

DIMENSION.

PROOF: BOTH ISOMORPHIC TO SAME $\mathbb{R}^n$. $\square$

IF WE HAVE A BASIS S, HOW DO WE SEE WHAT $[v]_S$ IS IF WE ARE GIVEN v IN THE USUAL BASIS?

GO BACK TO OUR EXAMPLE:

$$S = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

NOTICE:

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}[v]_S$$

$$= \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1\\-1\\3\end{bmatrix}$$

$$= \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{VECTOR IN STD BASIS}$$

$$= [v]_{\{e_1, e_2, e_3\}}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = ?$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

NOTICE:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = [v]_S .$$

:)

WHAT IS

$$P_{T \leftarrow S} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

THE COLUMNS ARE THE ELEMENTS OF S, IN ORDER, WRITTEN IN TERMS OF $T = \{e_1, e_2, e_3\}$.

IF WE WRITE T IN TERMS OF S, WE GET

$$[e_1]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[e_2]_S = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$[e_3]_S = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$P_{S \leftarrow T} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{T \leftarrow S} \cdot [v]_S = [v]_T$$

$$P_{S \leftarrow T} \cdot [v]_T = [v]_S.$$

TRANSITION MATRICES

LAST TIME WE SAW A TRANSITION MATRIX.

RECALL: LET $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$v_1 \quad v_2 \quad v_3 \qquad\qquad w_1 \quad w_2 \quad w_3$

NOTE THAT

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \end{bmatrix}_T$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_2 \end{bmatrix}_T$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_3 \end{bmatrix}_T$$

WE SAW AN EXAMPLE OF

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \end{bmatrix}_S = \begin{bmatrix} v \end{bmatrix}_T$$

NAMELY

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ [v_1]_T & [v_2]_T & [v_3]_T \\ | & | & | \end{bmatrix} \begin{bmatrix} v \end{bmatrix}_S = \begin{bmatrix} v \end{bmatrix}_T$$

IN GENERAL: $\dim V = n$.

LET $S = \{v_1, \dots, v_n\}$ AND $T = \{w_1, \dots, w_n\}$

ORDERED BASES.

LET $P_{T \leftarrow S}$ BE THE MATRIX

WHOSE jth COLUMN IS $[v_j]_T$.

i.e.

BOOK SOMETIMES CALLS THIS $Q_{T \leftarrow P}$

$$P_{T \leftarrow S} = \begin{bmatrix} | & & | \\ [v_1]_T & \cdots & [v_n]_T \\ | & & | \end{bmatrix}$$

NOW, IF

$$w = b_1 v_1 + b_2 v_2 + \dots + b_n v_n$$

$$[w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

NOW NOTE THAT

$$[w]_T = [b_1 v_1 + \dots + b_n v_n]_T$$

$$= [b_1 v_1]_T + \dots + [b_n v_n]_T$$

$$= b_1 [v_1]_T + \dots + b_n [v_n]_T = P_{T \leftarrow S} \cdot [w]_S.$$

INTERCHANGING T AND AND S WE HAVE

$$P_{S\leftarrow T} = \begin{bmatrix} | & & | \\ [w_1]_S & \cdots & [w_n]_S \\ | & & | \end{bmatrix} [v]_T = [v]_S .$$

$$P_{S\leftarrow T} = P_{T\leftarrow S}^{-1} .$$

EX. $S = \left\{ \overset{v_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \overset{v_2}{\begin{pmatrix}1\\1\\0\end{pmatrix}}, \overset{v_3}{\begin{bmatrix}1\\1\\1\end{bmatrix}} \right\}$, $T = \left\{ \overset{w_1}{\begin{bmatrix}1\\0\\1\end{bmatrix}}, \overset{w_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \overset{w_3}{\begin{bmatrix}-1\\0\\0\end{bmatrix}} \right\}$

LET'S FIND $P_{T\leftarrow S} = \begin{bmatrix} | & | & | \\ [v_1]_T & [v_2]_T & [v_3]_T \\ | & | & | \end{bmatrix}$

WHAT ARE

$[v_1]_T$, $[v_2]_T$, $[v_3]_T$?

SOLVE THE LINEAR SYSTEMS

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

ALL COEFF. MATRICES ARE SAME,

SO WE CAN SOLVE ALL AT SAME TIME

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 - R_1 \to R_3}{\sim} \left[\begin{array}{ccc|c|c|c} 1 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right]$$

$$\overset{R_1 + R_3 \to R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{array}{c|c|c} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{array}}_{P_{T \leftarrow S}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$P_{T \leftarrow S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

CONSIDER $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ← VECTOR IN USUAL BASIS e_1, e_2, e_3

$[v]_S = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$ FROM LAST TIME.

SO $[v]_T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

Let's check!

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So $[v]_T = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ ✓

RANK.

LAST TIME WE DEFINED A LINEAR MAP $L: V \to W$ TO BE A FUNCTION SUCH THAT

OR "LINEAR TRANSFORMATION"

$$L(u+v) = L(u) + L(v).$$

AND $L(cu) = cL(u)$.

IF V AND W ARE BOTH $\mathbb{R}^n$,

SO $L: \mathbb{R}^n \to \mathbb{R}^n$,

THEN L IS GIVEN BY A MATRIX:

THERE IS AN $n \times n$ MATRIX A SUCH THAT

$$L(v) = A \cdot v.$$

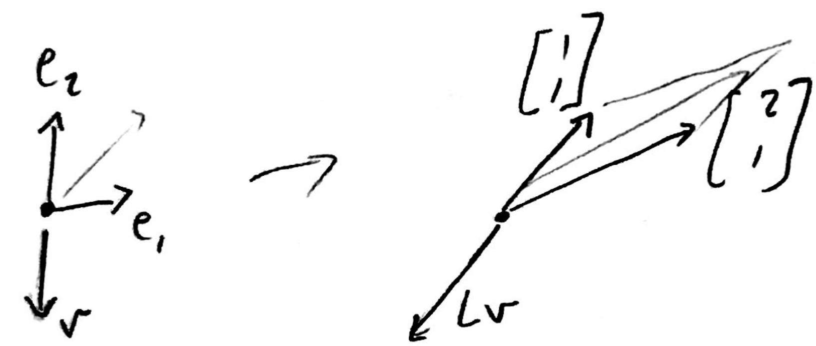
WHAT IS A?

$$A = \begin{bmatrix} | & & | \\ L(e_1) & \cdots & L(e_n) \\ | & & | \end{bmatrix}$$ = MATRIX WHOSE COLUMNS ARE $L(e_i)$.

THIS IS HOW WE'VE THOUGHT ALL ALONG.

EX.

$$L(v) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot v$$



NOTICE THAT THIS L IS AN ISOMORPHISM.

DEF: THE IMAGE OF $L: V \to W$

$$L(V) = \text{IMAGE}(L) = \{\underbrace{L(v)}_{\in W} \mid v \in V\}$$

$$L(V) = \text{SPAN}\{L(v_1), \dots, L(v_n)\}$$

WHEN $v_1, \dots, v_n$ IS A BASIS.

DEF THE RANK OF L

IS THE DIMENSION OF $L(V)$.

DEF. RANK OF AN $n \times m$ MATRIX

A IS RANK OF $L: \mathbb{R}^m \to \mathbb{R}^n$

GIVEN BY $L(v) = A \cdot v$.

IN THIS CASE

$L(V)$ IS THE SPAN OF THE COLUMNS

OF A.

AND IS ALSO CALLED THE COLUMN SPACE.

EX.

$$L : \mathbb{R}^3 \to \mathbb{R}^3$$

$$L(v) = A \cdot v$$

WHERE $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$L(\mathbb{R}^3) = \{\underline{0}\}$$

RANK OF L IS 0.

" " A IS 0

EX $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

RANK 1.

THIS IS ALSO CALLED THE "COLUMN RANK"

DIMENSION OF "ROW SPACE" IS

THE "ROW RANK" ([illegible])

(THE ROW RANK OF A IS THE RANK OF THE LINEAR TRANSFORMATION $M : \mathbb{R}_m \to \mathbb{R}_m$

$$M(v) = v \cdot A$$)

ALL OF THIS MAKES SENSE FOR MATRICES THAT AREN'T SQUARE.

$m \times n$ MATRIX A GIVES

A MAP $L : \mathbb{R}^n \to \mathbb{R}^m$

BY $L(v) = A \cdot v$

AND ANOTHER ONE

$M : \mathbb{R}^m \to \mathbb{R}^n$

$M(v) = v \cdot A$.

IF B IS SQUARE AND $\det B \neq 0$,

THEN COLUMN RANK OF BA = COL RANK OF A

AND " " OF AB = COL RANK OF A.

SAME FOR ROW RANK.

PROOF:

$w_1, \ldots, w_r$ BASIS OF COLUMN SPACE

SO $w_1, \ldots, w_r$ ARE LINEAR INDEPENDENT.

BUT THEN $Bw_1, \ldots, Bw_r$

ARE STILL LINEARLY INDEPENDENT:

IF $\underline{0} = a_1 B w_1 + \ldots\ a_r B w_r$

THEN $\underline{0} = B^{-1}\underline{0} = B^{-1}(a_1 Bw_1 + \ldots + a_r Bw_r)$

$$= B^{-1}(a_1 Bw_1) + \ldots + B^{-1}(a_r Bw_r)$$

$$= B^{-1}Ba_1 w_1 + \ldots + B^{-1}Ba_r w_r$$

$$= a_1 w_1 + \ldots + a_r w_r.$$

$\Rightarrow a_i = 0$ FOR ALL i.

SO $Bw_1, \ldots, Bw_r$ IS A BASIS FOR SPAN $L(BA)$. ✓

SO

ROW OPERATIONS DON'T CHANGE ROW RANK OR COLUMN RANK

COL OPERATIONS DON'T CHANGE COLUMN RANK. OR ROW RANK.

ROW RANK IS # OF NONZERO ROWS IN REDUCED ROW ECHELON FORM

THM ROW RANK = COLUMN RANK.

PROOF: A MATRIX.

ROW REDUCE TO REDUCED ROW ECHELON FORM. (DOESN'T CHANGE ROW RANK)

CAN PERFORM COL. OPERATIONS KEEPING IN RED. ROW ECHELON FORM (DOESN'T CHANGE COL. RANK)

TO GET

$$\begin{bmatrix} I_R & \boxed{\text{JUNK}} \\ 0 & \end{bmatrix}$$

COL RANK IS R

FIRST R COLUMNS BASIS FOR COL. SPACE

NONZERO ROWS IS ROW RANK = R. □

THE SPACE OF ALL VECTORS X

SUCH THAT

$$A\underline{x} = \underline{0}$$

IS CALLED THE

KERNEL OF A

IS SOLUTION SPACE OF THE HOMOG. SYSTEM.

OR KERNEL OF $L: \mathbb{R}^n \to \mathbb{R}^m$

$$L(v) = A \cdot v.$$

DEF THE DIMENSION OF THE KERNEL IS THE NULLITY.

A IS EQUIVALENT TO

$$m\Bigg\{\underbrace{\begin{bmatrix} I_r & \boxed{} \\ \bigcirc & \end{bmatrix}}_{n}$$

NULLITY $= n - r$.

