A LOWER BOUND ON VOLUMES OF END-PERIODIC MAPPING TORI

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Abstract. We provide a lower bound on the volume of the compactified mapping torus of a strongly irreducible end-periodic homeomorphism $f : S \to S$. This result, together with work of Field, Kim, Leininger, and Loving [FKLL23], shows that the volume of $\overline{M}_f$ is comparable to the translation length of f on a connected component of the pants graph $\mathcal{P}(S)$, extending work of Brock [Bro03b] in the finite-type setting on volumes of mapping tori of pseudo-Anosov homeomorphisms.

1. Introduction

A central theme in the post-geometrization study of 3-manifolds is to clarify the relationship between geometric and topological features of a manifold. Fibered hyperbolic 3-manifolds provide a particularly rich class of examples in this vein, as their topology is completely determined by the monodromy homeomorphism $f\colon S \to S$ of their fiber surface S, which realizes the manifold as a mapping torus M_f. When S is of finite type, the isotopy class of f is an element of the mapping class group of S, and the abundant collection of actions of this group provide a wealth of information about the geometry of M_f. The current paper is motivated by such a connection due to Brock [Bro03b], who showed that the hyperbolic volume of M_f is comparable to the translation length $\tau(f)$ of f acting on the pants graph $\mathcal{P}(S)$. More precisely, he shows that there are constants K_1 and K_2, depending only on the topology of S, such that

$$K_2\tau(f) \leq \mathrm{Vol}(M_f) \leq K_1\tau(f). \tag{1.1}$$

When S has infinite type and $f\colon S \to S$ is a *strongly irreducible end-periodic* homeomorphism, earlier work of Field, Kim, Leininger, and Loving [FKLL23] provides an analogous upper bound on the hyperbolic volume of the *compactified* mapping torus with its totally geodesic boundary structure. Namely,

$$\mathrm{Vol}(\overline{M}_f) \leq C_1\tau(f)$$

for a controlled constant C_1. We complete the analogy with (1.1) here by establishing the following lower bound.

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Theorem 1.1. *For any surface S with finitely many ends, each accumulated by genus, and any strongly irreducible end-periodic homeomorphism $f\colon S \to S$, we have*

$$C_2\tau(f) \leq \mathrm{Vol}(\overline{M}_f),$$

where the constant C_2 depends only on the capacity *of f.*

The *capacity* of f is a pair of numbers that describes the topology of a subsurface of S where f is "interesting" (*i.e.* where f does not simply act by translation)—see Section 2.3. Since the surface S has infinite topological type, the dependence of C_2 on the capacity of f serves as a substitute for the dependence of Brock's K_2 on the topology of the finite-type fiber.

Remark. We note that this relationship between volumes of hyperbolic 3-manifolds and distance in the pants graph has also been explored in a different setting by Cremaschi–Rodríguez-Migueles–Yarmola [CRMY22] who give upper and lower bounds analogous to those of Brock.

There are two pieces of data naturally associated to any f-invariant component $\Omega \subset \mathcal{P}(S)$:

(1) the translation length of f on Ω, denoted $\tau_\Omega(f)$, and
(2) an induced pants decomposition P_Ω of $\partial\overline{M}_f$.

See Section 7 for a detailed description. The following theorem provides a component that (coarsely) optimizes both of these, and ties the action of f on $\mathcal{P}(S)$ to a bounded length pants decomposition of $\partial\overline{M}_f$.

Theorem 1.2. *Given $f\colon S \to S$, a strongly irreducible end-periodic homeomorphism, there is a component $\Omega \subset \mathcal{P}(S)$ and $E > 0$ (depending on the capacity), so that each curve in $P_\Omega \subset \partial\overline{M}_f$ has length at most E, and so that $\tau_\Omega(f) \leq E\tau(f)$.*

We expect this theorem to be more generally useful in future analysis of the hyperbolic geometry surrounding depth-one foliations.

Historical notes and future directions. End-periodic homeomorphisms are an important class of homeomorphisms of infinite-type surfaces, due in large part to their connection with depth-one foliations of 3-manifolds. Indeed, after collapsing certain trivial product foliation pieces, a co-oriented, depth-one foliation of a 3-manifold is obtained by gluing together finitely many compactified mapping tori of end-periodic homeomorphisms. Such foliations (and more generally finite-depth foliations) were studied in detail by Cantwell and Conlon [CC81], and arise in Gabai's analysis [Gab83, Gab83, Gab87] of the Thurston norm [Thu86b]. In [Thu86b], Thurston also observed that depth-one foliations occur naturally as limits of foliations by fibers in sequences of cohomology classes limiting to the boundary of the cone on a fibered face of the Thurston norm ball.

In unpublished work, Handel and Miller began a systematic study of end-periodic homeomorphisms using laminations in the spirit of the modern interpretation of Nielsen's approach to the Nielsen–Thurston Classification (see Gilman [Gil81], Miller [Mil82], Handel–Thurston [HT85], and Casson–Bleiler [CB88]). Some aspects of this work were described and developed by Fenley in [Fen89, Fen97], and more recently expanded upon by Cantwell, Conlon, and Fenley in [CCF21]. The analogy between strongly irreducible end-periodic homeomorphisms and pseudo-Anosov homeomorphisms was further strengthened by work of Patel–Taylor showing that many end-periodic homeomorphisms admit loxodromic actions on various arc and curve graphs of infinite-type surfaces [PT].

Recent work of Landry, Minsky, and Taylor [LMT] further studies the behavior of Thurston's depth-one foliations [Thu86b] arising from the boundaries of the fibered faces. In particular, using how the lifts of a first return map act on the boundary circle of a depth one leaf lifted to the universal cover, they relate the invariant laminations for end-periodic homeomorphisms to the laminations of the pseudo-Anosov flow associated to the fibered face. Moreover, using veering triangulations [Ago11, Gué16], they show that any compactified mapping torus appears in the boundary of some fibered face of some fibered 3–manifold.

Fenley [Fen92] provided the first connection between the hyperbolic geometry of a 3-manifold and its depth-one foliations, proving that when the end-periodic monodromies are irreducible, the depth-one leaves admit Cannon–Thurston maps from the compactified universal covers $\overline{\mathbb{H}}^2 \to \overline{\mathbb{H}}^3$. This is an analogue of Cannon and Thurston's seminal work in the finite-type case (circulated as a preprint for decades before appearing in [CT07]). Unlike Cannon and Thurston's map, Fenley's boundary map is not surjective, but rather surjects the limit set of the compactified mapping torus, which is a Sierpinski carpet. In the course of his arguments, Fenley provides a quasi-isometric comparison between the hyperbolic metric and a (semi)-metric defined by the foliation, which also parallels Cannon and Thurston's approach.

The comparison between the hyperbolic metric and the metric defined by the fibration, as studied by Cannon and Thurston, was greatly elaborated on by Minsky [Min93] to provide uniform estimates depending on the injectivity radius of the 3-manifold and the genus of the fiber. Building on this, and the deep machinery developed by Masur–Minsky [MM99, MM00], Minsky [Min10] and Brock–Canary–Minsky [BCM12] constructed combinatorial, uniformly biLipschitz models for fibered hyperbolic 3-manifolds.

