

[www.math.wisc.edu/~kent/MATH340.FALL.2017.HTML](http://www.math.wisc.edu/~kent/MATH340.FALL.2017.HTML)

2x 25% MIDTERMS 10/17 & 11/16

25% FINAL 12/16 7:25 PM - 9:25 PM

25% QUIZZES IN DISCUSSION.

GOING FOR A COUPLE WEEKS. GERMAN, PAUL.

CHAPTER ONE.

LINEAR ALGEBRA.

- SOLVE SYSTEMS OF LINEAR EQUATIONS.

- IMAGE COMPRESSION (SVD CAN COMPRESS SIZE OF RGB IMAGES  $768 \times 1024$

- CAN USE IT TO BY FACTOR OF 18.)

SHOW ONLY RATIONAL FUNCTION HAS PARTIAL FRACTION DECOM.

- GOOGLE PAGES RANK ALGORITHM.

CHAPTER 1.

SYSTEMS OF LINEAR EQUATIONS.

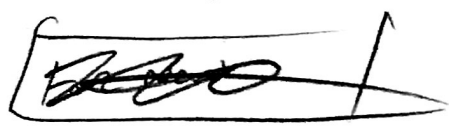
EX.

UNKNOWN  $x_1, x_2$   $\rightarrow$

SUBSCRIPTS  
WILL BE USE,  
LATER WHEN  
WE HAVE  
HUNDREDS OF  
XS.

$$x_1 - 3x_2 = -7$$

$$2x_1 + x_2 = 7$$



THERE IS A UNIQUE SOLUTION HERE.

NAMELY  $x_1 = 2$  AND  $x_2 = 3$ .

EX.

$$8x_1 - 3x_2 = 7$$

$$3x_1 - 2x_2 = 0$$

$$10x_1 - 2x_2 = 14$$

ALSO HAS A UNIQUE SOLUTION.

IN FACT THE SAME ONE,

$x_1 = 2$  AND  $x_2 = 3$ .



How to solve?

3

Q.

$$\textcircled{1} \quad x + 2y + 3z = 6$$

$$\textcircled{2} \quad 2x - 3y + 2z = 14$$

$$\textcircled{3} \quad 3x + y - z = -2$$

} NOTE  
ORDER OF  
VARIABLES

WE CAN ELIMINATE X.

WE COULD SOLVE FOR X AND IN  $\textcircled{1}$

AND PLUG INTO  $\textcircled{2}$ .

AND PLUG INTO  $\textcircled{3}$ .

BETTER THING TO DO IS

MULT.  $\textcircled{1}$  BY 2 AND SUBTRACT  
FROM  $\textcircled{2}$ .

AND  
MULT.  $\textcircled{1}$  BY 3 AND SUBTRACT  
FROM  $\textcircled{3}$ .



2 TIMES ① IS

$$2x + 4y + 6z = 12$$

SUBTRACT FROM

$$\textcircled{2} \quad 2x - 3y + 2z = 14$$

AND GET

$$-7y - 4z = 2$$

& SUBTRACTING 3 TIMES ① FROM 3 PIVOTS

$$-5y - 10z = -20.$$

SO NOW SYSTEM:

$$-7y - 4z = 2$$

$$-5y - 10z = -20.$$

DIVIDE SECOND BY -3

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$$-7y - 4z = 2$$

$$y + 2z = 4$$

$$\leadsto \textcircled{*} y + 2z = 4$$

$$\textcircled{**} -7y - 4z = 2$$

ELIMINATE  $y$ .

ADD 7 TIMES  $\textcircled{*}$  TO  $\textcircled{**}$

$$y - 10z = 30$$

$$\leadsto z = 3.$$

$$z = 3 \text{ AND } \textcircled{*} y + z = 4$$

$$\Rightarrow y = -2.$$

w/  $\textcircled{1}$ , we have

$$\textcircled{x = 1.}$$

THIS ENTIRE PROCESS PRODUCED  
A LINEAR SYSTEM

$$x + 2y + 3z = 6$$

$$y + 2z = 4$$

$$z = 3$$

MUCH EASIER  
TO SOLVE.

CONSIDER (A)  $x - 3y = -7$

(B)  $2x - 6y = 7$ .

ELIMINATE  $x$ .

SUBTRACT 2 · (A) FROM (B).

~

$$0 = 21.$$

ABSURD.

NO SOLUTION

$$\text{Use } \textcircled{C} \quad x + 2y - 3z = -4$$

$$\textcircled{D} \quad 2x + y - 3z = 4.$$

ELIMINATE  $x$ . from  $\textcircled{C}$

$$\leadsto -3y + 3z = 12$$

$$y = z - 4.$$

So  $z$  CAN BE ANYTHING

$$\textcircled{C} \leadsto x + 2(z - 4) - 3z = -4$$

$$x = z + 4$$

INFINITELY MANY SOLUTIONS

IN GENERAL,

LINER EQUATION:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b.$$

$x_i$  CALLED UNKNOWNS.

$a_i$  CONSTANTS CALLED COEFFICIENTS.

A SOLUTION IS A SEQUENCE

$$s_1, \dots, s_n$$

OF #S S.T.

$$a_1 s_1 + \dots + a_n s_n = b.$$

---

IF WE HAVE  $m$  LINER EQUATIONS  
IN  $n$  UNKNOWN,

SAY WE HAVE A

SYSTEM OF  $m$  LINER EQUATIONS

IN  $n$  UNKNOWNS.

A solution of the linear system  
 is a set,  $s_1, \dots, s_n$  that  
 solves all simultaneously.

---

IF NO SOL<sup>n</sup>, INCONSISTENT,  
 OTHERWISE CONSISTENT.

---

EASILY WRITTEN:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

IF  $b_1 = b_2 = \dots = b_m = 0$

SYSTEM IS HOMOGENEOUS.

ALWAYS  
 HAS  
TRIVIAL  
 SOLUTION.

TWO SYSTEMS ARE EQUIV. IF  
OF SAME  
TYPE

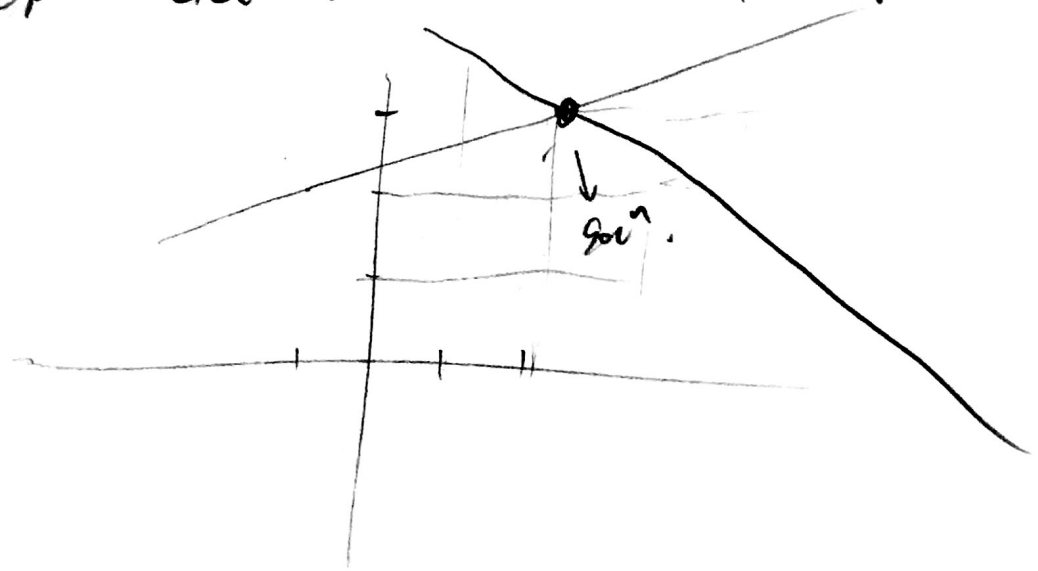
SAME SET OF SOLUTIONS.

CAN USE ELIMINATION TO TRY AND SOLVE.  
(ALWAYS WORKS.)

WE SAW ONE, NO, or MANY  
SOLUTIONS.

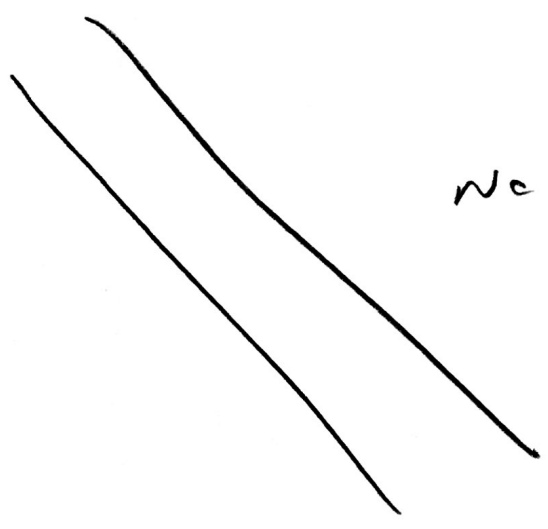
THIS IS ALWAYS CASE.

OUR VERY FIRST EXAMPLE WAS A PAIR  
OF EQUATIONS FOR LINES.





IF TWO LINES PARALLEL, BUT  
NOT SAME:

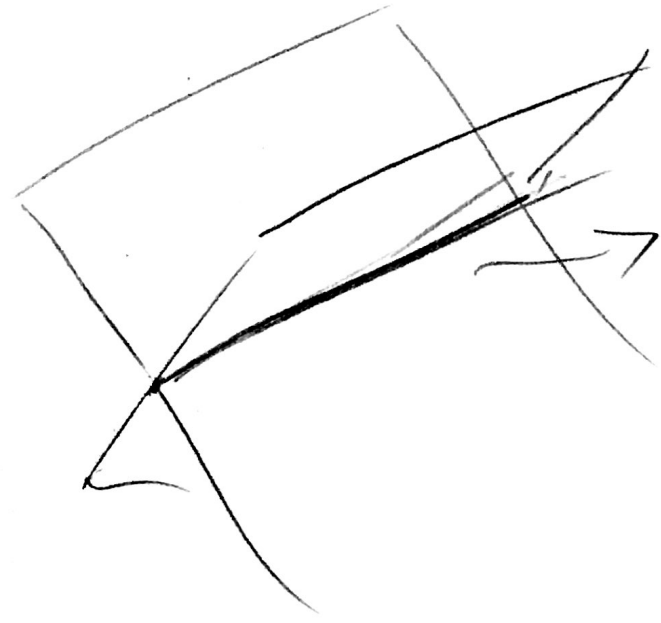


NO SOLUTION.

OUR INFINITE # SOL<sup>n</sup> IS.

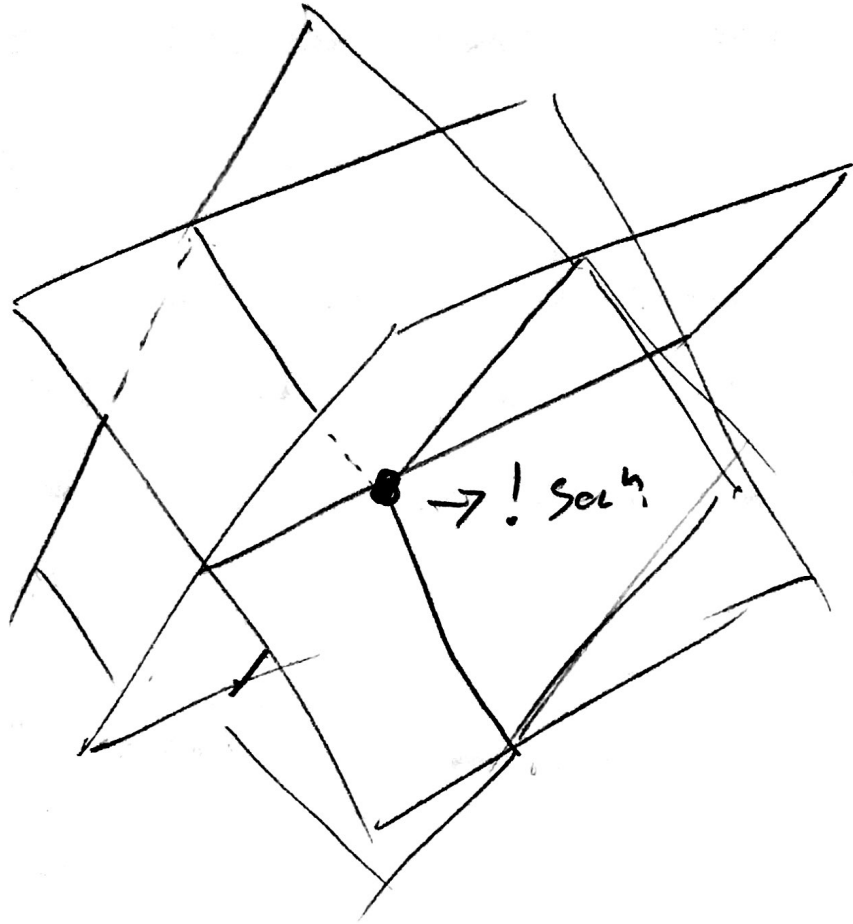
WE HAVE 2 EQ<sup>n</sup>s FOR PLANS:

NOT PARALLEL



LINE OF SOLUTIONS.

THREE PLANES:



# 1.2 MATRICES.

KEEPING TRACK OF OUR OPERATIONS

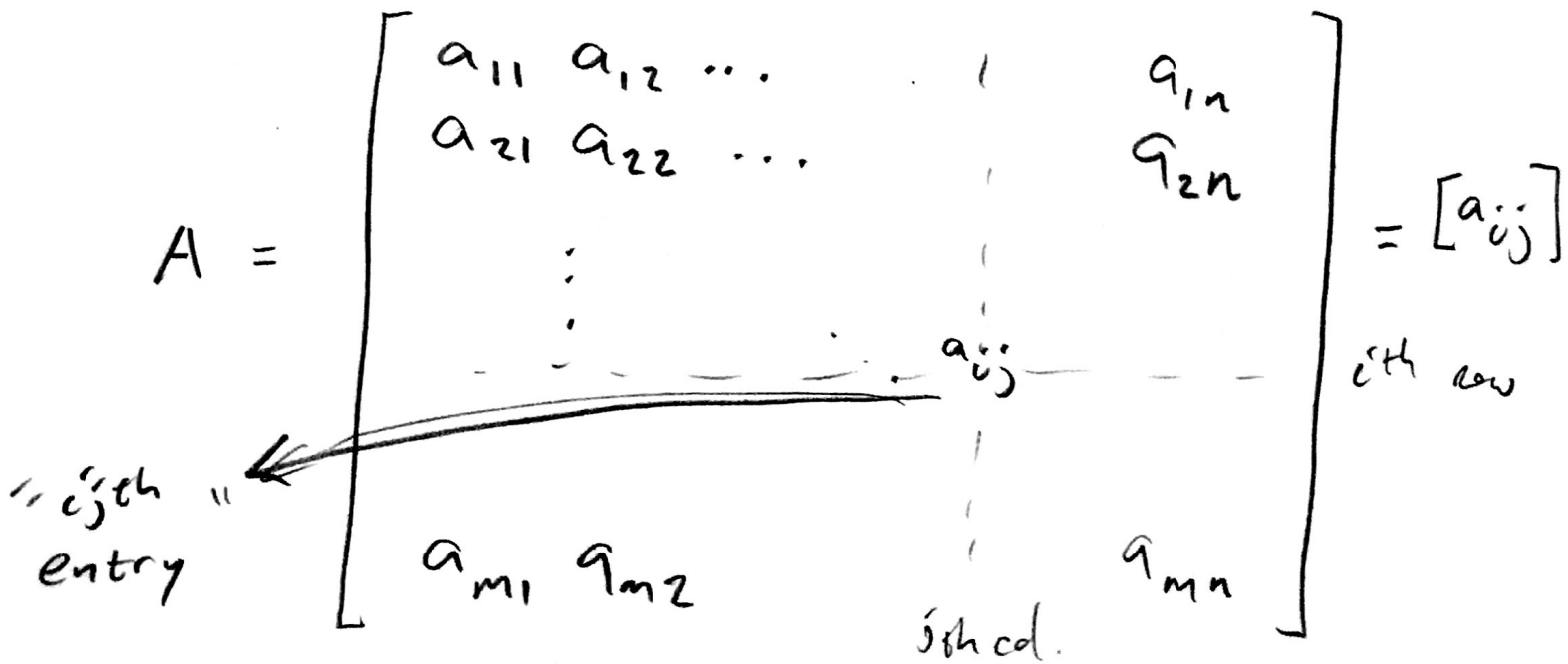
DURING ELIMINATION IS CUMBERSOME.

OBSERVE: WE ONLY MANIPULATE THE COEFFICIENTS  $a_{ij}$  AND THE  $b_i$ .

WE CAN MORE EASILY KEEP TRACK BY FOCUSING ONLY ON THE  $a_{ij}$  AND  $b_i$ .

DEF An  $m \times n$  <sup>columns</sup> MATRIX  $A$  IS AN ARRAY <sub>rows</sub>

OF  $m \cdot n$  NUMBERS,  $m$  ROWS,  $n$  COLUMNS.



$i$ th row

$$[a_{i1} \ a_{i2} \ \dots \ a_{in}]$$

$j$ th column

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

IF  $m=n$ ,  $A$  IS SQUARE.

THE TERMS

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \dots & \\ & & & a_{nn} \end{bmatrix}$$

(MAIN)  
→ DIAGONAL.

Ex.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \quad 2 \times 3$$

$$B = \begin{bmatrix} 7 & 5 & -\pi \\ 6 & 1 & 1 \\ \frac{1}{2} & 1 & 0 \end{bmatrix} \quad 3 \times 3$$

$$C = [4] \quad 1 \times 1$$

row<sub>i</sub>(A)  
i<sup>th</sup> row

col<sub>j</sub>(A)  
j<sup>th</sup> col.

rows AND columns ARE VECTORS.

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$


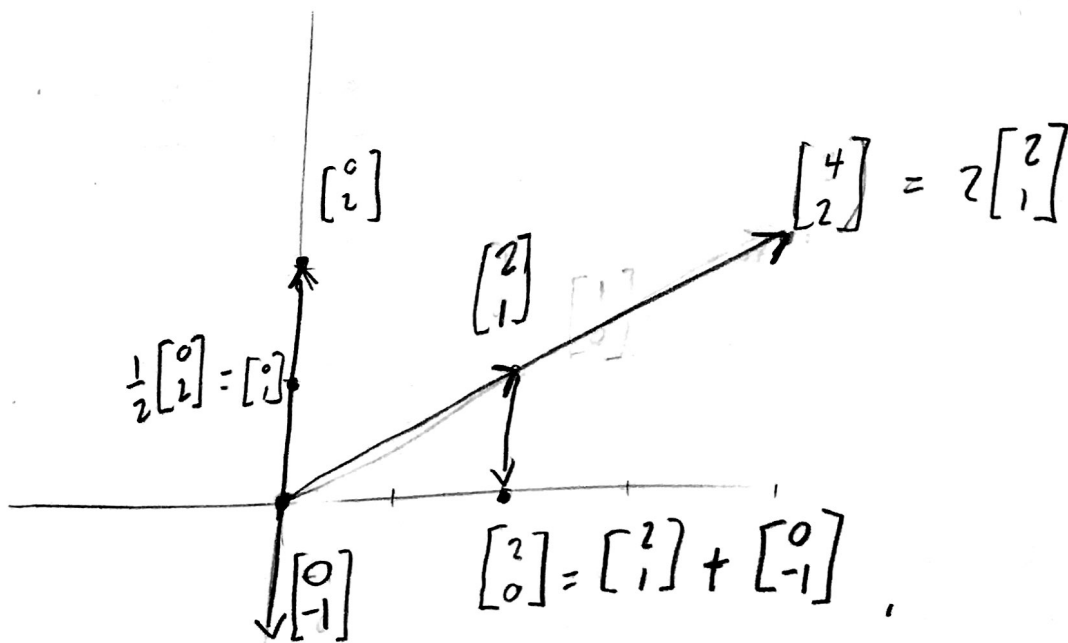
column vector

$$\underline{v} = [1 \ 2 \ 7 \ -6] \quad \text{row vector.}$$

MOSTLY USE column vectors.

WHILE THE NOTATION OF COLUMNS AND ROWS OF NUMBERS ARE CONVENIENT COMPUTATIONAL TOOLS, IT IS IMPORTANT TO HAVE AS MANY GEOMETRIC PICTURES:

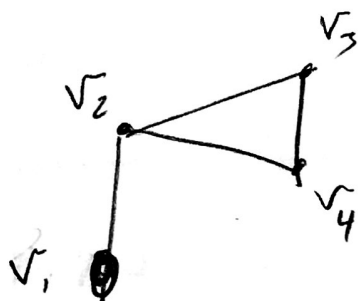
A VECTOR IS AN "ARROW" THAT MAY BE SCALED (MULTIPLIED BY A NUMBER) AND VECTORS MAY BE ADDED (BY ADDING THEIR COORDINATES.)

Ex

A VERY USEFUL APPLICATION OF MATRICES COMES UP IN THE STUDY OF GRAPHS.

A GRAPH IS A COLLECTION OF VERTICES (NODES) CONNECTED BY EDGES.



THIS KIND OF THING IS IMPORTANT IN GOOGLE'S PAGE-RANK ALGORITHM.

INCIDENCE MATRIX

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \left[ \begin{array}{ccccc} & v_1 & v_2 & v_3 & v_4 \\ v_1 & 0 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 1 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 0 & 1 & 1 & 0 \end{array} \right]$$

ENTRY IS

1 IF

CONNECTED BY EDGE

0 OTHERWISE.

CAN EASILY COMPUTE # EDGES BETWEEN VERTICES USING THIS MATRIX. (LATER).

EX ANOTHER USE OF MATRICES COMES UP IN IMAGE COMPRESSION.

IMAGE IS A (BIG) MATRIX OF RGB COLOR VALUES OR INTENSITY VALUES.

LINEAR ALGEBRA CAN STORE THIS INFO EFFICIENTLY.

MATRIX ADDITION

$A = [a_{ij}]$      $B = [b_{ij}]$     BOTH  $m \times n$ .

$A + B = [a_{ij} + b_{ij}]$

So  $A + B = C = [c_{ij}]$  where  $c_{ij} = a_{ij} + b_{ij}$

WE ONLY WRITE  $A + B$  IF  $A$  AND  $B$  SAME SIZE.

ALSO HAVE  $A - B$ .



## SCALAR MATRICES:

$$A = [a_{ij}]$$

$\lambda$  is a scalar #,

$$\lambda A = [c_{ij}] \quad \text{where } c_{ij} = \lambda a_{ij}.$$

EX So  $7 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 14 & 7 \end{bmatrix}.$

SI

EX CAN THINK OF A STORE'S INVENTORY AS A VECTOR. (EACH COORD. CORRESPONDS TO AN ITEM AND ENTRY IS THE # OF THAT ITEM.)

IF  $\underline{u}$  IS INVENTORY VECTOR  
AND  $\underline{v}$  IS VECTOR FOR AMOUNT SOLD,  
RESULTING INVENTORY IS

$$\underline{u} - \underline{v}$$

REVIEW SUMMATION      NOTATION

$$\sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n.$$

(505 PAGES 17 FOR A REVIEW.)

A LINEAR COMBINATION OF  $m \times n$  MATRICES

IS A SUM

$$\sum_{i=1}^{\ell} c_i A_i \quad \text{WHERE } c_1, \dots, c_{\ell} \text{ \#s}$$

$$A_1, \dots, A_{\ell} \text{ } m \times n \text{ MATRICES.}$$

MOST COMMON SITUATION IS WHEN THE

$A_i$  ARE VECTORS:

$$7 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \pi \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

SOME MORE NOTATION.DEF

$$A = [a_{ij}] \quad \underline{m \times n} \text{ - MATRIX,}$$

$$A^T = [a_{ji}] \quad \text{IS THE } \underline{\text{TRANSPOSE}} \text{ OF } A,$$

$n \times m$   $\swarrow$  INTERCHANGES ROWS & COLUMNS.

EX.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 5 \end{bmatrix}$$

### 1.3 MATRIX MULTIPLICATION.

DOT PRODUCT (INNER PRODUCT)

$$\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\underline{a} \cdot \underline{b} = a_1 b_1 + \dots + a_n b_n = \sum_{i=1}^n a_i b_i$$

EX       $\underline{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$       AND       $\underline{v} = \begin{bmatrix} 3 \\ 5 \\ -1 \\ 2 \end{bmatrix}$

$$\begin{aligned} \underline{u} \cdot \underline{v} &= 2 \cdot 3 + (-1) \cdot 5 + 0 \cdot (-1) + 1 \cdot 2 \\ &= 6 - 5 + 0 + 2 \\ &= 3 \end{aligned}$$

# MATRIX MULTIPLICATION

DEFINITION

$$A = [a_{ij}] \quad m \times p$$

$$B = [b_{ij}] \quad p \times n$$

$AB$  IS  $m \times n$  MATRIX  $C = [c_{ij}]$

WHERE  $c_{ij}$  IS THE DOT PRODUCT OF THE  $i$ th ROW OF  $A$  AND THE  $j$ th ROW OF  $B$ .

EX.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 5 & 1 \cdot 3 + 1 \cdot 7 \\ 0 \cdot 2 + 1 \cdot 5 & 0 \cdot 3 + 1 \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}$$

$$\begin{aligned}
 \text{So } c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ip}b_{pj} \\
 &= \sum_{k=1}^p a_{ik}b_{kj} .
 \end{aligned}$$

Now SQ. 6X .

$$\left[ \begin{array}{ccc|cc}
 1 & -2 & 3 & 1 & 4 \\
 4 & 2 & 1 & 3 & -1 \\
 0 & 1 & -2 & -2 & 2
 \end{array} \right]$$

$$= \left[ \begin{array}{cc|cc}
 1-6 & -6 & 4+2+6 & & \\
 4+6 & -2 & 16-2+2 & & \\
 0+3 & +4 & 0-1-4 & & 
 \end{array} \right] = \left[ \begin{array}{cc}
 -11 & 12 \\
 8 & 16 \\
 7 & -5
 \end{array} \right] .$$

# IMPORTANT EXAMPLES

MATRIX TIMES A VECTOR

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}$$

WE WERE MOTIVATED BY LINEAR SYSTEMS TO THINK ABOUT MATRICES, WHICH ARE MORE IMPORTANT. "TRANSFORMATIONS" ARE ALSO IMPORTANT. CAUTIONS: MORE IMPORTANT.

2x2 MATRIX IS A TRANSFORMATION

OF PLANE:  
E.G.  $\begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$

STRETCHES X-AXIS  
SHRINKS Y-AXIS.

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

ROTATES PLANE.

3x3 MATRIX TRANSFORMATION OF  $\mathbb{R}^3$  ETC.

- BA MAY NOT BE DEFINED. (EVEN IF AB IS.)
- IF BA IS DEFINED, THEN  $m=n$ .  
AB IS  $m \times m$  AND BA IS  $n \times n$ .
- IF A AND B ARE SQUARE, THEN BOTH AB AND BA ARE DEFINED BUT COULD BE DIFFERENT!

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

## RETURN TO LINEAR SYSTEMS.

m EQS IN n UNKNOWNNS.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

CAN ENCODE THIS IN

$$A \underline{x} = \underline{b} \quad \text{WHERE}$$

$$A = [a_{ij}] , \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} , \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} .$$

"COEFFICIENT  
MATRIX"

(A HOMOGENEOUS SYSTEM IS  $A \underline{x} = \underline{0}$ .)



WE WILL ALSO USE WHAT'S CALLED

THE AUGMENTED MATRIX OF THE SYSTEM,

$$[A \mid b]$$

$$= \left[ \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

5X

$$-4x + y + z = 2$$

$$2x + 3y - z = 3$$

$$x + y + z = 4$$

$$\begin{bmatrix} -4 & 1 & 0 \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$A$                        $x$                        $b$

USEFUL OBSERVATION:

(IN THIS EX)

$$\underline{Ax} = x \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So  $\underline{Ax} = \underline{b}$  IS CONSISTENT

IFF  $\underline{b}$  IS A LINEAR COMBO  
OF COLUMNS OF  $A$ .



ADD

PAGE 3

PROPERTIES OF MATRIX & MULT.

---

ADD

✓ a)  $A + B = B + A$

✓ b)  $A + (B + C) = (A + B) + C$

c)  $A + O = A$       where  $O$  zero mx,  $n$   
 $(A + -A = O)$

MULT.

a)  $A(BC) = (AB)C$

b)  $(A+B)C = AC + BC$

c)  $C(A+B) = CA + CB$

SCALAR MULT.

a)  $r(sA) = (rs)A$

b)  $(r+s)A = rA + sA$

c)  $r(A+B) = rA + rB$

d)  $A(rB) = rAB = (rA)B$

TRANSPOSES

$$a) (A^T)^T = A$$

$$b) (A+B)^T = A^T + B^T$$

$$c) (AB)^T = B^T A^T$$

$$d) (rA)^T = rA^T$$

$$\rightarrow \underline{EX.} \quad A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 5 \\ 7 & -3 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix} = (AB)^T$$

## WARNINGS

$$AB = 0$$

DOES NOT MEAN ANY OF  
THE MATRICES IS ZERO.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

ALSO

$AB = AC$  DOESN'T MEAN  $B = C$ .

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$$



SOME SPECIAL MATRICES.

$n \times n$   $A = [a_{ij}]$  IS DIAGONAL IF  $a_{ij} = 0$  WHEN  $i \neq j$ .

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  DIAGONAL.

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$  NOT DIAGONAL.

$A$  IS SCALAR IF DIAGONAL AND ALL DIAGONAL ENTRIES EQUAL.

Ex.  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$n \times n$   $I_n = [\delta_{ij}]$   $\delta_{ij} = 1$  IF  $i = j$   
(ALSO JUST  $I$ )  $\delta_{ij} = 0$  IF  $i \neq j$

IS  $n \times n$  IDENTITY MATRIX.

IF  $A$  IS  $m \times n$

$$\underline{I}_m A = A$$

$$A \underline{I}_n = A.$$

Powers.  $A$   $n \times n$ .

$$A^p = \underbrace{A \cdot A \cdots A}_{p \text{ TIMES}}$$

$$A^0 = \underline{I}_n.$$

$$A^p A^q = A^{p+q} \text{ AND } (A^p)^q = A^{pq}$$

WARNING  $(AB)^p = A^p B^p$  ONLY WHEN  $AB = BA$

UPPER TRIANGULAR

$$\text{Ex. 6.} \begin{bmatrix} 1 & 5 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = [a_{ij}]$$

upper triang.

$$\text{IF } a_{ij} = 0 \text{ IF } i > j,$$

LOWER TRIANGULAR

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 7 & 2 & 1 \end{bmatrix}$$

$$a_{ij} = 0 \text{ IF } i < j.$$

A is SYMMETRIC IF  $A = A^T$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \checkmark$$

(show symmetric IF  $A^T = -A$ .)



[NOTE TO SELF,  
SKIP OVER PARTITIONED MATRICES.]

---

NONSINGULARITY,

WE SAW ABOVE THAT  $AB = AC$

DOES NOT MEAN THAT  $B = C$ .

---

IF WE HAVE NUMBERS  $a \neq 0$  AND  $b \neq c$  AND  $c \neq 0$

THEN  $ab = ac$  IMPLIES

THAT  $b = c$ .

$$\text{OR } \frac{1}{a}(ab) = \frac{1}{a}(ac)$$

$$\Rightarrow b = c,$$

BUT WE CAN'T DIVIDE BY A MATRIX!

SO WE CAN'T DO

$$\frac{1}{A} AB = \frac{1}{A} AC.$$

BUT SOMETIMES, WE SORTA CAN!

WATCH!

CONSIDER  $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ .

AND  $B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$

$$BA = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 + 3 & -3 + 3 \\ 2 - 2 & 3 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

SO IF WE HAD  $C, D$  WITH

$$AC = AD, \text{ THEN } BAC = BAD \Rightarrow C = D, \quad \Rightarrow I_2 C = I_2 D$$

WE CAN  $B$  AN INVERSE OF  $A$ .

AND WRITE  $A^{-1} = B$ .

YOU CAN CHECK THAT  $AA^{-1} = I_2$ .

---

DEF.  $A$   $n \times n$  IS INVERTIBLE (NONSINGULAR)

IF THERE IS AN  $n \times n$  MATRIX  $B$

$$\text{s.t. } AB = BA = I_n.$$

$B$  IS CALLED AN INVERSE OF  $A$

AND IS WRITTEN  $A^{-1}$ .