$=$ # FREE VARIABLES

$L = A \cdot v$

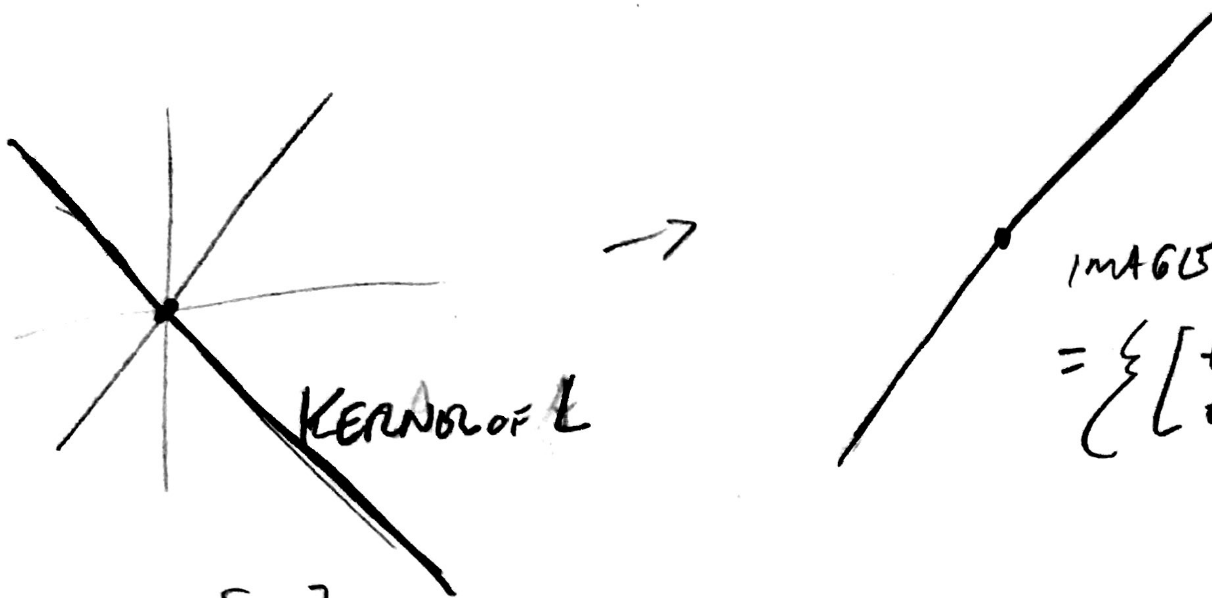


IMAGE OF L $= \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

kernel is $\left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

SO IF YOU "CRUSH" AWAY A k-DIMENSIONAL THING, "WHAT'S LEFT" HAS DIMENSION $n-k$.

BACK TO 4.7

HOMOGENEOUS SYSTEMS.

A $m \times n$ MATRIX.

HOW TO FIND A BASIS FOR KERNEL OF A ("NULL SPACE")

SUPPOSE RANK OF A IS R.

NOTE $R \leq n$.

$[A|\underline{0}]$ IS EQUIVALENT TO

$$\begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad}^{I_R} \qquad\qquad\qquad\qquad\qquad \\ [B|\underline{0}] = \left[\begin{array}{ccccc ccc|c} 1 & 0 & 0 & \cdots & 0 & b_{1R+1} & \cdots & b_{1n} & 0 \\ 0 & 1 & 0 & & 0 & b_{2R+1} & \cdots & b_{2n} & 0 \\ 0 & 0 & 1 & & 0 & \vdots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & & & & 0 \\ 0 & 0 & 0 & \cdots & 1 & b_{RR+1} & \cdots & b_{Rn} & 0 \\ & & & & & & & & \vdots \\ & & & 0 & & & & & \vdots \\ & & & & & & & & 0 \end{array}\right] \end{array}$$

$p = n - r =$ NULLITY.

SO A SOLUTION $\underline{x}$ TO $A\underline{x} = 0$ LOOKS LIKE

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -b_{1r+1}s_1 - b_{1r+2}s_2 - \cdots - b_{1n}s_p \\ \vdots \\ -b_{rr+1}s_1 - b_{rr+2}s_2 - \cdots - b_{rn}s_p \\ \left.\begin{matrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{matrix}\right\} \text{ARBITRARY} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{bmatrix} = s_1 \underbrace{\begin{bmatrix} -b_{1r+1} \\ \vdots \\ -b_{rr+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{= v_1} + s_2 \underbrace{\begin{bmatrix} -b_{1r+2} \\ \vdots \\ -b_{rr+2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{= v_2} + \cdots + s_p \underbrace{\begin{bmatrix} -b_{1n} \\ \vdots \\ -b_{rn} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{= v_p}$$

$\{v_1, \ldots, v_p\}$ IS A BASIS FOR KERNEL OF A.

CLEARLY SPAN AND

EXAMINING ROWS $r+1, \ldots, n$

CAN SEE LIN. IND.

EX.

$$A = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

GENERAL SOLUTION LOOKS LIKE

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s_1 - s_2 - s_3 \\ -4s_1 - 0s_2 - 3s_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

$$= s_1 \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

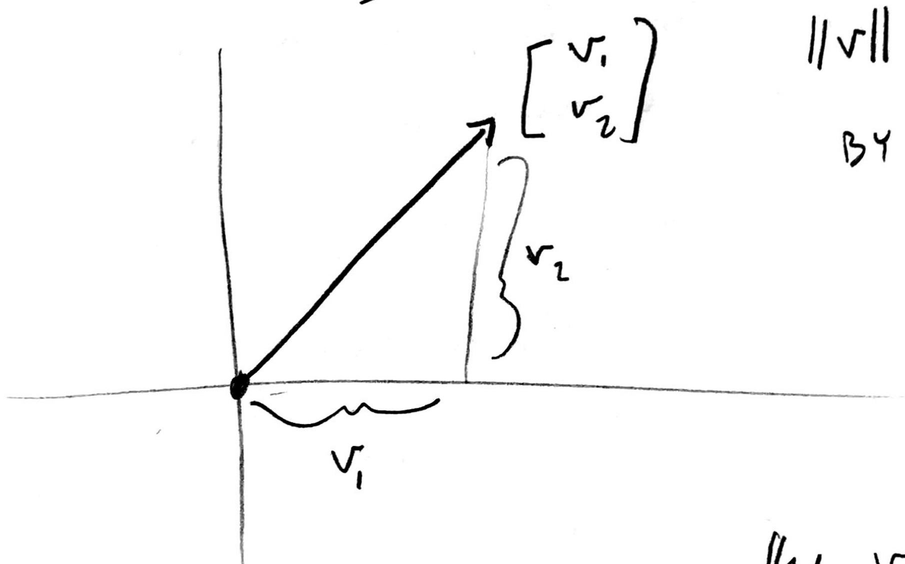
BASIS FOR KERNEL OF A.

CHAPTER 5. (SKIP 5.2) INNER PRODUCT SPACES.

5.1 LENGTH AND DIRECTION.

A VECTOR v IN $\mathbb{R}^2$ OR $\mathbb{R}^3$ IS VISUALIZED AS A LINE SEGMENT WITH A DIRECTION.

IT HAS A **LENGTH** $\|v\|$: OR "**SIZE**"

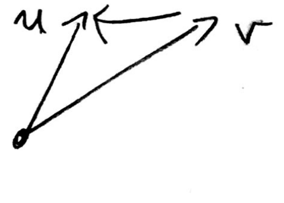


$$\|v\| = \sqrt{v_1^2 + v_2^2}$$

BY PYTHAGOREAN THM.

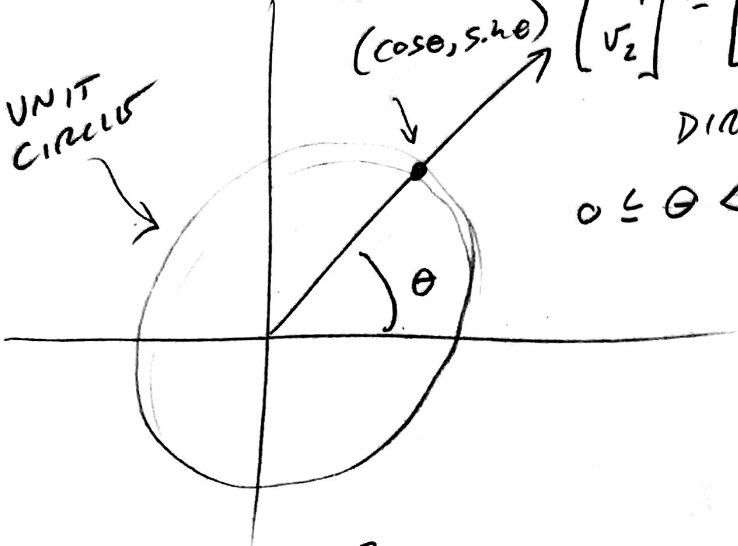
THE **SIZE** OF A DIFFERENCE $\|u - v\|$ MEASURES THE **DISTANCE** BETWEEN u AND v:

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v$$



$$\|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

DIRECTION:



$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \|v\| \cos\theta \\ \|v\| \sin\theta \end{bmatrix}$$

DIRECTION DETERMINED BY $0 \leq \theta < 2\pi$.

WHAT IS θ?

WELL $\cos\theta$ IS x COORDINATE OF POINT ON UNIT CIRCLE AT ANGLE θ.

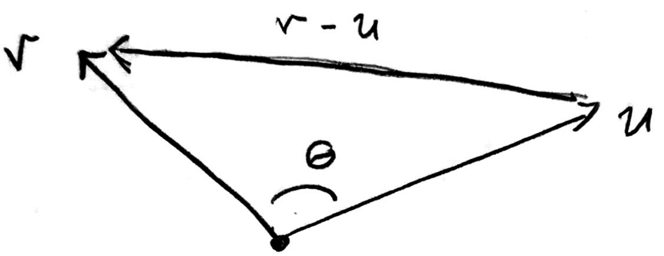
$\sin\theta$ IS y COORD. THERE.

SO $v = \begin{bmatrix} \|v\| \cos\theta \\ \|v\| \sin\theta \end{bmatrix}$.

IF u AND v ARE VECTORS IN $\mathbb{R}^2$,

SO $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$,

THEN u AND v MAKE AN ANGLE $\theta \leq \pi$:



APPLYING THE LAW OF COSINES TO THIS TRIANGLE:

$$\|v - u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta ,$$

$$\text{SO } \cos\theta = \frac{\|u\|^2 + \|v\|^2 - \|v - u\|^2}{2\|u\|\|v\|}$$

$$= \frac{u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^3}{2\|u\|\|v\|}$$

$$- \frac{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}{2\|u\|\|v\|}$$

$$= \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{\|u\| \|v\|}$$

$$= \frac{u \cdot v}{\|u\| \|v\|}.$$

SAME THING IN $\mathbb{R}^3$ or $\mathbb{R}^n$:

$$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|} \qquad 0 \leq \theta \leq \pi$$

EX. $u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\cos\theta = \frac{1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1}{\sqrt{1^2+1^2+0^2}\sqrt{0^2+1^2+1^2}} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

NOTICE THAT

$$\|v\|^2 = v \cdot v$$

AND ANGLE BETWEEN u AND v

GIVEN BY

(*)
$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

DOT PRODUCT SEEMS IMPORTANT.

ALSO CALLED THE STANDARD INNER PRODUCT.

NOTE THAT, BY (*),

TWO VECTORS ARE ORTHOGONAL,

OR MEET AT RIGHT ANGLES,

IFF $u \cdot v = 0$.

EX.

$$\begin{bmatrix} 7 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 7 \end{bmatrix} = 0,$$

THM $u, v, w \in \mathbb{R}^n$.

① $u \cdot u \geq 0$ AND $u \cdot u = 0$ IFF $u = \underline{0}$.

② $u \cdot v = v \cdot u$

③ $(u+v) \cdot w = u \cdot w + v \cdot w$

④ $cu \cdot v = c(u \cdot v)$

DEF. A VECTOR OF LENGTH 1 IS A UNIT VECTOR,

GIVEN $v \neq 0$,

$$u = \frac{1}{\|v\|} v$$

IS THE UNIT VECTOR IN THE DIRECTION OF v.

EX.

$$\mathbf{v} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\|v\| = \sqrt{4 + 16}$$
$$= 2\sqrt{5}$$

SO $u = \frac{1}{2\sqrt{5}} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$

IS A UNIT VECTOR.

5.3 INNER PRODUCT SPACES.

NOTE THE NOTATION!

DEF. V A VECTOR SPACE.

AN INNER PRODUCT ON V IS A FUNCTION THAT ASSIGNS A # (u,v) TO ANY PAIR OF VECTORS u, v AND SATISFIES THE FOLLOWING:

1. $(u,u) \geq 0$ AND $(u,u) = 0$ IFF $u = \underline{0}$.
2. $(v,u) = (u,v)$
3. $(u+v, w) = (u,w) + (v,w)$
4. $(cu, v) = c(u,v)$

V w/ $(\,,\,)$ IS "INNER PRODUCT SPACE"

EX. STANDARD DOT PRODUCT.

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$(u, v) = u \cdot v = u_1 v_1 + \cdots + u_n v_n.$$

or $(u, v) = u \cdot v = u^T v$ ← MATRIX MULT.

EX. LET V BE OF DIMENSION n WITH BASIS S.

THEN

$$(u, v) = [u]_S \cdot [v]_S$$

(CAN TALK ABOUT ANGLE BETWEEN POLYNOMIALS!!)
(DEPENDS ON S.)