Brock's volume estimates [Bro03b] above were used to prove his analogous volume estimates in terms of the Weil–Petersson translation length on Teichmüller space, but with less control over the constants. A more direct proof of the Weil–Petersson upper bound, with explicit constants, was proved by Brock–Bromberg [BB16] and Kojima–McShane [KM18] using renormalized volume techniques.

The techniques developed here, and in [FKLL23], combined with forthcoming work of Bromberg, Kent, and Minsky [BKM], provides the framework to prove volume estimates for closed hyperbolic 3-manifolds with depth-one foliations. One might ultimately hope for a uniform biLipschitz model, but the tools needed to guarantee one seem considerably more difficult. First steps in this direction are taken by Whitfield in [Whi], where she extends a result of Minsky [Min00, Theorem B] to the infinite-type setting to produce short curves whose lengths are bounded in terms of subsurface projections. From a Teichmüller-theoretic perspective, the fact that end-periodic homeomorphisms can be made to act isometrically near the ends suggests the possibility of an action on a Teichmüller space with finite translation length, and it is natural to wonder on the relation of such a length to the volume.

Finally, some important motivation for this work comes from the study of big mapping class groups. In particular, [AIM, Problem 1.7] asks for a characterization of big mapping classes whose mapping tori admit complete hyperbolic metrics. One hope is that better understanding the geometry of end-periodic mapping tori will provide some insights into giving a complete solution to this problem.

Comparison with finite-type case. We briefly outline here Brock's strategy for his lower bound [Bro03b], point out the complications that arise when adapting the strategy to our setting, and discuss how we address these challenges.

Brock's proof involves controlling the number and location of bounded length curves in the mapping torus, as each bounded length curve provides a definite contribution to the volume [Bro03a, Lemma 4.8]. To produce bounded length curves, Brock works in the infinite cyclic cover $S \times \mathbb{R}$ of the mapping torus, and constructs an interpolation between a *simplicial hyperbolic surface* [Can96, Bon86] homotopic to the inclusion of S and the image of this surface under the generator of the deck group. The deck group acts like f on the S factor, and the interpolation produces a sequence of bounded length pants decompositions, starting with one on the initial surface and ending with its f–image in the translate. He then shows that *the number of curves* arising in this sequence provides an upper bound on distance in the pants graph, and hence a bound on the translation length of f [Bro03a, Lemma 4.3].

The interpolation between the two simplicial hyperbolic surfaces can overlap significantly with its translates by the deck transformation, leading to an overestimate in the number of bounded length curves in the mapping torus (as many curves may project to the same curve). To account for this, Brock first situates a neighborhood of the entire interpolation inside some fixed, but *uncontrolled* number n_0 of consecutive translates of a fundamental domain for the covering action. The concatenation of any $j > 0$ consecutive translates of the interpolation produces a path between a pants decomposition and its image under f^j, all of whose curves are of bounded length and situated inside $j + n_0$ translates of the fundamental domain. The number of curves that occur in the sequence of pants decompositions is an upper bound on j times the translation length, and a lower bound on $j + n_0$ times the volume. Thus, taking $j \to \infty$ and dividing both quantities by j, the n_0 term disappears, proving the required lower bound on volume in terms of pants translation length.

Our proof of Theorem 1.1 involves a similar strategy. Notably, Brock's lower bound on volume in terms of the number of bounded length curves still serves as

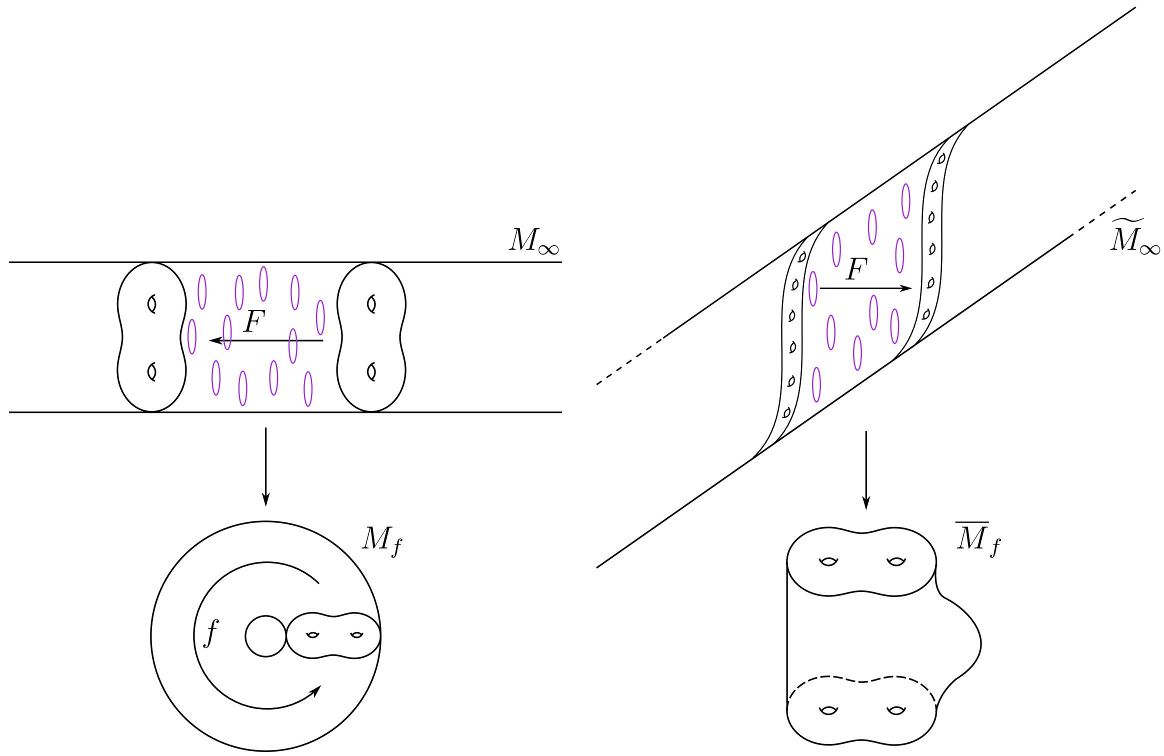

FIGURE 1. In the figure on the right, when we "interpolate" from left to right through $\widetilde{M}_\infty$ the "core" of f slides along S as we zip along the bottom boundary and unzip along the top boundary. In both cases, F is the covering transformation given by $F(x,t) = (f(x), t-1)$.

the primary mechanism for controlling the volume. We also carry out many of the arguments in the infinite cyclic cover, though in our situation this cover does not have finitely generated fundamental group. See Figure 1 for a cartoon of the differences between the infinite cyclic covers in the two settings.

Several of the ideas in Brock's proof break down in fundamental ways in our setting, and we discuss these in turn. We make extensive use of pleated and simplicial hyperbolic surfaces in the compactified mapping torus with totally geodesic boundary as well as in its infinite cyclic cover. The infinite-type setting demands some care, but no serious issues arise here.