(IT IS UNIQUE)

NOT ALL MATRICES ARE INVERTIBLE  
 BUT "MOST" ARE,

FACTS, (SUB BOOK FOR MOST)

- IF  $A$  AND  $B$  ARE NONSINGULAR,  
 THEN  $(AB)^{-1} = B^{-1}A^{-1}$ .

PROOF:

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= AB\cancel{B}^{-1}A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I. \end{aligned}$$

AND

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}A^{-1}AB \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I. \end{aligned}$$

□

Also,  $(A^{-1})^{-1} = A$ .

$$(A^{-1})^T = (A^T)^{-1}$$

CHECK OUT

THIS IDEA HAS SURPRISING POWER.

BACK TO LINEAR SYSTEMS:

SUPPOSE WE HAVE SYSTEM

$$A \underline{x} = \underline{b}$$

IF  $A$  IS INVERTIBLE:

$$A^{-1}A \underline{x} = A^{-1}\underline{b}$$

$$I \underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1}\underline{b}$$

IS A FORMULA FOR A SOLUTION.

NOTE THAT  
THIS GIVES  
A UNIQUE  
SOLN SO WILL  
ONLY WORK  
IN THAT  
CASE!

$$\underline{\text{Ex}} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\text{If } \underline{b} = \begin{bmatrix} 8 \\ 6 \end{bmatrix},$$

then  $A\underline{x} = \underline{b}$  HAS UNIQUE

SOLUTION

$$\underline{x} = A^{-1}\underline{b}$$

$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

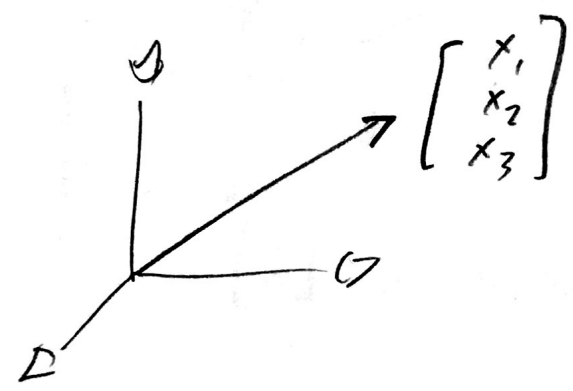
$$= \begin{bmatrix} -10 \\ 9 \end{bmatrix}.$$



RETURN TO THE IDEA OF TRANSFORMATIONS.

LET  $\mathbb{R}^n$  BE THE COLLECTION OF  
ALL  $n$ -VECTORS  $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  WRITTEN AS COLUMNS.

AGAIN, THINK GEOMETRICALLY



VECTOR IS AN ARROW.

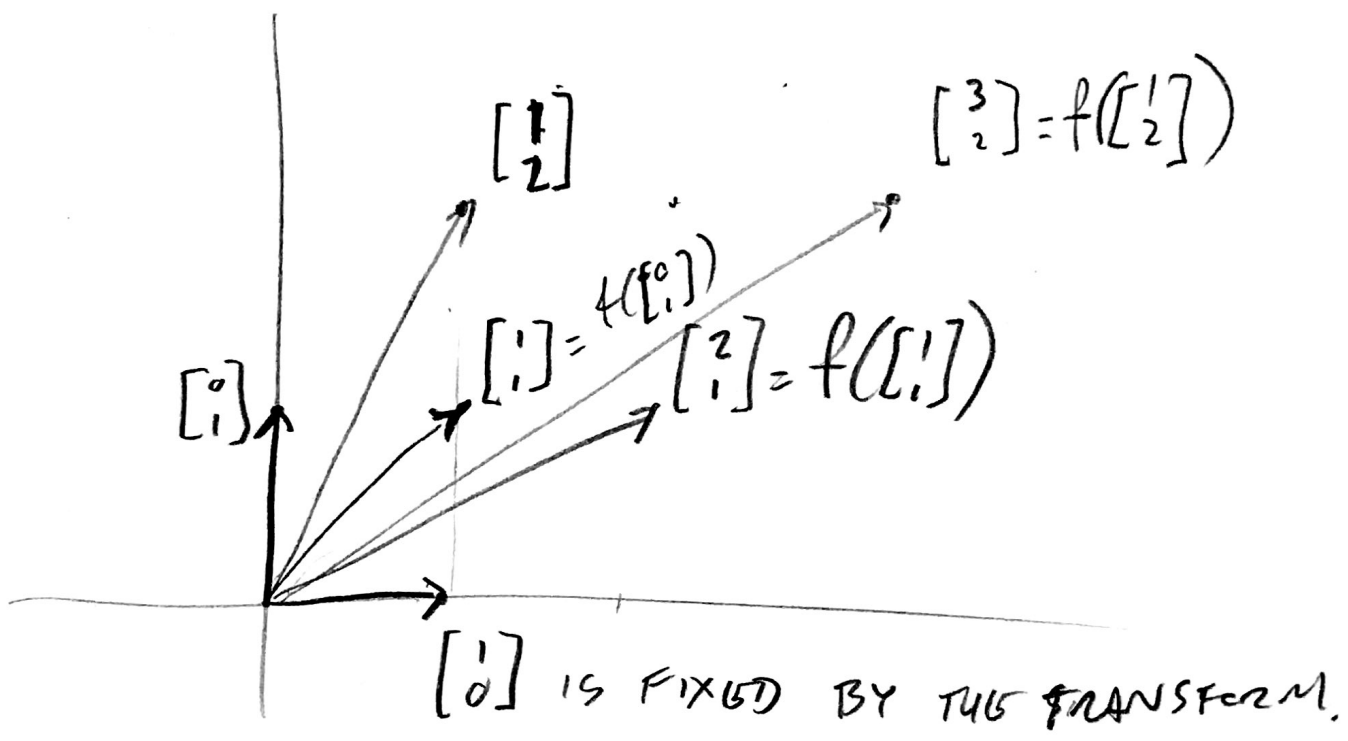
AN  $n \times n$  MATRIX  $A$  WILL DEFINE A

TRANSFORMATION  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$

+ DEFINED BY  $f(\underline{v}) = A\underline{v}$ .

THIS IS A FUNCTION WHOSE INPUT IS A VECTOR  $\underline{v}$  AND WHOSE OUTPUT IS A VECTOR  $A\underline{v}$ .

B.G. 
$$\begin{matrix} & A & \underline{v} & f(\underline{v}) \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} x+y \\ y \end{bmatrix} \end{matrix}$$



MORE GENERALLY, YOU CAN HAVE <sup>LINER</sup> TRANSFORMATIONS

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

GIVEN BY AN  $m \times n$  MATRIX  $A$ .

$$\begin{matrix} f(\underline{v}) = A \cdot \underline{v} \rightarrow m\text{-VECTOR,} \\ \downarrow \\ n\text{-VECTOR} \end{matrix}$$



EX. SCALAR MATRIX LIKE

DILATION  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

JUST STRETCHES EVERY VECTOR (EXCEPT 0).

OR LIKE

CONTRACTION  $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

SHRINKS EVERYTHING.

EX THE ZERO MATRIX JUST ZAPS EVERYTHING TO 0.

EX  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  IS A REFLECTION.

EX AND AS MENTIONED

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

ROTATION.  
SEE PAGE 61  
FOR DERIVATION.

QX TO REFLECT IN A LINE

IF YOU WANT TO REFLECT IN A LINE L THAT ISN'T THE X-AXIS:

① FIRST ROTATE L TO X-AXIS

USING SOME ROTATION MATRIX

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

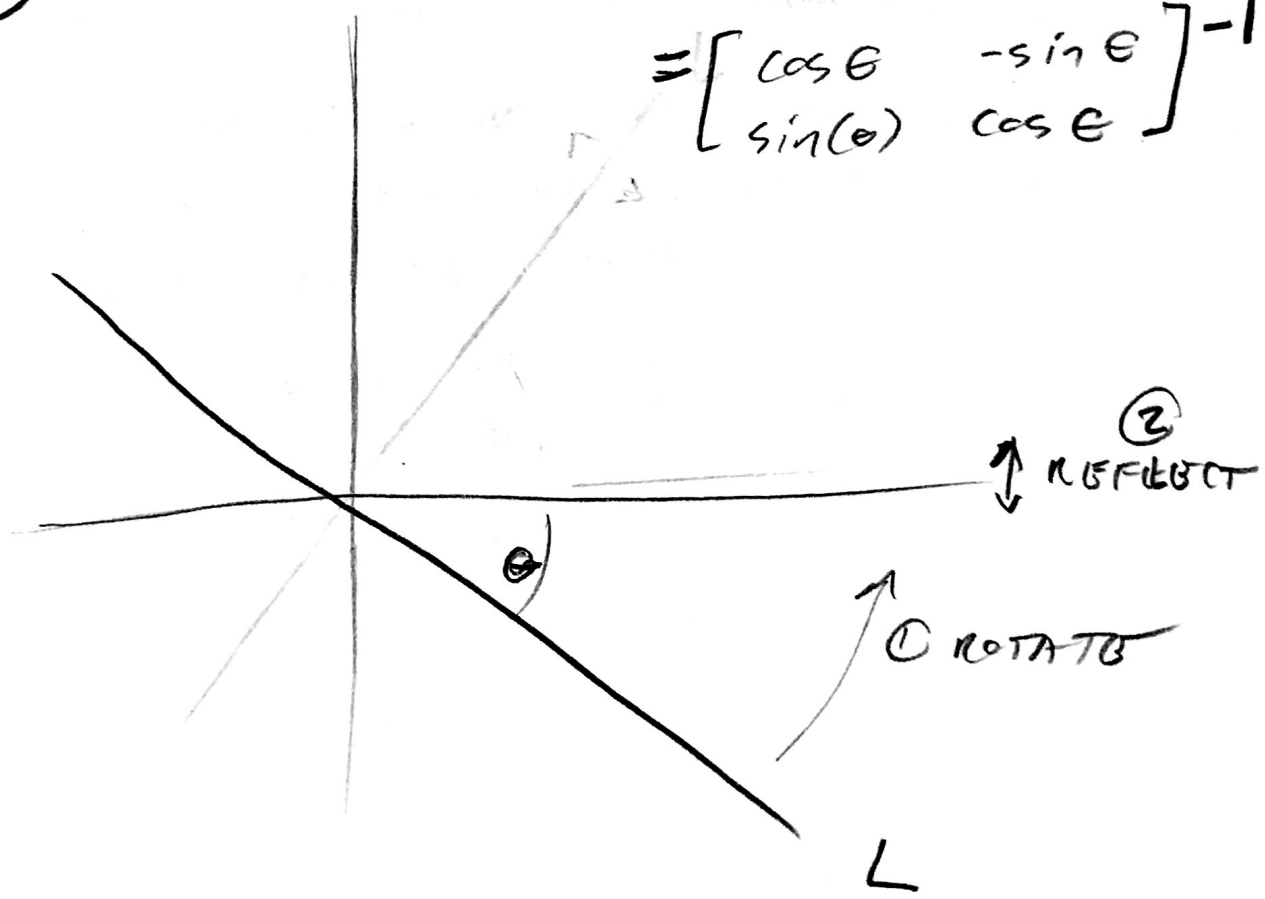
② REFLECT IN X-AXIS USING

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

③ ROTATE BACK W/

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1}$$



So REFLECTION IN L IS TRANSFORM  $R_\theta^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_\theta$ .

CHAPTER 2. SOLVING LINEAR SYSTEMS

2.1 Gaussian form of a matrix.

IN THIS SECTION WE STUDY MOVES THAT PUT A MATRIX IN A SPECIAL FORM. THIS PROCEDURE WILL STREAMLINE THE PROCESS OF ELIMINATION WE SAW BEFORE.

---

AT BEGINNING OF SEMESTER, WE USED ELIMINATION TO CHANGE A LINEAR SYSTEM INTO A SYSTEM OF THE FORM

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3 \end{aligned}$$

THIS IS EASY TO SOLVE, (EACH EQ. HAS 1 VARIABLE WITH COEFFICIENT 1, THE COEFFICIENT MATRIX LOOKS NICE;

$$\begin{bmatrix} \textcircled{1} & 2 & 3 \\ 0 & \textcircled{1} & 2 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

THIS MATRIX IS IN <sup>ROW</sup> REFORM <sub>FORM</sub>

Def An  $m \times n$  MATRIX IS IN ROW REFORM IF

- ALL ZERO ROWS (IF ANY), AT BOTTOM.
- IF A ROW IS NONZERO, ITS FIRST NONZERO ENTRY IS 1. (THE "LEADING 1")
- THE LEADING 1 OF A NONZERO ROW IS TO THE RIGHT OF ANY LEADING 1 IN THE PREVIOUS ROW.

IF A MATRIX SATISFIES a) - c) AND

d) THE LEADING ENTRY OF A NONZERO ROW IS THE ONLY NONZERO ENTRY IN ITS COLUMN,  
 THEN MATRIX IS IN REDUCED ROW REFORM

FORM,

Ex.

$$B = \begin{bmatrix} 1 & 0 & 2 & 5 & 6 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

↓  
row operation

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REDUCED  
row operation.

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

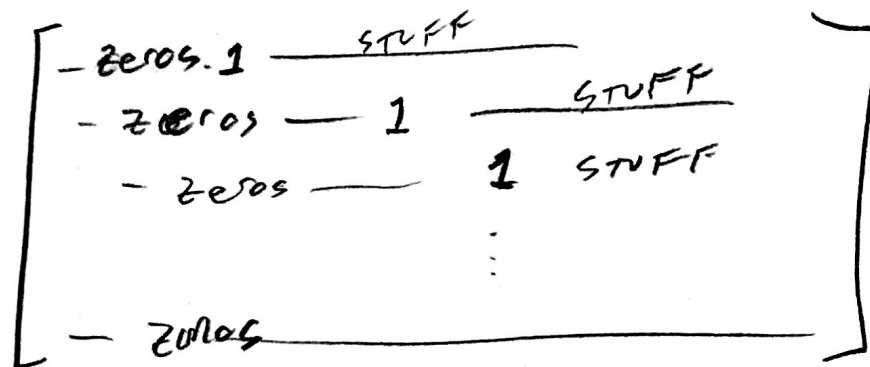
NOTHING

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

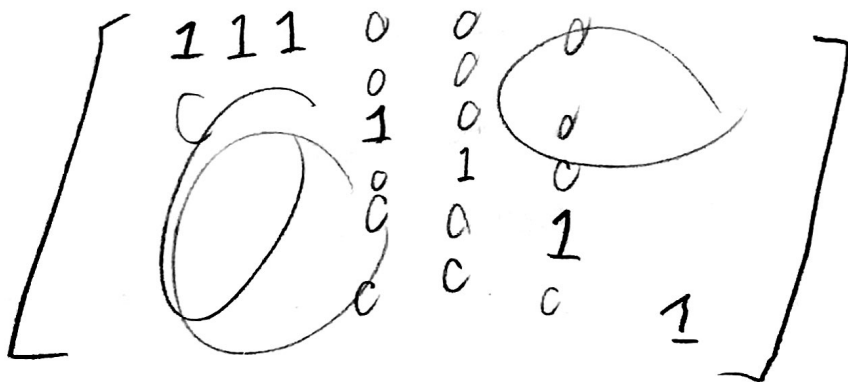
NOTHING.

bc TO ASCII:

BC4620:



REDUCED BC4620



OPERATIONS.ELEMENTARY ROW OPERATIONS:

TYPE I. INTERCHANGES TWO ROWS

TYPE II. MULTIPLY A ROW BY A NONZERO NUMBER

TYPE III. ADD A MULTIPLE OF ONE ROW TO ANOTHER.

(ELEMENTARY COLUMN OPERATIONS ARE  
SAME BUT W/ COLUMNS.)

NOTE: IF WE PERFORM THESE OPERATIONS  
ON THE AUGMENTED MATRIX OF A LINEAR  
SYSTEM, WE GET THE AUGMENTED MATRIX  
OF AN EQUIV. SYSTEM.

NOTATION TYPE I: INTERCHANGES ROWS (COLS)  
 $i$  AND  $j$

$$r_i \leftrightarrow r_j$$

$$(c_i \leftrightarrow c_j)$$

II, REPLACES row  $i$  by  $k$  TIMES  
row  $i$

$$\begin{array}{ccc} k r_i & \rightarrow & r_i \\ \text{NEW} & & \text{OLD} \\ \text{row} & & \text{row} \end{array}$$

$$\left( \begin{array}{l} k c_i \rightarrow c_i \\ \text{for cols} \end{array} \right)$$

III REPLACES row  $j$  WITH

$k$  TIMES row  $i$  + row  $j$

$$\begin{array}{ccc} k r_i + r_j & \rightarrow & r_j \\ \text{NEW} & & \text{OLD} \\ \text{row} & & \text{row} \end{array}$$

WE USE THESE AS SUBSCRIPTS

SO GIVEN  $A$ ,  $A_{r_1 \leftrightarrow r_3}$

MEANS THE MATRIX AFTER  
INTERCHANGING ROWS 1 AND 3,



$$\text{Ex. } A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}$$

$$B = A_{r_1 \leftrightarrow r_3} = \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$C = A_{\frac{1}{3}r_3 \rightarrow r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

$$D = A_{-2r_2 + r_3 \rightarrow r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix} \rightarrow \text{row 2 starts}$$

WE SAY  $B$  IS row EQUIVALENT TO  $A$   
 IF  $B$  CAN BE PRODUCED BY APPLYING  
 A FTS SEQUENCE OF ROW OPERATIONS.

$$\underline{\text{Ex}} \quad A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -2 & 2 & 3 \end{bmatrix}$$

$$B = A \begin{matrix} 2r_3 + r_2 \rightarrow r_2 \\ \end{matrix} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & -3 & 7 & 8 \\ 1 & -2 & 2 & 3 \end{bmatrix}$$

$$C = B \begin{matrix} r_2 \leftrightarrow r_3 \\ \end{matrix} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & -2 & 2 & 3 \\ 4 & -3 & 7 & 8 \end{bmatrix}$$

$$D = C \begin{matrix} 2r_1 \rightarrow r_1 \\ \end{matrix} = \begin{bmatrix} 2 & 4 & 8 & 6 \\ 1 & -2 & 2 & 3 \\ 4 & -3 & 7 & 8 \end{bmatrix}$$

$$\text{Say } D = A \begin{matrix} 2r_3 + r_2 \rightarrow r_2 \\ r_2 \leftrightarrow r_3 \\ 2r_1 \rightarrow r_1 \end{matrix}$$

Ex. 1.2.2. A row eq. A.

IF A row eq. B THEN B row equivalent to A.

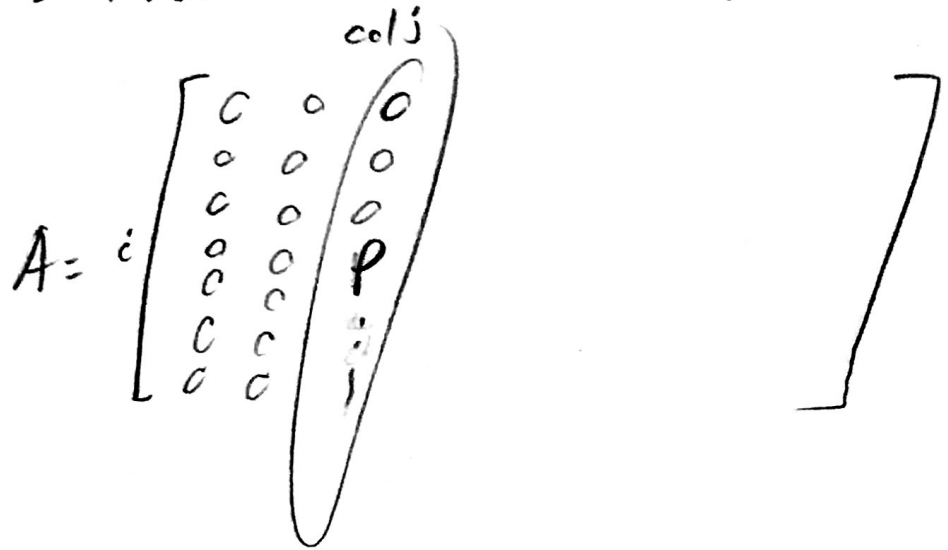
IF A row eq. to B row eq. C, THEN A row eq. C.

Thm, EVERY MATRIX IS EQUIVALENT TO A MATRIX IN ROW REDUCED FORM.

Proof

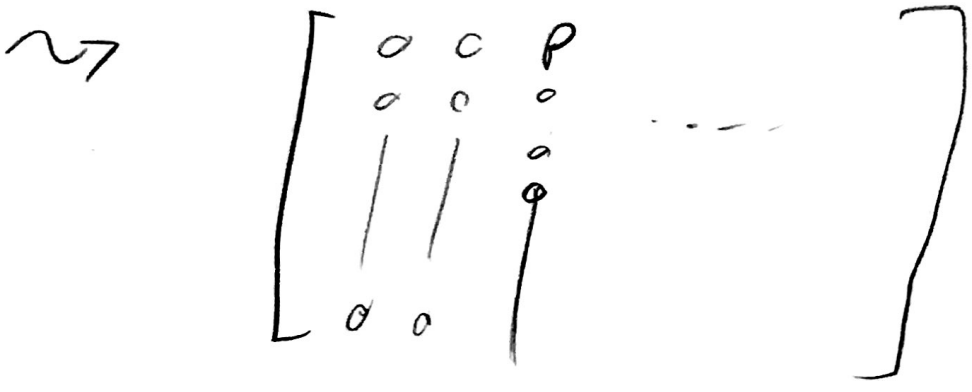
EXAMPLE FIRST.

① FIND FIRST NON ZERO COLUMN, call it col j

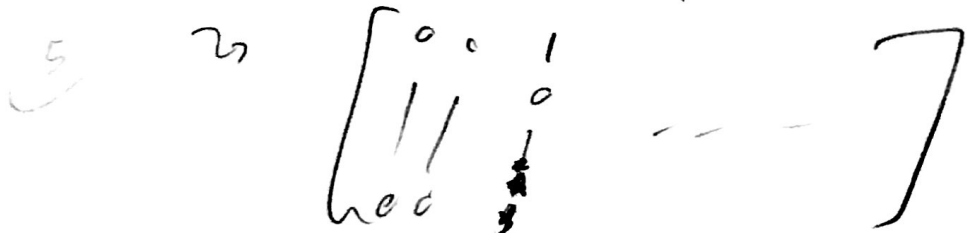


② CALL THIS COLUMN'S FIRST non zero entry P. (THIS "PIVOT") IT IS IN SOME row, i SAY.

③ SWAP row 1 AND i.

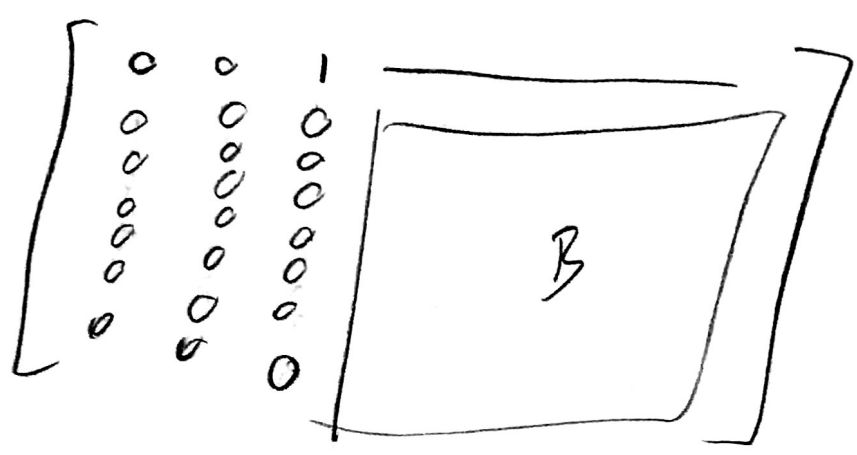


④ DIVIDE row 1 by 1/P.



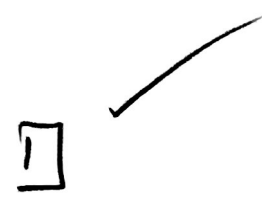
⑤ IF THERE IS A NONZERO ENTRY IN COL  $j$   
 BELOW row 1, WE CAN CLEAR IT TO  
 ZERO BY SUBTRACTING A MULTIPLE  
 OF row 1.

RESULT:



⑥ REPEAT WITH B.

THIS ENDS QUANTITATIVELY.



EX.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\leadsto \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\leadsto \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \checkmark$$

BACK TO PROOF

THEN

THEM ~~THE~~ ADDITION ROW EQUIV. TO A MATRIX IN REDUCED REFORM FORM.

PROOF. FIRST USE ROWS FROM PREVIOUS THM AND THEN CLEAR ABOUT PIVOTS.

IN EX.

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \square$$

## 2.2 LINEAR SYSTEMS.

THM  $A\underline{x} = \underline{b}$   $C\underline{x} = \underline{d}$

LINEAR SYSTEMS.

THESE ARE EQUIVALENT IF THE  
AUGMENTED MATRICES ARE

ROW EQUIVALENT,

PROPS. CORR. MOVES ON MATRICES  
CORRESPOND TO MOVES ON SYSTEMS.

ALGORITHM TO SOLVE  $A\underline{x} = \underline{b}$ .

PUT THE AUGMENTED MATRIX INTO

ROW OR COL. RED. (Gauss form  
(GAUSS CRIM.) (GAUSS-JORDAN CRIM.)

AND THEN SOLVE USING BACK-SUBST.

FOR NEW SYSTEM.

$$\begin{array}{l} \text{6x} \\ x + 2y + 3z = 9 \\ 2x - y + z = 8 \\ 3x \quad \quad - z = 3 \end{array}$$

$$[A | \underline{b}] = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\begin{array}{l} \sim \\ \sim \end{array} \quad \begin{array}{l} z = 3 \\ x + 2y + 3z = 9 \\ 0 \quad y + z = 2 \end{array}$$

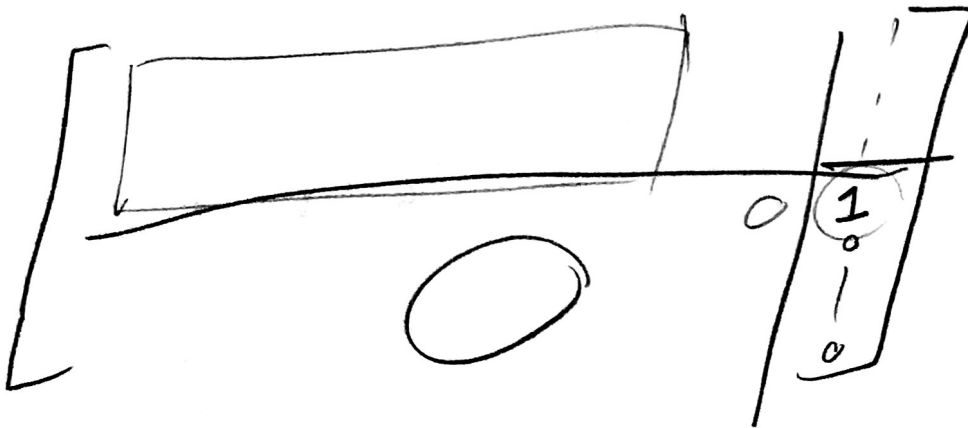
$$\begin{array}{l} \sim \\ \sim \\ \sim \end{array} \quad \begin{array}{l} z = 3 \\ y = 1 \\ x = 2 \end{array}$$

UNIQUE SOLUTION.

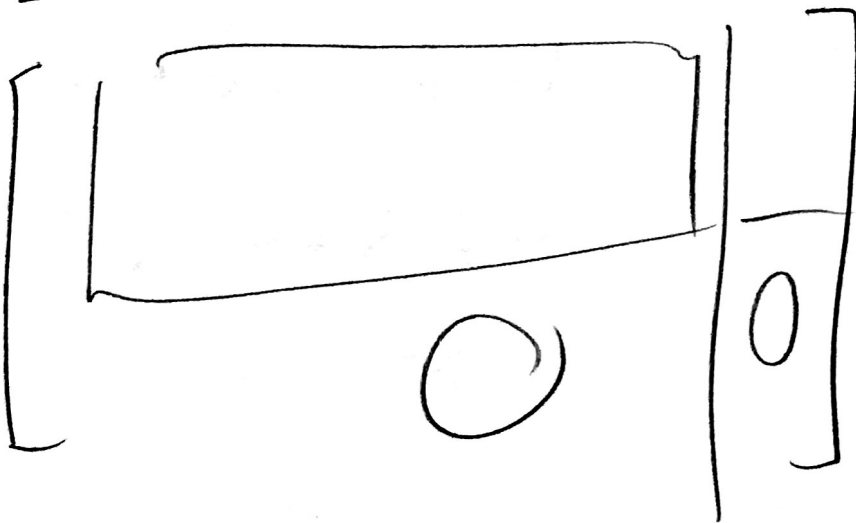


WHEN THERE ARE NO SOLUTIONS,

YOU GET A MATRIX LIKE



INFINITELY MANY SOLUTIONS:



BACK SUBSTITUTE AND SOME VARIABLES  
 COULD BE ANY THING.

GG,

x ANY TIME.

$$\left[ \begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

$$w \quad x \quad y \quad z$$

$$\begin{aligned} z &= 1 \\ y &= 1 \end{aligned}$$

$$w + x + z = 1$$

$$\Rightarrow w + x = 0$$

$$w = -x$$

---

## HOMOGENEOUS SYSTEMS

$$A\underline{x} = \underline{0}$$

THEM IF THERE ARE MORE EQUATIONS  
THAN EQS, THERE IS A NONTRIVIAL SOL<sup>n</sup>.

PROOF. NO PIVOT ELEMENTS  
i.e. THERE IS A COLUMN w/ NO PIVOT.  
THAT VARIABLE IS UNDETERMINED.  $\square$

HOMOGE. AND NONHOMOGE.

$$A\underline{x} = \underline{b} \neq \underline{0} \rightsquigarrow \text{A HOMOGE. SYSTEM. } A\underline{x} = \underline{0}$$

A IF  $\underline{x}_p$  IS A SOL<sup>n</sup> TO  $A\underline{x} = \underline{b}$

AND  $\underline{x}_h$  IS A SOL<sup>n</sup> TO  $A\underline{x} = \underline{0}$

THEN  $\underline{x}_h + \underline{x}_p$  IS A SOL<sup>n</sup> TO  $A\underline{x} = \underline{b}$ .

$$A(\underline{x}_h + \underline{x}_p) = A\underline{x}_h + A\underline{x}_p = \underline{0} + \underline{b}$$

ALLSOLNS OF  $AX = b$  ARE LIKE THIS.

PROOF ; EXERCISE 29. IN 2.2.

## 2.3 ELEMENTARY MATRICES.

FINDING THE INVERSE OF A MATRIX.

DEF. AN  $n \times n$  ELEMENTARY MATRIX OF TYPE I, II, III IS A MATRIX OBTAINED FROM  $I_n$  BY AN ELEM. MOVE OF TYPE I, II, III.

EX  $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ c & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$   $\downarrow$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$   $\downarrow$   $\text{--- II}$

$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thm

$A$   $m \times n$   
matrix.

$B$  matrix obtained by an  $m \times m$  row ops.  
from  $I_m$  by performing

$\mathcal{B}$   $m \times m$  " " " " " " " "  
same row ops.

Thm  $B = \mathcal{B}A$ .

Proof: Exercise 1.

Ex.  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathcal{B}A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

BY USING MULT. BY ELEM. MATRICES  
 INSTEAD OF ROWS, WE CAN WRITE  
 OUR ROW REDUCTION USING MX MULT.

Thm A AND B  $m \times n$   $m \times s$ .

A row eq. B IFF THERE  
 ARE ELEMENTARY MATRICES  
 $E_1, \dots, E_k$  s.t.

$$B = E_k E_{k-1} \dots E_2 E_1 A. \quad \checkmark$$

Thm ELEMENTARY MATRICES ARE  
 INVERTIBLE AND INVERSES ARE  
 ELEM. OF SAME TYPE.

Def row. G

Thm  $A$  invertible iff  $A$  is prod. of elementary matrices.

65

Lemma. IF  $A$   $n \times n$  AND  $A\underline{x} = \underline{0}$   
HAS ONLY TRIVIAL SOL<sup>n</sup>, THEN  $A$   
IS ROW EQUIVALENT TO  $\underline{I}$ .

Proof. IF  $B$  IS ROW EQ. TO  $A$   
AND  $B\underline{x} = \underline{0}$  HAS ONLY TRIV. SOL<sup>n</sup>,  
THEN  $B = \underline{I}$ . SINCE IT CAN'T HAVE  
ANY ZERO ROWS.  $\square$

PROOF OF THM.