EX. V VECTOR SPACE OF ALL CONTINUOUS FUNCTIONS ON $[0,1]$.

FOR $f, g \in V$,

DEFINE

$$(f,g) = \int_0^1 f(t)g(t)\,dt$$

$$(f,f) = \int_0^1 (f(t))^2\,dt \geq 0 \quad \checkmark$$

$$(f,g) = \int_0^1 f(t)g(t)\,dt = \int_0^1 g(t)f(t)\,dt = (g,f)$$

$$\begin{aligned}(f+g,h) &= \int_0^1 (f(t)+g(t))h(t)\,dt \\ &= \int_0^1 f(t)h(t)\,dt + \int_0^1 g(t)h(t)\,dt \\ &= (f,h) + (g,h).\end{aligned}$$

$$\begin{aligned}(cf,g) &= \int_0^1 cf(t)g(t)\,dt = c\int_0^1 f(t)g(t)\,dt \\ &= c\,(f,g).\end{aligned}$$

e.g. $f = t - 1$

$g = t^2 + 2t + 1$

$$(f,g) = \int_0^1 (t-1)(t^2+2t+1)\,dt$$

$$= \int_0^1 t^3 + 3t^2 + t - t^2 - 2t - 1\,dt$$

$$= \int_0^1 t^3 + 2t^2 - t + 1\,dt$$

$$= \left(\frac{t^4}{4} + \frac{2t^3}{3} - \frac{t^2}{2} + t\right)\Bigg|_0^1$$

$$= \frac{1}{4} + \frac{2}{3} - \frac{1}{2} + 1 - 0$$

$$= \frac{3}{4} + \frac{2}{3}$$

$$= \frac{9}{12} + \frac{8}{12} = \frac{17}{12}.$$

EX $\underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$(u, v) = u_1 v_1 - u_2 v_1 - u_1 v_2 + 3 u_2 v_2$$

CHECK ① (OTHERS EXERCISE).

$$\begin{aligned}(u, u) &= u_1^2 - 2u_1 u_2 + 3u_2^2 \\ &= u_1^2 - 2u_1 u_2 + u_2^2 + 2u_2^2 \\ &= (u_1 - u_2)^2 + 2u_2^2 \geq 0.\end{aligned}$$

0 ONLY WHEN $u_1 = u_2$ AND $u_2 = 0$. ✓

ON FINITE DIM^l V, EVERY INNER PRODUCT IS COMPLETELY DETERMINED BY A MATRIX!

LET $\{u_1, \ldots, u_n\}$ BE ORDERED BASIS FOR V.

LET $(\, , \,)$ BE AN INNER PRODUCT.

DEFINE $c_{ij} = (u_i, u_j)$

AND LET $C = [c_{ij}]$. $\longrightarrow$ "MATRIX OF THE INNER PRODUCT."

① C IS SYMMETRIC MATRIX.

② C DETERMINES (v, w) FOR ANY v, w.

PROOF: ① ✓

② $v = \sum_{i=1}^{n} a_i u_i$

$w = \sum_{i=1}^{n} b_i u_i$

$$[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad [w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$(v, w) = \left(\sum_{i=1}^{n} a_i u_i, w \right)$$

$$= \sum_{i=1}^{n} (a_i u_i, w)$$

$$= \sum_{i=1}^{n} a_i (u_i, w)$$

$$= \sum_{i=1}^{n} a_i \left(u_i, \sum_{j=1}^{n} b_j u_j \right)$$ → CAREFUL WITH INDICES.

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j (u_i, u_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j (u_i, u_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j$$

$$= [v]_S^T \cdot C \cdot [w]_S$$

$$= [a_1 a_2 \cdots a_n] \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & \ddots & \vdots \\ c_{n1} & \cdots & & c_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= [a_1 \cdots a_n] \begin{bmatrix} c_{11} b_1 + c_{12} b_2 + \cdots + c_{1n} b_n \\ c_{21} b_1 + c_{22} b_2 + \cdots + c_{2n} b_n \\ \vdots \\ c_{n1} b_1 + c_{n2} b_2 + \cdots + c_{nn} b_n \end{bmatrix}$$

$$= [v]_S \cdot C [w]_S$$

$$= C [v]_S \cdot [w]_S .$$

NOTE THAT HERE WE HAVE

$$C\underline{x} \cdot \underline{y} = \underline{x} \cdot C\underline{y}.$$

IT IS IMPORTANT THAT C IS SYMMETRIC.

IN GENERAL, IF A $n \times n$, STANDARD DOT PRODUCT SATISFIES

$$A\underline{x} \cdot \underline{y} = \underline{x} \cdot A^T\underline{y}.$$

C SATISFIES.

$$\underline{x}^T C \underline{x} > 0 \quad \forall \underline{x} \neq \underline{0}.$$

~~DEF. $n \times n$ AN MATRIX A IS POS~~

__DEF.__ AN $n \times n$ SYMMETRIC MATRIX A IS __POSITIVE DEFINITE__ IF

$$\underline{x}^T A \underline{x} > 0 \quad \forall \underline{x} \neq \underline{0}.$$

(SUCH A MATRIX IS NON-SINGULAR.)

IF C IS ANY POSITIVE DEFINITE MATRIX, THEN IT DEFINES AN INNER PRODUCT:

$$(v, w)$$
$$= \left([v]_S, C[w]_S \right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j \, .$$

$\boxed{\text{MORE LATER.}}$

EX. $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$x^T C \underset{\sim}{x} = [x_1 \; x_2] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= [x_1 \; x_2] \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$$
$$= (2x_1^2 + x_1 x_2) + (x_2 x_1 + x_2^2)$$
$$= 2x_1^2 + 2x_1 x_2 + x_2^2$$
$$= x_1^2 + (x_1 + x_2)^2 > 0 \text{ IF } \underline{x} \neq \underline{0}.$$

THM CAUCHY-SCHWARZ INEQUALITY,

IN AN INNER PRODUCT SPACE,

$$|(u,v)| \le \|u\|\|v\|.$$

PROOF: IF $u = \underline{0}$, $\|u\| = 0$ AND $(u,v) = 0$. ✓

SUPPOSE $u \neq \underline{0}$. LET $r \in \mathbb{R}$ AND CONSIDER

$ru + v$.

$0 \le (ru+v, ru+v)$

$= (u,u)r^2 + 2r(u,v) + (v,v)$

$= ar^2 + 2br + c$ FOR SOME a, b, c

$$\left(\begin{aligned} a &= (u,u) = \|u\|^2 \\ b &= (u,v) \\ c &= (v,v) = \|v\|^2 \end{aligned}\right)$$

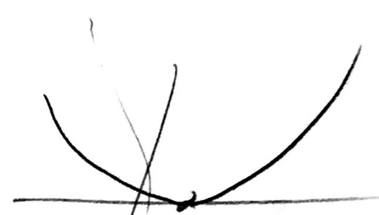
FIX u, v.

THEN $p(r) = ar^2 + 2br + c$ IS

QUADRATIC POLYNOMIAL.

ALSO IT IS NONNEG.

SO IT HAS AT MOST 1 REAL ROOT.



BY QUADRATIC FORMULA $b^2 - ac \le 0$

SO $b^2 \le ac$. SO $b \le \sqrt{a}\sqrt{c}$. ✓ □

AMAZING POWER OF LINEAR ALGEBRA:

$$\left|\int_0^1 f(t)g(t)\,dt\right|^2 \leq \left(\int_0^1 f(t)^2\,dt\right)\cdot\left(\int_0^1 g(t)^2\,dt\right)$$

ex.

$$\left|\int_0^1 e^t\, t^{100}\,dt\right|^2 \leq \int_0^1 e^{2t}\,dt \int_0^1 t^{100}\,dt$$

$$= \frac{1}{2}e^{2t}\Big|_0^1 \cdot \frac{t^{101}}{101}\Big|_0^1$$

$$= \left(\frac{e^2}{2} - \frac{1}{2}\right)\cdot\frac{1}{101}$$

TRIANGLE INEQUALITY.

$$\|u+v\| \le \|u\| + \|v\|.$$

PROOF:

$$\|u+v\|^2 = (u+v,\ u+v)$$
$$= (u,u) + 2(u,v) + (v,v)$$
$$= \|u\|^2 + 2(u,v) + \|v\|^2$$

$(u,v) \le |(u,v)| \le \|u\|\,\|v\|$ BY C.S.

SO $$\|u+v\|^2 \le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$
$$= (\|u\| + \|v\|)^2.$$

✓

□

DEF. IN INNER PROD. SPACES:
DISTANCE IS $\|u-v\|$.

u, v ORTHOGONAL IF $(u,v) = 0$.

ANGLE GIVEN BY

$$\cos\theta = \frac{(u,v)}{\|u\|\|v\|}.$$

<u>DEF.</u> A COLLECTION OF VECTORS IS ORTHONORMAL IF EVERY PAIR IS ORTHOGONAL AND THEY ALL HAVE UNIT LENGTH.

<u>EX.</u> STD. BASIS FOR $\mathbb{R}^4$.

<u>$\cancel{EX}$</u>

<u>SA.</u>

THM IF $S = \{u_1, \dots, u_k\}$

IS AN ORTHONORMAL SET AND DOESN'T CONTAIN $\underset{\sim}{0}$,
THEN S IS LINEARLY INDEPENDENT.

PROOF: SUPPOSE NOT.

THEN

SOME $u_i = \sum_{j \neq i} a_j u_j$.

THEN $0 = (u_i, u_i) = \left(\sum_{j \neq i} a_j u_j, u_i\right)$

$= \sum_{j \neq i} a_j (u_j, u_i)$

$= 0$.

$\Rightarrow u_i = \underset{\sim}{0}$. THIS IS A CONTRADICTION.

$\square$

5.4.

CONSIDER $\mathbb{R}^3$ WITH THE USUAL INNER PRODUCT AND CONSIDER THE BASIS:

$$T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

THESE ARE PAIRWISE ORTHOGONAL BUT IT ISN'T AN <u>ORTHONORMAL</u> BASIS CUZ THEY AREN'T ALL UNIT VECTORS.

$$S = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

IS ORTHONORMAL.

CONSIDER $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ WHAT'S $[v]_S$?

$$v = \frac{3}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

THM IF $S = \{u_1, \dots, u_n\}$

IS AN ORTHONORMAL BASIS IN AN INNER PRODUCT SPACE, THEN

$$v = c_1 u_1 + \dots + c_n u_n$$

WHERE

$$c_i = (v, u_i).$$

PROOF:

$v = c_1 u_1 + \dots c_i u_i \dots + c_n u_n$ FOR SOME c_i

SINCE S IS A BASIS.

$$(v, u_i) = (c_1 u_1 + \dots + c_n u_n, u_i)$$

$$= c_1 (u_1, u_i) + \dots + c_i (u_i, u_i) + \dots + c_n (u_n, u_i)$$

$$= c_i (u_i, u_i) = c_i$$

$\square$

SO WE DON'T HAVE TO SOLVE A SYSTEM OF EQ^S TO FIND $[v]_S$. !!

WE CAN ALWAYS FIND AN ORTHONORMAL BASIS USING A PROCESS CALLED

THE GRAM - SCHMIDT PROCESS :

THM IF $W \neq \{0\}$ IS A SUBSPACE OF V, THERE IS AN ORTHONORMAL BASIS $T = \{w_1, \ldots, w_m\}$ FOR W.

PROOF:

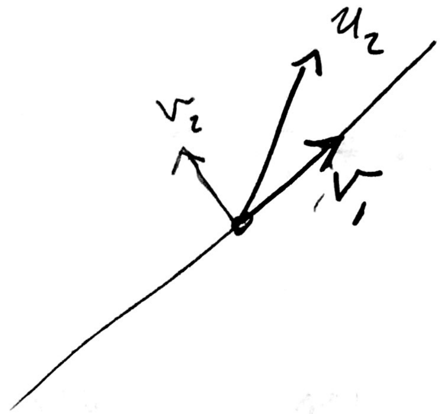
WE FIRST FIND AN ORTHOGONAL BASIS $T^* = \{v_1, \ldots, v_m\}$ FOR W.

PICK ANY BASIS $\{u_1, \dots, u_m\}$.

OUR FIRST VECTOR IN T^* IS u_1.