The first real obstacle we encounter is that the bound on pants distance in terms of the number of curves that appear in all of the pants is not immediately applicable as it relies on the work of Masur and Minsky [MM99, MM00], where the constants depend on the topological type of the surface. While our pants decompositions contain infinitely many curves, a finite sequence of pants moves takes place on a finite-type subsurface. Fixing the capacity of f provides an initial bound on the topological type of this subsurface, but even under this condition, two additional issues arise. First, iteration of the map increases the size of this subsurface linearly in the power, and so the strategy of iterating and taking limits is not viable. The second issue is that there is no universal bound on the length of curves in a pants decomposition of a finite-type surface with boundary.

To address the second issue, we construct *minimally well-pleated surfaces* in the compactified mapping torus, which send "as much of the (infinite-type) surface as possible" into the boundary—see Section 3.2 and Section 4. Appealing to Basmajian's collar lemma [Bas94], the bound on capacity produces *a priori* bounds on the length of the boundary of a minimal core—see Lemma 4.1. After passing to a uniform power, we may enlarge the core to "support" the interpolation through simplicial hyperbolic surfaces, which *does* have a bound on the length of its boundary—see Section 5.3. This enlarged surface is of bounded topological type, again thanks to bounded capacity, and, in this setting, there is a uniform bound on the length of a bounded length pants decomposition—see Theorem 2.9.

As we cannot iterate and take a limit as Brock does, we address the first issue by essentially gaining control on the "uncontrolled" constant n_0 in Brock's proof. Interestingly, the feature of end-periodic homeomorphisms that forces the capacity to grow linearly under powers is, along with strong-irreducibility, what comes to the rescue. Namely, as all of our bounded length curves are homotopic into the enlargement of the core, we can find a uniform power (depending only on the capacity) so that for all higher powers the images of these bounded length curves are *not* homotopic into this subsurface (see Lemma 3.1). In particular, no two curves in our bounded length set project to the same curve in the mapping torus for this uniform power, and this guarantees that they all contribute to the volume.

Outline of the paper. We begin in Section 2 with preliminaries on end-periodic homeomorphisms and their mapping tori, including the definition of capacity. In Section 3 we establish key *topological* features of a strongly irreducible end-periodic homeomorphism acting on a core, and describe how the core sits in the compactified mapping torus. The details for the pleated surface technology we need, and the resulting uniform *geometric* features for a strongly irreducible end-periodic homeomorphism are described in Section 4. The kinds of simplicial hyperbolic surfaces we will use, as well as our applications of these, are described in Section 5. We assemble all the ingredients into the proof of Theorem 1.1 in Section 6. Finally, in Section 7, we prove Theorem 1.2.

2. Preliminaries

In this section we set some notation, recall some of the facts that we will need, particularly from [FKLL23], and define the notion of "capacity."

2.1. **End-periodic homeomorphisms.**

We restrict our attention to surfaces of infinite-type with finitely many ends, each accumulated by genus, and without boundary. The interested reader can find a more general discussion of end-periodic homeomorphisms in [Fen92, Fen97, CCF21].

A homeomorphism of an infinite-type surface S is *end-periodic* if there is an $m > 0$ such that, for each end E of S, there is a neighborhood U_E of E so that either

(i) $f^m(U_E) \subsetneq U_E$ and the sets $\{f^{nm}(U_E)\}_{n>0}$ form a neighborhood basis of E; or
(ii) $f^{-m}(U_E) \subsetneq U_E$ and the sets $\{f^{-nm}(U_E)\}_{n>0}$ form a neighborhood basis of E.

We say that E is an *attracting end* in the first case, and a *repelling end* in the second. The neighborhoods U_E are *nesting neighborhoods* of the ends, and when convenient we assume (as we may) that we have chosen disjoint nesting neighborhoods for distinct ends. We denote the union of the neighborhoods of the attracting ends U_+ and write U_- for the union of the neighborhoods of the repelling ends. If $f^{\pm 1}(U_\pm) \subset U_\pm$, and $\partial \overline{U}_\pm$ is a union of simple closed curves, then we say that $U_\pm$ are *tight nesting neighborhoods*. Every end-periodic homeomorphism admits tight nesting neighborhoods. For instance, the good nesting neighborhoods from [FKLL23] are a particular example of tight nesting neighborhoods, with the additional assumption that each component of $\overline{U}_\pm$ has a single boundary component.

A compact subsurface $Y \subset S$ is a *core* for f if $S - Y$ is a disjoint union of tight nesting neighborhoods U_+ and U_-. Given a core Y, define the *junctures* $\partial_+ Y$ and $\partial_- Y$ to be the boundary components meeting U_+ and U_-, respectively. Note that there are infinitely many choices of cores for f (as a given core can always be enlarged).

Given a core Y for f, a hyperbolic metric on S for which $f|_{U_+} \colon U_+ \to U_+$ and $f^{-1}|_{U_-} \colon U_- \to U_-$ are isometric embeddings is said to be *compatible with* Y. Adjusting f by an isotopy if necessary, there are always metrics which are compatible with a given core Y, see [Fen97].

Define

$$\mathcal{U}_+ = \bigcup_{n \geq 0} f^{-n}(U_+) \quad \text{and} \quad \mathcal{U}_- = \bigcup_{n \geq 0} f^{n}(U_-),$$

which are the *positive* and *negative escaping sets for* f, respectively. We note that any choice of nesting neighborhoods will give rise to the same escaping sets $\mathcal{U}_\pm$ (depending only on the homeomorphism f). With these assumptions, the restrictions $\langle f \rangle|_{\mathcal{U}_\pm}$ act cocompactly on $\mathcal{U}_\pm$ with quotients $S_\pm = \mathcal{U}_\pm / \langle f \rangle$, which may be disconnected, see, *e.g.* [FKLL23, Lemma 2.4].

In the following, *curve* and *line* refer to proper homotopy classes of essential simple closed curves and essential properly embedded lines, respectively. A curve α is called *reducing* with respect to an end-periodic homeomorphism f if there exists $m, n \in \mathbb{Z}$ with $m < n$ and such that $f^n(\alpha)$ is contained in a nesting neighborhood of an attracting end and $f^m(\alpha)$ is contained in a nesting neighborhood of a repelling end.

Definition 2.1 (Strong irreducibility)**.** An end-periodic homeomorphism, $f : S \to S$, is *strongly irreducible* if it has no *periodic curves*, no *periodic lines*, and no *reducing curves.*

2.2. **Mapping tori and their compactifications.** We define a partial compactification of $S \times (-\infty, \infty)$ inside $S \times [-\infty, \infty]$ by

$$\widetilde{M}_\infty = \{(x,t) \in S \times [-\infty,\infty] \mid x \in \mathcal{U}_\pm \text{ if } t = \pm\infty\},$$

and define $F\colon \widetilde{M}_\infty \to \widetilde{M}_\infty$ by $F(x,t) = (f(x), t-1)$, where $\pm\infty - 1 = \pm\infty$. The group $\langle F \rangle$ acts properly discontinuously and cocompactly on $\widetilde{M}_\infty$, see, *e.g.* [FKLL23, Lemma 3.2]. The quotient $p\colon \widetilde{M}_\infty \to \overline{M}_f = \widetilde{M}_\infty/\langle F \rangle$ is a compact manifold with boundary naturally homeomorphic to $S_- \cup S_+$ and whose interior is the mapping torus M_f of f which we call the *compactified mapping torus*. The manifold $\overline{M}_f$ is particularly nice when f is strongly irreducible.