IF  $A$  IS PROD  $B_1 \dots B_k$   $B_i$  elem,  
THEN  $A^{-1} = B_k^{-1} \cdot B_{k-1}^{-1} \dots B_1^{-1}$ .

IF  $A$  IS INVERTIBLE, THEN

$$A\underline{x} = \underline{0} \Rightarrow A^{-1}(A\underline{x}) = A^{-1}\underline{0} = \underline{0}$$

$$\text{So } \underline{I}_n \underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0}.$$

So  $A\underline{x}$  HAS ONLY TRIV. SOL<sup>n</sup>.

BY Lemma, DONE.  $\square$

Cor.  $A$  INVERTIBLE IFF  
 $A$  ROW EQUIV. TO  $I_n$ .

Thm  $Ax = 0$  ( $A^{n \times n}$ ) IFF  $A$  SINGULAR.

THIS GIVES US AN ALGORITHM TO  
 FIND  $A^{-1}$ .

WRITE  $A = \underbrace{B_k B_{k-1} \dots B_1}_{\text{circled}} A I = I_n$   
 So  $A = B_1^{-1} B_2^{-1} \dots B_k^{-1}$   
 Thus  $A^{-1} = \underbrace{B_k B_{k-1} \dots B_1}_{\text{circled}}$

GAUSS METHOD.

$$B_k \dots B_1 [A \mid I_n] \rightsquigarrow [I \mid A^{-1}]$$

ELEMENTARY  
 MOVES ON LEFT TURNING  $A$  INTO  $I$   
 GIVES  $A^{-1}$  ON RIGHT.



Ex. (sub 7)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array} \right]$$

- Don't mult.
- Do row operations directly.

$\sim$

$-5r_1 + r_3 \rightarrow r_3$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array} \right]$$

$\sim$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array} \right]$$

$\sim$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array} \right]$$

GSX GSX

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\left[ \begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array} \right]$$

субста!

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

thm  $A$  SINGULAR IFF

EQ.  $B$  w/ row of zeroes.

I'VE BEEN SAYING  $B = A^{-1}$  EVEN IF WE ONLY HAVE  
 $AB = I$  OR  $BA = I$ .

thm, IF  $A, B$   $n \times n$  AND

$$AB = I, \text{ THEN } BA = I.$$

PROOF: SEE TEXT.

IDEA. IF  $AB = I$  THEN  $A$  IS NON-SING.

IF NOT,  $A \sim C$  row of zeroes.

$$\text{SO } C = U_1 \dots U_r A$$

row of zeroes.

THEN  $U_1 \dots U_r AB$  HAS

row of zeroes.

SO  $AB$  row EQ. to  $CB$

SO  $AB$  IS SINGULAR, BY COROLL.

BUT  $AB = I$ .  $\times$

SO  $A$  HAS AN INVERSE  $A^{-1}$ .

$$A^{-1}AB = A^{-1} \Rightarrow B = A^{-1}.$$

2.3  
DEF.

$A, B$  EQUIVALENT

(NOT JUST "ROW EQ.")

IF.  $B$  OBTAINED FROM  $A$  BY FITS  
SQ. OF ROWS, ROW OR COL. OPERATIONS.

Thm.  $A$  NZERO.

$A$  EQ TO

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$$

□

Thm  $A$  ROW.  $B$

$$\text{IFF } B = PAQ$$

FOR NONSINGULAR  $P, Q$ .

Thm  $A$  INVERTIBLE IFF  $A$  ROW.  $I$ .

# CHAPTER 3

## DETERMINANTS

CAN ASSOC. #S TO MATRIX,

E.G.

TRACE OF  $n \times n$  MATRIX  $A = [a_{ij}]$

IS

$$\text{TR}(A) = \sum_{i=1}^n a_{ii}$$

ANOTHER VERY IMPORTANT #

IS THE DETERMINANT.

LET  $A$  BE  $n \times n$  MATRIX,

START WITH  $n=2$ .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

LET'S THINK ABOUT  $A$  AS A TRANSFORMATION,

LET'S USE PARTICULAR #s.

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

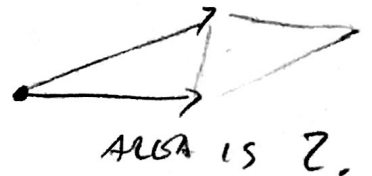
$$B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



$$B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

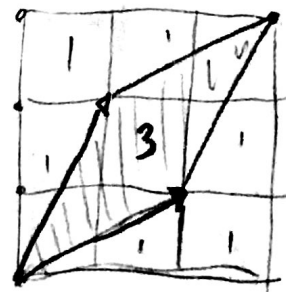
$$C \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



$$C \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$D \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$$D \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$



DEFINING

$$\text{DET}(A) = ad - bc$$

NOTICES!

$$\text{DET}(B) = ?$$

$$\text{DET}(C) = ?$$

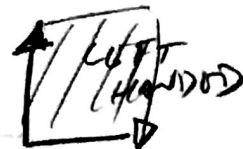
$$\text{DET}(D) = ?$$

Unit square

VS

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{DET } B = -1$$



AREA IS

ONE BUT BLENDED



IF  $\det A = 0$ , PARALLEL

WE WILL DEFINE  $\det(A)$

FOR ARBITRARY  $n \times n$   $A$ .

$|\det(A)|$  IS VOLUME OF PARALLELEPIPED.

NOTICE THAT  $A$  WILL BE INVERTIBLE  
IFF  $\det A \neq 0$ .

JUST STRAIGHTEN OUT THE VECTORS.

TO DEFINE IT WE NEED TO WORK  
PERMUTATIONS.

$$S = \{1, \dots, n\}$$

IN BOOK  
 $i_1, \dots, i_n$

A PERMUTATION IS A REARRANGEMENT  $\checkmark$  OF  
TERMS.

PRECISELY,  $\sigma: S \rightarrow S$ .

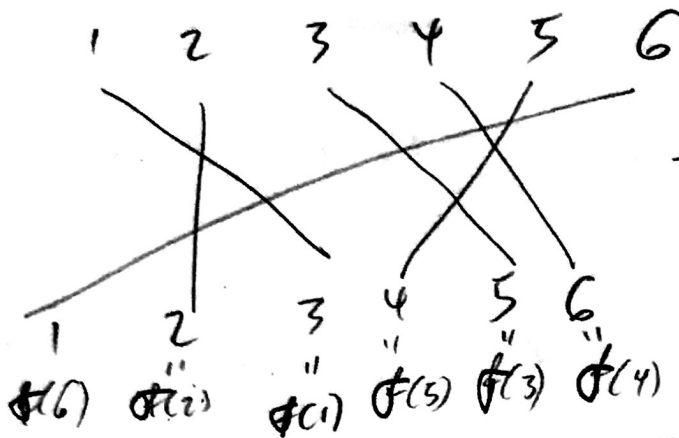
$$i_k = \sigma(k).$$

EX.

$$\begin{aligned} \sigma(1) &= 4 \\ \sigma(2) &= 2 \\ \sigma(3) &= 3 \\ \sigma(4) &= 1. \end{aligned}$$

THINK THIS WAY

3, 2, 5, 6, 4, 1.



→ ANTICLOCKWISE

DRAW  
SO THAT IN  
CORRECT  
POSITION.

How MANY PERMUTATIONS ARE THERE

WHICH  $S = \{1, 2, 3\}$ .

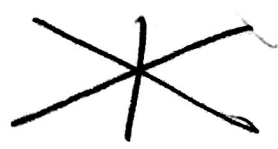
How MANY LISTS OF LENGTH 3

FROM  $S = \{1, 2, 3\}$

3 choices 2 1

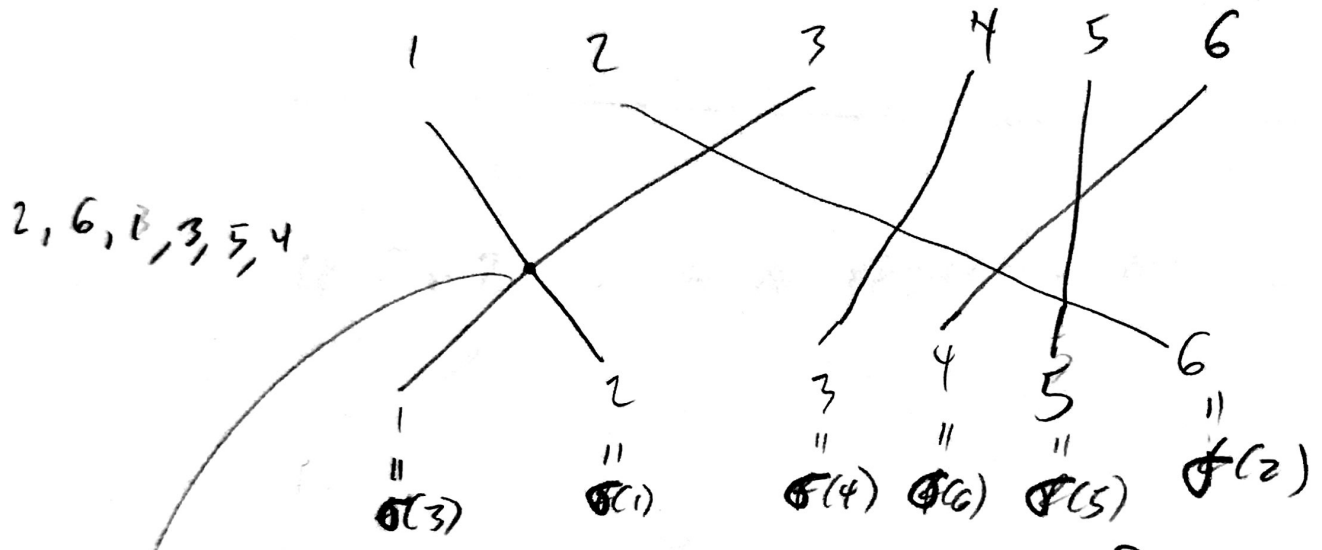


NOT CIRCLES



# TYPES OF PERMUTATIONS.

LOOK AT A BRAD PICTURE.



WHAT DOES A CROSSING MEAN?

IT MEANS THAT A NUMBER SHOWS UP BEFORE A SMALLER ONE.

CAUSED AN INVERSION.

MEANS THAT 2 SHOWS UP BEFORE 1.

THEIR ORDER IS FLIPPED.

# CROSSINGS IS # OF INVERSIONS.

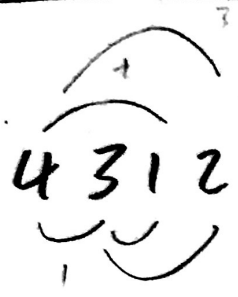
P IS EVEN IF EVEN.  
 A IS ODD IF ODD.

n/2

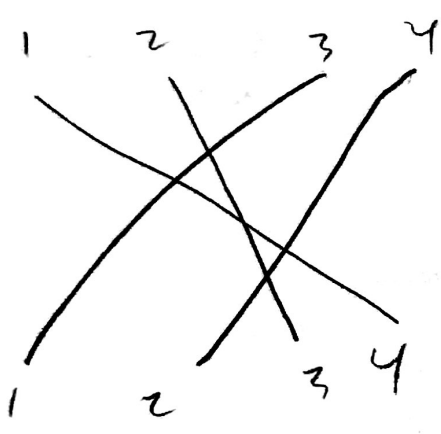
PAIRS ARE

$\frac{n!}{2}$  EVEN PERMS

AND  $\frac{n!}{2}$  ODD PERMS



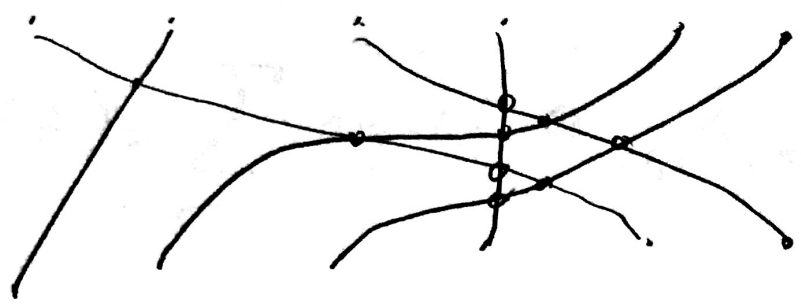
IS A PERMUTATION,



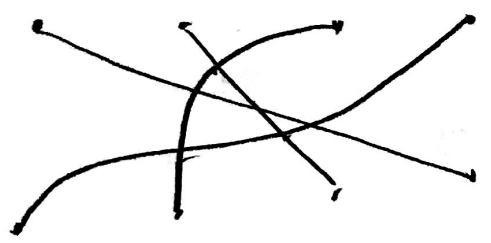
ODD.



9  
ODD



6



$$\text{sign}(\sigma) = 1 \text{ IF } \sigma \text{ EVEN}$$

$$= -1 \text{ IF } \sigma \text{ ODD.}$$

Def.  $A = [a_{ij}]$

$$|A| = \det(A) = \sum_{\sigma} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}.$$

ex  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

(DIFFER  
US SAID  
 $a_{11}a_{22} - a_{12}a_{21}$ )

WHAT ARE THE PERMUTATIONS OF  $\{1, 2\}$ ?

12 AND 21.  
EVEN ODD  
 $\sigma(1)=1 \quad \sigma(2)=2$        $\tau(1)=2 \quad \tau(2)=1$ .

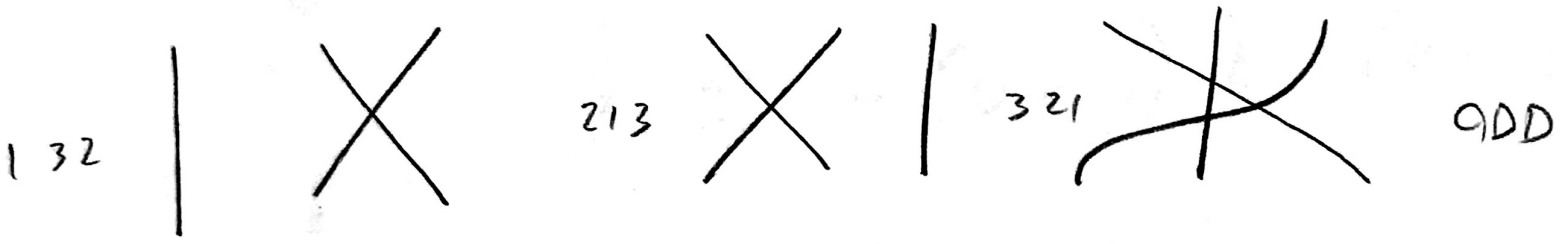
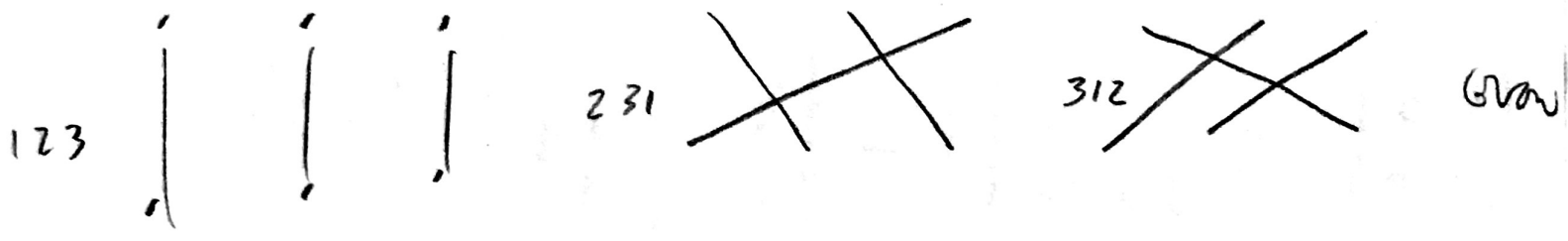
||

X

$$\det(A) = + a_{1\sigma(1)} a_{2\sigma(2)} - a_{1\tau(1)} a_{2\tau(2)}$$

$$= a_{11} a_{22} - a_{12} a_{21}.$$

ex.  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$



$\det(A) = \begin{array}{|c|} \hline a_{11} a_{22} a_{33} \\ \hline \end{array} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}$

$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Qx

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} + 5 \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= (2 - 3) - (0 - 9) + 5(0 - 3)$$

$$= -1 + 9 + 15$$

$$= -7$$



### 3.2 PROPERTIES.

Thm  $\det(A) = \det(A^T)$

of slp.

USEFUL!

OBV. FOR  $2 \times 2$ ,  
COMES BACK LATER.

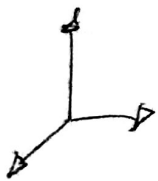
Thm IF WE OBTAIN  $B$  FROM  $A$  BY SWITCHING  
 ROWS, THEN  $\det(B) = -\det(A)$ .

ex.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$      $\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$ .

Geometric IDEA. IF WE SWITCH TWO COLUMNS  
 SIGN CHANGES. (MIRROR IMAGE).

CHANGING TWO ROWS  
 CHANGING TWO COLUMNS OF TRANSPOSE,  
 SO SAME EFFECT.



THINK OF THE EFFECT OF  
 THE ELEMENTARY MATRICES OF TYPE I.  
 E.G.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

# PROOF THAT $\det(A^T) = \det(A)$ .

GIVEN A PERMUTATION  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ .

THERE IS AN INVERSE PERMUTATION  $\sigma^{-1}$  THAT UNDOES  $\sigma$ .

## EXAMPLE.

INVERSE OF 312 IS 231.

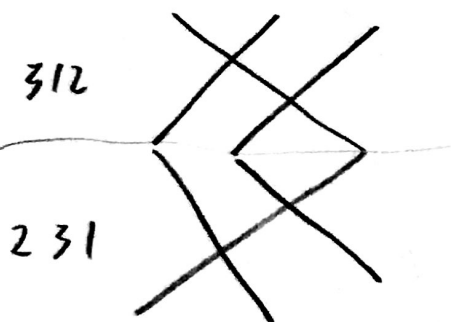
EASIER TO SEE WITH DISCRETS

$$\sigma(1) = 3 \quad \sigma(2) = 1 \quad \sigma(3) = 2$$

$$1 = \sigma^{-1}(3) \quad 2 = \sigma^{-1}(1) \quad 3 = \sigma^{-1}(2)$$

$$\text{So } \sigma^{-1}(1) = 2, \quad \sigma^{-1}(2) = 3, \quad \sigma^{-1}(3) = 1.$$

PICTURE:



NOTICE  $\sigma^{-1}$  HAS SAME SIGN AS  $\sigma$ .

SINCE # OF CROSSINGS SAME

CONSIDER  $B = A^T$ .  $B = [b_{ij}] = [a_{ji}]$

$b_{13} b_{21} b_{32}$  IN DETERM. OF  $\det(A^T)$

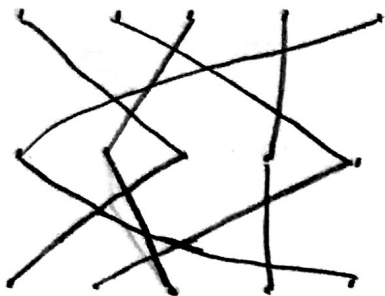
$$= a_{31} a_{12} a_{23}$$

$$= a_{12} a_{23} a_{31}$$

$$= a_{1, \sigma^{-1}(1)} a_{2, \sigma^{-1}(2)} a_{3, \sigma^{-1}(3)}$$

35241

53142

PERMUTATION IS SAME SIGN.

$$[b_{ij}] = B = A^T$$

$$b_{ij} = a_{ji}$$

$$b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)} b_{4\sigma(4)} b_{5\sigma(5)}$$

$$= b_{13} b_{25} b_{32} b_{44} b_{51}$$

$$= a_{31} a_{52} a_{23} a_{44} a_{15}$$

$$= a_{15} a_{23} a_{31} a_{44} a_{52}$$

) NO ORDER

$$= a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} a_{3\sigma^{-1}(3)} a_{4\sigma^{-1}(4)} a_{5\sigma^{-1}(5)}$$

$$B = [b_{ij}] = [a_{ji}] = A^T$$

$$\det(B) = \sum_{\sigma} \text{sign}(\sigma) b_{1\sigma(1)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma^{-1}} \text{sign}(\sigma^{-1}) a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}$$

$$= \sum_{\tau} \text{sign}(\tau) a_{1\tau(1)} \cdots a_{n\tau(n)} = \det(A) \quad \square$$

Thm IF TWO rows (or columns)  
ARE EQUAL, THEN  $\det(A) = 0$ .

Pr, INTERCHANGES rows to GET B.

~~$\det(A)$~~

$$\text{So } \det(B) = -\det(A).$$

BUT  $A = B$ , so

$$\det(A) = -\det(A).$$

$$\text{So } \det(A) = 0.$$

□

Thm IF A HAS A ZERO ROW,

$$\det(A) = 0.$$

Proof SUPPOSES row  $i$  IS ZERO.

EVERY TERM OF THIS SUM DEFINING

$\det(A)$  INVOLVES A TERM FROM row  $i$ :

$$\det(A) = \sum_{\sigma} \text{sign}(\sigma) a_{1\sigma(1)} \cdots \underbrace{a_{i\sigma(i)}}_0 \cdots a_{n\sigma(n)}$$

□

$$\text{ex } \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

THM, IF OBTAIN B FROM A  
BY MULTIPLYING A ROW BY  $k$ , (OR COLUMN)

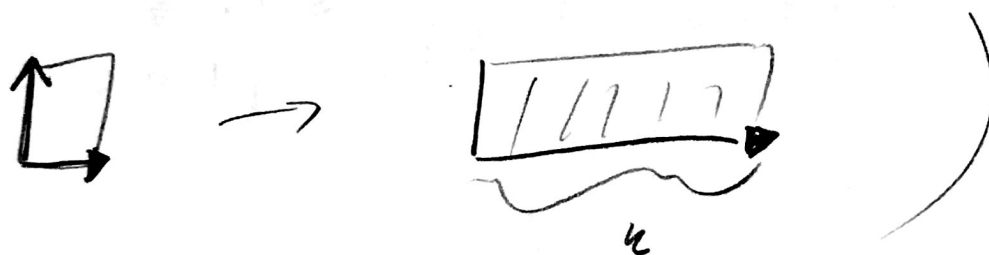
$$\text{THEN } \det(B) = k \det(A).$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \checkmark$$

(THINK COLUMNS. THINK PARALLELOGRAM STRETCHING)

BY  $k$ :



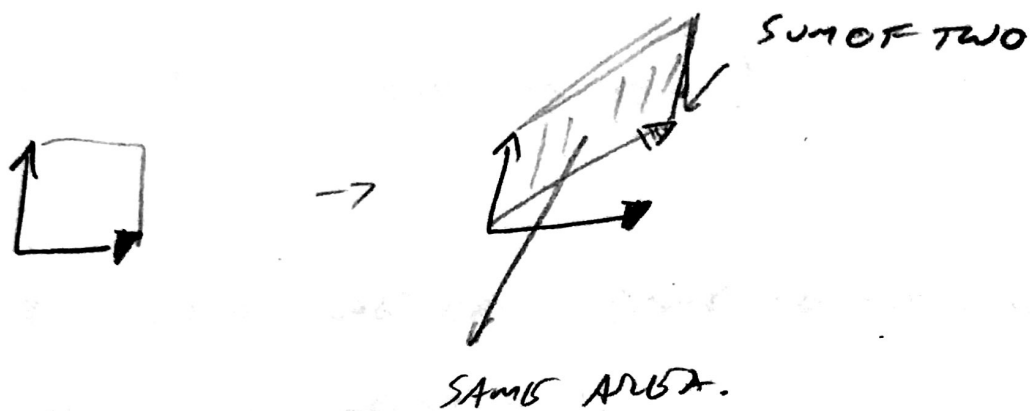
Pf

$$\begin{aligned} & \sum \text{sign}(\sigma) b_{1\sigma(1)} \dots b_{r\sigma(r)} \dots b_{n\sigma(n)} \\ &= \sum \text{sign}(\sigma) a_{1\sigma(1)} \dots (k a_{r\sigma(r)}) \dots a_{n\sigma(n)} \\ &= k \sum \text{sign}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)} \\ &= k \det(A). \end{aligned}$$

□

QUM TYPE III Row operations DON'T

CHANGE DETERMINANT. D



Ex

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

or

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

or

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

Thm If  $A = [a_{ij}]$  is upper (lower)  
TRIANGULAR,

$$\text{THEN } \det A = a_{11} a_{22} \dots a_{nn}$$

(follows from previous thm.)

USING THESE WE CAN COMPUTE DETERMINANT  
USING row operations.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det = 1$$

↑ TYPE I MULT.  
row 1 BY 2.

$$\sim \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

$$\det = \frac{1}{2}$$

↑ TYPE III

$$\sim \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\det = \frac{1}{2}$$

(PRODUCT OF DIAGONALS)

THM IF  $B$  IS ELEMENTARY

$$\det(BA) = \det(B)\det(A).$$

PF. USE THMS ABOUT CHANGE IN DETERMINANT UNDER ROW OPERATIONS.  $\square$

THM  $A$  INVERTIBLE IFF  $\det(A) \neq 0$ .

PF. IF  $A$  IS INVERTIBLE THEN

$$A = B_1 \dots B_n \quad \text{WHERE } B_i \text{ ELEM.}$$

$$\text{SO } \det(A) = \det(B_1) \dots \det(B_n) \neq 0.$$

IF  $A$  SINGULAR,  $A$  ROW EQUIV. TO

$B$  WITH ROW OF ZEROS.

$$A = B_1 \dots B_n B$$

$$\begin{aligned} \text{HENCE } \det(A) &= \det(B_1)\det(B_2) \dots \det(B_n)\det(B) \\ &= 0. \end{aligned}$$

$\square$



Cor. $Ax = 0$  HAS NONZERO SOLUTIONIFF  $A$  SINGULAR.Thm IF  $A$  AND  $B$  ARE  $n \times n$ 

$$\text{then } \det(AB) = \det(A)\det(B).$$

Proof: Suppose  $A$  IS SINGULAR,

THEN RHS IS ZERO.

ALSO  $A$  IS ROW EQUIV. TO  $C$  WITH  
ROW OF ZEROS.

$$C = U_n \dots U_1 A.$$

$$\text{so } CB = U_n \dots U_1 AB.$$

HAS  
ROW OF  
ZEROS  $\Rightarrow AB$  IS SINGULAR.

$$\text{so } \det(AB) = 0.$$

IF  $A$  INVERTIBLE,

$$\text{then } A = U_1 \dots U_n$$

$$\text{so } \det(AB) = \det(U_1 \dots U_n) \det(B) \checkmark.$$

Cor.  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

Proof:

$$1 = \det(I) \det(AA^{-1}) = \det(A) \det(A^{-1}).$$

□

### 3.3 COFACTOR EXPANSION.

DEF. LET  $A = [a_{ij}]$ .

LET  $M_{ij}$  BE THIS  $(n-1) \times (n-1)$  SUBMATRIX

OBTAINED BY DELETING THE  $i$ TH ROW AND THE  $j$ TH COLUMN.

$\det(M_{ij})$  IS THE MINOR OF  $a_{ij}$ .

THIS COFACTOR  $A_{ij}$  OF  $a_{ij}$  IS

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

EX

$$A = \begin{bmatrix} 5 & 7 & 1 \\ 2 & 6 & 4 \\ 2 & -1 & 0 \end{bmatrix} \quad M_{12} = \begin{bmatrix} 2 & 4 \\ 2 & 0 \end{bmatrix}$$

$$\det(M_{12}) = -8$$

$$A_{12} = (-1)(-8)$$

$$M_{22} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\det(M_{22}) = -2$$

$$A_{22} = (+1)(-2) = -2$$

THIS SIGN IN FRONT OF  $\det(M_{ij})$   
 IN COFACTOR CAN BE REMEMBERED  
 BY PUTTING +S AND -S IN A  
 CHESSBOARD PATTERN IN THE MATRIX:

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix}$$

Thm  $A = [a_{ij}]$   $n \times n$  mx.

"EXPANSION ALONG  
 $i$ th row"

Thm  $\det(A) = a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}$   
 (just like  $3 \times 3$  case!)

AND

$$\det(A) = a_{1j} A_{1j} + \dots + a_{nj} A_{nj}$$

"EXPANSION ALONG  $j$ th column."

SHOULD PICK ROW OR COLUMN WITH  
LOTS OF ZEROS.

□

Ex.

$$\begin{vmatrix} 2 & 5 & 7 & 1 \\ 6 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 10 & 1 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$= 0 \begin{vmatrix} 5 & 7 & 1 \\ 3 & 0 & 0 \\ 0 & 10 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 7 & 1 \\ 6 & 0 & 0 \\ 0 & 10 & 1 \end{vmatrix}$$

$$+ 2 \begin{vmatrix} 2 & 5 & 1 \\ 6 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 5 & 7 \\ 6 & 3 & 0 \\ 0 & 0 & 10 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & 5 & 1 \\ 6 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$= 2 \cdot (6 - 30)$   
 $= 48$

$$= 2 \left( 0 \cdot \begin{vmatrix} 5 & 1 \\ 3 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 1 \\ 6 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} \right)$$

CAN USE ROW OPERATIONS  
TO INTRODUCE ZEROS:

$$\begin{vmatrix} 1 & 2 & 5 & 6 \\ 7 & 1 & 1 & 1 \\ 22 & 1 & 1 & 1 \\ -1 & 5 & 5 & 6 \end{vmatrix}$$

SUBTRACT ROW  
2 FROM ROW 3,

$$\begin{matrix} -r_2 + r_3 \rightarrow r_3 \\ = \end{matrix} \begin{vmatrix} 1 & 2 & 5 & 6 \\ 7 & 1 & 1 & 1 \\ 15 & 0 & 0 & 0 \\ -1 & 5 & 5 & 6 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$= +15 \begin{vmatrix} 2 & 5 & 6 \\ 1 & 1 & 1 \\ 5 & 5 & 6 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{matrix} -r_1 + r_3 \rightarrow r_3 \\ = \end{matrix} 15 \begin{vmatrix} 2 & 5 & 6 \\ 1 & 1 & 1 \\ 3 & 0 & 0 \end{vmatrix}$$

$$= 15 \left( 3 \begin{vmatrix} 5 & 6 \\ 1 & 1 \end{vmatrix} \right)$$

$$= 45 \begin{vmatrix} 5 & 6 \\ 1 & 1 \end{vmatrix} =$$



WHY IS DETERMINANT THE SIGNED AREA FOR  $2 \times 2$  MATRICES?

BOOK USES COFACTOR EXPANSION TO SHOW THIS.

THINK GEOMETRICALLY:

SUPPOSE  $A$  IS A  $2 \times 2$  MATRIX.

LET  $R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  BE A ROTATION

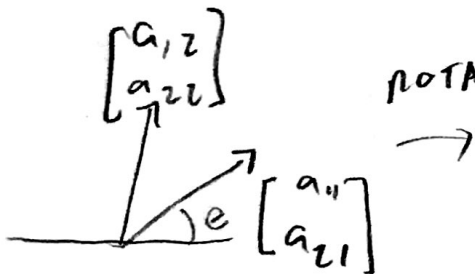
MATRIX. NOTE  $\det R_\theta = 1$  BY PYTHAGOREAN THM,

$$\text{SO } \det(R_\theta A) = \det(R_\theta) \det(A) = 1 \cdot \det(A) = \det(A).$$

$$\text{FOR SOME } \theta, R_\theta A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

THINK:

$$\text{IF } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$



GET TWO NOW VECTORS  
AND 1ST ONE

HAS 0 SECOND COORD.



NOTATION ALSO DOESN'T CHANGE AREA.