SO LET $v_1 = u_1$. ✓

WE NOW LOOK FOR VECTOR IN SPAN$\{v_1, u_2\}$ THAT IS ORTHOGONAL TO v_1:



WHATEVER v_2 IS, IT WILL BE A LINEAR COMBO OF v_1 AND u_2:

$$v_2 = a_1 v_1 + a_2 u_2$$

WE WANT $(v_1, v_2) = 0$

$$\text{SO } 0 = (v_1, a_1 v_1 + a_2 u_2)$$

DEFINE

$$= a_1 \underbrace{(v_1, v_1)}_{\neq 0} + a_2 (v_1, u_2)$$

SO $$a_1 = -a_2 \frac{(u_2, v_1)}{(v_1, v_1)}.$$

CAN CHOOSE a_2.

SET $a_2 = 1$,

SO $$a_1 = -\frac{(u_2, v_1)}{(v_1, v_1)}.$$

LET $$v_2 = a_1 v_1 + u_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1$$

$\neq 0$

SO v_1 AND v_2 ARE ORTHOGONAL.

NOW WE WANT v_3 IN SPAN $\{v_1, v_2, u_3\}$.

SUCH THAT v_3 IS ORTHOGONAL TO

v_1 AND v_2.

$$v_3 = b_1 v_1 + b_2 v_2 + b_3 u_3.$$

WANT TO CHOOSE b_i SO THAT

$$(v_3, v_2) = (v_3, v_1) = 0.$$

$$\begin{aligned} 0 = (v_3, v_1) &= (b_1 v_1 + b_2 v_2 + b_3 u_3, v_1) \\ &= b_1 (v_1, v_1) + b(v_2, v_1) + b_3 (u_3, v_1) \\ &= b_1 (v_1, v_1) + 0 + b_3 (u_3, v_1) \end{aligned}$$

$$\begin{aligned} 0 = (v_3, v_2) &= (b_1 v_1 + b_2 v_2 + b_3 u_3, v_2) \\ &= 0 + b_2 (v_2, v_2) + b_3 (u_3, v_1) \end{aligned}$$

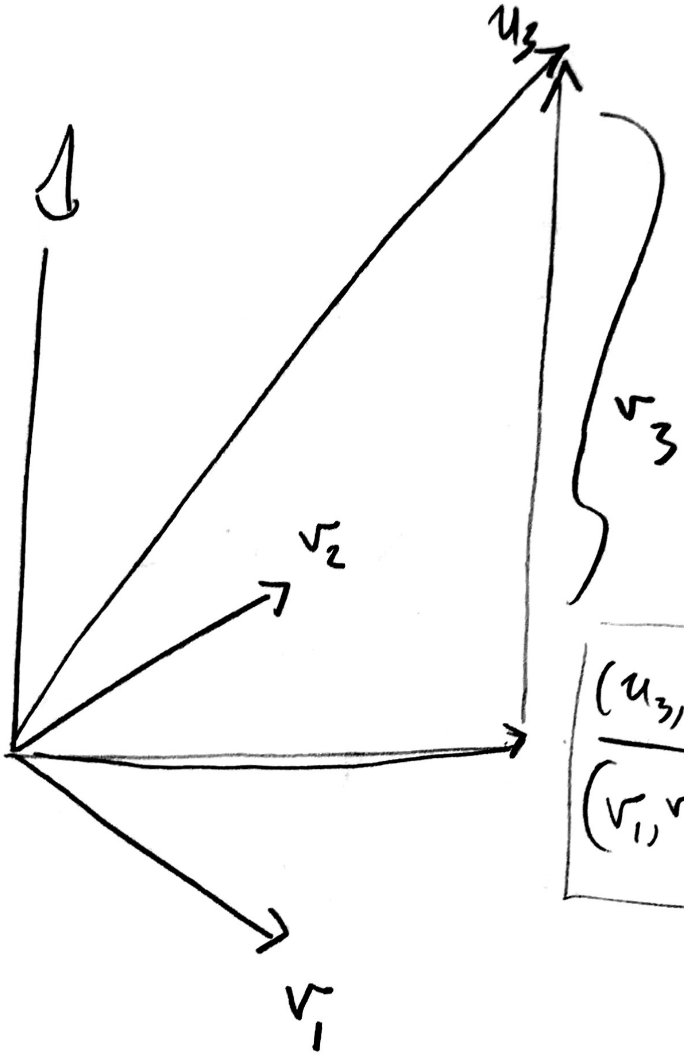
NOW
$$b_1 = -b_3 \frac{(u_3, v_1)}{(v_1, v_1)}$$

$$b_2 = -b_3 \frac{(u_3, v_2)}{(v_2, v_2)}$$

FREE b_3 TO CHOOSE, SET $b_3 = 1$.

So

$$v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2$$



SHADOW OF u_3 IN $v_1 v_2$ PLANE.

$$\boxed{\frac{(u_3, v_1)}{(v_1, v_1)} v_1 + \frac{(u_3, v_2)}{(v_2, v_2)} v_2}$$

KEEP GOING

$$v_4 = u_4 - \frac{(u_4, v_1)}{(v_1, v_1)} v_1 - \frac{(u_4, v_2)}{(v_2, v_2)} v_2 - \frac{(u_4, v_3)}{(v_3, v_3)} v_3.$$

$$\vdots$$

$$T^* = \{v_1, \dots, v_m\}.$$

$$T = \{w_1, \dots, w_m\}$$

$$w_i = \frac{v_i}{\|v_i\|}.$$

$\square$

EX. GRAM SCHMIDT STARTING WITH

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

$$v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \overbrace{\frac{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\0\\0\end{bmatrix}}^{\text{PROJECTION OF } \begin{bmatrix}1\\1\\0\end{bmatrix} \text{ TO SPAN } v_1}$$

$$= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{1}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \underbrace{\frac{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\text{PROJ. to SPAN}\{v_1, v_2\}}.$$

$$= \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

DOESN'T ALWAYS GIVE STANDARD BASIS

EX. P_3 WITH $(p,q) = \int_0^1 p(t)q(t)\,dt$.

FIND ORTHONORMAL BASIS FOR SUBSPACE

W OF POLYS WITH 0 CONSTANT COEFFICIENT.

$S = \{t^2, t\}$ IS A BASIS.

FIND ORTHONORMAL ONE.

$v_1 = u_1 = t^2$.

$$v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1$$

$$= t - \frac{\int_0^1 t \cdot t^2\,dt}{\int_0^1 t^2 \cdot t^2\,dt} \cdot t^2$$

$$= t - \frac{\frac{1}{4}t^4\Big|_0^1}{\frac{1}{5}t^5\Big|_0^1} t^2 = t - \frac{\frac{1}{4}}{\frac{1}{5}} \cdot t^2 = t - \frac{5}{4}t^2$$

$$\|v_2\|^2 = (v_2, v_2) = \int_0^1 \left(t - \frac{5}{4}t^2\right)^2 dt$$

$$= \frac{1}{48}$$

$$\|v_2\| = \frac{1}{\sqrt{48}}$$

$$\|v_1\| = \frac{1}{\sqrt{5}}$$

So $\left\{ \sqrt{5}\, t^2, \sqrt{48}\left(t - \frac{5}{4}t^2\right) \right\}$

IS AN ORTHONORMAL BASIS. $\square$

IF WE HAD CHOSEN $u_1 = t$, $u_2 = t^2$,

WE WOULD GET

$$\left\{ \sqrt{3}\, t, \sqrt{30}\left(t^2 - \frac{1}{2}t\right) \right\}.$$

IF WE CHOOSE AN ORTHONORMAL BASIS THEN INNER PRODUCT BECOMES ~~THE~~ USUAL INNER PRODUCT IN COORDINATES:

<u>THM</u>: LET S BE ORTHONORMAL BASIS IN AN INNER PRODUCT SPACE.

THEN $\langle v, w \rangle = [v]_S \cdot [w]_S$

USUAL DOT PRODUCT.

$\square$

Orthogonal complements.

Consider a vector space V and two subspaces U, W.

Let $U + W = \{ u + w \mid u \in U, w \in W \}$.

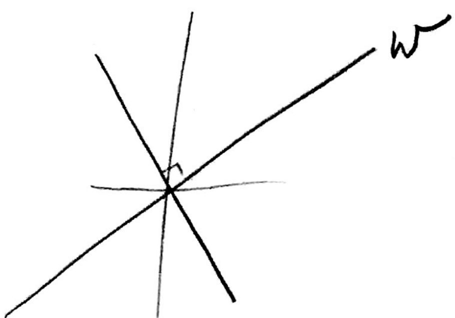
If $U \cap W = \{\underline{0}\}$,

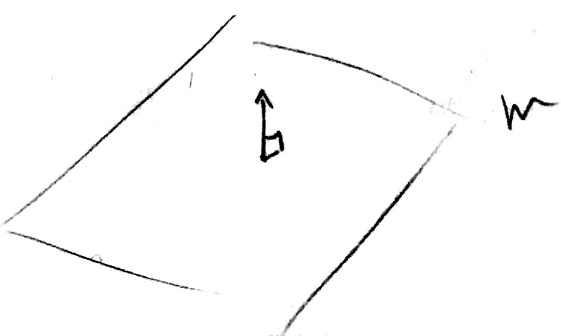
we write $U \oplus W$. (the direct sum)

If $V = U \oplus W$ we say V is direct sum of U and W.

In this case every $v \in V$ is expressed uniquely as $v = u + w$ for $u \in U$ and $w \in W$.

Given a subspace $W \subset V$ it is often desirable to fill W out to V in this way, i.e. find a U s.t. $W \oplus U = V$.





DEF. LET $W \subset V$ BE A SUBSPACE.

A VECTOR $u \in V$ IS ORTHOGONAL TO W IF IT IS ORTHOGONAL TO EVERY VECTOR IN W.

$$W^{\perp} = \{ u \in V \mid u \text{ ORTHOGONAL TO } W \}$$

IS THE ORTHOGONAL COMPLEMENT OF W IN V.

EX $w = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$ IN $\mathbb{R}^3$ WITH DOT PRODUCT.

$W = \text{SPAN}\{w\}$.

$$W^{\perp} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix} = 0 \right\}$$

$$= \underbrace{\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| 4x - y - 2z = 0. \right\}}_{\text{EQUATION OF A PLANE FROM SCHOOL.}}$$

THM ① $W^\perp$ IS A SUBSPACE. (CHECK CLOSED UNDER ADD. & MULT.

② $W \cap W^\perp = \{\underline{0}\}$ ✓

EX IF YOU'D LIKE TO SEE IF $u \in W^\perp$, THEN IT IS ENOUGH TO CHECK THAT u IS ORTHOGONAL TO A SPANNING SET S OF W.

EX. P_3 WITH $(p, q) = \int pq$.

$W = \text{SPAN}\{1, t^2\}$.

FIND BASIS FOR $W^\perp$.

$p(t) = a t^3 + b t^2 + ct + d \in W^\perp$.

IN PARTICULAR $(p, 1) = 0 = (p, t^2)$.

$$(p(t), 1) = \int_0^1 (at^3 + bt^2 + ct + d)\,dt = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0$$

$$(p(t), t^2) = \int_0^1 (at^5 + bt^4 + ct^3 + dt^2)\,dt = \frac{a}{6} + \frac{b}{5} + \frac{c}{4} + \frac{d}{3} = 0$$

SOLVE:

GET

$a = 3r + 16s$, $b = -\frac{15}{4}r - 15s$, $c = r$, $d = s$.

$$p(t) = (3r+16s)t^3 + \left(-\frac{15}{4}r - 15s\right)t^2 + rt + s$$

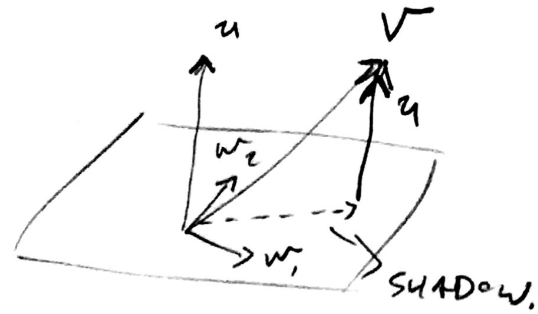
$$= r\underbrace{\left(3t^3 - \frac{15}{4}t^2 + t\right)}_{v_1} + s\underbrace{\left(16t^3 - 15t^2 + 1\right)}_{v_2}.$$

SPAN $\{v_1, v_2\} = W^\perp$.

THEY ARE LIN. INDEP. SINCE THEY ARE NOT MULTIPLES OF ANOTHER.