Theorem 2.2. [FKLL23, Proposition 3.1] *Let $f : S \to S$ be a strongly irreducible, end-periodic homeomorphism of a surface with finitely many ends, all accumulated by genus. Then $\overline{M}_f$, is a compact, irreducible, atoroidal, acylindrical* 3*-manifold, with incompressible boundary.*

Together with Thurston's Geometrization Theorem for Haken manifolds and Mostow Rigidity [Thu86a, McM92, Mor84], the above result implies the following theorem.

Theorem 2.3. *If $f : S \to S$ is a strongly irreducible, end-periodic homeomorphism of a surface with finitely many ends, all accumulated by genus, then $\overline{M}_f$ admits a convex hyperbolic metric σ_0 with totally geodesic boundary, which is unique up to isometry.*

Whenever discussing metric properties of $\overline{M}_f$, we will assume it is equipped with the convex hyperbolic metric σ_0, and may simply refer to it as *the* hyperbolic metric on $\overline{M}_f$ (due to the uniqueness statement), without specific reference to its name. The metric σ_0 pulls back to a complete hyperbolic metric on $\widetilde{M}_\infty$ for which $\mathcal{U}_\pm \times \{\pm\infty\}$ is totally geodesic.

Given a core Y with tight nesting neighborhoods $U_\pm$, we choose a hyperbolic metric μ on S so that the "inclusions" $U_\pm \to U_\pm \times \{\pm\infty\}$ into $\partial\widetilde{M}_\infty$ are isometric embeddings. Then, after adjusting f by an isotopy on $U_\pm$ if necessary, μ is compatible with Y. We say that such a metric is *induced by the metric on* $\widetilde{M}_\infty$.

2.3. **Euler characteristic, complexity, and capacity.** Given a compact surface Z of genus g with n boundary components, there are two measures of the "size" of Z; $\chi(Z)$, the Euler characteristic, and the complexity $\xi(Z) = 3g - 3 + n$. We note that $\xi(Z)$, when positive, is the maximal number of essential, pairwise disjoint, pairwise non-isotopic simple closed curves on Z, *i.e.* the number the of curves in a pants decomposition of Z. Since $\chi(Z) = 2 - 2g - n$, we have the following elementary fact for all Z with $\xi(Z) \geq 0$,

$$|\chi(Z)| - 1 \leq \xi(Z) \leq \tfrac{3}{2}|\chi(Z)|.$$

For Z closed (and genus at least 1) the second inequality is an equality. We extend both of these quantities to disconnected surfaces, additively over the components (which is natural for the Euler characteristic), and observe that when all components have $\xi \geq 0$ (the only case of interest for us), the inequality on the right still holds. We use all of this in what follows without explicit mention.

Given an end-periodic homeomorphism $f\colon S \to S$, we define the *core characteristic of* f to be

$$\chi(f) = \max_{Y \subset S} \chi(Y),$$

where the maximum is taken over all cores $Y \subset S$ for f. Informally, a core is a subsurface where curves from the repelling end get "hung up" under forward iteration of f (or where curves from the attracting end get hung up under backward iteration). Thus, $\chi(f)$ measures the minimal size of the subsurface where that behavior occurs. Any core Y with $\chi(Y) = \chi(f)$ will be called a *minimal core*.

Remark. Note that it is always possible to choose a core Y so that each component of U_+ or U_- meets Y in a single simple closed curve (see [FKLL23, Corollary 2.5] and the discussion preceding it). In this case, connectivity of S implies connectivity of Y. However, cores (especially minimal cores) need not be connected, as the example in Figure 2 illustrates.

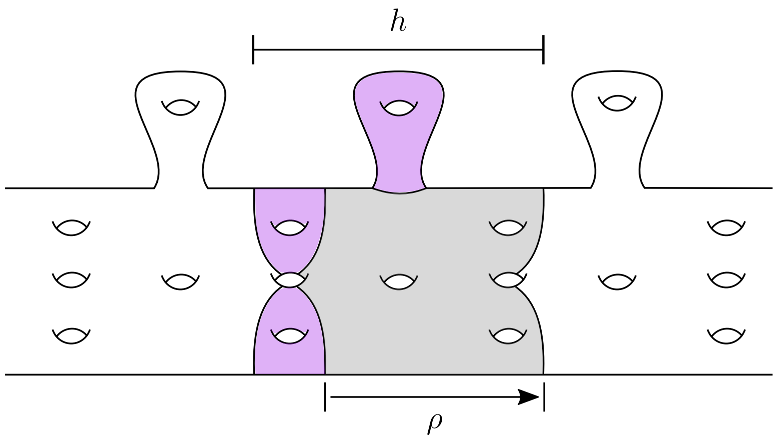

FIGURE 2. The homeomorphism ρ generates a covering action with quotient a genus 6 surface, and h is a partial pseudo Anosov supported on the subsurface which is the shaded (purple and gray) subsurface with genus 6 and four boundary components. The disconnected shaded subsurface (purple in the figure) is a core for $f = h \circ \rho$, regardless of what h is. Note that one of the components of this core has a single boundary component that faces U_+, and thus no component that faces U_-. We can remove this component, and what remains is still a core for f. For h "sufficiently complicated", $f = h \circ \rho$ will be strongly irreducible.

Given a core Y for a strongly irreducible end-periodic homeomorphism $f\colon S \to S$, some components of Y may be disjoint from either $\partial_- Y$ or $\partial_+ Y$. We call such a component *imbalanced*, and say that Y is *balanced* if there are no imbalanced components. The next lemma says that these imbalanced components can always be safely ignored. While the existence of imbalanced components does not affect any of the arguments in this paper, it can be helpful in developing intuition to assume there are none.

Lemma 2.4. *If $f\colon S \to S$ is a strongly irreducible end-periodic homeomorphism and Y is a core for f, then there is a subsurface $Y' \subset Y$ which is balanced. In particular, any minimal core is balanced.*

Proof. Let Y be a core for f and suppose there is an imbalanced component disjoint from $\partial_- Y$. We first show that we can remove at least one such component to obtain a new core for f. To that end, let $Y_0 \subset Y$ be the union of all imbalanced components with $\partial_- Y_0 = \emptyset$. Since $f(U_+) \subset U_+$, and the components of $\{f^k(U_+)\}_{k>0}$ determine a neighborhood basis for the attracting ends, it follows that $f^{-n}(\partial_+ Y)$ has no transverse intersections with $\partial_+ Y$, for all $n \geq 1$. Now set Y_1 to be the (possibly empty) intersection $Y_1 = Y \cap f(Y_0)$. Observe that $f^{-1}(Y_1) = f^{-1}(Y) \cap Y_0$, is a (possibly empty) subsurface of Y_0 whose boundary components are either components of $\partial_+ Y_0$ or else are contained in the interior of Y_0 and hence $\partial f^{-1}(Y_1) \subset f^{-1}(\partial_+ Y)$. Consequently, Y_1 is a union of components of Y_0.