SO WE CAN PROVE THM FOR MATRICES

LIKE

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

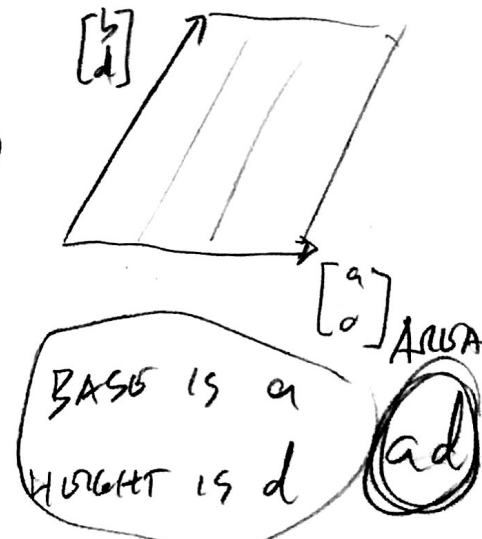
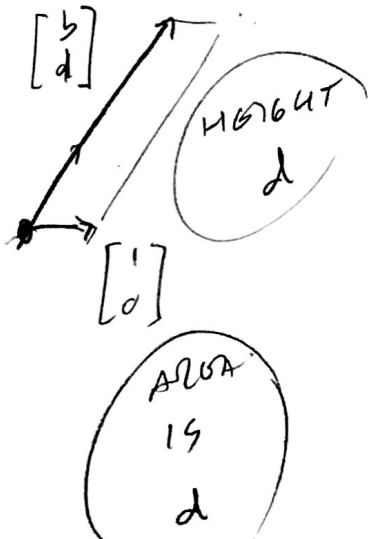
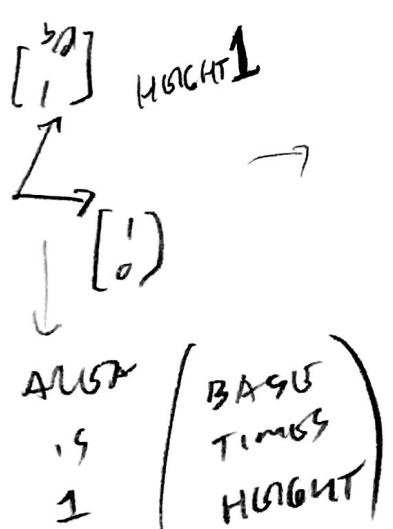
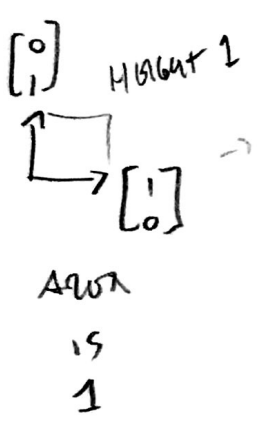
ASSUMES  $\det(A) = ad - 0$   
 $= ad \neq 0$

THIS MATRIX IS COLUMN EQUIV. TO I :

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \xrightarrow{\frac{1}{a} \cdot C_1} \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} \xrightarrow{\frac{1}{d} \cdot C_2} \begin{bmatrix} 1 & b/d \\ 0 & 1 \end{bmatrix} \xrightarrow{\begin{matrix} C_2 - \frac{b}{d} C_1 \\ C_2 - \frac{b}{d} C_1 \end{matrix}} I$$

REVERSING THIS:

$$I \rightarrow \begin{bmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$





### 3.4. INVERSES AGAIN

DEF<sup>n</sup>  $A = [a_{ij}] \quad n \times n.$

$A_{ij}$  is the  $i_j$ <sup>th</sup> cofactor.

then the ADJOINT of  $A$  is

$$\text{adj } A = [A_{ij}]^T = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

EX



$$\underline{\text{Qx}}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A_{11} = + \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3$$

$$A_{12} = - \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = -2$$

$$A_{13} = + \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} = 3$$

$$A_{21} = - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2$$

$$A_{22} = + \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

$$A_{23} = - \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = -2$$

$$A_{31} = + \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$$

$$A_{32} = - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2$$

$$A_{33} = + \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad 100$$

NOTE  $\det A$ :

$$= 1 \cdot \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= 3 - 4 + 3 = \textcircled{2}$$

$$\therefore [A_{ij}] = \begin{bmatrix} 3 & -2 & 3 \\ -2 & 2 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\text{adj} A = [A_{ji}]^T$$

$$= \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$

MAGIC:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \text{det} A \cdot I$$

!!!

thm,

$$A \operatorname{adj} A = (\operatorname{adj} A) A = (\det A) I$$

$$\Rightarrow A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

WHY DOES THIS WORK??

$$A \operatorname{adj} A$$

$$= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & \dots & A_{j1} & \dots & A_{n1} \\ \vdots & & \vdots & & \vdots \\ A_{1n} & \dots & A_{jn} & \dots & A_{nn} \end{bmatrix}$$

$i$ 'th entry is

$$a_{i1} A_{j1} + a_{i2} A_{j2} + \dots + a_{in} A_{jn}.$$

if  $i=j$  this is  $\det A$ .

IF NOT,

THEN

WHAT IS

$$a_{i_1} A_{j_1} + \dots + a_{i_n} A_{j_n} ?$$

CLAIM THIS IS 0.

WHY?

Form  $B$  BY REPLACING

new  $j$  WITH NEW  $i$ ,

$$\det B = 0$$

(TWO OF ITS ROWS ARE EQUAL!)

$$\text{BUT } B_{jk} = A_{jk}$$

i.e. COFACTORS OF  $A$  AND  $B$   
ARE THE SAME AT ROW  $j$ .

SO

$$0 = \det B$$

$$= b_{j_1} B_{j_1} + \dots + b_{j_n} B_{j_n}$$

$$= a_{i_1} B_{j_1} + \dots + a_{i_n} B_{j_n}$$

$$= a_{i_1} A_{j_1} + \dots + a_{i_n} A_{j_n}.$$

□

3.5. CRAMER'S RULES.

$\det A \neq 0$ .

USE ADJOINT TO FIND  $A^{-1}$ .

THEN ! SOLN TO  $A\underline{x} = \underline{b}$

IS  $A^{-1}\underline{b}$ .

---

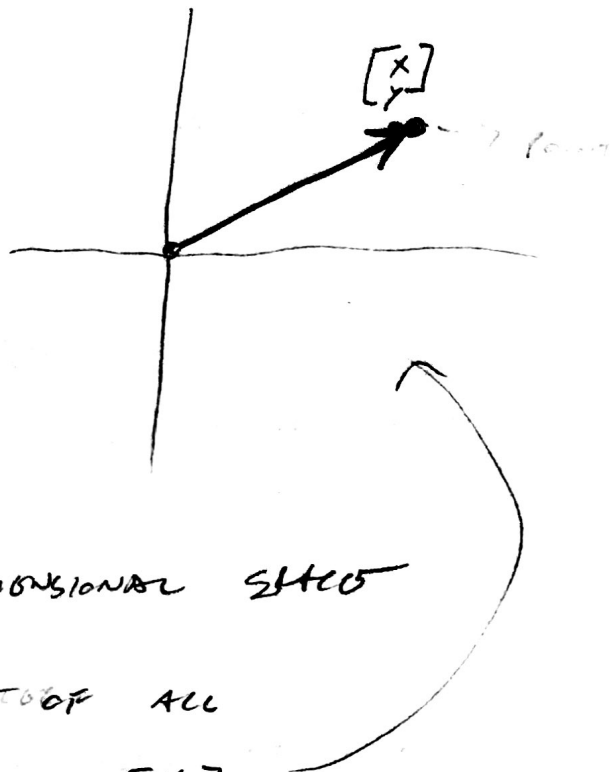
CHAPTER 4.

VECTOR SPACES.



## CHAPTER 4.

### VECTOR SPACES.



AS WE'VE BEEN DOING,

WE THINK OF TWO DIMENSIONAL SPACE

$\mathbb{R}^2$  AS THE SET OF ALL

$2 \times 1$  COLUMN VECTORS:  $\begin{bmatrix} x \\ y \end{bmatrix}$ .

WE WILL USE UNDERLINES TO REPRESENT VECTORS:

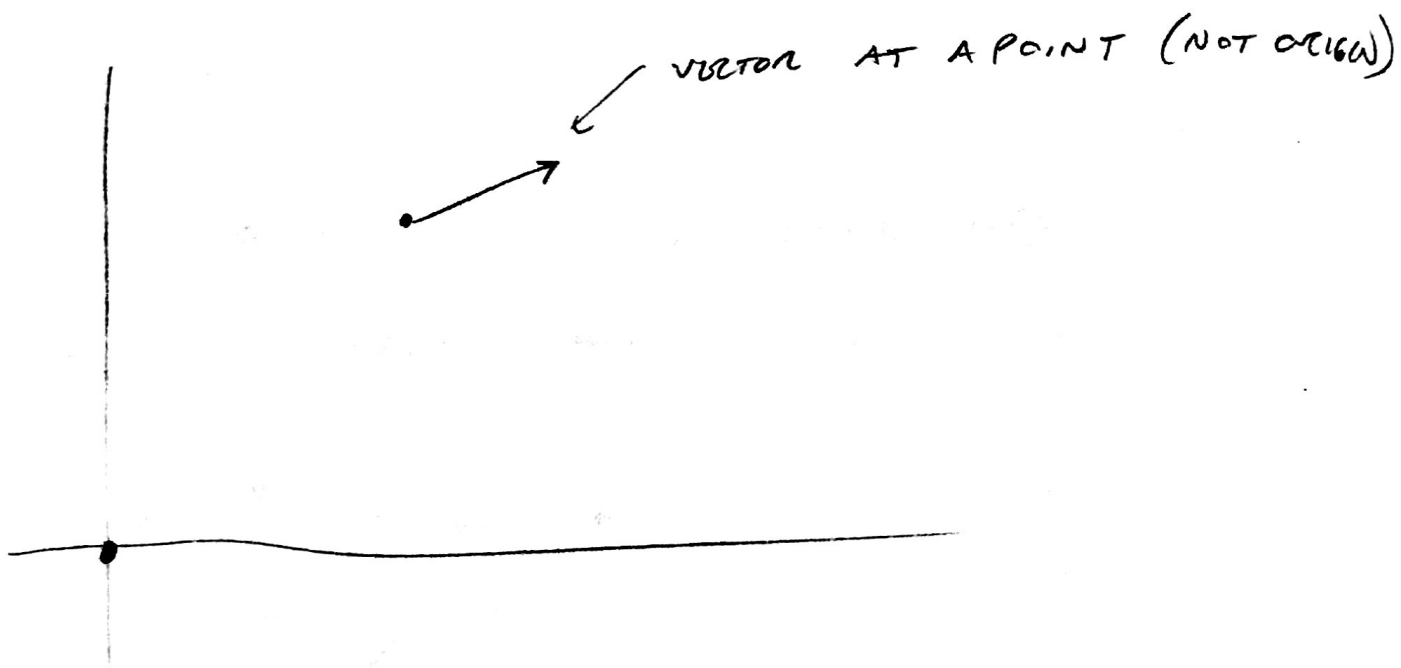
$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

THE BOOK USES BOLD TYPE, AND A LOT OF PEOPLE WRITE  $\vec{x}$ .

vec

IT IS USEFUL TO DISTINGUISH BETWEEN  
VECTORS AND POINTS.

(WE OFTEN WOULD LIKE TO TALK  
ABOUT A VELOCITY VECTOR AT A POINT,  
FOR EXAMPLE.)



WE WILL WRITE  $P(x, y)$ ,  $(x, y)$ , ETC.

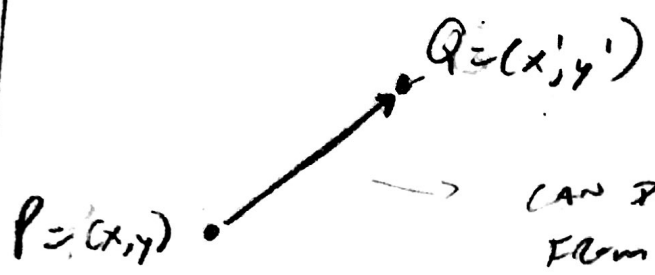
FOR POINTS IN THE PLANE

AND

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

FOR VECTORS (START AT ORIGIN)

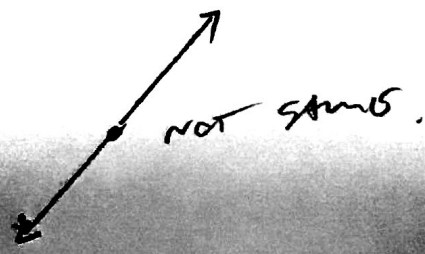
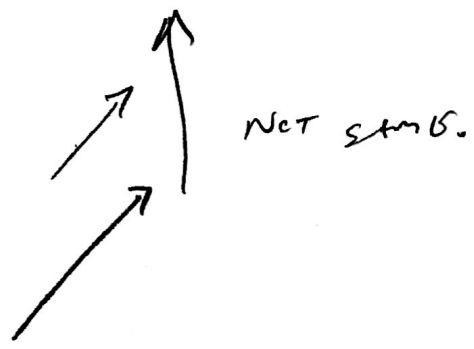
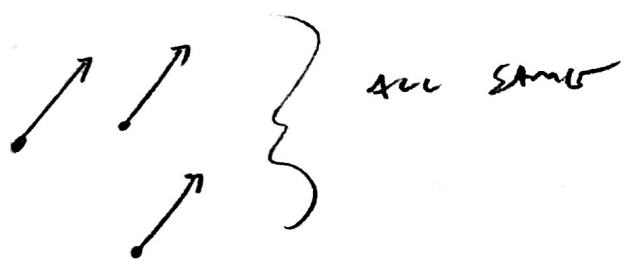
Book  
writes  
 $P(x, y)$



→ CAN DRAW A VECTOR  
FROM P TO Q,  
(A DIRECTED LINE SEGMENT  
FROM P TO Q DENOTES  
 $\vec{PQ}$ )

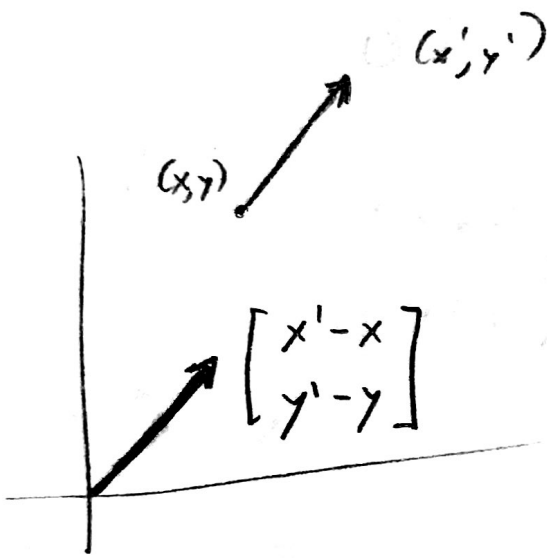
WE THINK OF TWO VECTORS BEING EQUAL

IF THEY HAVE SAME LENGTH AND DIRECTION.

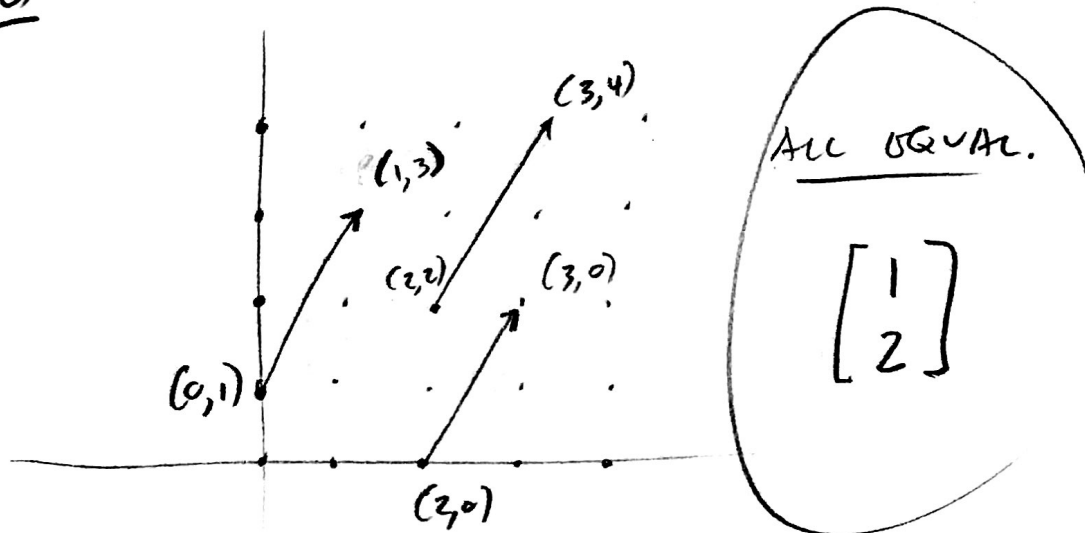




CAN TRANSLATE SUCH A VECTOR TO ORIGIN  
 WITHOUT CHANGING ITS DIRECTION  
 TO GET A VECTOR WITH SAME  
 DIRECTION AND MAGNITUDE AT ORIGIN,



EX



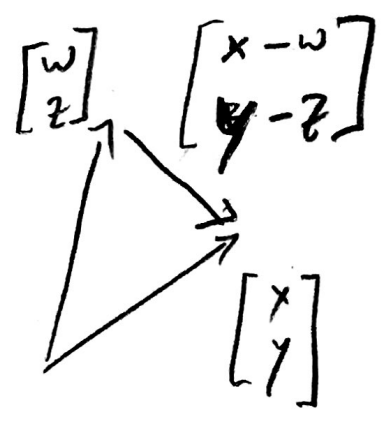
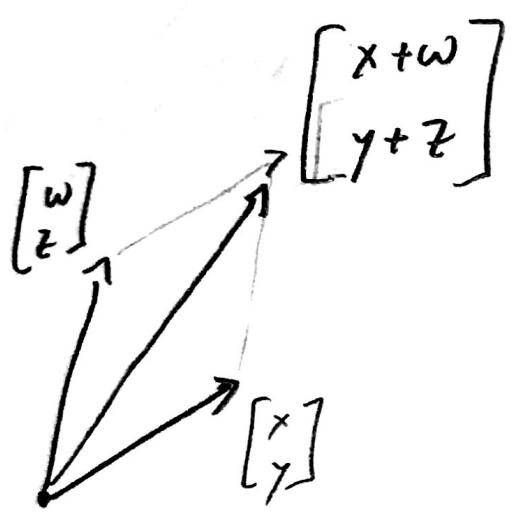
ALL EQUAL.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

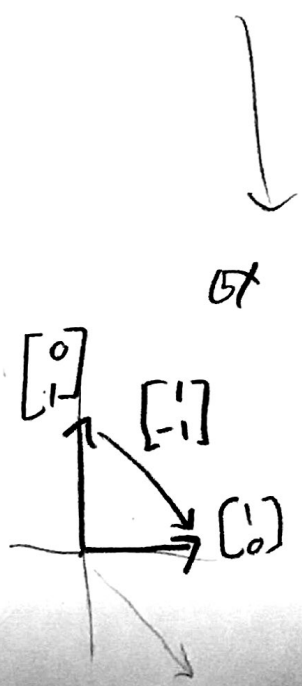
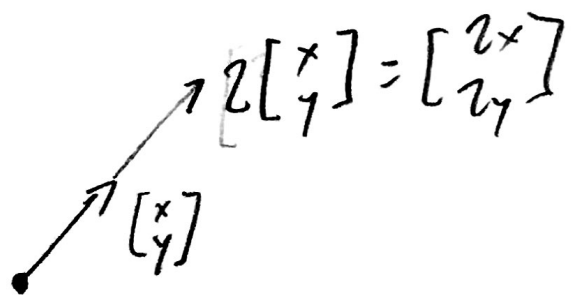
RECALL VECTOR ADDITION HAS A

NICE PICTURES:  $\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} w \\ z \end{bmatrix}$$

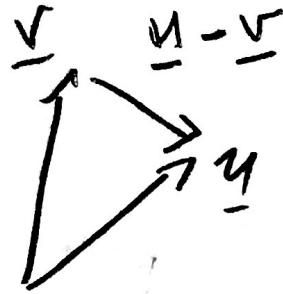
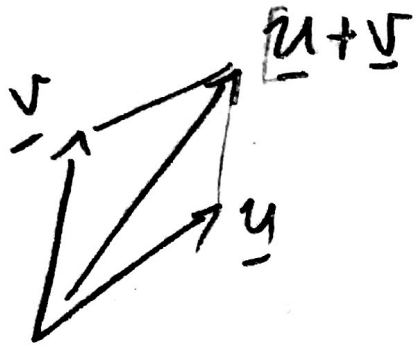


SCALAR MULTIPLICATION

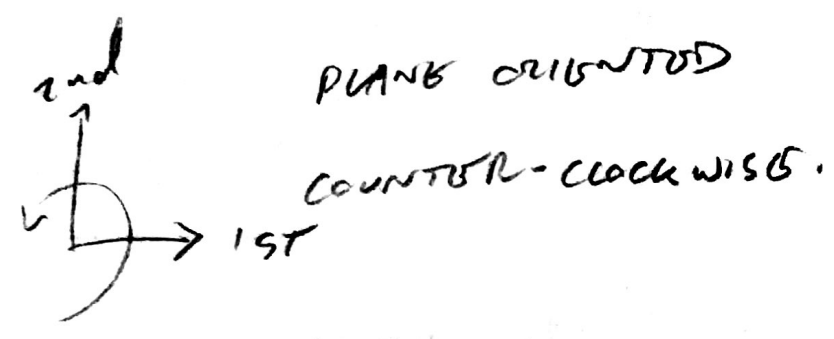


writing

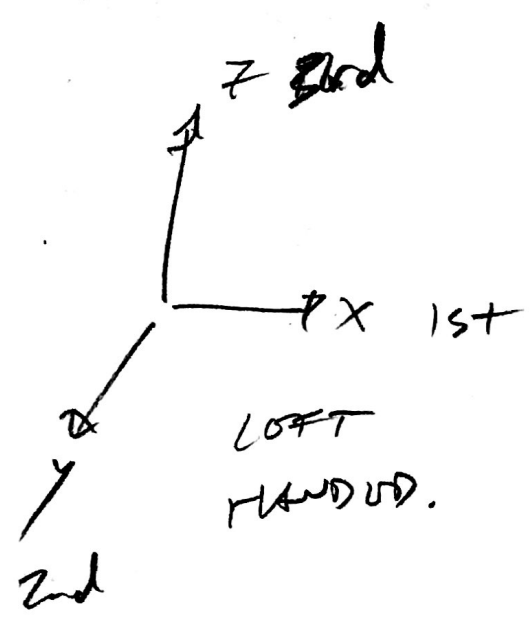
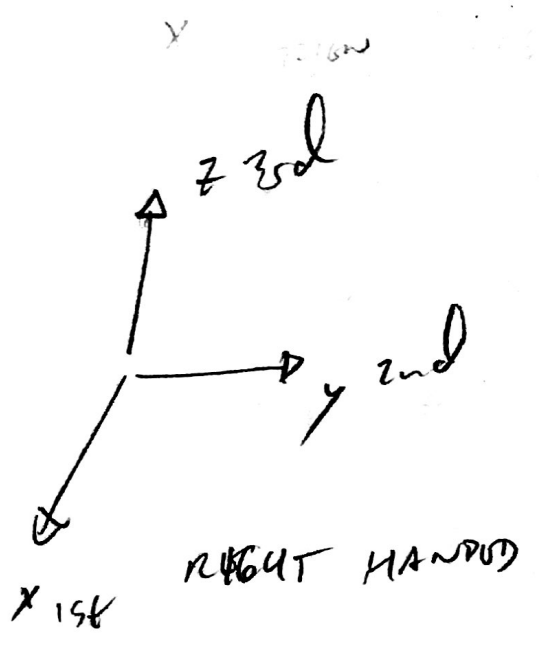
$$\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \underline{v} = \begin{bmatrix} w \\ z \end{bmatrix}$$



IN PLANE, IT IS IMPORTANT TO REMEMBER  
"X COMES FIRST"



IN SPACE, WE HAVE RIGHT HAND RULE:



HAVE SAME DISCUSSION OF VECTORS  
IN SPACE, AND FOR  $\mathbb{R}^n$  = SPACE OF  
 $n \times 1$  COLUMN VECTORS.

VECTOR SPACES.

Def A REAL VECTOR SPACE IS A SET  $V$   
WITH TWO OPERATIONS  $+$  AND  $\cdot$ ,  
(BOOK CIRCLES THESE, BUT I DON'T CARE.)  
SATISFYING THE FOLLOWING RULES: UNLESS IT WOULD BE CONFUSING,

A. IF  $u, v \in V$  THEN  $u + v \in V$ .  
( $V$  IS "CLOSED UNDER ADDITION.")

- ①  $u + v = v + u$
- ②  $u + (v + w) = (u + v) + w$
- ③ THERE IS AN ELEMENT  $0 \in V$  SUCH THAT  
 $u + 0 = 0 + u = u$  FOR ALL  $u$  IN  $V$
- ④ FOR EACH  $u$  THERE IS  $-u$  IN  $V$   
SUCH THAT  $u + -u = -u + u = 0$ .

B. IF  $\underline{u}$  IN  $V$  AND  $c \in \mathbb{R}$ ,

$c \cdot \underline{u}$  IN  $V$  ( $V$  IS "CLOSED UNDER MULTIPLICATION.")

$$(5) \quad c \cdot (\underline{u} + \underline{v}) = c \cdot \underline{u} + c \cdot \underline{v}$$

$$(6) \quad (c + d) \cdot \underline{u} = c \cdot \underline{u} + d \cdot \underline{u}$$

$$(7) \quad c \cdot (d \cdot \underline{u}) = (cd) \cdot \underline{u}$$

$$(8) \quad 1 \cdot \underline{u} = \underline{u}$$

ELEMENTS OF  $V$  ARE VECTORS

ELEMENTS OF  $\mathbb{R}$  ARE SCALARS.

EXAMPLES.

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mid x_i \in \mathbb{R} \right\}$$

EX  $\{ A \mid A \text{ is an } m \times n \text{ matrix} \}$

WITH MATRIX ADDITION.

EX.  $\mathbb{R}$ .

EX.  $\mathbb{R}_n$  row vectors  
 $[x_1, \dots, x_n]$ .

~~EX.~~

QX

$V = \{ A \mid A \text{ } 2 \times 2 \text{ matrix of trace } 0 \}$   
 WITH MATRIX ADDITION.

Recall  $\text{tr}(A) = \text{SUM OF DIAGONAL}$   
 $\text{ENTRIES,}$

So

$$\text{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} r & s \\ t & p \end{bmatrix} = \begin{bmatrix} a+r & b+s \\ c+t & d+p \end{bmatrix}$$

$$\begin{aligned} \text{tr}(\text{RHS}) &= a+r + d+p \\ &= a+d + r+p \\ &= \text{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \text{tr} \begin{bmatrix} r & s \\ t & p \end{bmatrix}. \end{aligned}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is zero vector.}$$



6X POLYNOMIALS.

if

$$P(t) = a_n t^n + \dots + a_0$$

if  $a_n \neq 0$ , DEGREE OF  $P$  IS  $n$ .

$$P_n = \{ \text{POLYS} \mid \text{DEGREE OF } P \leq n \}$$

WITH ADDITION OF FUNCTIONS

AND USUAL SCALAR MULT. IS VECTOR SPACE.

~~As~~ A VECTOR SPACE,

THERE ISN'T MUCH DIFFERENCE  
BETWEEN  $P_n$  AND  $\mathbb{R}^{n+1}$ .

EX  $V$  SET OF ALL CONTINUOUS  
REAL VALUED FUNCTIONS ON  $\mathbb{R}$ .

NONEX  $V$  SET OF REAL MULTIPLES  
OF FUNCTIONS OF THE FORM

$$e^{kx} \text{ WHERE } k \in \mathbb{R}.$$

DEFINING

$$ce^{kx} \oplus de^{lx} = cd e^{(k+l)x}$$

$$r \odot ce^{kx} = (rc)e^{kx}$$

CLOSED UNDER BOTH OPERATIONS.

NOTICE THAT THIS <sup>CONSTANT</sup> FUNCTION

$$\mathbb{1} = e^{0x} \text{ IS IN } V$$

$$\text{AND } ce^{kx} \oplus \mathbb{1} = \mathbb{1} \oplus ce^{kx} = ce^{kx}$$

AND SO  $\mathbb{1}$  IS THE ZERO VECTOR  $\underline{\underline{0}}$ .

ALSO NOTICE THAT THE CONSTANT  
FUNCTION  $0 = 0e^{kx}$  IS IN  $V$ ,

BUT  $0$  HAS NO ADDITIVE INVERSE!

$$0 \oplus ce^{kx} = 1$$

$$\Rightarrow ce^{lx} \oplus ce^{kx} = 1$$

$$\Rightarrow (c \cdot c) e^{(l+k)x} = 1$$

$$\Rightarrow 0 = 1. \quad \text{FALSE.}$$

SO NOT A VECTOR SPACE.

Thus  $\forall$  v.s.

a)  $0 \cdot \underline{u} = \underline{0}$

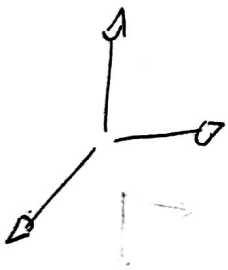
b)  $c \cdot \underline{0} = \underline{0}$

c) IF  $c \cdot \underline{u} = \underline{0}$  THEN

either  $c = 0$  or  $\underline{u} = \underline{0}$ .

d)  $(-1) \cdot \underline{u} = -\underline{u}$

## SUBSPACES.



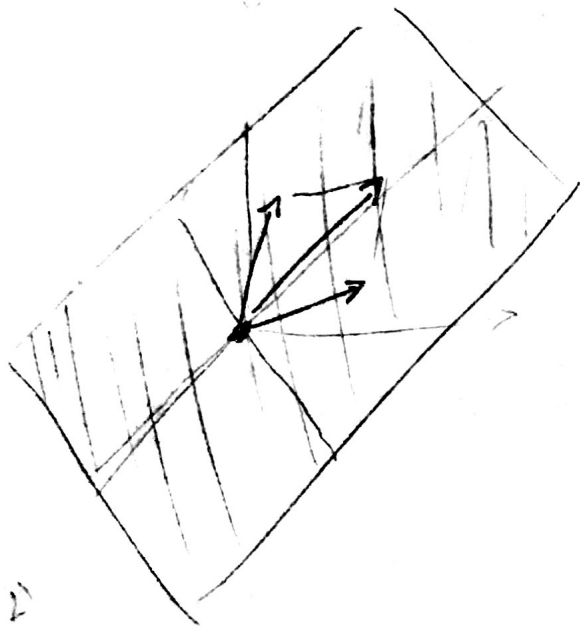
NOTICE THAT IF WE

ADD OR SCALE VECTORS

IN  $xy$ -PLANE YOU STAY IN THIS

$xy$  PLANE.

LET  $W$  BE A PLANE THROUGH ORIGIN.



IF WE HAVE TWO  
VECTORS  $v$  AND  $w$   
IN  $W$ ,  
THEIR SUM IS  
STILL IN  $W$ .

(IMAGINE ROTATING  
SPACE SO THAT  $v$  AND  $w$   
ARE LYING ON THE  
"FLOOR.")

DEFINITION. LET  $V$  BE A VECTOR SPACE  
AND LET  $W$  BE A NONEMPTY  
SUBSET OF  $V$ .  
IF  $W$  IS A VECTOR SPACE  
WITH THE OPERATIONS OF  $V$ ,  
THEN  $W$  IS A SUBSPACE.

Thm  $W \subset V$  WITH  $V$  A VECTOR SPACE.

THEN  $W$  IS A SUBSPACE IFF

THE FOLLOWING HOLD:

- a) IF  $u, v \in W$ , THEN  $u + v \in W$
- b) IF  $u \in W$ , THEN  $c \cdot u \in W$ .

Ex.  $V$  IS A SUBSPACE OF ITSELF.

$\{0\}$  IS A SUBSPACE.

Ex. LET  $V$  BE THE VECTOR SPACE OF ALL  $2 \times 2$  MATRICES WITH MATRIX ADD. AND SCALAR MULT.

LET  $W$  BE THE SET OF  $2 \times 2$  MATRICES OF TRACE 0.

$W$  IS A SUBSPACE.

Ex  $P_2$  BE THE SET OF ALL POLYNOMIALS OF DEGREE  $\leq 2$ .

$P_2$  IS A SUBSPACE OF THE VECTOR SPACE

$P$  OF ALL POLYNOMIALS.

NON EX

LET  $W$  BE THE SUBSET OF  $P$   
CONSISTING OF ALL POLYS OF  
DEGREE EXACTLY 3.

$W$  IS NOT A SUBSPACE, SINCE

$$(x^3 + x^2 + 2) - (x^3 + x^2 + 1)$$

HAS DEGREE 1.

HOW CAN WE BUILD SUBSPACES?

LET  $V$  BE A VECTOR SPACE.

LET  $\underline{u} \in V$ .

① THEN  $W = \{ c\underline{u} \mid c \in \mathbb{R} \}$  IS A SUBSPACE.

② LET  $\underline{u}, \underline{v} \in V$ ,

$$\text{THEN } W = \{ a\underline{u} + b\underline{v} \mid a, b \in \mathbb{R} \}$$

IS A SUBSPACE.

$$\begin{aligned} & (a\underline{u} + b\underline{v}) + (c\underline{u} + d\underline{v}) \\ &= (a+c)\underline{u} + (b+d)\underline{v} \in W. \end{aligned}$$

AND  $c(a\underline{u} + b\underline{v}) = ca\underline{u} + cb\underline{v} \in W$  ✓

---

DEF. Let  $\underline{v}_1, \dots, \underline{v}_n$  be vectors in a vector space  $V$ .