THM W BE A FTE DIMENSIONAL SUBSPACE OF AN INNER PRODUCT SPACE. THEN

$V = W \oplus W^\perp$,



PROOF:

SUPPOSE $\dim W = m$.

BY GRAM-SCHMIDT, THERE IS AN ORTHONORMAL BASIS FOR W, SAY $S = \{w_1, \dots, w_m\}$.

LET $v \in V$.

DEFINE

$$\underbrace{w = (v, w_1)w_1 + (v, w_2)w_2 + \dots + (v, w_m)w_m}_{\text{SHADOW OF } v \text{ ON } W.} \in W$$

"PROJECTION OF v TO W."

DEFINE
$$u = v - w.$$

$u \in W^{\perp}$ SINCE

$$\begin{aligned}
(u, w_i) &= (v - w, w_i) \\
&= (v, w_i) - (w, w_i) \\
&= (v, w_i) - \left((v, w_1)w_1 + \dots + (v, w_m)w_m, w_i\right) \\
&= (v, w_i) - 0 - 0 - \dots - (v, w_i)(w_i, w_i) - 0 - \dots - 0. \\
&= (v, w_i) - (v, w_i) \cdot 1. \\
&= 0.
\end{aligned}$$

So u orthogonal to everything in W.

So $u \in W^\perp$.

Now,

$$v = w + u$$

and so $V = W + W^\perp$

So $V = W \oplus W^\perp$ since $W \cap W^\perp = \{0\}$.

Thm, If W fte dim^{l} subspace of V.

then $(W^\perp)^\perp = V$.

Proof (skip).

Geometric meaning of fundamental subspaces.

Thm A $m \times n$ matrix.

1. The kernel of A is the orthogonal complement
(null space)

or the row space.

2. The image of A (column space) is

the orthogonal complement of the

kernel of A^T.

$$A : \mathbb{R}^n \to \mathbb{R}^m$$
$$\cup$$
$$\ker A$$

PROOF: a)

NOTE THAT THE DIMENSIONS ARE CORRECT.

THE ROW SPACE HAS DIMENSION r = RANK

ITS COMPLEMENT HAS DIMENSION $n-r$,

WHICH IS THE DIMENSION OF THE KERNEL.

LET $\underline{x} \in$ KERNEL $A \subset \mathbb{R}^n$. $(A\underline{x} = \underline{0})$.

LET $v_1, \dots, v_m$ BE ROWS OF A.

$$A\underline{x} = \begin{bmatrix} v_1 x \\ \vdots \\ v_m x \end{bmatrix} = \begin{bmatrix} v_1 \cdot \underline{x} \\ v_2 \cdot \underline{x} \\ \vdots \\ v_m \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0}.$$

$\underline{x}$ ORTHOGONAL TO EACH v_i AND

HENCE EVERYTHING IN THE ROW SPACE.

SO $\underline{x}$ IS IN COMPLEMENT OF ROW SPACE.

SO KER A IS CONTAINED IN COMPLEMENT OF ROW SPACE.

CONVERSELY IF $\underline{x}$ IS ORTHOGONAL TO EVERYTHING IN ROW SPACE, IT IS ORTH. TO ROWS AND SO

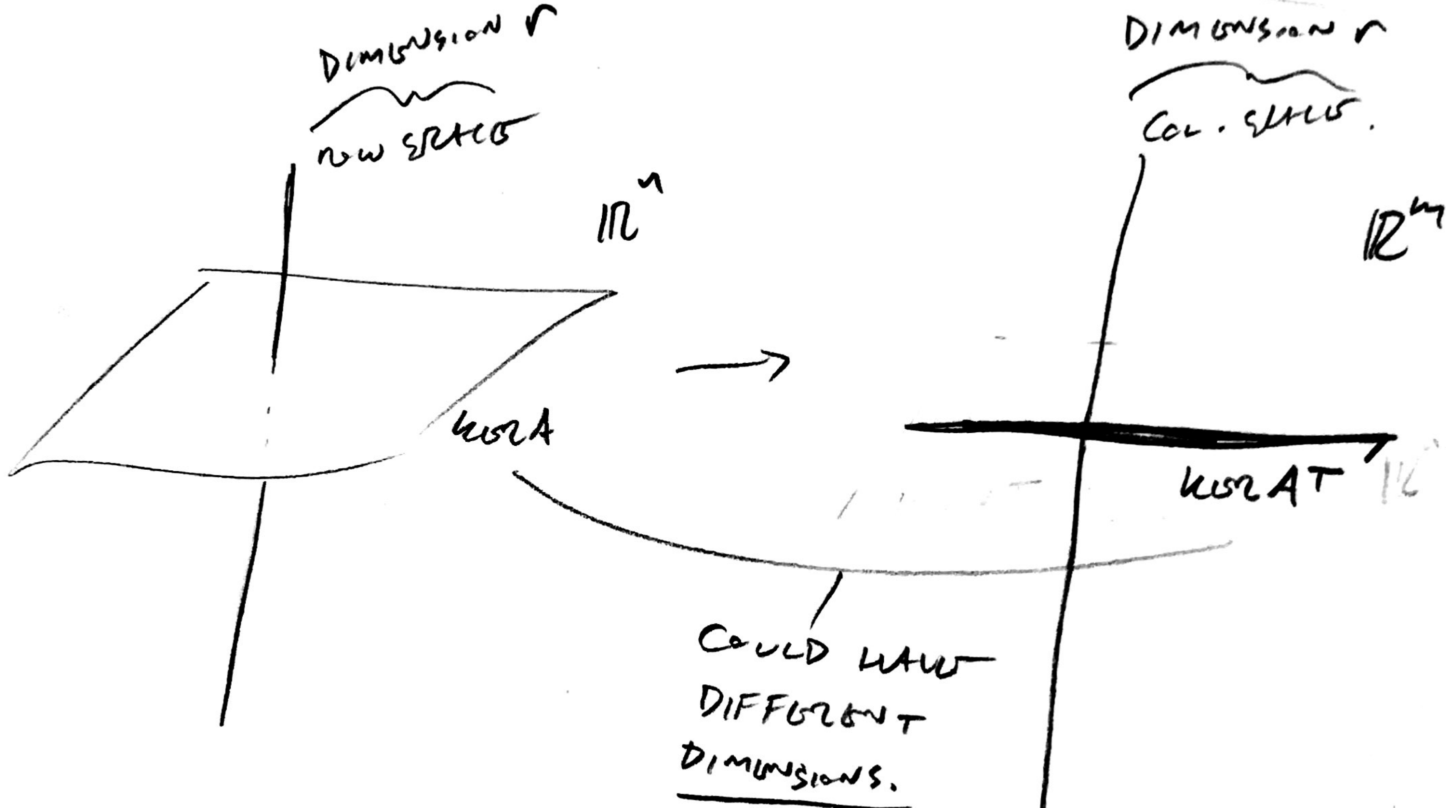
$$A\underline{x} = \begin{bmatrix} v_1 x \\ \vdots \\ v_m x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0}. \checkmark$$

b) REPLACE A WITH A^T. ✓

∎

WARNING:

DRAWINGS IN BOOK ARE TERRIBLE!
(THEY SHOW KERNEL A AND ROW SPACE INTERSECTING IN A LINE.)



EX $A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -3 & 2 & 7 \\ 0 & 2 & -6 & 4 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -3 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$A\underline{x} = 0$

$\rightsquigarrow S = \left\{ \begin{bmatrix} -2 \\ +3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} +1 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

BASIS FOR KERNEL. (SEE 4.7).

$T = \left\{ [1\ 0\ 2\ 1\ -1], [0\ 1\ -3\ 2\ 7] \right\}$

IS A BASIS FOR THE ROW SPACE

(NOT ALWAYS JUST THE SUBSET OF ROWS.)

S AND T ARE ORTHOGONAL. (THINKING OF ELTS OF S AS ROW VECTORS)

OR ELTS OF T AS COLS.

PROJECTIONS AGAIN:

W FTE DIM^L SUBSPACE WITH ORTHONORMAL BASIS $S = \{w_1, \ldots, w_m\}$. GIVEN $v \in V$, THERE ARE UNIQUE $w \in W$ AND $u \in W^\perp$ S.T.

$$v = w + u.$$

$$w = \underbrace{(v, w_1)w_1 + (v, w_2)w_2 + \cdots + (v, w_m)w_m}_{\text{PROJ}_W v}$$

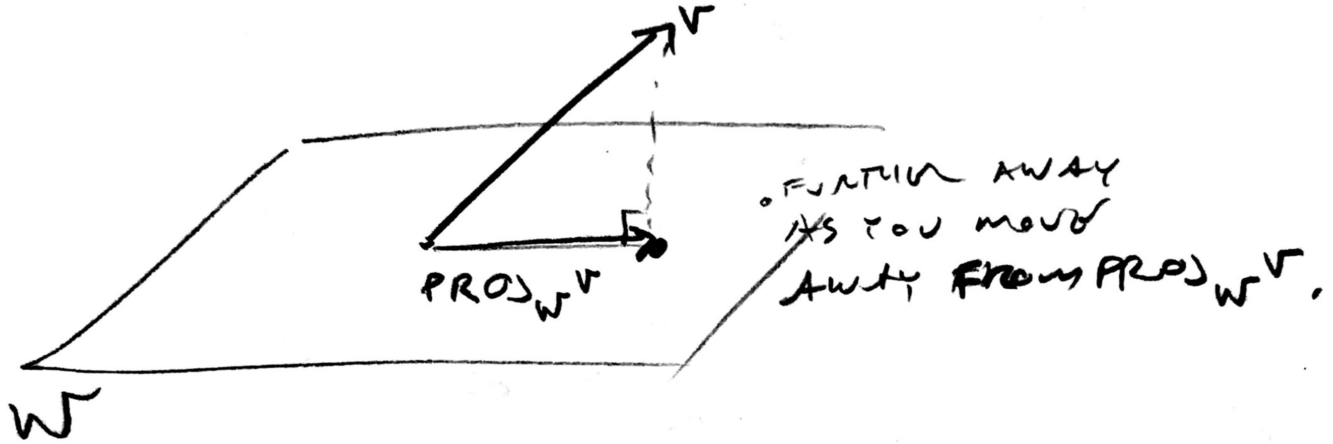
"SHADOW"

ORTHOGONAL PROJECTION OF v TO W.

IF $\{w_1, \ldots, w_m\}$ IS ORTHOGONAL

$$\text{PROJ}_W v = \frac{(v, w_1)}{(w_1, w_1)} w_1 + \cdots + \frac{(v, w_m)}{(w_m, w_m)} w_m .$$

NOTE THAT $\text{PROJ}_W v$ IS CLOSEST POINT ON W TO v:



SO DISTANCE FROM v TO W IS

$$\|v - \text{proj}_W v\|.$$

15X

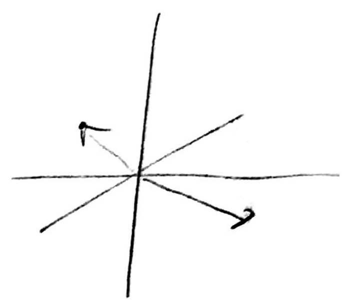
WHAT IS THE DISTANCE FROM

$\begin{bmatrix}1\\1\\1\end{bmatrix}$ TO PLANE $2x - y + 2z = 0$.

$W = \left\{\begin{bmatrix}2\\-1\\2\end{bmatrix}\right\}$ THEN PLANE IS $W^{\perp}$.

WHAT IS A BASIS FOR $W^{\perp}$?

$\begin{bmatrix}1\\0\\-1\end{bmatrix} \in W^{\perp}$. SO IS $\begin{bmatrix}1\\2\\0\end{bmatrix}$.



LIN. INDEPENDENT AND SPAN.

NOT ORTHOGONAL: $\begin{bmatrix}1\\0\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\0\end{bmatrix} = 1$.