We note that Y_1 is a proper subsurface of Y_0 since otherwise, f^{-1} would simply permute the boundary components of Y_0, creating a periodic curve, contradicting the strong irreducibility of f. Continuing in a similar way, we see that $Y_2 = Y \cap f(Y_1)$ is a union of components of Y_1. In fact, this subsurface Y_2 must be a proper subsurface of Y_1 by a similar argument as above. Continuing inductively, we find a nested sequence

$$Y_0 \supset Y_1 \supset Y_2 \supset \cdots$$

defined by $Y_{j+1} = Y \cap f(Y_j)$, for all $j \geq 0$. Furthermore, $Y_j \neq Y_{j+1}$ if $Y_j \neq \emptyset$.

As the areas of the subsurfaces in this sequence always decrease by a multiple of π, there is some smallest $n \geq 1$ so that $Y_n = \emptyset$ (and then $Y_j = \emptyset$ for all $j \geq n$). It follows that $Y_{n-1} \neq \emptyset$ is a union of components of Y_0 and $f(Y_{n-1}) \cap Y = \emptyset$. Since $f(\partial_+ Y) \subset \overline{U}_+$, it follows that $f(Y_{n-1}) \subset U_+$. Therefore, $W = Y - Y_{n-1}$ must also be a core. Indeed,

$$S - W = U_- \sqcup (U_+ \cup Y_{n-1}),$$

and since $f(Y_{n-1}) \subset U_+$, we have

$$f^{-1}(U_-) \subset U_- \text{ and } f(U_+ \cup Y_{n-1}) \subset U_+ \subset U_+ \cup Y_{n-1},$$

meaning that U_- and $U_+ \cup Y_{n-1}$ are tight nesting neighborhoods.

We thus have a new core and have reduced the number of imbalanced components. Repeating this procedure finitely many times we can remove all imbalanced components with non-empty positive boundary. Likewise, repeating for the union Y_0' of imbalanced components with $\partial_+ Y_0' = \emptyset$, and replacing f with f^{-1} in the arguments above, we can remove all imbalanced components. □

For additional intuition, we record the following.

Lemma 2.5. *If $Y \subset S$ is a balanced core for a strongly irreducible, end-periodic homeomorphism and $\mathcal{U}_0 \subset \mathcal{U}_\pm$ is a component defined by a component $U_0 \subset S - Y$, then $\partial \overline{U}_0$ separates $\mathcal{U}_0$ into two components, each a neighborhood of an end of $\mathcal{U}_0$.*

Proof. Suppose $\mathcal{U}_0 \subset \mathcal{U}_+$, for concreteness (the other case follows by replacing f with f^{-1}). By construction, $U_0 \subset \mathcal{U}_0 - \partial U_0$ is an unbounded component which is a neighborhood of an end of $\mathcal{U}_0$. Since $\mathcal{U}_0$ has two ends, there is at least one other component $U_0' \subset \mathcal{U}_0 - \partial U_0$, which is necessarily unbounded. We must show that $\mathcal{U}_0 - \partial U_0 = U_0 \cup U_0'$.

Suppose there is another component, $U_1 \subset \mathcal{U}_0 - \partial U_0$, different from U_0, U_0'. Observe that U_1 is the interior of a compact subsurface $\overline{U}_1$ with

$$\partial \overline{U}_1 \subset \partial U_0 \subset \partial_+ Y.$$

We claim that $\partial Y - \partial U_0$ cannot intersect $\overline{U}_1$. Indeed, it is disjoint from ∂U_0, so if it intersected $\overline{U}_1$, it would necessarily be contained in it. But every component $\alpha \subset \partial Y - \partial U_0$ faces a neighborhood of another end of S (different than U_0), and so α is contained in a component $\mathcal{U}_\pm$ *different* than $\mathcal{U}_0$. The claim implies $\overline{U}_1$ is a component of Y with $\partial \overline{U}_1 \subset \partial U_0 \subset \partial_+ Y$; thus, $\overline{U}_1$ is an imbalanced component, which is a contradiction. The lemma follows. $\square$

For any end-periodic homeomorphism $f \colon S \to S$, we also define the *end complexity of f* to be

$$\xi(f) = \xi(\partial \overline{M}_f).$$

Note that in [FKLL23], the right-hand side is the notation for this complexity, but because it will appear often later, we have adopted this short-hand.

Given any core Y for f with tight nesting neighborhoods $U_\pm$, set

$$\Delta_\pm = \overline{U_\pm - f^{\pm 1} U_\pm},$$

which are compact subsurfaces in $\overline{U}_\pm$. We note that $\Delta_\pm \subset \mathcal{U}_\pm$ serves as a fundamental domain for the restricted action of $\langle f \rangle|_{\mathcal{U}_\pm}$, therefore $\chi(\Delta_\pm) = \chi(S_\pm)$ (see the proof of [FKLL23, Corollary 2.5]). Therefore,

$$|\chi(\Delta_\pm)| = |\chi(S_\pm)| = \tfrac{2}{3}\xi(S_\pm),$$

since $S_\pm$ are closed. From the same corollary, $\xi(S_+) = \xi(S_-)$, and thus

$$\xi(f) = \xi(\partial \overline{M}_f) = \xi(S_+) + \xi(S_-) = 2\xi(S_+) = 3|\chi(\Delta_\pm)|.$$

Thus, $\xi(f)$ can be thought of as measuring the amount of translation of f on the ends of S; an alternative perspective on this is explained in [FKLL23, Corollary 2.8] which connects to work of Aramayona-Patel-Vlamis [APV20].

Taken together, $\chi(f)$ and $\xi(f)$ provide a measure of the size of the subsurface where f does something "interesting". More precisely, f is "translating" from the negative end into the positive end, $\xi(f)$ measures "how much" translation is happening and $\chi(f)$ measures how large of a subsurface that translation is trying to pass through. These two quantities thus serve as a substitute for the genus, Euler characteristic, or complexity of a finite-type surface.

Definition 2.6. For any end-periodic homeomorphism f, we call the pair $(\chi(f), \xi(f))$, the *capacity* of f.

We also note that since $\xi(f) = \frac{3}{2}|\chi(\partial\overline{M}_f)|$ and since for any $n > 0$, $\overline{M}_{f^n}$ is an n–fold cover of $\overline{M}_f$, we have

$$\xi(f^n) = \tfrac{3}{2}|\chi(\partial\overline{M}_{f^n})| = n\tfrac{3}{2}|\chi(\partial\overline{M}_f)| = n\xi(f).$$

Thus raising to powers increases end-complexity in a predictable way. On the other hand, a core for f is also a core for f^n, and hence core characteristic is non-decreasing under raising to powers.

2.4. **The pants graph and Bers pants decompositions.** A *pants decomposition* on S is a multicurve P in S such that $S - P$ is a collection of three-holed spheres (*i.e.* pairs of pants). An *elementary move* on a pants decomposition P replaces a single curve in P with a different one intersecting it a minimal number of times, producing a new pants decomposition P'. There are two types of elementary moves corresponding to whether the complexity one subsurface in which the elementary move takes place is a one-holed torus or a four-holed sphere. This is illustrated in Figure 3.