Then

$\underline{v}$  is a linear combination of

$\underline{v}_1, \dots, \underline{v}_n$  if

$$\begin{aligned} \underline{v} &= a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_n \underline{v}_n \\ &= \sum_{i=1}^n a_i \underline{v}_i. \end{aligned}$$



Ex, every vector in  $\mathbb{R}^n$  is a linear combination of

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Let  $\underline{v} \in \mathbb{R}^n$ .

$$\text{then } \underline{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^n x_i \underline{e}_i.$$

Ex, every vector in  $\mathcal{P}_2$  is a linear combination of

$$t^2, t, 1.$$

Ex.

Let  $A$  be an  $m \times n$  matrix.

Consider the homogeneous system

$$A\underline{x} = \underline{0}. \quad (*)$$

Let  $W$  be the set of solutions to  $(*)$ .

$W$  is a subspace:

$$\text{If } A\underline{u} = \underline{0} \text{ and } A\underline{v} = \underline{0}$$

$$\text{then } A(c\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0} \\ = \underline{0}$$

$$\text{and } A(c\underline{u}) = cA\underline{u}$$

$$\text{so if } A\underline{u} = \underline{0} \text{ so does } A(c\underline{u}).$$

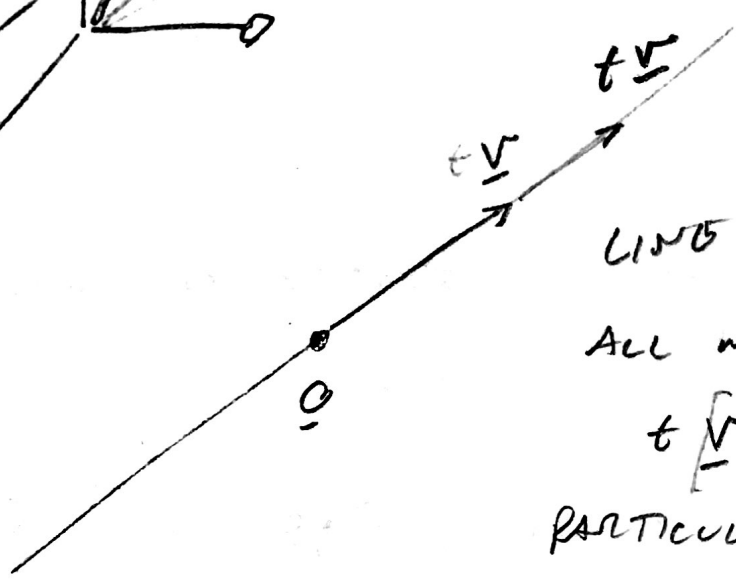
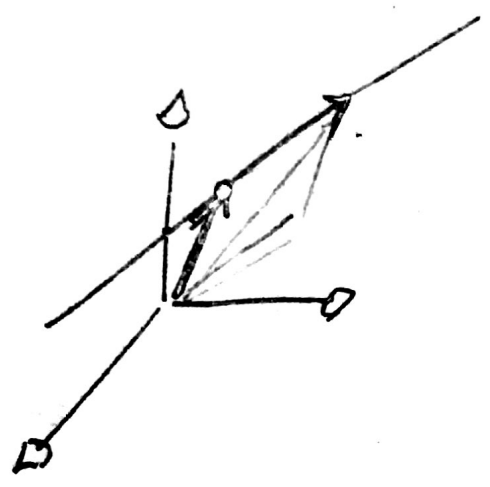
$W$  is called the null space of  $A$ .

(or the "kernel" of  $A$ .)

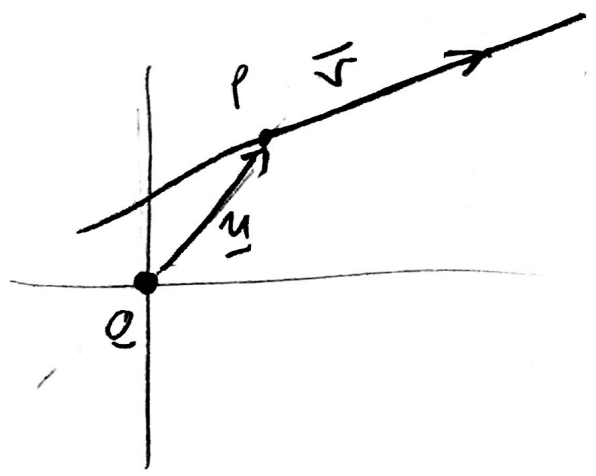
(If  $\underline{b} \neq \underline{0}$ , the set of sol<sup>n</sup>s  
to  $A\underline{x} = \underline{b}$  is not a subspace.)

PARAMETRIC EQUATIONS  
FOR LINES.

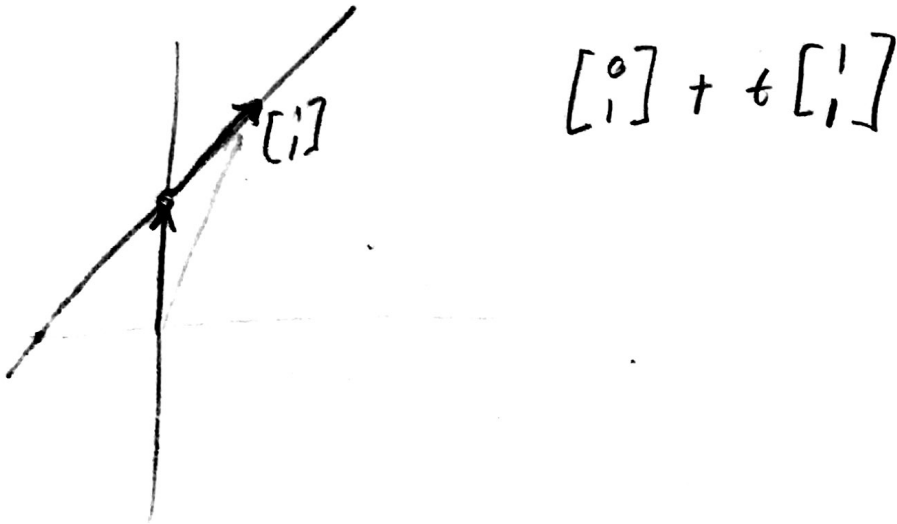
WHAT IS A LINE?



LINE AT 0 IS  
ALL MULTIPLES  
tv OF A  
PARTICULAR VECTOR.



LINE THROUGH ANOTHER  
POINT LOOKS LIKE  
ALL VECTORS  
u + tv FOR FIXED  
u AND v.



## SPAN

A NONZERO VECTOR  $\underline{v}$  DETERMINES  
A LINE  $\{t\underline{v} \mid t \in \mathbb{R}\}$ .

TWO <sup>NONZERO</sup> VECTORS  $\underline{u} \neq \underline{v} \neq \underline{0}$  GIVE A PLANE  
 $\{t\underline{u} + s\underline{v} \mid t, s \in \mathbb{R}\}$ .

Def. LET  $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\} \subset V$ .

THEN  $\text{SPAN } S = \text{SPAN}\{\underline{v}_1, \dots, \underline{v}_k\}$

$$= \left\{ \sum_{i=1}^k a_i \underline{v}_i \mid a_i \in \mathbb{R} \right\}$$

(ALSO WORKS FOR INFINITE  $S$ ,  
WHERE WE TAKE ALL FINITE LINEAR COMBS.)

Ex.  $S = \{t^2, t, 1\} \subset \mathcal{P}_2$ .

$\text{SPAN } S = \mathcal{P}_2$ .

Ex  $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\text{SPAN } S = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \right\}$ .

Thm  $\text{SPAN } S$  IS A SUBSPACE.

Proof. COMBINE LINEAR TERMS. ✓

Ex.  $S = \{t^2, t\}$

$\text{SPAN} = \{at^2 + bt \mid a, b \in \mathbb{R}\}$ .

---

Def  
IF  $\text{SPAN } S = V$  WE SAY

$S$  SPANS  $V$  OR  $V$  IS SPANNED BY  $S$ .

AND CALL  $S$  A SPANNING SET.

Ex, Let  $\mathcal{P}$  be v.s. of All Polynomials.

$$S = \{1, t, t^2, t^3, \dots\}$$

is a spanning set,

every spanning set of  $\mathcal{P}$  is infinite!

(because  $\mathcal{P}$  contains polynomials of arbitrarily large degrees.)

How do we tell if a given  $\underline{v}$  is in span  $S$ ?

Ex: Let  $S = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$

is  $\underline{v} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \in \text{span } S$ ?

i.e. are there  $a_1, a_2$  s.t.

$$a_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} ?$$

THIS IS A LINEAR SYSTEM

$$\left[ \begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 3 & -7 \end{array} \right]$$

$\leadsto$   
 row.  
 reduced  
 form

$$\left[ \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array} \right]$$

$$a_1 = 2 \quad a_2 = -3.$$

so  $\underline{v} \in \text{span } S,$

QX. IN  $\mathcal{P}$ .

$$S = \left\{ \begin{array}{l} \underline{v}_1 = 2t^2 + t + 2 \\ \underline{v}_2 = t^2 - 2t \\ \underline{v}_3 = 5t^2 - 5t + 2 \\ \underline{v}_4 = -t^2 - 3t - 2 \end{array} \right\}$$

Is  $\underline{v} = t^2 + t + 2 \in \text{span } S$ ?

$$\begin{aligned} & a_1 (2t^2 + t + 2) \\ & + a_2 (t^2 - 2t) \\ & + a_3 (5t^2 - 5t + 2) \\ & + a_4 (-t^2 - 3t - 2) \\ & = t^2 + t + 2. \end{aligned}$$

$$2a_1 + a_2 + 5a_3 - a_4 = 1$$

$$a_1 + 2a_2 - 5a_3 - 3a_4 = 1$$

$$2a_1 + 0a_2 + 2a_3 - 2a_4 = 2.$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \text{RREF.} \\ \text{Row} \\ \text{Ech.} \\ \text{Form} \end{array}$$

NO SOLUTION

$\underline{v}$  NOT IN  $\text{span } S$ .



Ex  $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ,  $\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ ,  $\underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  <sup>131</sup>

Do these span  $\mathbb{R}^3$ ?

ARBITRARY VECTOR.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & a \\ 2 & 0 & 1 & b \\ 1 & 2 & 0 & c \end{array} \right]$$

ROW REDUCE MATRIX AND SOLUTION OF

$$a_1 = \frac{-2a + 2b + c}{3}$$

$$a_2 = \frac{a - b + c}{3}$$

$$a_3 = \frac{4a - b - 2c}{3}$$

So, YES,  $\mathbb{R}^3 = \text{SPAN} \{ \underline{v}_1, \underline{v}_2, \underline{v}_3 \}$

LINEAR INDEPENDENCE.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\text{SPAN } S = \mathbb{R}^2 = \text{SPAN} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

SO WE DON'T NEED  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  TO GET  $\mathbb{R}^2$ .

ALSO NOTICE

$$\text{SPAN} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$$

SO WE DON'T NEED  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\text{AND } \text{SPAN} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2.$$

SINCE  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  IS IN  $\text{SPAN} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ ,

WE CAN WRITE  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  AS A <sup>LINEAR</sup> COMBINATION OF  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  AND  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

NAMELY

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

DEFN A COLLECTION  $\left\{ \underline{v}_1, \dots, \underline{v}_k \right\}$  OF NON-ZERO VECTORS IS

LINEARLY DEPENDENT IF WE CAN WRITE ONE OF THEM AS

NON-ZERO LINEAR COMBINATION OF THE OTHERS:

$$\underline{v}_i = a_1 \underline{v}_1 + \dots + a_{i-1} \underline{v}_{i-1}$$

$$+ a_{i+1} \underline{v}_{i+1} + \dots + a_k \underline{v}_k$$

WE SAY  $\{v_{-1}, \dots, v_{-k}\}$  IS  
LINEARLY INDEPENDENT IF IT IS  
NOT LINEARLY DEPENDENT.

so,  $S = \{v_{-1}, \dots, v_{-k}\}$  IS LINEARLY DEPENDENT

IF IT IS REDUNDANT, i.e.,

WE CAN REMOVE ONE VECTOR  
 AND NOT CHANGE THE SPAN.

ACT DEF.

$S = \{v_{-1}, \dots, v_{-k}\}$  LINEARLY DEPENDENT

IF THERE IS A NONTRIVIAL  
 LINEAR COMBINATION OF THESE  $v_{-i}$   
 THAT EQUALS  $0$ , i.e.,

$$0 = \sum_{i=1}^k a_i v_{-i}$$

FOR SOME  
 $a_i$  NOT ALL  
 ZERO.

USING THIS DEFINITION,

$S$  IS LINEARLY INDEPENDENT

IF  $\underline{0} = \sum a_i v_i$  IMPLIES THAT

$$a_1 = a_2 = \dots = a_k = 0.$$

EX. Are  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 17 \\ 6 \\ 11 \end{bmatrix}$  LINEARLY DEPENDENT?

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 17 \\ 6 \\ 11 \end{bmatrix} = \underline{0}$$

$$\left[ \begin{array}{ccc|c} 1 & 7 & 17 & 0 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 11 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 7 & 17 & 0 \\ 0 & -14 & -28 & 0 \\ 0 & -20 & -40 & 0 \end{array} \right]$$

$$a_1 = -7a_2 - 17a_3 = 14k - 17k = -3k$$

$$a_2 = -2k$$

$$a_3 = k$$

$$\text{So } k=1 \quad \begin{bmatrix} 1 & 7 & 17 \\ 2 & 0 & 6 \\ 3 & 1 & 11 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} = \underline{0}$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 7 & 17 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 7 & 17 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Q

$$v_1 = t^2 + t + 2$$

$$v_2 = 2t^2 + t$$

$$v_3 = 3t^2 + 2t + 2$$

linearly  
dependent.

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \underline{0} \quad ?$$

$$\begin{aligned} & a_1 (t^2 + t + 2) \\ & + a_2 (2t^2 + t) \\ & + a_3 (3t^2 + 2t + 2) \end{aligned}$$

$$= \underline{0} \quad ?$$

$$a_1 + 2a_2 + 3a_3 = 0$$

$$a_1 + a_2 + 2a_3 = 0$$

$$2a_1 + 0a_2 + 2a_3 = 0$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -4 & -4 & 0 \end{array} \right]$$

linearly

THM Let  $v_1, \dots, v_n \in \mathbb{R}^n$ ,

$v_1, \dots, v_n$  <sup>are</sup> ALSO LINEARLY INDEPENDENT

IFF  $\det \underbrace{\begin{bmatrix} v_1 & \dots & v_n \\ \vdots & & \vdots \end{bmatrix}}_{\text{MATRIX}} \neq 0.$

WHOSE COLS ARE THE  $v_i$ 's.

Proof. IF THEY ARE LINEARLY INDEPENDENT,

THEN  $\begin{bmatrix} v_1 & \dots & v_n \\ \vdots & & \vdots \end{bmatrix}$  IS ROW EQUIVALENT

TO  $I_n$ . (BECAUSE THE HOMOGENEOUS SYSTEM HAS ONLY THE TRIVIAL SOLUTION)

SO  $\det$  IS  $\neq 0$ .

OTW: IF  $\det \neq 0$  THEN

$\begin{bmatrix} v_1 & \dots & v_n \\ \vdots & & \vdots \end{bmatrix}$  IS ROW EQ. TO  $I$ ,

AND SO THE COLLECTION

IS LINEARLY INDEPENDENT.

Thm Let  $R \subseteq S \subseteq V$ ,

if  $S$  is linearly independent

then so is  $R$ .

if  $R$  is linearly dependent

so is  $S$ .

Proof: think about it in terms  
of redundancy:

if  $S$  is not redundant,  
how could  $R$  be?

if  $R$  is redundant,  
then so is  $S$ !





NOTE:

IF  $\underline{0} \in S$ , THEN  $S$  IS LINEARLY  
DEPENDENT.

$$\underline{0} = \sum_{\substack{v \in S \\ v \neq \underline{0}}} 0 \cdot v_1 + \dots + 0 \cdot v_k$$

THEM IF  $v_1, \dots, v_k$  ARE LINEARLY  
DEPENDENT, THEN THERE IS AN  
 $i$  S.t.  $v_i \in \text{SPAN}\{v_1, \dots, v_{i-1}\}$ ,  
AND  $v_1, \dots, v_{i-1}$  ARE LINEARLY INDEPENDENT.

PROOF: LET  $i$  BE THE FIRST  $i$  SUCH  
THAT  $\{v_1, \dots, v_i\}$  IS L.D.

□

## BASES AND DIMENSION.

DEF.

$S \subset V$  IS A BASES OF  $V$

IF 1)  $\text{SPAN } S = V$

2) (EVERY FINITE SUBSET OF)

$S$  IS LINEARLY INDEPENDENT.

EX

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

IS A BASIS FOR  $\mathbb{R}^n$ .

Ex  $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

IS ALSO A BASIS,

Ex.  $S = \{1, t, t^2, t^3\}$   $\swarrow$  STD BASIS.  
IS A BASIS  
FOR  $P_3$ .

SO IS  $\{1, 2+t, t^2, 1+t^2+t^3\}$

Ex Let  $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a+d=0 \right\}$

CLAIM:  $S = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

IS A BASIS.

Proof:

A matrix in  $W$  looks like

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

So span  $S = W$ .

Also: IF

$$0 = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

THEN

$$0 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \Rightarrow a = 0, b = 0, c = 0.$$

✓

DEF 1

$V$  IS FINITE DIMENSIONAL

IF IT HAS A FINITE BASIS, AND  
 THE DIMENSION OF  $V$  IS # OF VECTORS IN BASIS,  
INFINITE DIMENSIONAL OTHERWISE.

WAIT. Do two DIFFERENT BASES HAVE  
 SAME SIZE?

WE'LL COME BACK TO THIS SOON.

THEM IF  $\{v_1, \dots, v_n\}$  IS A BASIS OF  $V$ ,  
 THEN EVERY  $v$  MAY BE WRITTEN  
 AS A LINEAR COMB OF THESE  $v_i$   
 IN A UNIQUE WAY.

ProofTHE  $v_i$  span so

$$v = \sum a_i v_i$$

$$\text{IF } v = \sum b_i v_i$$

$$\begin{aligned} \text{THEN } \underline{0} = v - v &= \sum a_i v_i - \sum b_i v_i \\ &= \sum (a_i - b_i) v_i \end{aligned}$$

$\Rightarrow \underline{0}$  IS A LINEAR COMB

OF THE  $v_i \Rightarrow$

COEFFICIENTS  $(a_i - b_i)$

ARE ALL zero.

SINCE THE  $v_i$  ARE LINEARLY  
INDEPENDENT.

SO  $a_i = b_i$  FOR ALL  $i$ .



Thm  $V$  v.s.

$$S = \{v_1, \dots, v_m\} \subset V$$

with  $\text{SPAN } S = V$ .

then some subset of  $S$  is a basis.

Proof:

By Exercise 14,

there is a first  $i$  such that

$v_1, \dots, v_{i-1}$  is linearly independent

and  $v_1, \dots, v_i$  is not.

$$\text{So } \text{SPAN} \{v_1, \dots, v_{i-1}\} = \text{SPAN} \{v_1, \dots, v_i\}.$$

delete  $v_i$  from  $S$  to get

a subset  $S_1 \subset S$

$$\parallel \\ \{u_1, \dots, u_{m-1}\}.$$

CONTINUE THIS WITH  $S_1$  TO GET

$$S_2 = \{w_1, \dots, w_{m-2}\} \subset S \text{ WITH SAME SPAN.}$$

THIS EVENTUALLY STOPS AT A BASIS.

□

Thm LET  $S = \{v_1, \dots, v_n\}$  BE A BASIS.

IF  $T = \{w_1, \dots, w_m\}$  IS LINEARLY INDEPENDENT, THEN  $m \leq n$ .

Proof:

$$T_1 = \{w_1, v_1, \dots, v_n\} \text{ LINEARLY DEPENDENT}$$

$$S_1 = \{w_1, v_1, \dots, v_{i-1}, v_{(i)}, \dots, v_n\}$$

WHERE  $v_i$  IS A LINEAR COMBO OF PREVIOUS VECTORS IN  $T_1$ .



$S_1$  SPANS.

LOT

$$T_2 = \{w_2, v_1, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$$

SOME VECTOR IN  $T_2$  IS A LIN. COMB OF PREVIOUS ONES.

NOT  $w_1$ . (COZ  $T$  IS LIN. INDEP.)

SO WE CAN DELETE ANOTHER  $v_j$  AND CONTINUE.

EACH TIME WE ADD A  $w$ ,

WE CAN DELETE A  $v$ , UNTIL

THE  $v$ 'S ARE GONE.

SO THERE MUST BE AS MANY

$v$ 'S AS  $w$ 'S. SO  $n \geq m$ .



Case

$$\text{IF } S = \{v_1, \dots, v_m\}$$

$$\text{AND } T = \{w_1, \dots, w_m\} \text{ ALSO}$$

BASES OF  $V$ .

THEN  $m = n$ .

SO DIMENSION MATCHES SOURCE!

Proof: BY PREVIOUS THM,  
 $m \leq n$  AND  $n \leq m$ .



DIMENSION.

DEF. LET  $S \subset V$ ,

$T \subset S$  IS A MAXIMAL INDEPENDENT  
SUBSET OF  $S$  IF IT IS LINEARLY INDEPENDENT  
 AND IS NOT PROPERLY CONTAINED  
 IN ANOTHER LINEARLY IND. SUBSET OF  $S$ ,  
 (i.e. IF  $T \subset T' \subset S$  WITH  
 $T'$  LINEARLY IND. THEN  $T' = T$ .)

OFTEN WE TAKE  $S = V$ . WE WILL  
 SEE THAT IF  $S = V$ , THEN A MAX. LIN.  
 IND. SET IS A BASIS.

Thm  $V$  HAS DIMENSION  $n$ ,

THEN A MAXIMALLY INDI. SUBSET OF  $V$   
CONTAINS  $n$  VECTORS.

P. Proof: LET  $S = \{v_1, \dots, v_k\}$  BE M.I. SUBSET  
OF  $V$ .

IF  $\text{SPAN } S = V$  THEN  $S$  IS A BASIS AND  $k = n$ .

IF  $\text{SPAN } S \neq V$  THEN THERE

IS A  $v$  THAT ISN'T IN  $\text{SPAN } S$ ,

BUT THEN  $\{v, v_1, \dots, v_k\}$

IS LINEARLY INDEPENDENT.

THIS CONTRADICTS MAXIMALITY  $\square$ .

Thm IF  $S$  IS A MINIMAL SPANNING SET,

THEN  $S$  HAS  $n$  ELEMENTS.

Proof. O.K.  $\square$

Thm. IF  $V$  HAS DIMENSION  $n$   
AND  $S \subset V$  HAS  $m > n$  ELEMENTS,  
THEN  $S$  IS LINEARLY DEPENDENT.

Thm IF  $S$  HAS  $< n$  ELEMENTS AND  
 $\dim V = n$ , THEN  $\text{SPANS} \neq V$ .

Thm IF  $S$  IS LINEARLY INDEPENDENT  
SUBSET OF A FINITE DIM<sup>d</sup> VECTOR SPACE,  
THEN  $S$  IS CONTAINED IN A BASIS,  
(MAY BE "FILLED OUT" TO A BASIS.)

Proof: IF  $S$  SPANS, IT IS A BASIS.

IF IT DOESN'T, THEN ADD A VECTOR  
 $w_1$  THAT ISN'T IN SPANS.

IF  $\{v_1, \dots, v_k, w_1\}$  IS A BASIS, D.N.O.

CONTINUE UNTIL  $\{v_1, \dots, v_k, w_1, \dots, w_{n-k}\}$

IS A BASIS. ✓



Thm ① IF  $V$  HAS DIMENSION  $n$   
AND  $\{v_1, \dots, v_n\}$  IS LINEARLY IND,  
THEN  $\{v_1, \dots, v_n\}$  IS A BASIS,

② IF  $\text{SPAN}\{v_1, \dots, v_n\} = V$ ,  
THEN  $\{v_1, \dots, v_n\}$  IS A BASIS,

Thm MAXIMALLY INDOP. SUBSET OF  $V$   
IS A BASIS.  $\square$

SKIP 4.7. FOR NOW.

4.8. COORDINATES.

AN ORDERED BASIS IS A BASIS

$\{v_1, v_2, \dots, v_n\}$  WHOSE ORDER MATTERS.

SO  $\{v_2, v_1, \dots, v_n\}$  IS A DIFFERENT  
ORDERED BASIS.

IF  $S = \{v_1, \dots, v_n\}$  IS AN ORDERED BASIS

WE CAN WRITE EACH  $v$  AS

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

UNIQUELY AND THE ORDER ALLOWS

US TO KEEP JUST THE COEFFICIENTS,

i.e. WE WRITE

$$[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

COORD. VECTOR OF  $v$  WITH RESPECT TO THE ORDERED BASIS  $S$ .

SO WE CAN PROCEED LIKE  $v$  IS  $\mathbb{R}^n$ .

Ex.  $S = \{1, t, t^2\}$  ORDBED BASIS  
OF  $P_2$ .

COORDINATES

$$[1 + 2t + 3t^2]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Ex.  $T = \{t, 1, t^2\}$

$$[1 + 2t + 3t^2]_T = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$



Ex  $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

WHAT IS  $[v]_S$ ?

WANT  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

SO GET  $\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$

$$\begin{array}{l} \boxed{c = 3} \\ \boxed{b + c = 2} \\ \Rightarrow \boxed{b = -1} \end{array}$$

$$a + b + c = 1$$

$$a - 1 + 3 = 1$$

$$a = -1$$

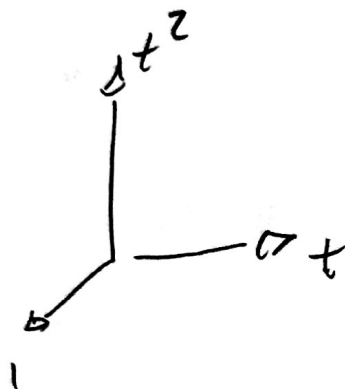
So

$$[v]_S = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

THIS ALLOWS US TO

PICTURE THE VECTOR SPACES:

$\mathbb{R}^2$  IS JUST LIKE  $\mathbb{R}^3$ :



ISOMORPHISMS.

LET  $S = \{v_1, \dots, v_n\}$  BE AN ORDERED BASIS

FOR  $V$ .

$$v = a_1 v_1 + \dots + a_n v_n$$

$$w = b_1 v_1 + \dots + b_n v_n$$

IN  $\mathbb{R}^n$ , WE HAVE

$$[v]_S \text{ AND } [w]_S.$$

NOTICE THAT  $[v+w]_S = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} = [v]_S + [w]_S$

AND

$$[cV]_S = c[v]_S$$


---

DEFINE A FUNCTION  $L: V \rightarrow \mathbb{R}^n$

BY  $L(v) = [v]_S.$

$L$  IS SURJECTIVE ("ONTO")

i.e. EVERY VECTOR IN  $\mathbb{R}^n$

IS THE IMAGE OF SOME  $v$  IN  $V$ ,

$L$  IS INJECTIVE ("1-1")

i.e. IF  $L(v) = L(w)$ , THEN  $v = w$ .

$L$  IS BISJECTIVE IF BOTH.

DEFINITION LET  $V$  AND  $W$  BE VECTOR SPACES,

A BISJECTION  $L: V \rightarrow W$

s.t. a)  $L(v+w) = L(v) + L(w)$

AND b)  $L(cv) = cL(v)$

IS AN ISOMORPHISM.

AND SAY  $V$  AND  $W$  ARE ISOMORPHIC.

Thm. IF  $V$  HAS DIMENSION  $n$ ,  
 THEN  $V$  IS ISOMORPHIC TO  $\mathbb{R}^n$ .

Pr.  $L: V \rightarrow \mathbb{R}^n$  GIVEN BY

$$L(v) = [v]_{\mathcal{B}} \quad \text{IS AN ISO.} \quad \square$$

Thm a)  $V$  ISOMORPHIC TO  $V$

b) IF  $V$  ISO. TO  $W$ , THEN  $W$  ISO. TO  $V$ .

c) IF  $U$  ISO TO  $V$  AND  $V$  ISO TO  $W$   
 THEN  $U$  ISO TO  $W$ .

Thm. TWO FTS DIM<sup>d</sup> V.S.S ARE  
 ISOMORPHIC IFF THEY HAVE SAME  
 DIMENSION.

Proof: BOTH ISOMORPHIC TO SAME  $\mathbb{R}^n$ .  $\square$

IF WE HAVE A BASIS  $S$ , HOW DO WE GET WHAT  $[v]_S$  IS IF WE ARE GIVEN  $v$  IN THE USUAL BASIS?

GO BACK TO OUR EXAMPLES:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

NOTICE:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} [v]_S$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ vector in STD BASIS}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\{e_1, e_2, e_3\}}$$



$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = ?$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

NOTES:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} v \end{bmatrix}_S.$$

☺

WHAT IS

$$P_{T \leftarrow S} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

THESE COLUMNS ARE  
THE COMPONENTS OF  $S$ ,  
IN ORDER, WRITTEN IN  
TERMS OF  $T = \{e_1, e_2, e_3\}$ .

IF WE WRITE  $T$  IN TERMS OF  $S$ ,

WE GET

$$[e_1]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[e_2]_S = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$[e_3]_S = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$P_{S \leftarrow T} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{T \leftarrow S} \cdot [v]_S$$

$$= [v]_T$$

$$P_{S \leftarrow T} \cdot [v]_T$$

$$= [v]_S$$

## TRANSITION MATRICES

LAST TIME WE SAW A TRANSITION MATRIX,  
RECALL:  $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ ,  $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$   
 $v_1 \quad v_2 \quad v_3 \quad w_1 \quad w_2 \quad w_3$

NOTE THAT

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ \phantom{v_1} \\ \phantom{v_1} \end{bmatrix}_T$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_2 \\ \phantom{v_2} \\ \phantom{v_2} \end{bmatrix}_T$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_3 \\ \phantom{v_3} \\ \phantom{v_3} \end{bmatrix}_T$$

WE SAW AN EXAMPLE OF

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \phantom{v} \\ \phantom{v} \end{bmatrix}_S = \begin{bmatrix} v \\ \phantom{v} \\ \phantom{v} \end{bmatrix}_T$$

NAMELY

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} v_1 \\ \phantom{v_1} \\ \phantom{v_1} \end{bmatrix}_T & \begin{bmatrix} v_2 \\ \phantom{v_2} \\ \phantom{v_2} \end{bmatrix}_T & \begin{bmatrix} v_3 \\ \phantom{v_3} \\ \phantom{v_3} \end{bmatrix}_T \end{bmatrix} \begin{bmatrix} v \\ \phantom{v} \\ \phantom{v} \end{bmatrix}_S = \begin{bmatrix} v \\ \phantom{v} \\ \phantom{v} \end{bmatrix}_T$$



IN GENERAL:  $\dim V = n$ .

LET  $S = \{v_1, \dots, v_n\}$  AND  $T = \{w_1, \dots, w_n\}$

ORDERED BASES.

LET  $P_{T \leftarrow S}$  BE THE MATRIX

WHOSE  $j$ TH COLUMN IS  $[v_j]_T$ .

i.e.

BOOK  
SOMETIMES  
CALLS  $\leftarrow$   
THIS Q.T.P.

$$P_{T \leftarrow S} = \begin{bmatrix} [v_1]_T & \dots & [v_n]_T \end{bmatrix}$$

Now, IF

$$w = b_1 v_1 + b_2 v_2 + \dots + b_n v_n$$

$$[w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Now note that

$$[w]_T = [b_1 v_1 + \dots + b_n v_n]_T$$

$$= [b_1 v_1]_T + \dots + [b_n v_n]_T$$

$$= b_1 [v_1]_T + \dots + b_n [v_n]_T = P_{T \leftarrow S} \cdot [w]_S$$

INTERCHANGING T AND S WE HAVE

$$P_{S \leftarrow T} = \begin{bmatrix} | & & | \\ [w_1]_S & \dots & [w_n]_S \\ | & & | \end{bmatrix} [v]_T = [v]_S.$$

$$P_{S \leftarrow T} = P_{T \leftarrow S}^{-1}$$

Ex.  $S = \left\{ \overset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}, T = \left\{ \overset{w_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{w_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{w_3}{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}} \right\}$

LET'S FIND  $P_{T \leftarrow S} = \begin{bmatrix} | & | & | \\ [v_1]_T & [v_2]_T & [v_3]_T \\ | & | & | \end{bmatrix}$

WHAT ARE

$$[v_1]_T, [v_2]_T, [v_3]_T ?$$

SOLOVE THIS LINEAR SYSTEMS

$$\left[ \begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right], \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right], \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array} \right]$$

ALL COEFF. MATRICES ARE SAME,

SO WE CAN SOLVE ALL AT SAME TIME

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_3 - R_1 \rightarrow R_3 \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array} \right]$$

$$R_1 + R_3 \rightarrow R_1 \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array} \right]$$

$P_{T \leftarrow S}$

$$P_{T \leftarrow S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

CONSIDER  $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  ✓ VECTOR IN USUAL BASIS  $e_1, e_2, e_3$

$$[v]_S = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \quad \text{FROM [LAST TIME].}$$

$$\text{SO } [v]_T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

write down!