GRAM SCHMIDT:

PROJECT $\begin{bmatrix}1\\2\\0\end{bmatrix}$ TO SPAN $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$.

$$\frac{\begin{bmatrix}1\\2\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\-1\end{bmatrix}}{\begin{bmatrix}1\\0\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\-1\end{bmatrix}} \begin{bmatrix}1\\0\\-1\end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 2 \\ 1/2 \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 2 \\ 1/2 \end{bmatrix} \right\}$$ <u>ORTHOGONAL</u> BASIS FOR $W^{\perp}$

$$\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{\frac{1}{4}+4+\frac{1}{4}}}\begin{bmatrix} 1/2 \\ 2 \\ 1/2 \end{bmatrix} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{\sqrt{2}}{3}\begin{bmatrix} 1/2 \\ 2 \\ 1/2 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \\ \frac{\sqrt{2}}{6} \end{bmatrix} \right\}$$ ORTHONORMAL BASIS.

$$\text{PROJ}_{W^\perp}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \left(\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}\right)\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix} + \left(\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}\sqrt{2}/6\\2\sqrt{2}/3\\\sqrt{2}/6\end{bmatrix}\right)\begin{bmatrix}\sqrt{2}/6\\2\sqrt{2}/3\\\sqrt{2}/6\end{bmatrix}$$

$$= 0\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix} + \left(\frac{\sqrt{2}}{6}+\frac{2\sqrt{2}}{3}+\frac{\sqrt{2}}{6}\right)\begin{bmatrix}\sqrt{2}/6\\2\sqrt{2}/3\\\sqrt{2}/6\end{bmatrix}$$

$$= \underline{0} + \left(\frac{6\sqrt{2}}{6}\right)\begin{bmatrix}\sqrt{2}/6\\2\sqrt{2}/3\\\sqrt{2}/6\end{bmatrix}$$

$$= \underline{0} + \begin{bmatrix}1/3\\4/3\\1/3\end{bmatrix}.$$

DISTANCE IS

$$\left\|\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1/3\\4/3\\1/3\end{bmatrix}\right\|$$

$$= \left\|\begin{matrix}+2/3\\-1/3\\+2/3\end{matrix}\right\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = 1 \;\therefore\; \checkmark$$

CHAPTER 6.

BACK TO LINEAR TRANSFORMATIONS.

A LINEAR TRANSFORMATION (LINEAR MAP)
AKA LINEAR OPERATOR

$L: V \to W$

$L(u+v) = L(u) + L(v)$

$L(cu) = c\,L(u).$

EX. $L(v) = Av$ $\quad$ A $m \times n$ MATRIX.

EX. REFLECTION IN X-AXIS.

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$$

EX REFLECTION IN LINE $x = y$

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}.$$

EX DILATION $\quad$ PICK $c \in \mathbb{R}$ $\quad c > 1$
$L: V \to V$

$L(v) = cv$

EX CONTRACTION $0 < c < 1$

$$L(v) = cv.$$

NON EX $L: \mathbb{R}^3 \to \mathbb{R}^3$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ 2y \\ z \end{bmatrix}$$

NOT LIN. TRANS.

$$L(\underline{0}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq 0.$$

LINEAR TRANS. $L(\underline{0}) = \underline{0}$

SINCE

$$\begin{aligned} L(\underline{0}) = L(v - v) &= L(v) + L(-v) \\ &= L(v) - L(v) \\ &= 0. \end{aligned}$$

Ex. $L : \mathbb{P}_1 \to \mathbb{P}_2$

$$L(p(t)) = t\,p(t)$$

$$L(p(t)+q(t)) = t(p(t)+q(t))$$
$$= t\,p(t) + t\,q(t)$$
$$= L(p(t)) + L(q(t)).$$

$$L(c\,p(t)) = t(c\,p(t))$$
$$= c\,t\,p(t)$$
$$= c\,L(p(t)).$$

Ex. $\mathbb{D}$ = ALL DIFFERENTIABLE FUNCTIONS.
$\mathcal{F}$ = ALL FUNCTIONS.

$L(f) = f'$. $\quad L : \mathbb{D} \to \mathcal{F}$

$$L(f+g) = (f+g)' = f' + g' = L(f) + L(g)$$
$$L(cf) = (cf)' = cf' = cL(f).$$

EX

$\mathbb{R}$ IS A VECTOR SPACE.

LET $\mathcal{S}$ BE ALL INTEGRABLE FUNCTIONS. $L: \mathcal{S} \to \mathbb{R}$

$$L(f) = \int_a^b f(x)\,dx =$$

$$L(f+g) = \int_a^b f(x) + g(x)\,dx$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$= L(f) + L(g)$$

$$L(cf) = \int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

$$= cL(f).$$

THM L IS COMPLETELY DETERMINED BY ITS VALUES ON A BASIS.

THM GIVEN $L: \mathbb{R}^n \to \mathbb{R}^m$ FORM THE MATRIX A WHOSE ith COLUMN IS $L(e_i)$:

$$A = \begin{bmatrix} | & & | \\ L(e_1) & \cdots & L(e_n) \\ | & & | \end{bmatrix}.$$

$$L(v) = Av.$$

PROOF. $v = a_1 e_1 + \dots + a_n e_n$

$$\begin{aligned} L(v) &= L(a_1 e_1 + \dots + a_n e_n) \\ &= a_1 L(e_1) + \dots + a_n L(e_n) \\ &= A \cdot v. \end{aligned}$$

$\square$

$L: V \to W$

$\ker L = \{ v \mid L(v) = \underline{0} \}$.

THM. ① KERNEL IS A SUBSPACE

② L IS ONE-TO-ONE IFF $\ker L = \{0\}$.

PROOF. ① EXERCISE

② $(\Rightarrow)$ SUPPOSE $\ker L \neq \{0\}$.

SO THERE ARE $x \neq y$ WITH $L(x) = L(y) = \underline{0}$.

THEN L IS NOT ONE-TO-ONE. ✓

$(\Leftarrow)$ SUPPOSE $\ker L = \{0\}$.

~~THEN L~~ SUPPOSE $L(v) = L(w)$

THEN $L(v) - L(w) = \underline{0}$.

$\Rightarrow L(v - w) = \underline{0}$.

SINCE $\ker L = \{0\}$, WE HAVE

$v - w = \underline{0}$

SO $v = w$. SO L IS 1-1.

$\square$

<u>THM</u> $L: V \to W$

$\dim \ker L + \dim \operatorname{image} L = \dim V$.

<u>THM</u> IF $\dim V = \dim W$,

THEN

① IF L IS 1-1, THEN IT IS ONTO.

② IF L IS ONTO, THEN IT IS 1-1.

<u>DEF.</u> $L: V \to W$ IS <u>INVERTIBLE</u>

IF $\exists\, L^{-1}: W \to V$ s.t.

$L^{-1}L = I_V$ AND $LL^{-1} = I_W$.

<u>THM</u> $L: V \to W$ IS INVERTIBLE IFF

IT IS A BIJECTION.

PROOF

IF $L : \mathbb{R}^n \to \mathbb{R}^n$

IS GIVEN BY A MATRIX A

(SO $L(v) = Av$)

AND L HAS AN INVERSE L^{-1},

THEN $L^{-1}(v) = A^{-1}v$.

L IS INVERTIBLE IFF

DET $A \neq 0$.

THM $L : V \to W$ IS INVERTIBLE

IFF $L(\{S\})$ IS LINEARLY INDEPENDENT

FOR EVERY LIN. IND. S.

SUMMARY OF THINGS WE KNOW:

TFAE:

1. A NONSINGULAR (INVERTIBLE)
2. $A\underline{x} = 0$ HAS ONLY TRIVIAL SOL^{n}
3. A ROW EQ. TO I
4. $A\underline{x} = \underline{b}$ HAS A UNIQUE SOL^{n} FOR ALL $\underline{b}$.
5. A PRODUCT OF ELEM. MATRICES.
6. $\det A \neq 0$
7. A HAS RANK n
8. ROWS (COLS) OF A LIN. IND.
9. DIMENSION OF KERNEL OF A IS ZERO
10. $L: \mathbb{R}^n \to \mathbb{R}^n$ DEFINED BY $L(v) = Av$

    IS 1-1 AND ONTO.

IF $L: V \to V$ IS INVERTIBLE

IFF L IS 1-1

IFF L IS ONTO.

6.3

MATRIX OF A LINEAR TRANSFORMATION,

GIVEN AN $m \times n$ MATRIX A

$\rightsquigarrow L : \mathbb{R}^n \to \mathbb{R}^m$ BY $L(v) = Av$,

NOW WE WANT TO GET A MATRIX

TO ENCODE AN ABSTRACT $L : V \to W$,

LET V BE OF DIMENSION n

W " " " m

LET $S = \{v_1, \ldots, v_n\}$ BE AN ORDERED BASIS OF V

LET $T = \{w_1, \ldots, w_m\}$ " " " " " W,

LET

$$A = \begin{bmatrix} | & | & & | \\ [L(v_1)]_T & [L(v_2)]_T & \cdots & [L(v_n)]_T \\ | & | & & | \end{bmatrix}$$

THIS MATRIX HAS PROPERTY

$$[L(\underline{x})]_T = A[x]_S$$

A IS THE ONLY MATRIX THAT DOES THIS.

Proof

$$\underline{x} = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

$$[\underline{x}]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

$$L(\underline{x}) = L(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$$
$$= a_1 L(v_1) + \cdots + a_n L(v_n)$$

$$[L(x)]_T = [a_1 L(v_1) + \cdots + a_n L(v_n)]_T$$
$$= a_1 [L(v_1)]_T + \cdots + a_n [L(v_n)]_T$$
$$= A[\underline{x}]_S .$$

SUPPOSE THERE IS ANOTHER MATRIX

B s.t.

~~A~~

A $\quad [L(x)]_T = B[x]_S$ FOR ALL X.

SUPPOSE kTH COLUMNS OF A AND B DIFFERENT,

$$[v_k]_S = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to k^{th} \text{ row.}$$

$$[L(v_k)]_T = A \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = k\text{th col of } A.$$

$$= B \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} = k\text{th col of } B$$

$\#$

$\square$

EX $L: P_2 \to P_1$

$L(p(t)) = p'(t)$.

$S = \{t^2, t, 1\}$ $T = \{t, 1\}$.

FIND MATRIX FOR L W.R.T. S AND T.

$L(t^2) = 2t = 2t + 0 \cdot 1$ $\quad [L(t^2)]_T = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$L(t) = ~~0 \cdot t~~ = 0 \cdot t + 1 \cdot 1$ $\quad [L(t)]_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$L(1) = 0 = 0t + 0 \cdot 1$ $\quad [L(1)]_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$p(t) = 5t^2 - 3t + 2$

$L(p(t)) = 10t - 3$.

$$[L(p(t))]_T = \begin{bmatrix} 10 \\ -3 \end{bmatrix}.$$

$$[p(t)]_S = \begin{bmatrix}5\\-3\\2\end{bmatrix}$$

$$A \cdot [p(t)]_S = \begin{bmatrix}2&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}5\\-3\\2\end{bmatrix}$$

$$= \begin{bmatrix}10\\-3\end{bmatrix}.$$

SAME L, DIFFERENT BASES

$S = \{t^2, t, 1\}$ $\quad T = \{t+1, t-1\}$

FIND A.

$$L(t^2) = 2t = a(t+1) + b(t-1)$$

$a = 1 \qquad b =$

$$\rightsquigarrow \quad a + b = 2$$
$$a - b = 0$$

$$\rightsquigarrow \quad a = 1 \quad b = 1.$$

So $[L(t^2)]_T = \begin{bmatrix}1\\1\end{bmatrix}.$

SIMILARLY:

$$L(t) = 1 = \frac{1}{2}(t+1) - \frac{1}{2}(t-1)$$

$$[L(t)]_T = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$$L(1) = 0 = 0(t+1) + 0(t-1) \quad [L(1)]_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix}.$$

$$[L(p(t))]_T = \begin{bmatrix} 1 & 1/2 & 0 \\ 1 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 13/2 \end{bmatrix}$$

So $L(p(t)) = \frac{7}{2}(t+1) + \frac{13}{2}(t-1)$

$= 10t - 3.$ ✓

<u>EX</u>.

$L: \mathbb{R}^3 \to \mathbb{R}^2$.

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \text{AND} \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

WHAT ARE THE T-COORDINATE VECTORS FOR THESE?

SOLVE:

$$\left[\begin{array}{cc|c|c|c} 1 & 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 5 & 3 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c|c|c} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

MATRIX OF L W.R.T. S AND T.

SO, TO FIND A W.R.T.

$S = \{v_1, \dots, v_n\}$ $\quad T = \{w_1, \dots, w_m\}$

~~FACE~~

$$\left[w_1 \, w_2 \, \cdots \, w_m \,\middle|\, L(v_1) \,\middle|\, \cdots \,\middle|\, L(v_n) \right]$$

TO REDUCED ROW ECHELON FORM.