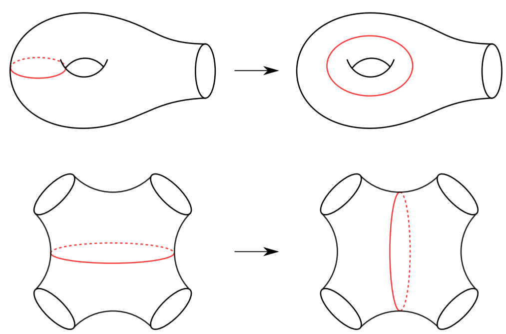

FIGURE 3. Elementary moves on pants decompositions.

Definition 2.7. The *pants graph*, $\mathcal{P}(S)$, is the graph whose vertices are (isotopy classes of) pants decompositions on S, with edges between pants decompositions that differ by an elementary move.

There is a path metric on (the components of) $\mathcal{P}(S)$ with respect to which the action of $\mathrm{Map}(S)$ on $\mathcal{P}(S)$ is isometric. It is defined as follows: an edge corresponding to an elementary move that occurs on a one-holed torus has length 1, and an edge corresponding to an elementary move that occurs on a four-holed sphere has length 2.

Brock proved in [Bro03b] that for finite-type surfaces Z, $\mathcal{P}(Z)$ is quasi-isometric to the Teichmüller space of Z, Teich(Z) equipped with the Weil–Petersson metric. In the infinite-type setting, we no longer have this correspondence between the pants graph and Teichmüller space, but, as we shall see, the pants graph still encodes important geometric data.

Definition 2.8. Given any end-periodic homeomorphism $f\colon S \to S$, we define the *asymptotic translation distance* of f on $\mathcal{P}(S)$ to be

$$\tau(f) = \inf_{P \in \mathcal{P}(S)} \liminf_{n\to\infty} \frac{d(P, f^n(P))}{n},$$

where this infimum is over all pants decompositions $P \in \mathcal{P}(S)$. Observe that $\tau(f^n) = n\tau(f)$ for all $n > 0$.

Note that $\mathcal{P}(S)$ is necessarily disconnected when S is of infinite type (see Branman [Bra20] for more on the pants graphs of infinite-type surfaces). In particular, for certain $P \in \mathcal{P}(S)$, $d(P, f^n(P))$ is infinite for all $n > 0$. Consequently, the infimum is effectively being taken over the union of connected components for which this distance is finite for some, hence infinitely many, $n > 0$ (in [FKLL23], such P were called f–asymptotic pants decompositions).

Throughout the paper it will be necessary to produce pants decompositions of bounded length. Bers proved that a closed hyperbolic surface of genus g admits a pants decomposition for which all the curves have length bounded by a constant depending only on the genus [Ber74, Ber85]. We will need a relative version of Bers result for surfaces with boundary, a short proof of a very concrete version of which was recently given by Parlier [Par].

Theorem 2.9. [Par, Theorem 1.1] *Let X be a hyperbolic surface, possibly with geodesic boundary, and of finite area. Then X admits a pants decomposition where each curve is of length at most*

$$L_B = \max\{\ell(\partial X), \mathrm{area}(X)\}.$$

3. Cores and topology

Let $f : S \to S$ be a strongly irreducible end-periodic homeomorphism. In this section, we will prove some additional topological information about f, a core $Y \subset S$ for f, and features of Y that are reflected in $\overline{M}_f$.

3.1. **Uniform power bounds.** After applying a sufficiently large power of f, some part of any curve $\alpha \in Y$ must leave Y (*c.f.* [Fen97, Theorem 2.7(iii)] and [LMT, Lemma 2.1]). We will need the following strengthening of this fact which provides a uniform power for which that behavior occurs.

Lemma 3.1. *Given a core $Y \subset S$ for a strongly irreducible end-periodic $f\colon S \to S$, then for all $k \geq 2\xi(Y)$ there are no closed, essential curves contained in $f^k(Y) \cap Y$.*

Proof. We prove the equivalent statement that there are no essential curves in $f^{-k}(Y) \cap Y$, since this introduces fewer total inverses in the proof.

We first make a definition and record an observation. We say that an essential, possibly disconnected, subsurface $Z \subset S$ is *lean* if no components are pants and no two annular components are homotopic. Given a lean subsurface $Z \subset S$, define

$$\zeta(Z) = (\xi(Z_0), y),$$

where $Z_0 \subset Z$ is the union of all non-annular components of Z, $\xi(Z_0)$ is the complexity of Z_0 (as defined in Section 2), and y is the number of annular components of Z. Observe that $\xi(Z_0)+y$ is the number of pairwise disjoint, non-parallel curves in Z (here the core curve of an annulus is considered an essential curve in the annulus). We consider such pairs as elements of $\mathbb{Z}^2$ with the dictionary order.

Suppose that $Z \subset Z'$ are lean subsurfaces of S (where we assume that any annular component of Z is either contained in an annular component of Z', or in a non-annular component of Z' in which it is non-peripheral). Then $\zeta(Z) \leq \zeta(Z')$, with equality if and only if there is a homeomorphism $h\colon S \to S$ isotopic to the identity so that $h(Z) = Z'$. Moreover, observe that if we write $\zeta(Z) = (\xi, y)$ and $\zeta(Z') = (\xi', y')$ and if Z' is the union of Z with the regular neighborhood of a multicurve, then $\zeta(Z) \leq \zeta(Z')$ implies that $\xi + y \leq \xi' + y'$.

Now, suppose to the contrary that there is a curve $\alpha \subset f^{-k}(Y) \cap Y$ for some $k \geq 2\xi(Y)$. Then, $f^k(\alpha) \subset Y$, and since $S - Y = U_+ \cup U_-$, it follows that $f^j(\alpha) \subset Y$ for all $j = 0, \ldots, k$, since $f(U_+) \subset U_+$.

For each $j = 0, \ldots, k$, let $Z_j \subset Y$ be the smallest lean subsurface filled by

$$\bigcup_{i=0}^{j} f^i(\alpha),$$

and write $\zeta(Z_j) = (\xi_j, y_j)$. Then $\zeta(Z_j) \leq \zeta(Z_{j+1}) \leq \zeta(Y) = (\xi(Y), 0)$ and $\xi_j + y_j \leq \xi_{j+1} + y_{j+1} \leq \xi(Y)$ for all $0 \leq j < k$. Observe that for any $j = 0, \ldots, k$, either $\zeta(Z_j) = \zeta(Z_{j+1})$, or one of the following strict inequalities must hold:

$$\xi_j < \xi_{j+1} \quad \text{or} \quad \xi_j + y_j < \xi_{j+1} + y_{j+1}.$$

This implies that the sequence of L^1–norms of the pairs $\{(\xi_j, \xi_j + y_j)\}_{j=0}^k$ is a non-decreasing sequence of integers from 1 to $2\xi_k + y_k \leq 2\xi(Y)$. But since $k \geq 2\xi(Y)$, there must be consecutive pairs $(\xi_j, \xi_j + y_j)$ and $(\xi_{j+1}, \xi_{j+1} + y_{j+1})$ whose L^1–norms are equal, and so, for this j, we have $\zeta(Z_j) = \zeta(Z_{j+1})$. In particular, there is a homeomorphism $h\colon S \to S$ isotopic to the identity so that $h(Z_j) = Z_{j+1}$.