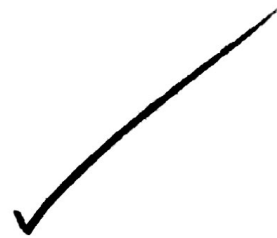
$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 3 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

so  $[v]_T = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$



RANK.

LAST TIME WE DEFINED A  
 LINEAR MAP  $L: V \rightarrow W$  TO  
 BE A FUNCTION SUCH THAT

$$L(u+v) = L(u) + L(v).$$

$$\text{AND } L(cu) = cL(u).$$

IF  $V$  AND  $W$  ARE BOTH  $\mathbb{R}^n$ ,

$$\text{SO } L: \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

THEN  $L$  IS GIVEN BY A MATRIX:

THAT IS AN  $n \times n$  MATRIX

$A$  SUCH THAT

$$L(v) = A \cdot v.$$

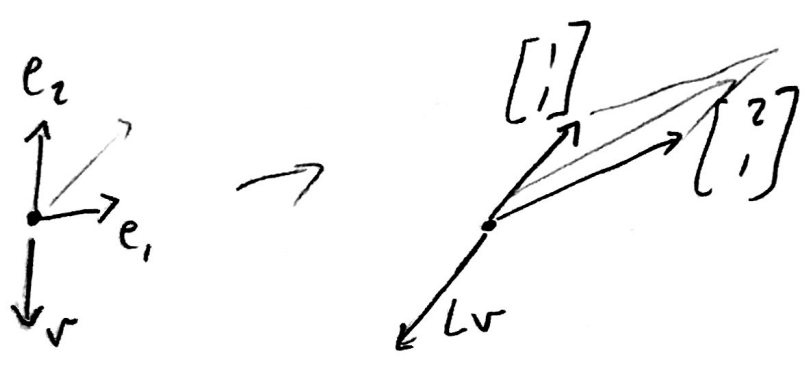
WHAT IS  $A$ ?

$$A = \begin{bmatrix} | & & | \\ L(e_1) & \dots & L(e_n) \\ | & & | \end{bmatrix} = \text{MATRIX WHOSE} \\ \text{COLUMNS ARE} \\ L(e_j).$$

THIS IS HOW WE'VE THOUGHT ALL ALONG.

EX.

$$L(v) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot v$$



NOTICE THAT THIS  $L$  IS AN ISOMORPHISM.

---

DEF: THE IMAGE OF  $L: V \rightarrow W$

$$L(V) = \text{IMAGE}(L) = \left\{ L(v) \mid v \in V \right\} \subseteq W$$

$$L(V) = \text{SPAN} \{ L(v_1), \dots, L(v_n) \}$$

WHERE  $v_1, \dots, v_n$  IS A BASIS.

DEF THE RANK OF  $L$  ~~IS THE DIMENSION OF  $L(V)$ .~~

IS THE DIMENSION OF  $L(V)$ .

DEF. RANK OF AN  $n \times n$  MATRIX

$A$  IS RANK OF  $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$

GIVEN BY  $L(v) = A \cdot v$ .

IN THIS CASE

$L(V)$  IS THE SPAN OF THE COLUMNS OF  $A$ .

AND IS ALSO CALLED THE COLUMN SPACE,

Ex.

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$L(v) = A \cdot v$$

where  $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$L(\mathbb{R}^3) = \{0\}$$

RANK OF  $L$  IS 0.

" "  $A$  IS 0

Ex  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

RANK 1.

THIS IS ALSO CALLED THE "COLUMN RANK"

DIMENSION OF "ROW SPACE" IS

THE "ROW RANK" (EQUIVALENT)



( THE NEW RANK <sup>OF A</sup> IS THE RANK OF  
 THE LINEAR TRANSFORMATION  $M: \mathbb{R}^m \rightarrow \mathbb{R}^m$   
 $M(v) = v \cdot A$  )

---

ALL OF THIS MAKES SENSE FOR MATRICES  
 THAT AREN'T SQUARE.

$m \times n$  MATRIX  $A$  GIVES  
 A MAP  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$\text{BY } L(v) = A \cdot v$$

AND ANOTHER ONE  
 $M: \mathbb{R}^m \rightarrow \mathbb{R}^n$

$$M(v) = v \cdot A.$$


---

IF  $B$  IS SQUARE AND  $\det B \neq 0$

THEN COLUMN RANK OF  $BA = \text{COL RANK OF } A$

AND " " OF  $AB = \text{COL RANK OF } A$ .

STAYS FOR ROW RANK.

Proof:

$w_1, \dots, w_n$  BASIS OF COLUMN SPACE

So  $w_1, \dots, w_n$  ARE LINEARLY INDEPENDENT.

COLUMNS OF  
BUT THEN  $Bw_1, \dots, Bw_n$

ARE STILL LINEARLY INDEPENDENT:

(AND)

$$\text{IF } \underline{0} = a_1 Bw_1 + \dots + a_n Bw_n$$

$$\begin{aligned} \text{THEN } \underline{0} &= B^{-1} \underline{0} = B^{-1}(a_1 Bw_1 + \dots + a_n Bw_n) \\ &= B^{-1}(a_1 Bw_1) + \dots + B^{-1}(a_n Bw_n) \\ &= B^{-1} B a_1 w_1 + \dots + B^{-1} B a_n w_n \\ &= a_1 w_1 + \dots + a_n w_n. \end{aligned}$$

$$\Rightarrow a_i = 0 \text{ FOR ALL } i.$$

SO  $Bw_1, \dots, Bw_n$  IS A BASIS FOR  $\text{Col}(BA)$ .

✓

So

Row operations DON'T CHANGE row rank  
OR column rank

Col operations DON'T CHANGE column rank,  
OR row rank.

~~Proof:~~

new rank is # of non zero rows in  
reduced row echelon form

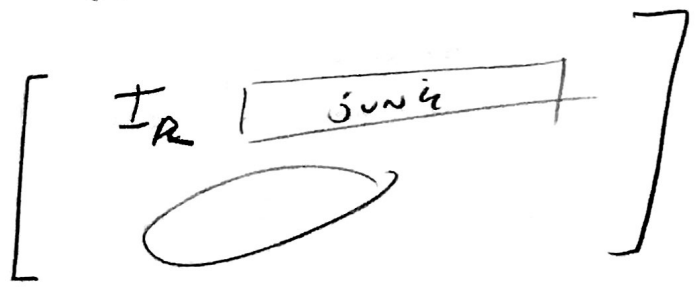
then new rank = column rank.

Proof: A matrix.

new reduced to reduced row echelon  
form. (doesn't change row rank)

can perform col. operations keeping in  
red. row echelon form (doesn't change)  
col. rank

to get



COL RANK IS R  
FIRST R COLUMNS  
BASIS FOR COL. SPACE  
# NONZERO rows  
IS ROW RANK  
= R. □

THE SPACE OF ALL VECTORS  $x$

SUCH THAT

$$Ax = 0$$

IS CALLED THE NULLSPACE OF  $A$

IS SOLUTION SPACE OF THE HOMOGENEOUS SYSTEM.

OR NULLSPACE OF  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$L(v) = A \cdot v.$$

DEF THE DIMENSION OF THE NULLSPACE IS THE NULLITY.

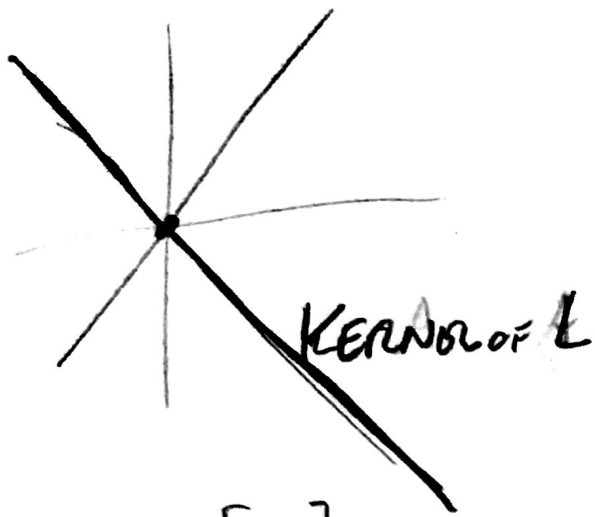
$A$  IS EQUIVALENT TO



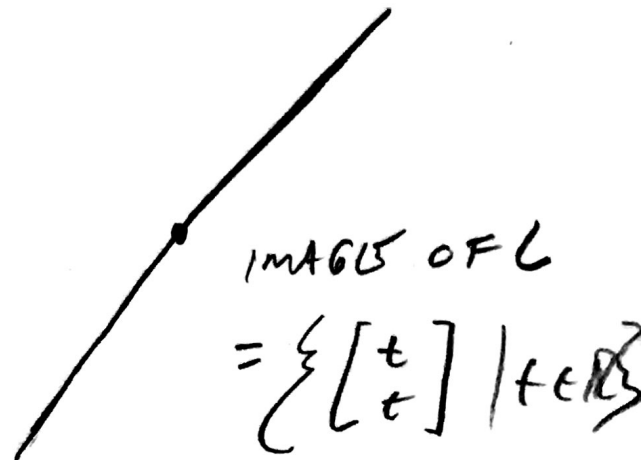
$$\begin{aligned} \text{NULLITY} &= n - r, \\ &= \# \text{ FREE VARIABLES} \end{aligned}$$

$$L = A \cdot v$$

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→



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

kernel is  $\left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \mid a \in \mathbb{R} \right\}$ .

So IF YOU "CRUSH" AWAY A  $k$ -DIMENSIONAL  
THING, "WHAT'S LEFT" HAS DIMENSION  
 $n-k$ .



$$\rho = n = r = \text{NULLITY}$$

So a solution  $x$  to  $Ax = 0$  looks like

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -b_{1,r+1}s_1 - b_{1,r+2}s_2 - \dots - b_{1,n}s_p \\ \vdots \\ -b_{r,r+1}s_1 - b_{r,r+2}s_2 - \dots - b_{r,n}s_p \\ s_1 \\ s_2 \\ \vdots \\ s_p \end{bmatrix}$$

} arbitrary

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{bmatrix} = s_1 \begin{bmatrix} -b_{1,r+1} \\ \vdots \\ -b_{r,r+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -b_{1,r+2} \\ \vdots \\ -b_{r,r+2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + s_p \begin{bmatrix} -b_{1,n} \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\parallel$   $v_1$                        $\parallel$   $v_2$                        $\parallel$   $v_p$

$\{v_1, \dots, v_p\}$  is a basis for kernel of  $A$ .

clearly span and

examining the pivot rows  $r+1, \dots, n$

can see lin. ind.

Ex.

$$A = \left[ \begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

General solution looks like

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s_1 - s_2 - s_3 \\ -4s_1 - 0s_2 - 3s_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

$$= s_1 \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

basis for kernel of A.



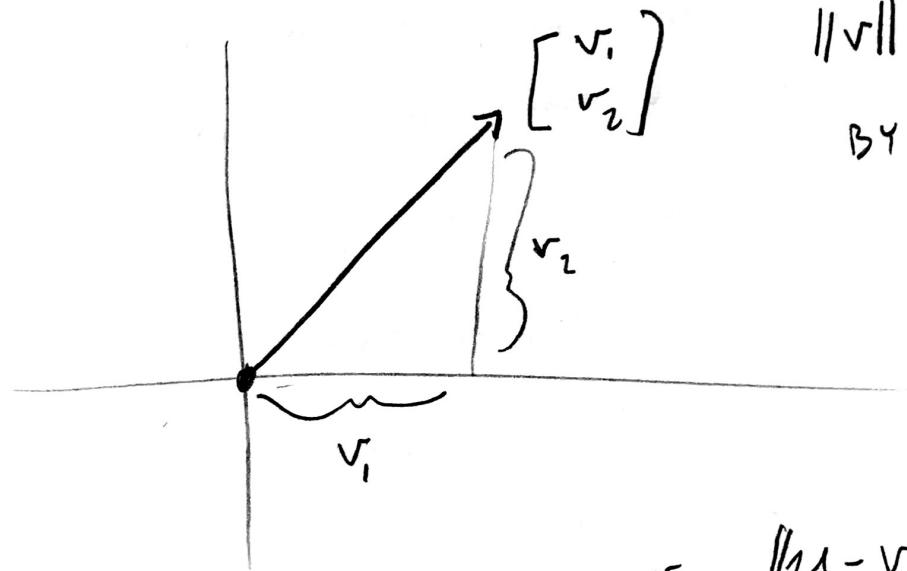
CHAPTER 5. (SKIP 5.2)

INNER PRODUCT SPACES.

5.1 LENGTH AND DIRECTION.

A vector  $v$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  is visualized as a line segment with a direction.

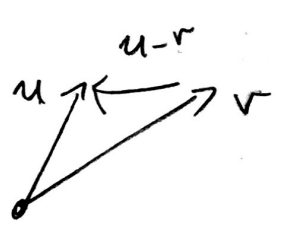
IT HAS A LENGTH  $\|v\|$  or "SIZE"



$$\|v\| = \sqrt{v_1^2 + v_2^2}$$
 BY PYTHAGOREAN THM.

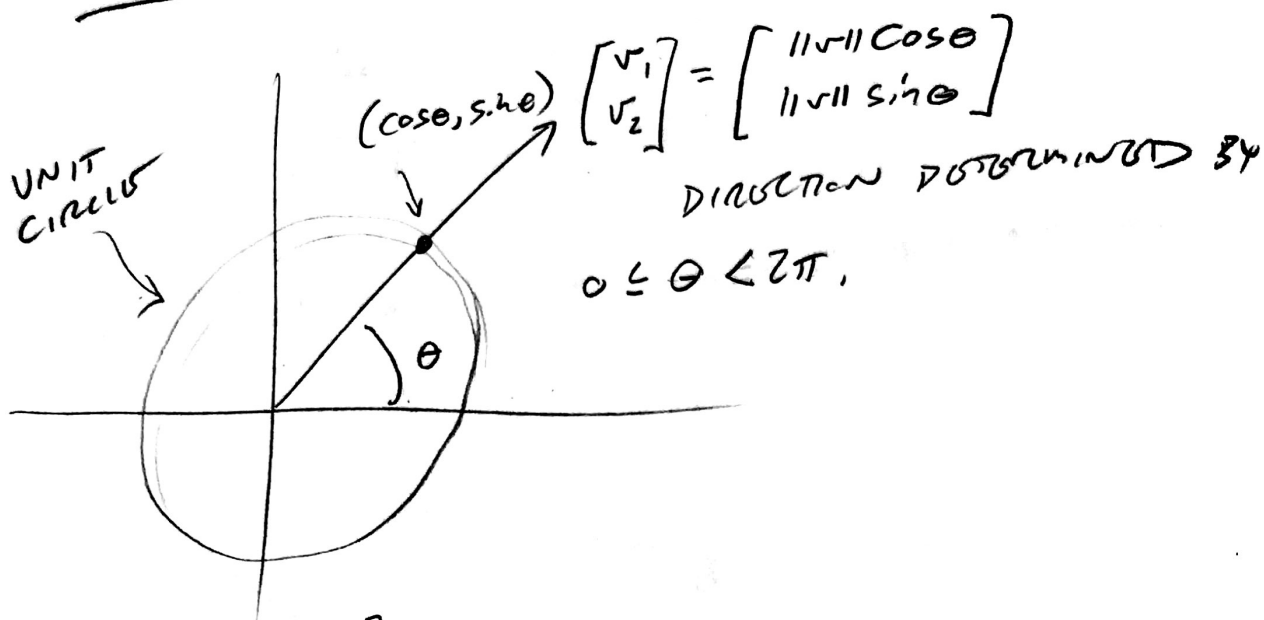
THE SIZE OF A DIFFERENCE  $\|u - v\|$  MEASURES THE DISTANCE BETWEEN  $u$  AND  $v$ ;

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v$$



$$\|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

DIRECTION:



WHAT IS  $\theta$ ?

WELL  $\cos \theta$  IS X COORDINATE OF POINT ON UNIT CIRCLE AT ANGLE  $\theta$ .

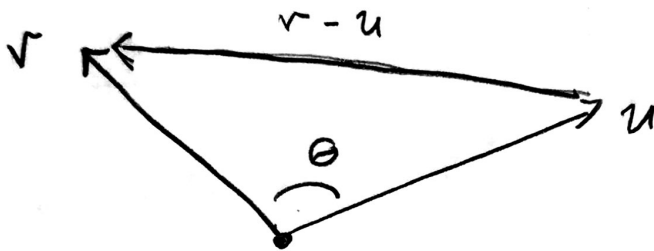
$\sin \theta$  IS Y COORD. VALUE.

SO 
$$v = \begin{bmatrix} \|v\| \cos \theta \\ \|v\| \sin \theta \end{bmatrix}$$

IF  $u$  AND  $v$  ARE VECTORS IN  $\mathbb{R}^3$ ,

$$\text{SO } u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

THEN  $u$  AND  $v$  MAKE AN ANGLE  $\theta \leq \pi$ :



APPLYING  
THE LAW OF COSINES TO THIS TRIANGLE:

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta,$$

$$\text{SO } \cos\theta = \frac{\|u\|^2 + \|v\|^2 - \|v-u\|^2}{2\|u\|\|v\|}$$

$$= \frac{u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2}{2\|u\|\|v\|}$$

$$= \frac{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}{2\|u\|\|v\|}$$

$$= \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{2 \|u\| \|v\|}$$

$$= \frac{u \cdot v}{\|u\| \|v\|}$$


---

SAME THING IN  $\mathbb{R}^3$  OR  $\mathbb{R}^n$ :

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} \quad 0 \leq \theta \leq \pi$$


---

EX.  $u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$   $v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\cos \theta = \frac{1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1}{\sqrt{1^2 + 1^2 + 0^2} \sqrt{0^2 + 1^2 + 1^2}} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

NOTICE THAT

$$\|v\|^2 = v \cdot v$$

AND ANGLE BETWEEN  $u$  AND  $v$   
GIVEN BY

$$(*) \quad \cos \theta = \frac{u \cdot v}{\|u\| \|v\|}$$

DOT PRODUCT SEEMS IMPORTANT.

ALSO CALLED THE STANDARD INNER PRODUCT,

NOTE THAT, BY  $(*)$ ,

TWO VECTORS ARE ORTHOGONAL,

OR MEET AT RIGHT ANGLES,

IFF  $u \cdot v = 0$ .

$$\underline{\text{Ex.}} \quad \begin{bmatrix} 7 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 7 \end{bmatrix} = 0,$$

Thm  $u, v, w \in \mathbb{R}^n$ .

- ①  $u \cdot u \geq 0$  AND  $u \cdot u = 0$  IFF  $u = \underline{0}$ .
- ②  $u \cdot v = v \cdot u$
- ③  $(u+v) \cdot w = u \cdot w + v \cdot w$
- ④  $c u \cdot v = c(u \cdot v)$

Def. A vector of length 1 is a UNIT VECTOR,

Given  $v \neq 0$ ;

$$u = \frac{1}{\|v\|} v \quad \text{is the } \underline{\text{UNIT VECTOR}}$$

IN THE DIRECTION OF  $v$ .

ex.

$$v = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\begin{aligned} \|v\| &= \sqrt{4 + 16} \\ &= 2\sqrt{5} \end{aligned}$$

$$\text{so } u = \frac{1}{2\sqrt{5}} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$$

is a unit vector.

### 5.3 INNER PRODUCT SPACES.

DEF.  $V$  A VECTOR SPACE.

AN INNER PRODUCT ON  $V$  IS A FUNCTION THAT ASSIGNS A #  $(u, v)$  TO ANY PAIR OF VECTORS  $u, v$  AND SATISFIES THE FOLLOWING:

- ①  $(u, u) \geq 0$  AND  $(u, u) = 0$  IFF  $u = \underline{0}$ .
- ②  $(v, u) = (u, v)$
- ③  $(u+v, w) = (u, w) + (v, w)$
- ④  $(cu, v) = c(u, v)$

NOTE THIS NOTATION!

$V$   
w/  
(, )  
is  
"INNER  
PRODUCT  
SPACE"

EX. STANDARD DOT PRODUCT.

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$(u, v) = u \cdot v = u_1 v_1 + \dots + u_n v_n.$$

$$\text{or } (u, v) = u \cdot v = u^T v \quad \left\{ \begin{array}{l} \text{matrix} \\ \text{mult.} \end{array} \right.$$

EX. Let  $V$  be of dimension  $n$   
with basis  $S$ .

then

$$(u, v) = [u]_S \cdot [v]_S$$

(can talk about ANGLES BETWEEN  
POLYNOMIALS !!  
(DEPENDS ON  $S$ ))



Ex.  $V$  vector space of all continuous functions on  $[0, 1]$ .

For  $f, g \in V$ ,

DEFINITION

$$(f, g) = \int_0^1 f(t)g(t) dt$$

$$(f, f) = \int_0^1 (f(t))^2 dt \geq 0 \quad \checkmark$$

$$(f, g) = \int_0^1 f(t)g(t) dt = \int_0^1 g(t)f(t) dt = (g, f)$$

$$\begin{aligned} (f+g, h) &= \int_0^1 (f(t)+g(t))h(t) dt \\ &= \int_0^1 f(t)h(t) dt + \int_0^1 g(t)h(t) dt \\ &= (f, h) + (g, h). \end{aligned}$$

$$\begin{aligned} (cf, g) &= \int_0^1 cf(t)g(t) dt = c \int_0^1 f(t)g(t) dt \\ &= c(f, g). \end{aligned}$$

$$\text{Ex. 9. } f = t - 1$$

$$g = t^2 + 2t + 1$$

$$(f, g) = \int_0^1 (t-1)(t^2+2t+1) dt$$

$$= \int_0^1 t^3 + 3t^2 + t - t^2 - 2t - 1 dt$$

$$= \int_0^1 t^3 + 2t^2 - t + 1 dt$$

$$= \left( \frac{t^4}{4} + \frac{2t^3}{3} - \frac{t^2}{2} + t \right) \Big|_0^1$$

$$= \frac{1}{4} + \frac{2}{3} - \frac{1}{2} + 1 - 0$$

$$= \frac{3}{4} + \frac{2}{3}$$

$$= \frac{9}{12} + \frac{8}{12} = \frac{17}{12}.$$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$(u, v) = u_1 v_1 - u_2 v_1 - 2u_1 v_2 + 3u_2 v_2$$

Case ① (other exercise).

$$\begin{aligned} (u, u) &= u_1^2 - 2u_1 u_2 + 3u_2^2 \\ &= u_1^2 - 2u_1 u_2 + u_2^2 + 2u_2^2 \\ &= (u_1 - u_2)^2 + 2u_2^2 \geq 0. \end{aligned}$$

① only when  $u_1 = u_2$  and  $u_2 = 0$ . ✓

---

ON A FINITE DIM<sup>l</sup> V, EVERY INNER PRODUCT IS COMPLETELY DETERMINED BY A MATRIX!

LET  $\{u_1, \dots, u_n\}$  BE ORTHOGONAL BASIS  
FOR  $V$ .

LET  $(, )$  BE AN INNER PRODUCT.

DEFINE  $c_{ij} = (u_i, u_j)$

AND LET  $C = [c_{ij}]$ .  $\rightarrow$  "MATRIX OF THIS  
INNER PRODUCT."

①  $C$  IS SYMMETRIC MATRIX.

②  $C$  DETERMINES  $(v, w)$

FOR ANY  $v, w$ .

Proof: ① ✓

$$\textcircled{2} \quad v = \sum_{i=1}^n a_i u_i$$

$$w = \sum_{i=1}^n b_i u_i$$

$$[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad [w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$(v, w) = \left( \sum_{i=1}^n a_i u_i, w \right)$$

$$= \sum_{i=1}^n (a_i u_i, w)$$

$$= \sum_{i=1}^n a_i (u_i, w)$$

$$= \sum_{i=1}^n a_i \left( u_i \sum_{j=1}^n b_j u_j \right)$$

→ CAREFUL WITH INDICES.

$$= \sum_{i=1}^n a_i \sum_{j=1}^n b_j (u_i u_j)$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i b_j (u_i u_j)$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i c_{ij} b_j$$

$$= [v]_s^T \cdot C \cdot [w]_s$$

$$[a_1, a_2, \dots, a_n] \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & \dots & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

||

$$[a_1, \dots, a_n] \begin{bmatrix} c_{11} b_1 + c_{12} b_2 + \dots + c_{1n} b_n \\ c_{21} b_1 + c_{22} b_2 + \dots + c_{2n} b_n \\ \vdots \\ c_{n1} b_1 + c_{n2} b_2 + \dots + c_{nn} b_n \end{bmatrix}$$

$$= [v]_s \cdot C [w]_s$$

$$= C [v]_s \cdot [w]_s$$

NOTE THAT HERE WE HAVE

$$C \underline{x} \cdot \underline{y} = \underline{x} \cdot C \underline{y}$$


---

IT IS IMPORTANT THAT  $C$  IS SYMMETRIC.

IN GENERAL, IF  $A$   $n \times n$ ,

STANDARD DOT PRODUCT SATISFIES

$$A \underline{x} \cdot \underline{y} = \underline{x} \cdot A^T \underline{y}$$


---

$C$  SATISFIES.

$$\underline{x}^T C \underline{x} > 0 \quad \forall \underline{x} \neq \underline{0}$$

~~DEF. AN  $n \times n$  SYMMETRIC MATRIX  $A$  IS POS~~

DEF. AN  $n \times n$  SYMMETRIC MATRIX  $A$   
IS POSITIVE DEFINITE IF

$$\underline{x}^T A \underline{x} > 0 \quad \forall \underline{x} \neq \underline{0}$$

(SUCH A MATRIX IS NON-SINGULAR.)

IF  $C$  IS ANY POSITIVE DEFINITE MATRIX, THEN IT DEFINES AN INNER PRODUCT:

$$\begin{aligned} (v, w) &= ([v]_S, [w]_S) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} c_{ij} b_j \end{aligned}$$

more later.

Ex.  $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$x^T C x = [x_1 \ x_2] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \ x_2] \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$$

$$= (2x_1^2 + x_1x_2) + (x_2x_1 + x_2^2)$$

$$= 2x_1^2 + 2x_1x_2 + x_2^2$$

$$= x_1^2 + (x_1 + x_2)^2 > 0 \text{ if } x \neq 0.$$

## THE CAUCHY-SCHWARZ INEQUALITY,

IN AN INNER PRODUCT SPACE,

$$|(u, v)| \leq \|u\| \|v\|.$$

Proof: IF  $u = 0$ ,  $\|u\| = 0$  AND  $(u, v) = 0$ . ✓

SUPPOSE  $u \neq 0$ . LET  $r \in \mathbb{R}$  AND CONSIDER

$$\|u + rv\|.$$

$$0 \leq (ru + rv, ru + rv)$$

$$= (u, u)r^2 + 2r(u, v) + (v, v)$$

$$= ar^2 + 2br + c$$

FIX  $u, v$ .

THEN  $p(r) = ar^2 + 2br + c$  IS

QUADRATIC POLYNOMIAL.

ALSO IT IS NONNEG.

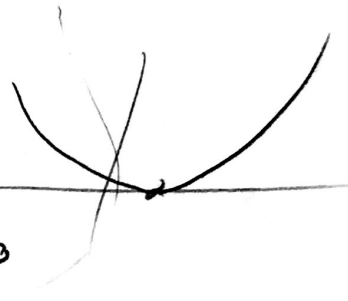
SO IT HAS AT MOST 1 REAL ROOT.

BY QUADRATIC FORMULA  $b^2 - ac \leq 0$

SO  $b^2 \leq ac$ . SO  $b \leq \sqrt{a}\sqrt{c}$ . ✓  $\square$

FOR SOME  $a, b, c$

$$\left( \begin{array}{l} a = (u, u) = \|u\|^2 \\ b = (u, v) \\ c = (v, v) = \|v\|^2 \end{array} \right)$$





AMAZING POWER OF LINEAR ALGEBRA:

$$\left| \int_0^1 f(x)g(x) dx \right|^2 \leq \left( \int_0^1 f(x)^2 dx \right) \cdot \left( \int_0^1 g(x)^2 dx \right)$$

ex.

$$\begin{aligned} \left| \int_0^1 e^x t^{100} dx \right|^2 &\leq \int_0^1 e^{2x} dx \int_0^1 t^{100} dx \\ &= \frac{1}{2} e^{2x} \Big|_0^1 \cdot \frac{t^{101}}{101} \Big|_0^1 \\ &= \left( \frac{e^2}{2} - \frac{1}{2} \right) \cdot \frac{1}{101} \end{aligned}$$

## TRIANGLE INEQUALITY.

$$\|u+v\| \leq \|u\| + \|v\|.$$

PROOF:

$$\begin{aligned} \|u+v\|^2 &= (u+v, u+v) \\ &= (u, u) + 2(u, v) + (v, v) \\ &= \|u\|^2 + 2(u, v) + \|v\|^2 \end{aligned}$$

$$(u, v) \leq |(u, v)| \leq \|u\| \|v\| \text{ BY C.S.}$$

$$\begin{aligned} \text{so } \|u+v\|^2 &\leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2. \end{aligned}$$



DEF. IN INNER PROD. SPACES:  
DISTANCE IS  $\|u-v\|$ .

$u, v$  ORTHOGONAL IF  $(u, v) = 0$ .

ANGLE GIVEN BY

$$\cos \theta = \frac{(u, v)}{\|u\| \|v\|}.$$

DEF.

A COLLECTION OF VECTORS IS

ORTHONORMAL IF EVERY PAIR

IS ORTHOGONAL AND THEY ALL

HAVE UNIT LENGTH.

EX.

STD. BASIS FOR  $\mathbb{R}^n$ .

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SA.

Thm If  $S = \{u_1, \dots, u_n\}$

is an ORTHONORMAL SET AND DOESN'T CONTAIN  $\underline{0}$ ,  
THEN  $S$  IS LINEARLY INDEPENDENT.

Proof: Suppose NOT.

Then

some  $u_i = \sum_{j \neq i} a_j u_j$ .

$$\begin{aligned} \text{then } \langle u_i, u_i \rangle &= \left\langle \sum_{j \neq i} a_j u_j, u_i \right\rangle \\ &= \sum_{j \neq i} a_j \langle u_j, u_i \rangle \\ &= 0 \end{aligned}$$

$\Rightarrow u_i = \underline{0}$ , THIS IS A CONTRADICTION.

□

5.4.

CONSIDER  $\mathbb{R}^3$  WITH THE USUAL INNER PRODUCT AND  
CONSIDER THE BASIS:

$$T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

THIS IS A PAIRWISE ORTHOGONAL BUT IT  
ISN'T AN ORTHONORMAL BASIS CUZ  
THEY AREN'T ALL UNIT VECTORS.

$$S = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

IS ORTHONORMAL.

CONSIDER  $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  WHAT'S  $[v]_S$ ?

$$v = \frac{3}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Thm IF  $S = \{u_1, \dots, u_n\}$

IS AN ORTHONORMAL BASIS IN AN INNER PRODUCT SPACE, THEN

$$v = c_1 u_1 + \dots + c_n u_n$$

WHERE

$$c_i = (v, u_i).$$

Proof:

$$v = c_1 u_1 + \dots + c_i u_i + \dots + c_n u_n \quad \text{FOR SOME } c_i$$

SINCE  $S$  IS A BASIS.

$$(v, u_i) = (c_1 u_1 + \dots + c_n u_n, u_i)$$

$$= c_1 (u_1, u_i) + \dots + c_i (u_i, u_i) + \dots + c_n (u_n, u_i)$$

$$= c_i (u_i, u_i) = \boxed{c_i} \quad \square$$

So we don't have to solve  
 a system of eqs to find  
 $[v]_{\mathcal{S}}$  !!