MATRIX ON RIGHT IS A.

CHAPTER 7 EIGENVECTORS & EIGENVALUES.

HERE IS A SIMPLE MATRIX:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

A DEFINES A LINEAR TRANSFORMATION

$$L: \mathbb{R}^2 \to \mathbb{R}^2$$

$$L(v) = Av.$$

NOTICE

$$Ae_1 = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2e_1$$

$$Ae_2 = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}e_2.$$

NOTE

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ FOR ANY } \lambda.$$

ex.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -2 + \frac{1-\sqrt{5}}{2} \\ -1 + \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\left(\frac{3+\sqrt{5}}{2}\right) \\ -\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$= \frac{3+\sqrt{5}}{2} \begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = \frac{3 - 2\sqrt{5} - 5}{2 \cdot 2}$$

$$= \frac{2 - 2\sqrt{5}}{4} = \frac{1-\sqrt{5}}{2},$$

EX. ROTATION:

NO VECTOR SUCH THAT

$$\underbrace{Av = \lambda v}$$

THIS SAYS THAT Av AND v ARE PARALLEL.

EX

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

DEF: $L: V \to V$ A LINEAR TRANSFORMATION

A NONZERO VECTOR v IS AN EIGENVECTOR OF L IF $L(v) = \lambda v$ FOR SOME $\lambda \in \mathbb{R}$.

λ IS CALLED THE ASSOCIATED EIGENVALUE.

EX. $L(x) = 2x$.

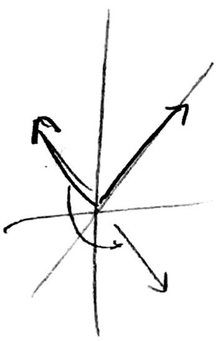
2 ONLY EIGENVALUE.

EVERY VECTOR AN EIGENVECTOR FOR 2.

EX.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$



REFLECTION IN LINE $y=x$.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ IS OBVIOUS FIXED,

EIGENVECTOR

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underset{\text{EIG. VALUE}}{1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ SORTA FIXED

EIG. VALUE

EIG. VECTOR.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \checkmark$$

EX, LET V BE ALL FUNCTIONS THAT ARE ∞'LY DIFFERENTIABLE.

$$L : V \to V$$

$$L(f) = f'.$$

DOES L HAVE EIGENVECTORS?

$$L(f) = \lambda f$$

$$f' = \lambda f$$

$$\frac{df}{dx} = \lambda f$$

EX. $\boxed{f(x) = k e^{\lambda x}}$

HOW DO WE FIND THESE?

EX. $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

WANT AND λ AND v S.T.

$$Av = \lambda v.$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$\leadsto$ $x + y = \lambda x$

$-2x + 4y = \lambda y$

$\leadsto$ $(1-\lambda)x + y = 0$

$-2x + (4-\lambda)y = 0$

$$\left[\begin{array}{cc|c} 1-\lambda & 1 & 0 \\ -2 & 4-\lambda & 0 \end{array}\right]$$

NONTRIVIAL SOL^n IFF

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda-1 & -1 \\ 2 & \lambda-4 \end{vmatrix} = 0$$

IFF $(\lambda-1)(\lambda-4)+2=0$

IFF $\lambda^2-5\lambda+6=0 = (\lambda-3)(\lambda-2)$

SO OUR ONLY POSSIBLE EIGENVALUES

ARE $\lambda_1 = 2$, $\lambda_2 = 3$.

WHAT ARE THE v?

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow$ $x + y = 2x$

$-2x + 4y = 2y$

$\Rightarrow$ $(1-2)x + y = 0$

$-2x + (4-2)y = 0$

$\Rightarrow$ $x - y = 0$

$2x - 2y = 0$

$x = y$ $\quad$ $y =$ ANYTHING. $= r$

$$v = \begin{bmatrix} r \\ r \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} r \\ r \end{bmatrix} = \begin{bmatrix} 2r \\ 2r \end{bmatrix}$$

$$= 2\begin{bmatrix} r \\ r \end{bmatrix}.$$

$\lambda_2 = 3$

$\rightsquigarrow$

$2x - y = 0$

$2x - y = 0$

$\Rightarrow \quad x = \frac{1}{2}y$

$y = s.$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} s/2 \\ s \end{bmatrix} = \begin{bmatrix} \frac{3s}{2} \\ -s + 4s \end{bmatrix} = 3 \begin{bmatrix} s/2 \\ s \end{bmatrix}.$$

DEFINITION

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & \lambda - a_{22} & & \vdots \\ & & \ddots & \\ -a_{n1} & & \cdots & \lambda - a_{nn} \end{bmatrix}$$

$p(\lambda) = \det(\lambda I - A)$ IS A POLYNOMIAL IN λ.

THE CHARACTERISTIC POLYNOMIAL.

$$\det(\lambda I - A)$$

$$= \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$$

$$a_n = \det(-A).$$
$$= (-1)^n \det A$$

ex $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -2 & 1 \\ -1 & \lambda & -1 \\ -4 & 4 & \lambda - 5 \end{bmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$p(\lambda) = (\lambda - 1)\begin{vmatrix} \lambda & -1 \\ 4 & \lambda - 5 \end{vmatrix}$$

$$+ 2 \begin{vmatrix} -1 & -1 \\ -4 & \lambda - 5 \end{vmatrix}$$

$$+ \begin{vmatrix} -1 & \lambda \\ -4 & 4 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda(\lambda - 5) + 5) + 2((-1)(\lambda - 5) - 5) + (-4 + \lambda 4)$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6.$$

THM EIGENVALUES OF A ARE

THE ROOTS OF CHARACTERISTIC POLY OF A.

<u>PROOF.</u>

$$Ax = \lambda x \iff A\underline{x} = (\lambda I)\underline{x}$$

$$\iff (A - \lambda I)\underline{x} = 0$$

THIS HAS NONTRIVIAL SOL^n

IFF $\det(A - \lambda I) = 0$.

SO λ IS A ROOT OF $\det(A - \lambda I)$. $\square$

<u>FINDING ROOTS IS HARD!</u>

BACK TO OUR EARLIER EXAMPLES:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda-2)(\lambda-1) - 1$$

$$= \lambda^2 - 3\lambda + 2 - 1$$

$$= \lambda^2 - 3\lambda + 1$$

ROOTS ARE $\dfrac{+3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{3+\sqrt{5}}{2}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = \frac{3+\sqrt{5}}{2} x \qquad 1)$$

$$x + y = \frac{3+\sqrt{5}}{2} y \qquad 2)$$

$$\left(2 - \frac{3+\sqrt{5}}{2}\right)x + y = 0$$

$$x + \left(1 - \frac{3+\sqrt{5}}{2}\right)y = 0$$

$$\rightsquigarrow \left[\begin{array}{cc|c} \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \end{array}\right].$$

2ND ROW IS A MULT. OF FIRST.

$y =$ ANYTHING

LET $y = \frac{1-\sqrt{5}}{2}$

$x = -1$. ✓

7.2. DIAGONALIZATION.

WE HAD THIS EXAMPLE:

$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ THIS MATRIX IS NICE!

TWO EIGENVALUES AND WE CAN SEE THEM!

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ ISN'T AS NICE.

EIGENVALUES $\frac{3+\sqrt{5}}{2}$ AND $\frac{3-\sqrt{5}}{2}$

$L : \mathbb{R}^2 \to \mathbb{R}^2$

$L(v) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} v.$

$L : \mathbb{R}^2 \to$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{-3-\sqrt{5}}{2} \\ -\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} = \underbrace{\frac{3+\sqrt{5}}{2}}_{\lambda_1} \underbrace{\begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}}_{v_1}$$

$$= \lambda_1 v_1$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2-\sqrt{5} \\ \frac{3-\sqrt{5}}{2} \end{bmatrix} = \underbrace{\frac{3-\sqrt{5}}{2}}_{\lambda_2} \underbrace{\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}}_{v_2}$$

$$= \lambda_2 v_2$$

$S = \{v_1, v_2\}$. NOTICE: $\boxed{S \text{ IS ORTHOGONAL!}}$

WITH RESPECT TO THIS BASIS, WHAT IS L?

$$[L(v)]_S = B[v]_S.$$

WHAT IS B?

$$[L(v_1)]_S = \left[\frac{3+\sqrt{5}}{2} v_1\right]_S = \frac{3+\sqrt{5}}{2} [v_1]_S = \frac{3+\sqrt{5}}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[L(v_2)]_S = \frac{3-\sqrt{5}}{2}[v_2]_S = \frac{3-\sqrt{5}}{2}\begin{pmatrix}0\\1\end{pmatrix}.$$

$$B = \begin{bmatrix}\frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2}\end{bmatrix}.$$

DEF. V n-DIMENSIONAL.

$L: V \to V$ DIAGONALIZABLE IS THERE IS A BASIS S SUCH THAT L IS REPRESENTED BY A DIAGONAL MATRIX.

DEF. TWO MATRICES A, B ARE SIMILAR (OR CONJUGATE) IF $B = P^{-1}AP$ FOR SOME INVERTIBLE P.

THM SIMILAR MATRICES HAVE SAME EIGENVALUES.

PROOF. (DIFFERENT THAN BOOK).

$\underline{x}$ EIGENVECTOR FOR $B = P^{-1}AP$.

SO $\lambda\underline{x} = B\underline{x} = P^{-1}AP\underline{x}$

$$\text{So} \quad \lambda\underline{x} = P^{-1}AP\underline{x}$$

$$\Rightarrow P\lambda\underline{x} = AP\underline{x}$$

$$\Rightarrow \lambda(P\underline{x}) = A(P\underline{x})$$

So $P\underline{x}$ IS AN EIGENVECTOR FOR A

W/ EIGENVALUE λ.

So B's EIG. VALUES ARE EIG. VALUES

OF A.

OTHER DIRECTION IS SIMILAR. $\square$

THM IF $S = \{v_1, \dots, v_n\}$

IS AN ORDERED BASIS OF EIGENVECTORS OF L

W/ EIG. VALUES $\lambda_1, \dots, \lambda_n$

THEN L IS DIAGONALIZABLE.

PROOF.

THE MATRIX OF L WITH RESPECT TO

S IS

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

$\square$

ALSO, IF L IS DIAGONALIZABLE,

GIVEN BY A DIAGONAL MATRIX D

W.R.T. S, THEN S IS A

BASIS OF EIGENVECTORS.

THE LINEAR TRANS. ASSOC. TO A

A MATRIX A IS DIAGONALIZABLE

IFF IT IS SIMILAR TO A DIAGONAL

MATRIX: $D = P^{-1}AP$.

IF T IS THE STANDARD BASIS AND

S IS A BASIS OF EIGENVECTORS

FOR A, THEN THE

$$P^{-1}_{T \leftarrow S} \, A \, P_{T \leftarrow S} \quad \text{IS}$$

THE MATRIX FOR A W.R.T. T.

THM $n \times n$ $A^{n \times n}$ IS SIMILAR TO A DIAGONAL MATRIX D

IFF A HAS n LIN. IND. EIG. VECTORS. $\square$

EX

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

EIGENVALUES

$\lambda_1 = 2 \qquad \lambda_2 = 3$

EIGENVECTORS

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

A IS SIMILAR TO $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, i.e. $\cancel{P^{-1}}$

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = P^{-1}AP$

WHAT IS $P_{T \leftarrow S}$?

$$P = P_{T \leftarrow S} = \left[\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_T \ \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right]_T \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

EX. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ DIAGONALIZABLE?

ONLY EIGENVECTORS ARE $\begin{bmatrix} r \\ 0 \end{bmatrix}$.

CAN'T MAKE A BASIS FROM THESE.

NOT DIAGONALIZABLE.

THM IF A HAS n DISTINCT EIGENVALUES, THEN A IS DIAGONALIZABLE.

PROOF. LET $\{\lambda_1, \dots, \lambda_n\}$ BE THE EIGENVALUES.

LET $S = \{x_1, \dots, x_n\}$ BE EIGENVECTORS

WE WILL SHOW THAT S IS LINEARLY INDEPENDENT.

<u>SUPPOSE NOT,</u>

SO WE ASSUME S REDUNDANT.

BY A PREVIOUS THM, SOME X_j

IS THE FIRST j S.T.