Since $f(Z_j) \subset Z_{j+1}$ and $\zeta(f(Z_j)) = \zeta(Z_j)$, there is a homeomorphism $h'\colon S \to S$ isotopic to the identity so that $h'(f(Z_j)) = Z_{j+1} = h(Z_j)$. Rewriting this, we have $h^{-1}h'f(Z_j) = Z_j$. But $h^{-1}h'f$ is isotopic to f, and we conclude that f preserves Z_j, and hence ∂Z_j, up to isotopy, contradicting the strong irreducibility of f. □

3.2. **Cores in the compactified mapping torus.** The suspension flow on M_f can be reparameterized and extended to a local flow (ψ_s) on $\overline{M}_f$. Fixing a core Y for f with the associated (tight) nesting neighborhoods, U_+ and U_-, we can define a homotopy of the inclusion $S \to \overline{M}_f$ along the flowlines, by flowing U_+ and U_- forward and backward, respectively, until they meet $\partial\overline{M}_f$. We do this, carrying along a small neighborhood of ∂Y in Y, but keeping the rest of Y fixed. If we let $h_t\colon S \to \overline{M}_f$, $t \in [0,1]$ denote the homotopy, then we can assume that h_t is injective for all $t \in [0,1)$. One way to think about this construction is via *spiraling neighborhoods* of the boundary; see *e.g.* [Fen97, §4],[LMT, §3.1]. We write

$Y_1 = h_1(Y) \subset \overline{M}_f$, and call Y_1 the Y–*viscera*. It is convenient to think of $h_1(S)$ as a branched surface in $\overline{M}_f$, transverse to (ψ_s).

Since the first return map of (ψ_s) to S is f, the result is a map of S into $\overline{M}_f$ which is embedded on the interior of Y, and for which U_+ and U_- map onto $\partial\overline{M}_f \cong S_+ \sqcup S_-$. After adjusting (precomposing) f by an isotopy supported in a small neighborhood of ∂Y, we may assume that $f(\overline{U}_+) \subset U_+$ and $f^{-1}(\overline{U}_-) \subset U_-$. Having done so, $Y_1 \subset \overline{M}_f$ is then properly embedded.

In what follows, we may need to adjust f by an isotopy which is the identity outside of Y. This does not affect the homeomorphism type of the pair $(\overline{M}_f, Y_1)$, and we use the same notation to denote the new pair.

A *boundary-compressing disk* for $Y_1 \subset \overline{M}_f$ (or more generally, for any properly embedded surface) is an embedded disk $D \subset \overline{M}_f$ such that $\partial D = \alpha \cup \beta$, where α is a properly embedded essential arc in Y_1, $\beta \subset \partial\overline{M}_f$, α and β intersect precisely in their endpoints, and the interior of D is disjoint from $Y_1 \cup \partial\overline{M}_f$. We can perform a homotopy of Y_1, rel ∂Y_1, "pushing across D" so that a neighborhood of α in Y_1 is mapped into $\partial\overline{M}_f$. See Figure 4.

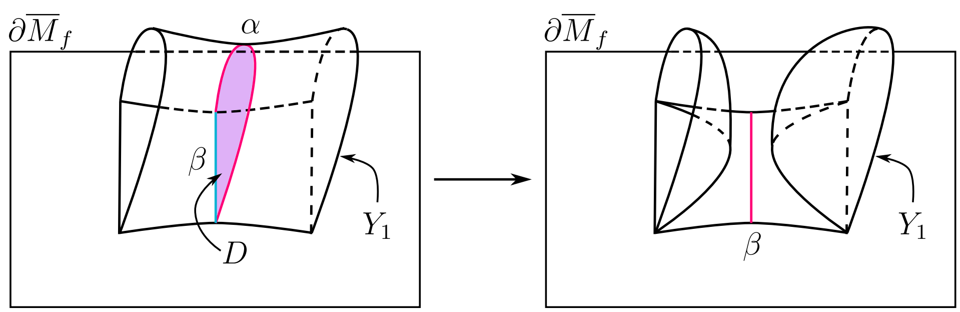

FIGURE 4. A boundary-compressing disk D (shaded purple) cobounded by an arc $\alpha \subset Y_1$ and an arc $\beta \subset \partial\overline{M}_f$. On the right of the figure is Y_1 after D is compressed.

A *flow-compressing disk* for Y_1 is a boundary-compressing disk $D \subset \overline{M}_f$ for Y_1 that is foliated by flowlines transverse to α and β. Given a flow-compressing disk D, if the arc β lies in $S_+ \subset \partial\overline{M}_f$, then $\alpha' = h_1^{-1}(\alpha)$ is an arc in Y, and $\beta = h_1(f(\alpha'))$. Since $h_1(f(\alpha')) \subset S_+$, it follows that $f(\alpha') \subset U_+$. Consequently, a flow-compressing disk is really determined by a properly embedded essential arc $\alpha' \subset Y$ that is mapped entirely into U_+ by f (or into U_- by f^{-1}). In this case, the homotopy of $h_1(S)$ obtained from the boundary compression of Y_1 along D can be carried out transverse to (ψ_s). See Figure 5. The part of the homotoped image of Y_1 that remains in the interior M_f, also determines a core for f which necessarily has larger Euler characteristic.

Lemma 3.2. *If $D \subset \overline{M}_f$ is a flow-compressing disk for the Y–viscera, then there is a core $Y' \subset Y$ for f such that $\chi(Y') > \chi(Y)$.*

Proof. Suppose that D intersects S_+ in the arc β (the proof in the case that $\beta \subset S_-$ is identical except with f replaced by f^{-1}).

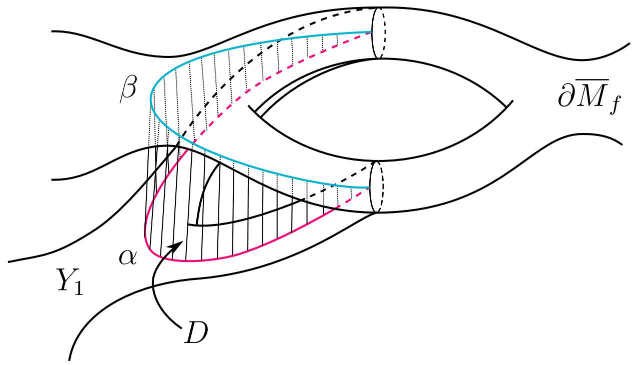

FIGURE 5. If a boundary-compressing disk D is foliated by flowlines transverse to α and β, as shown here, then we call it a flow-compressing disk. Here $\alpha = h_1(\alpha')$ and $\beta = h_1(f(\alpha'))$.

There is a small neighborhood N of D which is also a union of segments of flowlines, so that $X := N \cap Y_1$ is a regular neighborhood of α. Let $Y'' = \overline{Y_1 - X}$ and set $Y' = h_1^{-1}(Y'')$, where h_t is the homotopy defining Y_1. Observe that $\chi(Y') > \chi(Y)$ since Y is obtained from Y' by adding a 1–handle. We can use N to push X along flowlines until it lies entirely inside $\partial\overline{M}_f$. Concatenating this homotopy with h_t, we get a homotopy $g_t\colon S \to \overline{M}_f$ pushing along flowlines, so that $g_1^{-1}(\partial\overline{M}_f) = \overline{S - Y'}$.