---

We can always find an orthonormal  
 basis using a process called

the Gram-Schmidt process :

Thm If  $W \neq \{0\}$  is a subspace of  $V$ ,

there is an orthonormal basis

$$T = \{w_1, \dots, w_m\} \text{ for } W.$$

Proof:

We first find an orthogonal basis

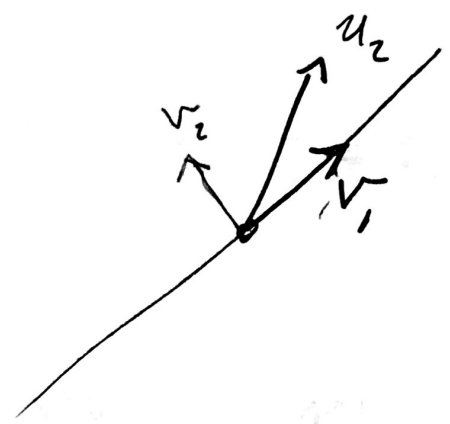
$$T^* = \{v_1, \dots, v_m\} \text{ for } W,$$

Pick any basis  $\{u_1, \dots, u_m\}$ .

our first vector in  $T^*$  is  $u_1$ .

So let  $v_1 = u_1$ . ✓

We now look for vector in span  $\{v_1, u_2\}$  that is orthogonal to  $v_1$ :



Whatever  $v_2$  is, it will be a linear combo of  $v_1$  and  $u_2$ :

$$v_2 = a_1 v_1 + a_2 u_2$$

We want  $(v_1, v_2) = 0$

$$\begin{aligned} \text{So } 0 &= (v_1, a_1 v_1 + a_2 u_2) \\ &= a_1 (v_1, v_1) + a_2 (v_1, u_2) \end{aligned}$$

#  
0

DEFINITION



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$$\text{So } a_1 = -a_2 \frac{(u_2, v_1)}{(v_1, v_1)}$$

CAN CHOOSE  $a_2$ .  
SET  $a_2 = 1$ .

$$\text{So } a_1 = - \frac{(u_2, v_1)}{(v_1, v_1)}$$

$$\text{LET } v_2 = a_1 v_1 + u_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1$$

SO  $v_1$  AND  $v_2$  ARE ORTHOGONAL,

NOW WE WANT  $v_3$  IN SPAN  $\{v_1, v_2, u_3\}$ .

SUCH THAT  $v_3$  IS ORTHOGONAL TO

$v_1$  AND  $v_2$ .

$$v_3 = b_1 v_1 + b_2 v_2 + b_3 u_3.$$

want to choose  $b_i$  so that

$$(v_3, v_2) = (v_3, v_1) = 0.$$

$$\begin{aligned} 0 = (v_3, v_1) &= (b_1 v_1 + b_2 v_2 + b_3 u_3, v_1) \\ &= b_1 (v_1, v_1) + b_2 (v_2, v_1) + b_3 (u_3, v_1) \\ &= b_1 (v_1, v_1) + 0 + b_3 (u_3, v_1) \end{aligned}$$

$$\begin{aligned} 0 = (v_3, v_2) &= (b_1 v_1 + b_2 v_2 + b_3 u_3, v_2) \\ &= 0 + b_2 (v_2, v_2) + b_3 (u_3, v_2) \end{aligned}$$

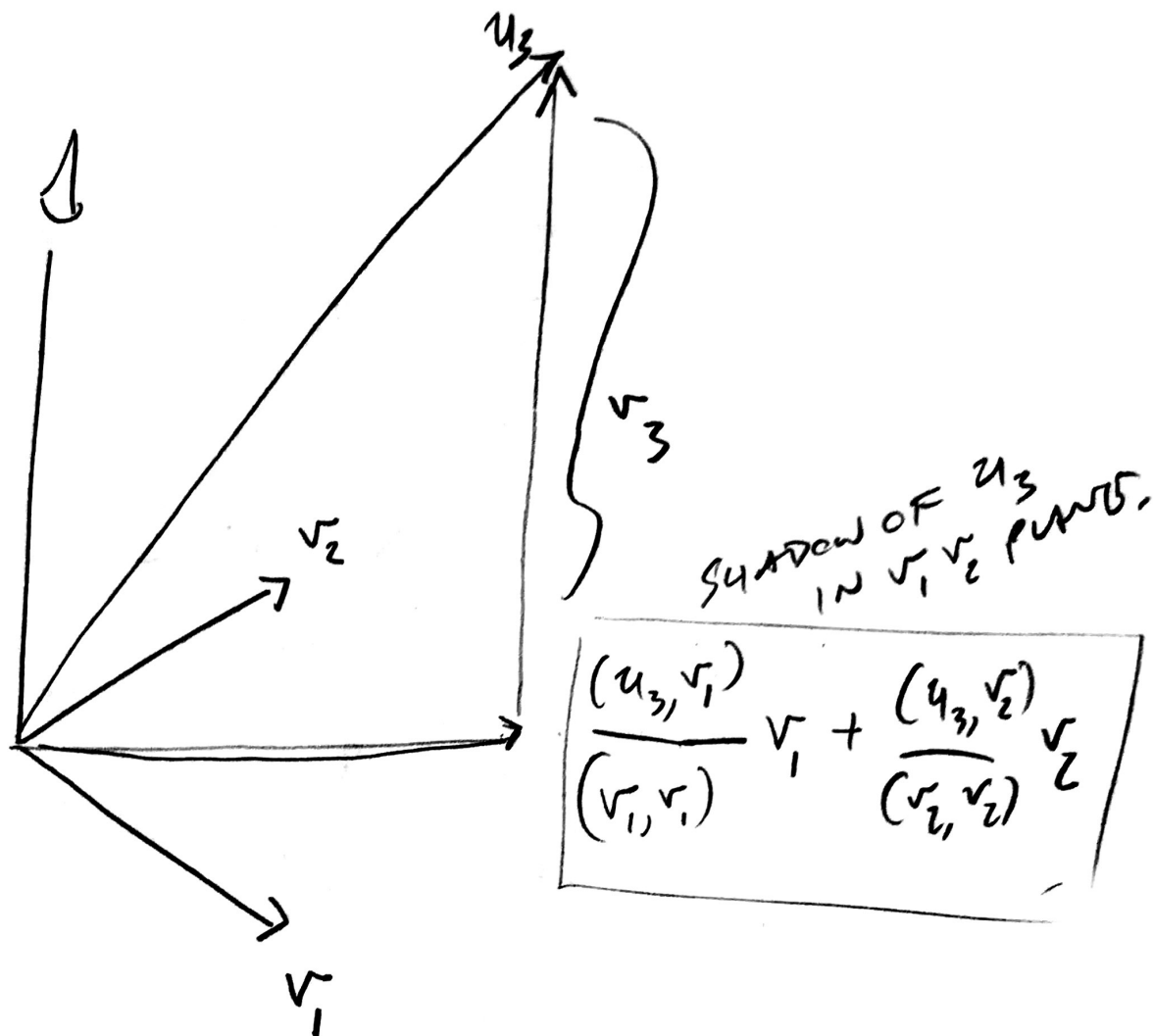
now  $b_1 = -b_3 \frac{(u_3, v_1)}{(v_1, v_1)}$

$$b_2 = -b_3 \frac{(u_3, v_2)}{(v_2, v_2)}$$

first to choose  $b_3$ , let  $b_3 = 1$ .

So

$$v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2$$



QUESTION 60156

$$v_4 = u_4 - \frac{(u_4, v_1)}{(v_1, v_1)} v_1 - \frac{(u_4, v_2)}{(v_2, v_2)} v_2 - \frac{(u_4, v_3)}{(v_3, v_3)} v_3,$$

⋮

$$T^* = \{v_1, \dots, v_m\}.$$

$$T = \{w_1, \dots, w_m\}$$

$$w_i = \frac{v_i}{\|v_i\|}.$$

□

Ex GRAM SCHMIDT STARTING WITH

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

PRODUCTION OF  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  TO GET  $v_1$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

PROD. TO GET  $\{v_1, v_2\}$ .

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

DOUBT ANSWERS GWS STANDARD PRIS

EX.  $P_3$  WITH  $(p, q) = \int_0^1 p(u)q(u)du$ .

FIND ORTHONORMAL BASIS FOR SUBSPACE

W OF POLYS WITH 0 CONSTANT COEFFICIENT,

$S = \{t^2, t\}$  IS A BASIS.

FIND ORTHONORMAL ONB.

$$v_1 = u_1 = t^2.$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1$$

$$= t - \frac{\int_0^1 t \cdot t^2 dt}{\int_0^1 t^2 \cdot t^2 dt} \cdot t^2$$

$$= t - \frac{\frac{1}{4}t^4 \Big|_0^1}{\frac{1}{5}t^5 \Big|_0^1} t^2 = t - \frac{\frac{1}{4}}{\frac{1}{5}} \cdot t^2 = t - \frac{5}{4}t^2$$

$$\|v_2\|^2 = (v_2, v_2) = \int_0^1 \left(t - \frac{5}{4}t^2\right)^2 dt$$

$$= \frac{1}{48}$$

$$\|v_2\| = \frac{1}{\sqrt{48}}$$

$$\|v_1\| = \frac{1}{\sqrt{5}}$$

So  $\left\{ \sqrt{5}t^2, \sqrt{48}\left(t - \frac{5}{4}t^2\right) \right\}$

IS AN ORTHONORMAL BASIS.  $\square$

IF WE HAD CHOSEN  $u_1 = t$ ,  $u_2 = t^2$ ,

WE WOULD GET

$$\left\{ \sqrt{3}t, \sqrt{30}\left(t^2 - \frac{1}{2}t\right) \right\}.$$

IF WE CHOOSE AN ORTHONORMAL BASIS  
 THEN INNER PRODUCT BOMAS USES  
 INNER PRODUCT IN COORDINATES:

THEM: LET  $S$  BE ORTHONORMAL BASIS  
 IN AN INNER PRODUCT SPACE.

$$\text{THEN } (v, w) = [v]_S \cdot [w]_S$$

↓  
 USUAL DOT PRODUCT.





## ORTHOGONAL COMPLEMENTS

CONSIDER A VECTOR SPACE  $V$  AND  
TWO SUBSPACES  $U, W$ .

LET  $U + W = \{u + w \mid u \in U, w \in W\}$ .

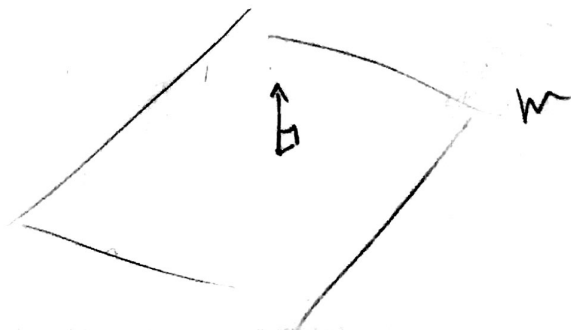
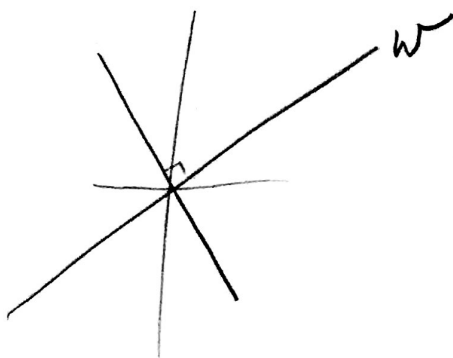
IF  $U \cap W = \{0\}$ ,

WE WRITE  $U \oplus W$ . (THIS DIRECT SUM)

IF  $V = U \oplus W$  WE SAY  $V$  IS  
DIRECT SUM OF  $U$  AND  $W$ .

IN THIS CASE EVERY  $v \in V$  IS EXPRESSED  
UNIQUELY AS  $v = u + w$  FOR  $u \in U$  AND  $w \in W$ .

GIVEN A SUBSPACE  $W \subset V$  IT IS OFTEN  
DESIRABLE TO FILL  $W$  OUT TO  $V$  IN  
THIS WAY, I.E. FIND A  $U$  S.T.  $W \oplus U = V$ .



DEF. LET  $W \subset V$  BE A SUBSPACE.

A VECTOR  $u \in V$  IS ORTHOGONAL TO  $W$   
IF IT IS ORTHOGONAL TO EVERY VECTOR  
IN  $W$ .

$$W^\perp = \{ u \in V \mid u \text{ ORTHOGONAL TO } W \}$$

IS THE ORTHOGONAL COMPLEMENT OF  $W$  IN  $V$ .

EX  $w = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$  IN  $\mathbb{R}^3$  WITH DOT PRODUCT.

$$W = \text{SPAN} \{ w \}.$$

$$W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 4x - y - 2z = 0. \right\}$$

EQUATION OF A PLANE  
FROM SCHOOL.

Thm (1)  $W^\perp$  is a subspace. (closed under add. & mult.)  
 (2)  $W \cap W^\perp = \{0\}$  ✓

Ex If you'd like to show if  
 $u \in W^\perp$ , then it is enough  
 to check that  $u$  is orthogonal to  
 a spanning set  $S$  of  $W$ .

Ex.  $P_3$  with  $(p, q) = \int pq$ .

$W = \text{span} \{1, t^2\}$ .

Find basis for  $W^\perp$ .

$p(t) = a t^3 + b t^2 + c t + d \in W^\perp$ ,

in particular  $(p, 1) = 0 = (p, t^2)$ .

$$(p(t), 1) = \int_0^1 (a t^3 + b t^2 + c t + d) dt = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0$$

$$(p(t), t^2) = \int_0^1 (a t^5 + b t^4 + c t^3 + d t^2) dt = \frac{a}{6} + \frac{b}{5} + \frac{c}{4} + \frac{d}{3} = 0$$

So we:

Get

$$a = 3r + 16s, \quad b = -\frac{15}{4}r - 15s, \quad c = r, \quad d = s.$$

$$p(t) = (3r + 16s)t^3 + \left(-\frac{15}{4}r - 15s\right)t^2 + rt + s$$

$$= r \underbrace{\left(3t^3 - \frac{15}{4}t^2 + t\right)}_{v_1} + s \underbrace{\left(16t^3 - 15t^2 + 1\right)}_{v_2}.$$

$$\text{SPAN } \{v_1, v_2\} = W^\perp.$$

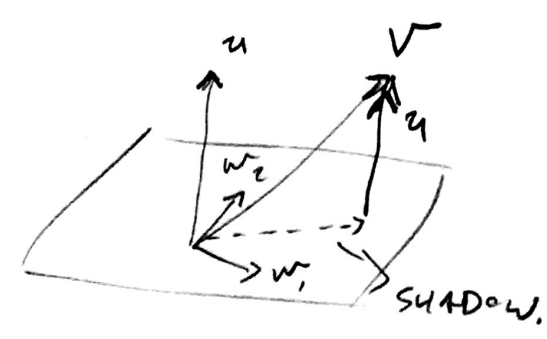
They are lin. indep. since they are not multiples of another.

Thus  $W$  is a finite dimensional subspace of an inner product space. Thus

$$V = W \oplus W^\perp,$$

Proof:

Suppose  $\dim W = m$ .



By GRAM-SCHMIDT, there is an orthonormal basis for  $W$ , say

$$S = \{w_1, \dots, w_m\}.$$

Let  $v \in V$ .

Define

$$w = (v, w_1)w_1 + (v, w_2)w_2 + \dots + (v, w_m)w_m \in W$$

SHADOW OF  $v$  ON  $W$ .  
"PROJECTION OF  $v$  TO  $W$ ."

Define

$$u = v - w.$$

$u \in W^\perp$  since

$$\begin{aligned}
 (u, w_i) &= (v - w, w_i) \\
 &= (v, w_i) - (w, w_i) \\
 &= (v, w_i) - ((v, w_1)w_1 + \dots + (v, w_m)w_m, w_i) \\
 &= (v, w_i) - 0 - 0 - \dots - (v, w_i)(w_i, w_i) - 0 - \dots - 0 \\
 &= (v, w_i) - (v, w_i) \cdot 1 \\
 &= 0.
 \end{aligned}$$

So  $u$  orthogonal to everything in  $W$ ,  
So  $u \in W^\perp$ .

Now,

$$V = W + u$$

and so  $V = W + W^\perp$

So  $V = W \oplus W^\perp$  since  $W \cap W^\perp = \{0\}$ .

Thm, If  $W$  is  $\ell$  dimensional subspace of  $V$ ,

then  $(W^\perp)^\perp = W$ .

Proof (sketch).

GEOMETRIC MEANING OF FUNDAMENTAL SUBSPACES

Thm A  $m \times n$  matrix.

① The kernel of  $A$  is the orthogonal complement  
(null space)  
of the row space.

② The image of  $A$  (column space) is  
the orthogonal complement of the  
kernel of  $A^T$ .

$$A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$\cup$   
 $\ker A$

Proof: a)

NOTE THAT THE DIMENSIONS ARE CORRECT.

THE ROW SPACE HAS DIMENSION  $r = \text{RANK}$

ITS COMPLEMENT HAS DIMENSION  $n - r$ ,

WHICH IS THE DIMENSION OF THE KERNEL.

LET  $\underline{x} \in \ker A \subset \mathbb{R}^n$ . ( $A\underline{x} = \underline{0}$ )

LET  $v_1, \dots, v_m$  BE ROWS OF  $A$ .

$$A\underline{x} = \begin{bmatrix} v_1 \cdot \underline{x} \\ \vdots \\ v_m \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} v_1 \cdot \underline{x} \\ v_2 \cdot \underline{x} \\ \vdots \\ v_m \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0}.$$

$\underline{x}$  ORTHOGONAL TO SPAN  $v_i$  AND

HENCE EVERYTHING IN THE ROW SPACE.

SO  $\underline{x}$  IS IN COMPLEMENT OF ROW SPACE.

SO  $\ker A$  IS CONTAINED IN COMPLEMENT OF  
ROW SPACE.

CONVERSELY IF  $\underline{x}$  IS ORTHOGONAL TO EVERYTHING  
IN ROW SPACE, IT IS ORTH. TO ROWS

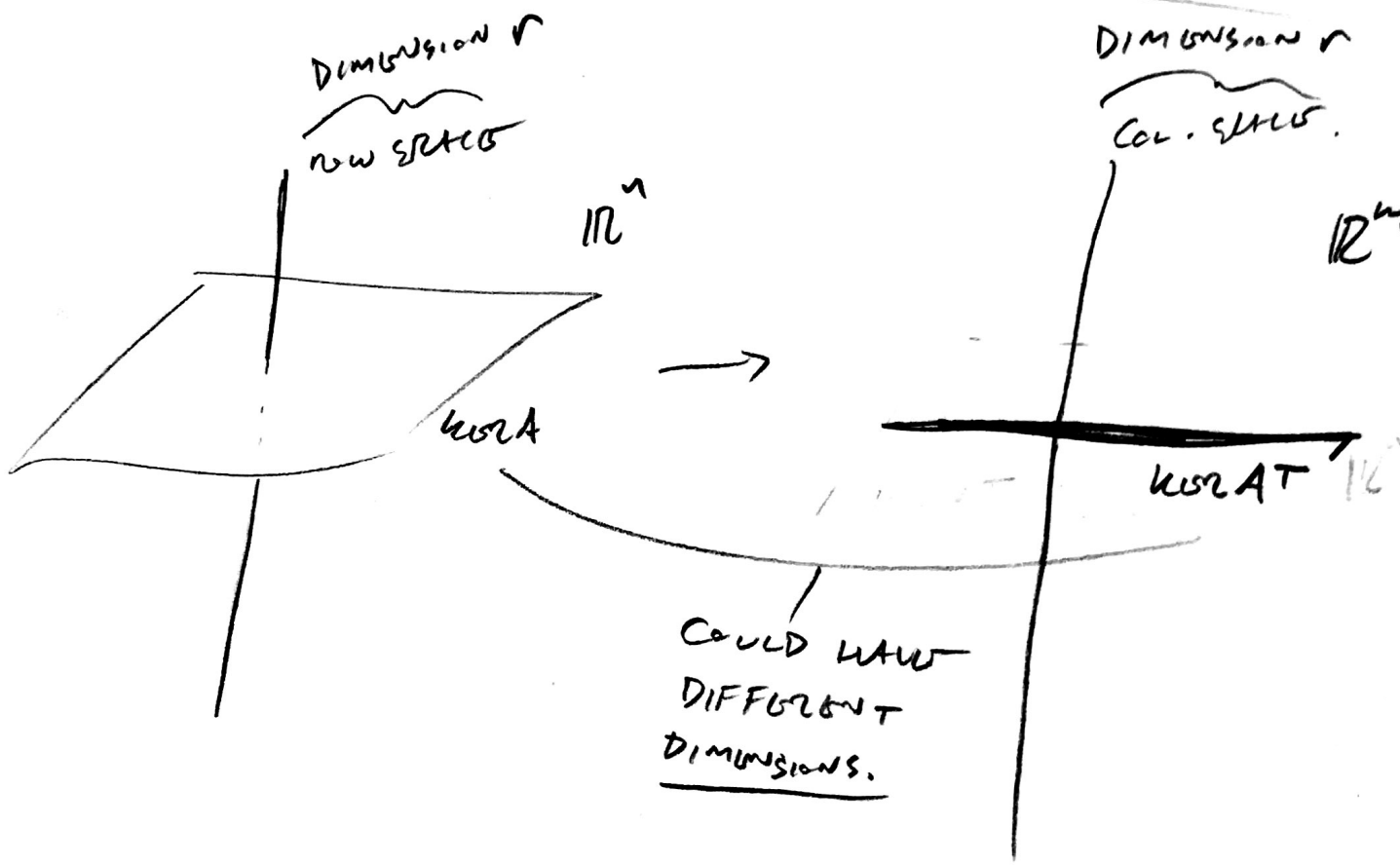
AND SO  $A\underline{x} = \begin{bmatrix} v_1 \cdot \underline{x} \\ \vdots \\ v_m \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0}$ . ✓

b) REPLACES A WITH  $A^T$ , ✓

□

WARNING:

DRAWINGS IN BOOK ARE TERRIBLE!  
(THEY SHOW  $KER(A)$  AND ROW SPACES  
INTERSECTING IN A LINE.)





$$\text{Ex } A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -3 & 2 & 7 \\ 0 & 2 & -6 & 4 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -3 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Bx = 0$$

$$\leadsto S = \left\{ \begin{bmatrix} -2 \\ +3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} +1 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

basis for nullspace. (see 4.7)

$$T = \left\{ [1 \ 0 \ 2 \ 1 \ -1], [0 \ 1 \ -3 \ 2 \ 7] \right\}$$

is a basis for row space

(not always just this subset of rows.)

S and T are orthogonal. (Thinking of  
 vts of S as row vectors)  
 or vts of T as cols.

## PROJECTIONS AGAIN:

$W$  FTS  $\dim^2$  SUBSPACE WITH ORTHONORMAL BASIS  $S = \{w_1, \dots, w_m\}$ . GIVEN  $v \in V$ , THERE ARE UNIQUE  $w \in W$  AND  $u \in W^\perp$  s.t.

$$v = w + u.$$

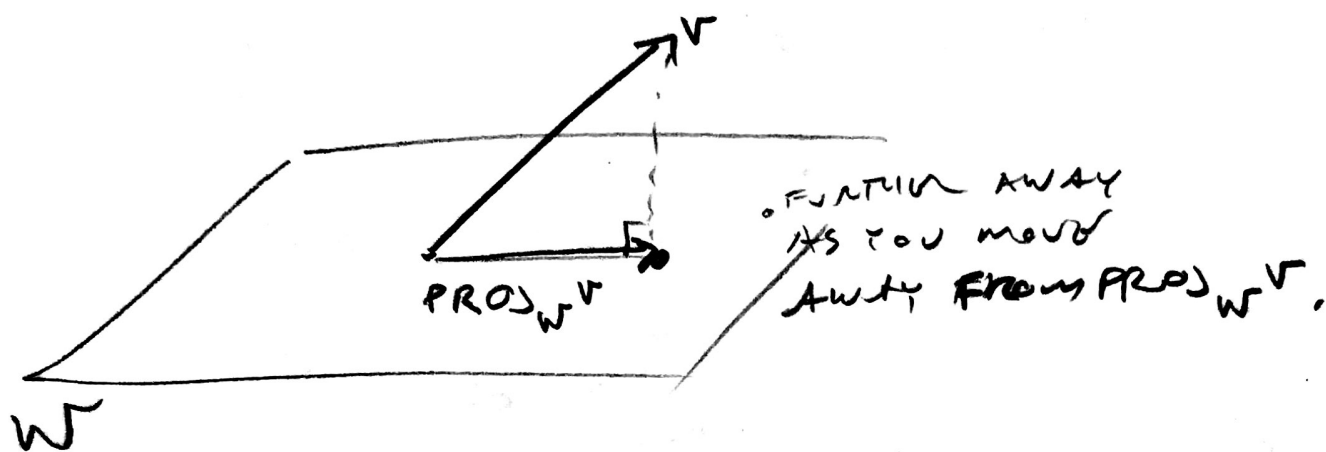
$$w = \underbrace{(v, w_1)w_1 + (v, w_2)w_2 + \dots + (v, w_m)w_m}_{\text{PROJ}_W v} \quad \text{"SHADOW"}$$

ORTHOGONAL PROJECTION OF  $v$  TO  $W$ .

IF  $\{w_1, \dots, w_m\}$  IS ORTHOGONAL

$$\text{PROJ}_W v = \frac{(v, w_1)}{(w_1, w_1)} w_1 + \dots + \frac{(v, w_m)}{(w_m, w_m)} w_m.$$

NOTE THAT  $\text{PROJ}_W v$  IS CLOSEST POINT  
ON  $W$  TO  $v$ .



SO DISTANCE FROM  $v$  TO  $W$  IS

$$\|v - \text{proj}_W v\|,$$

15x

WHAT IS THE DISTANCE FROM

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

TO PLANE

$$2x - y + 2z = 0,$$

$$W = \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right\}$$

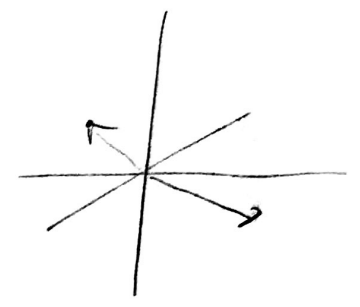
THEN PLANE IS  $W^\perp$ .

WHAT IS A BASIS FOR  $W^\perp$ ?

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in W^\perp, \text{ so is } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

LIN. INDEPENDENT AND SPAN.

NOT ORTHOGONAL:  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 1.$



GRAM SCHMIDT:

PROJECT  $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$  TO  $\text{SPAN} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ .

$$\frac{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ 2 \\ \frac{1}{2} \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 2 \\ \frac{1}{2} \end{bmatrix} \right\} \quad \underline{\text{ORTHOGONAL}} \quad \text{BASIS FOR } W^\perp$$

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{\frac{1}{4} + 4 + \frac{1}{4}}} \begin{bmatrix} \frac{1}{2} \\ 2 \\ \frac{1}{2} \end{bmatrix} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{\sqrt{2}}{3} \begin{bmatrix} \frac{1}{2} \\ 2 \\ \frac{1}{2} \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \\ \frac{\sqrt{2}}{6} \end{bmatrix} \right\} \quad \text{ORTHOGONAL BASIS.}$$

$$PR_{W^\perp} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} + \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2}/6 \\ 2\sqrt{2}/3 \\ \sqrt{2}/6 \end{bmatrix} \right) \begin{bmatrix} \sqrt{2}/6 \\ 2\sqrt{2}/3 \\ \sqrt{2}/6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \left( \frac{\sqrt{2}}{6} + \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{6} \right) \begin{bmatrix} \sqrt{2}/6 \\ 2\sqrt{2}/3 \\ \sqrt{2}/6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \left( \frac{6\sqrt{2}}{6} \right) \begin{bmatrix} \sqrt{2}/6 \\ 2\sqrt{2}/3 \\ \sqrt{2}/6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

DISTANCE IS

$$\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 4/3 \\ 1/3 \end{bmatrix} \right\|$$

$$= \left\| \begin{matrix} +2/3 \\ -1/3 \\ +2/3 \end{matrix} \right\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{1} = 1 \quad \checkmark$$

CHAPTER 6.

BACK TO LINEAR TRANSFORMATIONS.

A LINEAR TRANSFORMATION (LINEAR MAP)  
AKA LINEAR OPERATOR

$$L: V \rightarrow W$$

$$L(u+v) = L(u) + L(v)$$

$$L(cu) = cL(u).$$

EX.  $L(v) = Av$   $A$   $m \times n$  matrix.

EX. REFLECTION W/ X-AXIS.

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$$

EX REFLECTION IN LINE  $x=y$

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}.$$

EX DILATION PICK  $c \in \mathbb{R}$   $c > 1$   
 $L: V \rightarrow V$

$$L(v) = cv$$

bx CONTRACTION  $0 < c < 1$

$$L(v) = cv.$$

Non bx  $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ 2y \\ z \end{bmatrix}$$

NOT LIN. TRANS.

$$L(\underline{0}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \underline{0}.$$

LINEAR  
TRANS.

$$L(\underline{0}) = \underline{0}$$

SINCE ~~LINEAR~~

$$\begin{aligned} L(\underline{0}) &= L(v - v) = L(v) + L(-v) \\ &= L(v) - L(v) \\ &= \underline{0}. \end{aligned}$$



Ex.  $L: P_1 \rightarrow P_2$

$$L(p(t)) = t p(t)$$

$$L(p(t) + q(t)) = t(p(t) + q(t))$$

$$= t p(t) + t q(t)$$

$$= L(p(t)) + L(q(t)).$$

$$L(c p(t)) = t(c p(t))$$

$$= c t p(t)$$

$$= c L(p(t)).$$

Ex.  $\mathcal{D} =$  ALL DIFFERENTIABLE FUNCTIONS  
 $\mathcal{F} =$  ALL FUNCTIONS.

$$L(f) = f', \quad L: \mathcal{D} \rightarrow \mathcal{F}$$

$$L(f + g) = (f + g)' = f' + g' = L(f) + L(g)$$

$$L(cf) = (cf)' = cf' = cL(f).$$

62

$\mathbb{R}$  IS VECTOR SPACE.

LET  $\mathcal{S}$  BE ALL INTEGRABLE  
FUNCTIONS.  $L: \mathcal{S} \rightarrow \mathbb{R}$

$$L(f) = \int_a^b f(x) dx =$$

$$\begin{aligned} L(f+g) &= \int_a^b f(x) + g(x) dx \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx \\ &= L(f) + L(g) \end{aligned}$$

$$\begin{aligned} L(cf) &= \int_a^b cf(x) dx = c \int_a^b f(x) dx \\ &= cL(f). \end{aligned}$$

Thm  $L$  is completely determined  
by its values on a basis.

Thm Given  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$   
Form the matrix  $A$  whose  $i$ th  
column is  $L(e_i)$ :

$$A = \begin{bmatrix} | & & | \\ L(e_1) & \dots & L(e_n) \\ | & & | \end{bmatrix}$$

$$L(v) = Av.$$

Proof.  $v = a_1 e_1 + \dots + a_n e_n$

$$\begin{aligned} L(v) &= L(a_1 e_1 + \dots + a_n e_n) \\ &= a_1 L(e_1) + \dots + a_n L(e_n) \\ &= A \cdot v. \end{aligned}$$

□

$$L: V \rightarrow W$$

$$\text{Ker } L = \{v \mid L(v) = \underline{0}\}.$$

Thm. ①  $\text{Ker } L$  is a subspace

②  $L$  is one-to-one iff  $\text{Ker } L = \{0\}$ .

Proof. ① Exercise

② ( $\Rightarrow$ ) Suppose  $\text{Ker } L \neq \{0\}$ .

$\Rightarrow$  So there are  $x \neq y$  with  $L(x) = L(y) = \underline{0}$ .

Then  $L$  is not one-to-one.  $\checkmark$

( $\Leftarrow$ ) Suppose  $\text{Ker } L = \{0\}$ .

~~Suppose~~ Suppose  $L(v) = L(w)$

$$L(v) - L(w) = \underline{0}.$$

$$\Rightarrow L(v-w) = \underline{0}.$$

Since  $\text{Ker } L = \{0\}$ , we have

$$v-w = \underline{0}$$

so  $v=w$ . So  $L$  is 1-1.



$$\text{Thm } L: V \rightarrow W$$

$$\dim \ker L + \dim \text{range } L = \dim V.$$

Thm IF  $\dim V = \dim W$ ,

THEN

① IF  $L$  IS 1-1, THEN IT IS ONTO.

② IF  $L$  IS ONTO, THEN IT IS 1-1.