$\{X_1, \ldots, X_j\}$ IS REDUNDANT

AND ~~FOR~~ THIS j X_j IS

A LINEAR COMBO OF THE

$X_1, \ldots, X_{j-1}$ (WHICH ARE LIN. IND.)

$$X_j = a_1 X_1 + \ldots + a_{j-1} X_{j-1}$$

$$\text{SO } (\star) \quad \lambda_j X_j = \lambda_j a_1 X_1 + \cdots + \lambda_j a_{j-1} X_{j-1}.$$

BUT ALSO

$$A X_j = a_1 A X_1 + \cdots + a_{j-1} A X_{j-1}$$

$$(\star\star) \quad \lambda_j X_j = a_1 \lambda_1 X_1 + \cdots + a_{j-1} \lambda_{j-1} X_{j-1}$$

SUBTRACT (**) FROM (*)

$\leadsto$

$$0 = (\lambda_j - \lambda_1)a_1 x_1 + \dots + (\lambda_j - \lambda_{j-1})a_{j-1}x_{j-1}$$

BUT THEN ALL OF THESE COEFFS

ARE 0 SINCE THESE ARE

LINEARLY INDEPENDENT.

BUT SOME $a_k \neq 0$

SO FOR THAT k

$\lambda_j - \lambda_k = 0$ #. $\square$

SHOWS THAT EIG. VECT.S

FOR DISTINCT EIG. VALUES

ARE LINEARLY <u>IND</u>.

IF $p(\lambda)$ HAS "MULTIPLE" ROOTS

LIKE $p(\lambda) = (\lambda - 1)^2$

$\left(\text{LIKE } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ OR } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$

THEN ~~IT~~ YOU MAY OR MAY NOT BE DIAGONALIZABLE.

$$(\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_\ell)^{k_\ell}$$

k_i <u>MULTIPLICITY</u> OF λ_i.

<u>EX</u> $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

NOT DIAGONALIZABLE.

$p(\lambda) = \lambda(\lambda - 1)^2$

$(I - A)\underline{x} = \underline{0}$

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ SOLUTIONS ARE $\begin{bmatrix} 0 \\ n \\ 0 \end{bmatrix}$,

EX

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$p(\lambda) = \lambda(\lambda-1)^2$$

$$(I-A) = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{matrix}$$

EIG. VECTORS ARE

$$\begin{bmatrix} 0 \\ r \\ s \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ BASIS FOR THIS

SPACE OF EIGENVECTORS

(THE EIGENSPACE FOR $\lambda = 1$).

$\lambda = 0$

$0I - A = -A$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

Sol'ns $\begin{bmatrix} r \\ 0 \\ -r \end{bmatrix}$.

LIN. IND. EIG. VECTORS $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. ✓

DIAGONALIZABLE

7.3

DIAGONALIZATION FOR SYMMETRIC MATRICES.

REMARK: A MATRIX A IS SYMMETRIC

IF $A = A^T$.

IF A IS AN $n \times n$ MATRIX WITH n DISTINCT

EIGENVALUES THEN A IS DIAGONALIZABLE.

IF EIGENVALUES ARE NOT DISTINCT,

IT MAY OR MAY NOT BE.

BUT! EVERY SYMMETRIC MATRIX

IS DIAGONALIZABLE.

THM THE CHARACTERISTIC POLYNOMIAL

OF A SYMMETRIC $n \times n$ REAL MATRIX

HAS n REAL ROOTS.

SOME FACTS ABOUT COMPLEX #S:

WHAT ARE COMPLEX #S?

$$\mathbb{C} = \{x + iy \mid x, y \text{ real}\}$$

$$i^2 = -1.$$

USUALLY CALL COMPLEX #S

$$z = x + yi$$

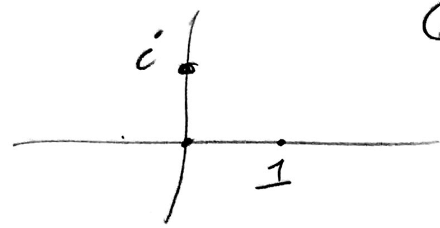
$$w = a + bi \qquad z = x + yi$$

$$\begin{aligned} wz &= (a+bi)\cdot(x+yi) \\ &= ax + bxi + ayi + byi^2 \\ &= ax - by + (bx+ay)i. \end{aligned}$$

So

$$\begin{aligned} (1+2i)(7-i) &= 7 + 14i - i + 2. \\ &= 9 + 13i. \end{aligned}$$

$\mathbb{C}$



IF $z = x + yi$

DEFINE $\bar{z} = x - yi$ COMPLEX CONJUGATE OF z.

z IS REAL IFF $z = \bar{z}$.

DEFINE $|z| = \sqrt{x^2 + y^2}$ THE MODULUS OF z.

NOTE $$z\bar{z} = (x + yi)(x - yi)$$
$$= x^2 + y^2 = |z|^2.$$

CAN TALK ABOUT COMPLEX VECTOR SPACES WHERE SCALARS ARE COMPLEX NUMBERS.

GIVEN A MATRIX $A = [a_{ij}]$ WITH COMPLEX ENTRIES, $\bar{A} = [\bar{a}_{ij}]$

AND $\bar{A}^T = [\bar{a}_{ji}]$ IS THE

CONJUGATE TRANSPOSE OF A.

IF $\underline{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ IS A VECTOR W/ CX ENTRIES

NOTE THAT

$$\bar{\underline{z}}^T z = [\bar{z}_1 z_1 + \bar{z}_2 z_2 + \cdots \bar{z}_n z_n]$$
$$= [|z_1|^2 + \cdots |z_1|^2]$$

IS NONZERO IF $\underline{z} \neq \underline{0}$.

PROOF OF THM.

LET λ BE A ROOT OF THE CHARACTERISTIC POLYNOMIAL OF A.

WE WILL SHOW $\bar{\lambda} = \lambda$.

$$A\underline{x} = \lambda \underline{x}$$

$$\bar{\underline{x}}^T A \underline{x} = \bar{\underline{x}}^T \lambda \underline{x} = \lambda \bar{\underline{x}}^T \underline{x}$$

TAKE CONJ. TRANSPOSE OF EACH SIDE

$$\bar{\underline{x}}^T \bar{A}^T \underline{x} = \bar{\lambda} \bar{\underline{x}}^T \underline{x}$$

So, since $\overbrace{\bar{A}^T = A^T}^{A \text{ real}}$ $\underbrace{= A}_{A \text{ symmetric}}$

we have

$$\underline{\bar{x}}^T A \underline{x} = \bar{\lambda} \underline{\bar{x}}^T \underline{x}.$$

So $$\lambda \underbrace{\underline{\bar{x}}^T \underline{x}}_{\neq 0} = \bar{\lambda} \underline{\bar{x}}^T \underline{x}$$

So $\lambda = \bar{\lambda}$. $\square$

<u>THM</u> IF A SYMMETRIC, AND

$$A\underline{x} = \lambda \underline{x} \quad \text{AND} \quad A\underline{y} = \mu \underline{y}$$

WITH $\mu \neq \lambda$,

THEN $\underline{x}$ AND $\underline{y}$ ARE ORTHOGONAL.

(W.R.T. STANDARD INNER PRODUCT.)

<u>PROOF</u>.

WE USE THE FACT:

$$\boxed{\text{IF } (\ ,\ ) \text{ IS THE DOT PRODUCT, THEN } (\underline{a}, A^T \underline{b}) = (A\underline{a}, \underline{b}).}$$

NOW,

$$\begin{aligned} \lambda(\underline{x}, \underline{y}) &= (\lambda \underline{x}, \underline{y}) \\ &= (A\underline{x}, \underline{y}) \\ &= (\underline{x}, A^T \underline{y}) = (\underline{x}, A\underline{y}) \\ &= (\underline{x}, \mu \underline{y}) \\ &= \mu(\underline{x}, \underline{y}). \end{aligned}$$

So $\lambda(\underline{x},\underline{y}) = \mu(\underline{x},\underline{y})$

BUT SINCE $\lambda \neq \mu$,

$(\underline{x},\underline{y}) = 0$. $\square$

Ex. $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

$p(\lambda) = (\lambda+2)(\lambda-4)(\lambda+1)$

$\lambda_1 = -2 \quad \lambda_2 = 4 \quad \lambda_3 = -1.$

EIGENVECTORS ARE SOL'NS

TO

$$\begin{bmatrix} \lambda & 0 & 2 \\ 0 & \lambda+2 & 0 \\ 2 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0}$$

$\lambda_1 = -2 \quad \rightsquigarrow \quad x_1 = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$

$$\lambda_2 = 4, \quad X_2 = \begin{bmatrix} -r/2 \\ 0 \\ r \end{bmatrix}$$

$$\lambda_3 = -1, \quad X_3 = \begin{bmatrix} 2r \\ 0 \\ r \end{bmatrix}$$

PICK

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

ALL ORTHOGONAL!

A CONJUGATE TO $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$P^{-1} A P = D$

WHERE $P = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

IF A SYMMETRIC, EIG. VECTORS FOR DISTINCT EIGENVALUES ARE ORTHOGONAL.

SO IF A IS SYMMETRIC, **AND** A IS DIAGONALIZABLE THEN THERE IS AN ORTHONORMAL BASIS $\{X_1, \ldots, X_n\}$ CONSISTING OF EIGENVECTORS

THEN $$P = \begin{bmatrix} | & | & & | \\ X_1 & X_2 & \cdots & X_n \\ | & | & & | \end{bmatrix}$$

IS SUCH THAT

$P^{-1}AP$ IS A DIAGONAL MATRIX D.

THIS P HAS A NICE PROPERTY:

$$P^T = \begin{bmatrix} X_1^T \\ \vdots \\ X_n^T \end{bmatrix}$$

AND ij ENTRY OF P^TP IS (X_i, X_j)

$(X_i, X_j) = 1$ IF $i = j$ AND 0 IF $i \neq j$.

i.e. $P^T P = I$.

So $P^T = P^{-1}$!

DEFINITION. A IS ORTHOGONAL

IF $A^{-1} = A^T$. (i.e. $A^T A = I$.)

IN EXAMPLE

$$\begin{bmatrix} 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

THM $^{n\times n}A$ ORTHOGONAL IFF COLS (ROWS) ARE ORTHONORMAL $\square$

THM $\det(A) = \pm 1$ IF A ORTHOGONAL.

THE ORTHOGONAL MATRICES ARE PRECISELY THE ISOMETRIES, i.e. THE LINEAR TRANSFORMATIONS THAT PRESERVE THE DISTANCE i.e. THE LINEAR TRANSFORMATIONS THAT PRESERVE THE INNER PRODUCT.

SINCE $(Ax, Ay) = (x, A^T A y) = (x, y)$.

IF A IS ORTHOGONAL.

IF $(Ax, Ay) = (x, y)$ FOR ALL x, y

THEN $(x, A^T A y) = (x, y)$ FOR ALL x, y

THIS IMPLIES THAT $A^T A = I$

(EXERCISE 7.3 #36).

THM IF A SYMMETRIC $n \times n$ MATRIX,

THEN A IS DIAGONALIZABLE.

WE WILL SKIP PROOF.

So FOR SYMMETRIC A

EACH EIGENSPACE IS AS BIG AS POSSIBLE.

i.e. EACH EIGENVALUE OF MULTIPLICITY k HAS A k-DIMENSIONAL SPACE OF EIGENVECTORS.

EX. $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

$p(\lambda) = (\lambda+2)^2(\lambda-4)$.

$\lambda_1 = -2 \qquad \lambda_2 = -2, \qquad \lambda_3 = 4.$

SOLVE

$(-2I - A)\underline{x} = 0$

$$\left[\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2-DIM^{l} SOL^{n} SPACE.

BASIS IS $x_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ AND $x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

NOT ORTHOGONAL, BUT

GRAM-SCHMIDT GIVES

$y_1 = x_1$ AND

$$y_2 = x_2 - \frac{(x_2, y_1)}{(y_1, y_1)} y_1$$

$$= \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

LET $y_2^* = 2y_2 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

NORMALIZE

$$z_1 = \frac{1}{\|y_1\|} y_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad z_2 = \frac{1}{\|y_2^*\|} y_2^* = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$\{z_1, z_2\}$ ORTHONORMAL BASIS

FOR THE EIGENSPACE FOR $\lambda = -2$.

FOR $\lambda = 4$, $\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} x = 0 \rightsquigarrow$

$$x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -1/\sqrt{6} & 1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{6} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

P ORTHOGONAL.