DEF. ~~DEF.~~

$L: V \rightarrow W$  IS INVERTIBLE

IF  $\exists L^{-1}: W \rightarrow V$  s.t.

$$L^{-1}L = I_V \quad \text{AND} \quad LL^{-1} = I_W,$$

Thm  $L: V \rightarrow W$  IS INVERTIBLE IFF

IT IS A BIsECTION.

Proof

IF  $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$

IS GIVEN BY A MATRIX  $A$

(SO  $L(v) = Av$ )

AND  $L$  HAS AN INVERSE  $L^{-1}$ ,

THEN  $L^{-1}(v) = A^{-1}v$ .

$L$  IS INVERTIBLE IFF

$\det A \neq 0$ .

Then  $L: V \rightarrow W$  IS INVERTIBLE

IFF  $L(\{S\})$  IS LINEARLY INDEPENDENT

FOR EVERY LIN. IND.  $S$ .

## SUMMARY OF THINGS WE KNOW:

TFAB:

1.  $A$  NONSINGULAR (INVERTIBLE)
  2.  $A\underline{x} = \underline{0}$  HAS ONLY TRIVIAL SOL<sup>s</sup>
  3.  $A$  ROW EQ. TO  $I$
  4.  $A\underline{x} = \underline{b}$  HAS UNIQUE SOL<sup>n</sup> FOR ALL  $\underline{b}$ .
  5.  $A$  PRODUCT OF ELEM. MATRICES.
  6.  $\det A \neq 0$
  7.  $A$  HAS RANK  $n$
  8. rows<sup>(cols)</sup> OF  $A$  L.I.; I.N.D.
  9. DIMENSION OF KERNEL OF  $A$  IS ZERO
  10.  $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$  DEFINED BY  $L(v) = Av$   
IS 1-1 AND ONTO.
- 

IF  $L: V \rightarrow V$  IS INVERTIBLE

IFF  $L$  IS 1-1

IFF  $L$  IS ONTO.

6.3

MATRIX OF A LINEAR TRANSFORMATION,GIVEN AN  $m \times n$  MATRIX  $A \mathbb{R}^n \rightarrow \mathbb{R}^m$ 

$$\leadsto L: \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ BY } L(v) = Av.$$

Now we want to get a matrix  
to encode an abstract  $L: V \rightarrow W$ ,

Let  $V$  be of dimension  $n$ 
 $W$  " " " "  $m$ 
Let  $S = \{v_1, \dots, v_n\}$  be an ordered basis of  $V$ Let  $T = \{w_1, \dots, w_m\}$  " " " " "  $W$ ,

Let

$$A = \begin{bmatrix} | & | & & | \\ [L(v_1)]_T & [L(v_2)]_T & \dots & [L(v_n)]_T \\ | & | & & | \end{bmatrix}$$

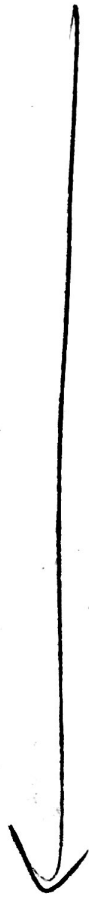


THIS MATRIX HAS PROPERTY

$$[L(x)]_T = A[x]_S$$

A IS THE ONLY MATRIX THAT DOES THIS.

Proof



$$\underline{x} = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

$$[\underline{x}]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

$$\begin{aligned} L[\underline{x}] &= L(a_1 v_1 + a_2 v_2 + \dots + a_n v_n) \\ &= a_1 L(v_1) + \dots + a_n L(v_n) \end{aligned}$$

$$\begin{aligned} [L(\underline{x})]_T &= [a_1 L(v_1) + \dots + a_n L(v_n)]_T \\ &= a_1 [L(v_1)]_T + \dots + a_n [L(v_n)]_T \\ &= A [\underline{x}]_S. \end{aligned}$$

SUPPOSE THERE IS ANOTHER MATRIX

B s.t.

#

A

$$[L(x)]_T = B[x]_S \quad \text{FOR ALL } x,$$

SUPPOSE  $k$ TH COLUMNS OF A AND B DIFFERENT,

$$[v_k]_S = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow k\text{th row.}$$

$$[L(v_k)]_T = A \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = k\text{th col of } A.$$

$$= B \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = k\text{th col of } B$$

#

□

$$\text{Ex } L: P_2 \rightarrow P_1$$

$$L(p(t)) = p'(t).$$

$$S = \{t^2, t, 1\} \quad T = \{t, 1\}.$$

FIND MATRIX FOR  $L$  W.R.T.  $S$  AND  $T$ .

$$L(t^2) = 2t = 2t + 0 \cdot 1$$

$$L(t) = 0 \cdot t = 0 \cdot t + 1 \cdot 1$$

$$L(1) = 0 = 0 \cdot t + 0 \cdot 1$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$[L(t^2)]_T = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$[L(t)]_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[L(1)]_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

---


$$p(t) = 5t^2 - 3t + 2$$

$$L(p(t)) = 10t - 3.$$

$$[L(p(t))]_T = \begin{bmatrix} 10 \\ -3 \end{bmatrix}.$$

$$[p(t)]_S = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$$

$$\begin{aligned} A \cdot [p(t)]_S &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \checkmark \end{aligned}$$

SAME  $L$ , DIFFERENT BASIS

$$S = \{t^2, t, 1\} \quad T = \{t+1, t-1\}$$

FIND A.

$$L(t^2) = 2t = a(t+1) + b(t-1)$$

$$\leadsto \begin{array}{l} a = 1 \\ a + b = 2 \end{array} \quad b =$$

$$a - b = 0$$

$$\leadsto a = 1 \quad b = 1.$$

$$\text{So } [L(t^2)]_T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

SIMILARLY:

$$L(t) = 1 = \frac{1}{2}(t+1) - \frac{1}{2}(t-1)$$

$$[L(t)]_T = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$L(1) = 0 = 0(t+1) + 0(t-1) \quad [L(1)]_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix},$$

---


$$[L(p(x))]_T = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ \frac{13}{2} \end{bmatrix}$$

$$\begin{aligned} \text{So } L(p(x)) &= \frac{7}{2}(t+1) + \frac{13}{2}(t-1) \\ &= 10t - 3. \end{aligned}$$

QX.

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^2,$$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

WHAT ARE THE T-COORDINATE VECTORS FOR THESE?

SOLVE:

$$\left[ \begin{array}{cc|cc|c} 1 & 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 5 & 3 \end{array} \right]$$

$$\leadsto \left[ \begin{array}{cc|ccc} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 \end{array} \right]$$

MATRIX OF  $L$   
W.R.T.  $S$  AND  $T$ .

So, TO FIND  $A$  W.R.T.

$$S = \{v_1, \dots, v_n\} \quad T = \{w_1, \dots, w_m\}$$

~~TABLE~~

$$\left[ w_1 \ w_2 \ \dots \ w_m \mid L(v_1) \mid \dots \mid L(v_n) \right]$$

TO REDUCED ROW ECHelon FORM.

MATRIX ON RIGHT IS  $A$ .



## CHAPTER 7 EIGENVECTORS & EIGENVALUES

HENCE IS A SIMPLY MATRIX:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

A DEFINES A LINEAR TRANSFORMATION

$$L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$L(v) = Av.$$

NOTICE

$$Ae_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2e_1,$$

$$Ae_2 = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}e_2.$$



NOTE

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ FOR ANY } \lambda.$$

bx.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -2 + \frac{1-\sqrt{5}}{2} \\ -1 + \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\left(\frac{3+\sqrt{5}}{2}\right) \\ -\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{3+\sqrt{5}}{2} \begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$



$$\left(\frac{3+\sqrt{5}}{2}\right) \left(\frac{1-\sqrt{5}}{2}\right) = \frac{3 - 2\sqrt{5} - 5}{2 \cdot 2}$$

$$= \frac{-2 - 2\sqrt{5}}{4} = \frac{1-\sqrt{5}}{2}$$

EX, ROTATION:



NO VECTOR SUCH THAT

$$Av = \lambda v$$

THIS SAYS THAT  $Av$  AND  $v$  ARE PARALLEL.

DEF:  $L: V \rightarrow V$  A LINEAR TRANSFORMATION

A NONZERO VECTOR  $v$  IS AN EIGENVECTOR

OF  $L$  IF  $L(v) = \lambda v$  FOR SOME  $\lambda \in \mathbb{R}$ .

$\lambda$  IS CALLED THE ASSOCIATED EIGENVALUE.

EX

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Ex.  $L(x) = 2x.$

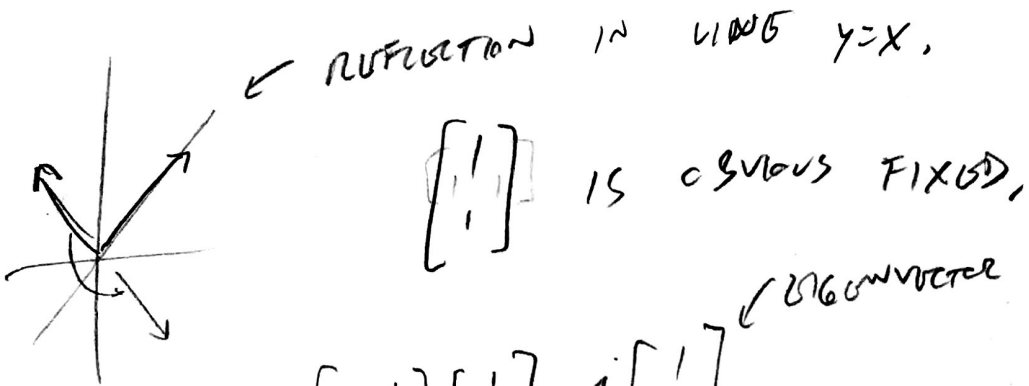
2 ONLY EIGENVALUE.

EVERY VECTOR AN EIGENVECTOR FOR 2.

Ex.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

EIG. VALUE

EIGENVECTOR

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

SO IT'S  
FIXED

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \checkmark$$

EIG. VALUE

EIG. VECTOR

Ex, let  $V$  be all functions  
that are  $\infty^y$  differentiable.

$$L: V \rightarrow V$$

$$L(f) = f'$$

Does  $L$  have eigenvectors?

$$L(f) = \lambda f$$

$$f' = \lambda f$$

$$\frac{df}{dx} = \lambda f$$

Ex,  $f(x) = ke^{\lambda x}$

How do we find these?

Ex.  $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

want  $\lambda$  and  $v$  s.t.

$$Av = \lambda v.$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\leadsto x + y = \lambda x$$

$$-2x + 4y = \lambda y$$

$$\leadsto (1-\lambda)x + y = 0$$

$$-2x + (4-\lambda)y = 0$$

$$\left[ \begin{array}{cc|c} 1-\lambda & 1 & 0 \\ -2 & 4-\lambda & 0 \end{array} \right]$$

$$\begin{vmatrix} \lambda-1 & -1 \\ 2 & \lambda-4 \end{vmatrix} = 0$$

NONTRIVIAL Sol<sup>n</sup>  
IFF

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$\text{IFF } (\lambda-1)(\lambda-4)+2=0$$

$$\text{IFF } \lambda^2-5\lambda+6=0 = (\lambda-3)(\lambda-2)$$

So our only possible eigenvalues

$$\text{are } \lambda_1=2, \lambda_2=3.$$

What are the  $v$ ?

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} \leadsto x+y &= 2x \\ -2x+4y &= 2y \end{aligned}$$

$$\begin{aligned} \leadsto (1-2)x + y &= 0 \\ -2x + (4-2)y &= 0 \end{aligned}$$

$$\begin{aligned} \leadsto x - y &= 0 \\ 2x - 2y &= 0 \end{aligned}$$

$$x=y \quad y = \text{anything} = r$$

$$v = \begin{bmatrix} r \\ r \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} r \\ r \end{bmatrix} &= \begin{bmatrix} 2r \\ 2r \end{bmatrix} \\ &= 2 \begin{bmatrix} r \\ r \end{bmatrix}. \end{aligned}$$

$$\lambda_e = 3$$

$\leadsto$

$$2x - y = 0$$

$$2x - y = 0$$

$$\Rightarrow x = \frac{1}{2}y$$

$$y = s,$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} s/2 \\ s \end{bmatrix} = \begin{bmatrix} \frac{3s}{2} \\ -s + 4s \end{bmatrix} = 3 \begin{bmatrix} s/2 \\ s \end{bmatrix}.$$

---



DEFINITION

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ \vdots & \lambda - a_{22} & & \vdots \\ -a_{n1} & & & \lambda - a_{nn} \end{bmatrix}$$

$p(\lambda) = \det(\lambda I - A)$  is a polynomial in  $\lambda$ .

this characteristic polynomial.

$$\det(\lambda I - A)$$

$$= \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{n-1} \lambda + a_n$$

$$\downarrow$$

$$a_n = \det(-A)$$

$$= (-1)^n \det A$$

$$\text{ex } A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -2 & 1 \\ -1 & \lambda & -1 \\ -4 & 4 & \lambda - 5 \end{bmatrix} \quad \begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

$$p(\lambda) = (\lambda - 1) \begin{vmatrix} \lambda & -1 \\ 4 & \lambda - 5 \end{vmatrix} + 1 \begin{vmatrix} \lambda - 2 \\ -1 & 4 \end{vmatrix}$$

$$+ 2 \begin{vmatrix} -1 & -1 \\ -4 & \lambda - 5 \end{vmatrix}$$

$$+ \begin{vmatrix} -1 & \lambda \\ -4 & 4 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda(\lambda - 5) + 5) + 2((-1)(\lambda - 5) - 5) + (-4 + \lambda 4)$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6.$$

THE EIGENVALUES OF  $A$   
 ARE

THE ROOTS OF CHARACTERISTIC POLY OF  $A$ .

Proof.

$$Ax = \lambda x \Leftrightarrow Ax = (\lambda I)x$$

$$\Leftrightarrow (A - \lambda I)x = 0$$

THIS HAS NONTRIVIAL SOL<sup>n</sup>

$$\text{IFF } \det(A - \lambda I) = 0.$$

SO  $\lambda$  IS A ROOT OF  $\det(A - \lambda I)$ .

□

---

FINDING ROOTS IS HARD!

BACK TO OUR EARLIER EXAMPLES:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 2)(\lambda - 1) - 1$$

$$= \lambda^2 - 3\lambda + 2 - 1$$

$$= \lambda^2 - 3\lambda + 1$$

roots ARE  $\frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{3 + \sqrt{5}}{2} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = \frac{3 + \sqrt{5}}{2} x \quad |$$

$$x + y = \frac{3 + \sqrt{5}}{2} y \quad \Rightarrow$$

$$\left(2 - \frac{3+\sqrt{5}}{2}\right)x + y = 0$$

$$x + \left(1 - \frac{3+\sqrt{5}}{2}\right)y = 0$$

$$\leadsto \left[ \begin{array}{cc|c} 1 - \frac{\sqrt{5}}{2} & 1 & 0 \\ 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \end{array} \right],$$

2ND row is a MULT. OF FIRST.

$y =$  ANYTHING.

$$\text{LET } y = \frac{1+\sqrt{5}}{2}$$

$$x = -1 \quad \checkmark$$

## 7.2. DIAGONALIZATION.

WE HAD THIS EXAMPLE:

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

THIS MATRIX IS NICE!

TWO EIGENVALUES AND WE  
CAN SOLVE THEM!

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

ISN'T AS NICE.

EIGENVALUES  $\frac{3+\sqrt{5}}{2}$  AND  $\frac{3-\sqrt{5}}{2}$

$$L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$L(v) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} v.$$

$L: \mathbb{R}^2 \rightarrow$ 

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{-3+\sqrt{5}}{2} \\ -\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} = \frac{3+\sqrt{5}}{2} \begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} \\ = \lambda_1 v_1$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2-\sqrt{5} \\ \frac{3-\sqrt{5}}{2} \end{bmatrix} = \frac{3-\sqrt{5}}{2} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \\ = \lambda_2 v_2$$

$$S = \{v_1, v_2\}$$

NOTE:

$S$  is ORTHOGONAL!

WITH RESPECT TO THIS BASIS, WHAT IS  $L$ ?

$$[L(v)]_S = B [v]_S$$

WHAT IS  $B$ ?

$$[L(v_1)]_S = \left[ \frac{3+\sqrt{5}}{2} v_1 \right]_S = \frac{3+\sqrt{5}}{2} [v_1]_S = \frac{3+\sqrt{5}}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[Lw_2]_S = \frac{3-\sqrt{5}}{2} [w_2]_S = \frac{3-\sqrt{5}}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$B = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix}.$$

DEF.  $V$   $n$ -DIMENSIONAL.

$L: V \rightarrow V$  DIAGONALIZABLE IS

THERE IS A BASIS  $S$  SUCH THAT

$L$  IS REPRESENTED BY A DIAGONAL MATRIX.

DEF. TWO MATRICES  $A, B$  ARE SIMILAR (OR

CONJUGATE) IF  $B = P^{-1}AP$

FOR SOME INVERTIBLE  $P$ .

THM SIMILAR MATRICES HAVE SAME EIGENVALUES.



PROOF. (DIFFERENT THAN BEFORE)

$\underline{x}$  EIGENVECTOR FOR  $B = P^{-1}AP$

$$\bullet \text{ SO } \lambda_{\underline{x}} = B_{\underline{x}} = P^{-1}AP_{\underline{x}}$$

$$\text{SO } \lambda_{\underline{x}} = P^{-1}AP_{\underline{x}}$$

$$\Rightarrow P\lambda_{\underline{x}} = AP_{\underline{x}}$$

$$\Rightarrow \lambda(P_{\underline{x}}) = A(P_{\underline{x}})$$

SO  $P_{\underline{x}}$  IS AN EIGENVECTOR FOR  $A$   
W/ EIGENVALUE  $\lambda$ .

SO  $B$ 'S EIG. VALUES ARE EIG. VALUES  
OF  $A$ .

OTHER DIRECTION IS SIMILAR,



Thm IF  $S = \{v_1, \dots, v_n\}$

IS AN <sup>ORDERED</sup> BASIS OF EIGENVECTORS OF  $L$   
 W/ EIG. VALUES  $\lambda_1, \dots, \lambda_n$   
 THEN  $L$  IS DIAGONALIZABLE.

Proof.

THE MATRIX OF  $L$  WRT RESPECT TO

$$S \text{ IS } \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

□

ALSO, IF  $L$  IS DIAGONALIZABLE,  
 GIVEN BY A DIAGONAL MATRIX  $D$   
 WRT.  $S$ , THEN  $S$  IS A  
 BASIS OF EIGENVECTORS.

THE LINEAR TRANS. ASSOC. TO A  
MATRIX A IS DIAGONALIZABLE

IFF IT IS SIMILAR TO A DIAGONAL  
MATRIX:  $D = P^{-1}AP$ .

IF  $T$  IS THE STANDARD BASIS AND  
 $S$  IS A BASIS OF EIGENVECTORS  
 FOR  $A$ , THEN THE

$$\begin{matrix}
 P^{-1} & A & P \\
 \text{TES} & & \text{TES}
 \end{matrix}$$

IS  
 THE MATRIX FOR  $A$  W.R.T.  $T$ .

THE  $n \times n$   $A$  IS SIMILAR TO A DIAGONAL MATRIX  $D$   
 IFF  $A$  HAS  $n$  LIN. IND. EIG. VECTORS  $\square$

---

QX

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

EIGENVALUES

$$\lambda_1 = 2 \quad \lambda_2 = 3$$

EIGENVECTORS

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

A is similar to  $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ , i.e.  $P^{-1}$

WHAT IS  $P_{T \leftarrow S}$  ?

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = P^{-1} A P$$

$$P = P_{T \leftarrow S} = \left[ \left[ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_T \quad \left[ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_T \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Ex.  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  DIAGONALIZABLE?

ONLY EIGENVECTORS ARE  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

CAN'T MAKE A BASIS FROM THESE.

NOT DIAGONALIZABLE.

Thm IF  $A$  HAS  $n$  DISTINCT  
EIGENVALUES, THEN  $A$  IS  
DIAGONALIZABLE.

Proof. Let  $\{\lambda_1, \dots, \lambda_n\}$  BE THE  
EIGENVALUES.

Let  $S = \{x_1, \dots, x_n\}$  BE EIGENVECTORS

WE WILL SHOW THAT  $S$  IS LINEARLY  
INDEPENDENT.

Express NOT,

So we assume  $S$  REDUNDANT.

By a previous theorem, some  $x_j$  is the first  $j$  s.t.

$\{x_1, \dots, x_j\}$  is redundant

and for this  $j$ ,  $x_j$  is

A linear combo of the

$x_1, \dots, x_{j-1}$  (which are lin. ind.)

$$x_j = a_1 x_1 + \dots + a_{j-1} x_{j-1}$$

So (A)  $\lambda_j x_j = \lambda_j a_1 x_1 + \dots + \lambda_j a_{j-1} x_{j-1}$ .

BUT ALSO

$$Ax_j = a_1 Ax_1 + \dots + a_{j-1} Ax_{j-1}$$

(\*\*\*)  $\lambda_j x_j = a_1 \lambda_1 x_1 + \dots + a_{j-1} \lambda_{j-1} x_{j-1}$ .

SUBTRACT  $(A)$  FROM  $(A)$

$\sim$

$$0 = (\lambda_j - \lambda_1) a_1 x_1 + \dots + (\lambda_j - \lambda_{j-1}) a_{j-1} x_{j-1}$$

BUT THEN ALL OF THESE COEFFS

ARE 0 SINCE THESE ARE  
LINEARLY INDEPENDENT,

BUT SOME  $a_k \neq 0$

SO FOR THAT  $k$

$$\lambda_j - \lambda_k = 0 \quad \#$$



SHOWS THAT EIG. VECTS  
FOR DISTINCT EIG. VALUES  
ARE LINEARLY IND.

IF  $P(\lambda)$  HAS "MULTIPLE" ROOTS

$$\text{like } P(\lambda) = (\lambda - 1)^2$$

$$\left( \text{like } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

THEN IT YOU MAY OR MAY NOT BE DIAGONALIZABLE.

$$(\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \dots (\lambda - \lambda_e)^{k_e}$$

$k_i$ : MULTIPLICITY OF  $\lambda_i$ .

EX  $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

NOT DIAGONALIZABLE.

$$P(\lambda) = \lambda(\lambda - 1)^2$$

$$(I - A)x = \underline{0}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

SOLUTIONS ARE  $\begin{bmatrix} c \\ c \\ 0 \end{bmatrix}$ .



Q

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$p(\lambda) = \lambda(\lambda-1)^2$$

$$(\underline{I} - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

GEN. VECTORS ARE

$$\begin{bmatrix} 0 \\ r \\ s \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{BASIS FOR THIS}$$

SPACE OF GEN VECTORS

(THIS GEN SPACE FOR  $\lambda=1$ )

$$\lambda = 0$$

$$0I - A = -A$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\text{Solns } \begin{pmatrix} r \\ 0 \\ -r \end{pmatrix}.$$

---

LIN. IND. B.G. VECTORS  $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$  ✓

DIAGONALIZABLE ✓

---

2.3

DIAGONALIZATION FOR SYMMETRIC MATRICES.

RECALL: A matrix  $A$  is SYMMETRIC  
IF  $A = A^T$ .

IF  $A$  IS AN  $n \times n$  MATRIX WITH  $n$  DISTINCT  
EIGENVALUES THEN  $A$  IS DIAGONALIZABLE.

IF EIGENVALUES ARE NOT DISTINCT,  
IT MAY OR MAY NOT BE.

BUT! EVERY SYMMETRIC MATRIX  
IS DIAGONALIZABLE.

THEM THE CHARACTERISTIC POLYNOMIAL  
OF A SYMMETRIC  $n \times n$  REAL MATRIX  
HAS  $n$  REAL ROOTS.

SOME FACTS ABOUT COMPLEX #S:

WHAT ARE COMPLEX #S?

$$\mathbb{C} = \{x + iy \mid x, y \text{ real}\}$$

$$i^2 = -1.$$

USUALLY CALL COMPLEX #S

$$z = x + yi$$

$$w = a + bi \quad z = x + yi$$

$$wz = (a + bi) \cdot (x + yi)$$

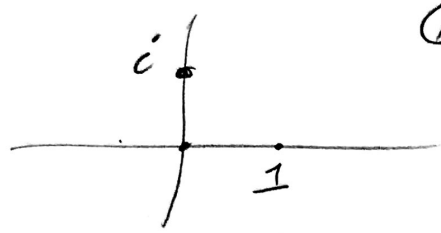
$$= ax + bx i + ay i + i^2 by i^2$$

$$= (ax - by) + i(bx + ay)i.$$

So

$$(1 + 2i)(7 - i) = 7 + 14i - i + 2.$$

$$= 9 + 13i.$$



IF  $z = x + yi$

DEFINES  $\bar{z} = x - yi$  COMPLEX CONJUGATE OF  $z$ .  
 $z$  IS REAL IFF  $z = \bar{z}$ .

DEFINES  $|z| = \sqrt{x^2 + y^2}$  THIS MODULUS OF  $z$ .

NOTE  $z\bar{z} = (x + yi)(x - yi)$   
 $= x^2 + y^2 = |z|^2$ .

CAN TALK ABOUT COMPLEX VECTOR  
 SPACES WHERE SCALARS ARE  
 COMPLEX NUMBERS.

GIVEN A MATRIX  $A = [a_{ij}]$  WITH COMPLEX  
 ENTRIES,  $\bar{A} = [\bar{a}_{ij}]$

AND  $\bar{A}^T = [\bar{a}_{ji}]$  IS THE

CONJUGATE TRANSPOSE OF  $A$ .

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IF  $\underline{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$  IS A VECTOR W/ CX UNITS

NOTE THAT

$$\begin{aligned} \bar{z}^T z &= [\bar{z}_1 z_1 + \bar{z}_2 z_2 + \dots + \bar{z}_n z_n] \\ &= [ |z_1|^2 + \dots + |z_n|^2 ] \end{aligned}$$

IS NONZERO IF  $\underline{z} \neq \underline{0}$ .

PROOF OF THM.

LET  $\lambda$  BE A ROOT OF THE CHARACTERISTIC POLYNOMIAL OF  $A$ .

WE WILL SHOW  $\bar{\lambda} = \lambda$ .

$$A \underline{x} = \lambda \underline{x}$$

$$\bar{x}^T A \underline{x} = \bar{x}^T \lambda \underline{x} = \lambda \bar{x}^T \underline{x}$$

TAKE CONJ. TRANSPOSE OF EACH SIDE

$$\bar{x}^T \bar{A}^T \bar{x} = \bar{\lambda} \bar{x}^T \bar{x}$$

$\underbrace{A \text{ real}}_{A \text{ symmetric}}$   
 So, since  $\bar{A}^T = A^T = A$

we have

$$\underline{\bar{x}}^T A \underline{x} = \bar{\lambda} \underline{\bar{x}}^T \underline{x}$$

$$\begin{array}{c}
 \text{So} \quad \lambda \underline{\bar{x}}^T \underline{x} = \bar{\lambda} \underline{\bar{x}}^T \underline{x} \\
 \underbrace{\quad}_{\neq 0}
 \end{array}$$

$$\text{So} \quad \lambda = \bar{\lambda} \quad \square$$

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Thm IF  $A$  SYMMETRIC, AND

$$A\underline{x} = \lambda\underline{x} \quad \text{AND} \quad A\underline{y} = \mu\underline{y}$$

WITH  $\mu \neq \lambda$ ,

THEN  $\underline{x}$  AND  $\underline{y}$  ARE ORTHOGONAL.  
(W.R.T. STANDARD INNER PRODUCT.)

PROOF.

WE USE THE FACT:

IF  $(, )$  IS THE DOT PRODUCT,

$$\text{THEN } (\underline{a}, A^T \underline{b}) = (A \underline{a}, \underline{b}).$$

NOW,

$$\lambda(\underline{x}, \underline{y}) = (\lambda \underline{x}, \underline{y})$$

$$= (A \underline{x}, \underline{y})$$

$$= (\underline{x}, A^T \underline{y}) = (\underline{x}, A \underline{y})$$

$$= (\underline{x}, \mu \underline{y})$$

$$= \mu(\underline{x}, \underline{y}).$$



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$$\text{So } \lambda(x, y) = \mu(x, y)$$

BUT SINCE  $\lambda \neq \mu$ ,

$$(x, y) = 0. \quad \square$$

Q.  $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

$$p(\lambda) = (\lambda + 2)(\lambda - 4)(\lambda + 1)$$

$$\lambda_1 = -2 \quad \lambda_2 = 4 \quad \lambda_3 = -1$$

UNBORN VECTORS AND SOLNS

TO

$$\begin{bmatrix} \lambda & 0 & 2 \\ 0 & \lambda + 2 & 0 \\ 2 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0}$$

$$\lambda_1 = -2 \quad \leadsto \quad x_1 = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$$

$$\lambda_2 = 4, \quad x_2 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ r \end{bmatrix}$$

$$\lambda_3 = -1, \quad x_3 = \begin{bmatrix} 2r \\ 0 \\ r \end{bmatrix}$$

pick

$$x_1 = \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

all orthogonal!

A conjugates to  $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$P^{-1}AP = D$$

where  $P = \begin{bmatrix} c & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ .

IF  $A$  SYMMETRIC, EIG. VECTORS FOR DISTINCT EIGENVALUES ARE ORTHOGONAL.

SO IF  $A$  IS SYMMETRIC, AND  $A$  IS DIAGONALIZABLE

THE THERE IS AN ORTHONORMAL BASIS  $\{x_1, \dots, x_n\}$  CONSISTING OF EIGENVECTORS

THEN  $P = [x_1 \ x_2 \ \dots \ x_n]$

IS SUCH THAT

$P^{-1}AP$  IS A DIAGONAL MATRIX  $D$ .

THIS  $P$  HAS A NICE PROPERTY:

$P^T = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$

AND  $ij$  ENTRY OF  $P^T P$  IS  $(x_i, x_j)$

$(x_i, x_j) = 1$  IF  $i=j$  AND  $0$  IF  $i \neq j$ .

i.e.  $P^T P = I$ .

So  $P^T = P^{-1}$ .

DEFINITION.  $A$  is ORTHOGONAL

IF  $A^{-1} = A^T$ . (i.e.  $A^T A = I$ .)

IN EXAMPLE

$$\begin{bmatrix} 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Thm  $n \times n$   $A$  ORTHOGONAL IFF COLS <sup>(ROWS)</sup> ARE ORTHONORMAL □

Thm  $\det(A) = \pm 1$  IF  $A$  ORTHOGONAL.

THE ORTHOGONAL MATRICES ARE PRECISELY

THE ISOMETRIES, i.e. THE LINEAR TRANSFORMATIONS THAT PRESERVE THE DISTANCES

i.e. THE LINEAR TRANSFORMATIONS  
 THAT PRESERVE THE INNER PRODUCT,

$$\text{SINCE } (Ax, Ay) = (x, A^T A y) = (x, y).$$

IF  $A$  IS ORTHOGONAL,

$$\text{IF } (Ax, Ay) = (x, y) \text{ FOR ALL } x, y$$

$$\text{THEN } (x, A^T A y) = (x, y) \text{ FOR ALL } x, y$$

THIS IMPLIES THAT  $A^T A = I$

(EXERCISE 7.3 #36)

THEM IF  $A$  SYMMETRIC  $n \times n$  MATRIX,

THEN  $A$  IS DIAGONALIZABLE.

WE WILL SUB PROOF.

For symmetric  $A$

each eigenvalue is as big as possible.

i.e. each eigenvalue of multiplicity  $k$  has a  $k$ -dimensional space of eigenvectors.

Ex.  $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$

$P(\lambda) = (\lambda + 2)^2(\lambda - 4)$ .

$\lambda_1 = -2 \quad \lambda_2 = -2, \quad \lambda_3 = 4.$

So we

$(-2I - A)x = 0$

$\left[ \begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

2-dim  $\lambda = -2$  space.

Basis is  $x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$  and  $x_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

NOT ORTHOGONAL, BUT

GRAM-SCHMIDT GIVES

$$y_1 = x_1 \quad \text{AND}$$

$$y_2 = x_2 - \frac{(x_2, y_1)}{(y_1, y_1)} y_1$$

$$= \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

let  $y_2^* = 2y_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

normalize

$$z_1 = \frac{1}{\|y_1\|} y_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$z_2 = \frac{1}{\|y_2^*\|} y_2^* = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$\{z_1, z_2\}$  ORTHONORMAL BASIS

For the EIGEN SPACES For  $\lambda = -2,$

For  $\lambda = 4,$

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} x = 0 \rightarrow$$

$$x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

P ORTHOGONAL.