Now observe that Y' is a core for f. Indeed, let W_+ and W_- be the unions of components of $S - Y'$ containing U_+ and U_-, respectively. Given $x \in W_+$, consider its maximal forward (ψ_s)–flowline in $\overline{M}_f$,

$$\ell_x^+ = \bigcup_{s \geq 0} \psi_s(x),$$

where the union is over all $s \geq 0$ for which $\psi_s(x)$ is defined. Every point of $\ell_x^+ \cap S$ is mapped by g_1 into S_+, and in particular, these points all lie in W_+. On the other hand, $\ell_x^+ \cap S$ is the union of forward images of x by f (since the first return map of (ψ_s) is f), and thus $f(x) \in W_+$. Therefore, $f(W_+) \subset W_+$. By similarly analyzing a backward flowline ℓ_x^-, we can see that $f^{-1}(W_-) \subset W_-$, proving that W_+ and W_- are tight nesting neighborhoods of the ends, and thus Y' is a core for f. □

The next lemma says that we can promote an arbitrary ∂–compressing disk to a flow-compressing disk.

Lemma 3.3. *If the Y–viscera in $\overline{M}_f$ is boundary compressible, then after adjusting f by an isotopy which is the identity outside Y, there is a flow-compressing disk $D \subset \overline{M}_f$.*

Proof. The local flow (ψ_s) defines a transverse orientation to Y_1. Suppose D is a compressing disk with boundary arcs $\alpha \subset Y_1$ and $\beta \subset \partial\overline{M}_f$. We assume that $\beta \subset S_+$, with the case $\beta \subset S_-$ proved by replacing f with f^{-1}. Since D is a compressing disk, $D \cap Y_1 = \alpha$, and so D must either be on the positive or negative side of Y near α.

If D is on the negative side, we observe that there are arcs $\alpha', \beta' \subset S$ meeting in their endpoints such that $h_1(\alpha') = \alpha$ and $h_1(\beta') = \beta$. See Figure 6. We can piece together a map of a disk $D' \to \overline{M}_f$ from the homotopy h_t and the disk D so that the boundary of D' maps homeomorphically to $\alpha' \cup \beta'$. We lift this to a map $D' \to \widetilde{M}_\infty$ so that the boundary of D' maps homeomorphically to $(\alpha' \cup \beta') \times \{0\}$ in $S \times \{0\}$. Projecting $\widetilde{M}_\infty$ onto the first factor, this in turn defines a homotopy from α' to β' in S, rel endpoints. However, β' is an arc in $\overline{U}_+$, while α' is an essential arc in Y. Since an essential arc in Y cannot be homotoped outside of Y, this is a contradiction.

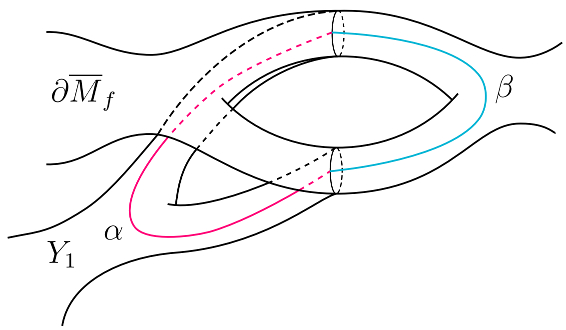

FIGURE 6. The arcs $\alpha \subset Y_1$ and $\beta \subset \partial\overline{M}_f$ when D is on the negative side of Y near α

From the previous paragraph, we may assume that D is on the positive side of Y near α. See Figure 7. We can again find arcs $\alpha', \beta' \subset S$ so that $h_1(\alpha') = \alpha$ and $h_1(\beta') = \beta$, but now these arcs do not meet at their endpoints. Instead, we can find such arcs so that the endpoints of β' are the first return points of the endpoints of α' by (ψ_s), *i.e.* the f–image of the endpoints of α'. Since flowing α' forward until it hits S, the image is precisely $f(\alpha')$, we see that $f(\alpha')$ is an arc with the same endpoints as β'.

Observe that if $f(\alpha') \subset U_+$, then $h_1(\alpha' \cup f(\alpha'))$ is the boundary of a flow compressing disk, and we are done. Because we have not imposed any constraints on the behavior of f inside of Y, we may not have $f(\alpha') \subset U_+$, in which case we claim we can adjust f by an isotopy supported in Y so that the new homeomorphism does send α' into U_+.

To find the required isotopy, we first observe that using the disk D, we can construct a homotopy, rel endpoints, from $f(\alpha')$ to β'. To do this, we consider the "rectangle" which is a union of the flowlines between α' and $f(\alpha')$ in $\overline{M}_f$, and drag this along via the homotopy h_t to define a disk $D' \subset \overline{M}_f$ whose boundary is the union of the two arcs $h_1(\alpha') = \alpha$ and $h_1(f(\alpha'))$, and whose interior is contained in a component of the complement of $\partial\overline{M}_f \cup Y_1$. Since both disks D and D' have their interiors in the same complementary component of $\overline{M}_f \cup Y_1$, their union $D \cup D'$ can

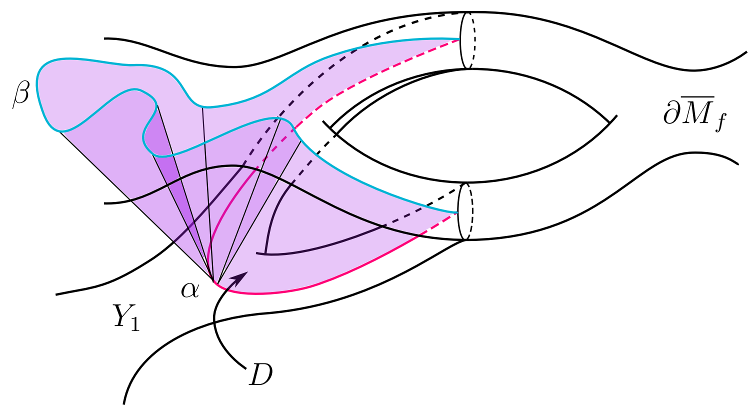

FIGURE 7. A boundary-compressing disk D, which is not foliated by flowlines.

be pushed forward via the flow into the branched surface $h_1(S)$, and then lifted back to S to define the homotopy, rel endpoints from $f(\alpha')$ to β'.

Now we assume, as we may, that $f(\partial Y)$ meets ∂Y transversely and minimally. Then since $\beta' \subset U_+$ is an arc disjoint from ∂Y and it is homotopic, rel endpoints, to $f(\alpha')$ which meets $f(\partial Y)$ only in its endpoints, it follows that we may postcompose f with an isotopy that is the identity outside $f(Y)$, so that $f(\alpha')$ is disjoint from ∂Y, and thus contained in U_+. This is equivalent to precomposing f by an isotopy that is the identity outside Y. Replacing f with this isotopic homeomorphism, $f(\alpha') \subset U_+$, and thus $\alpha = h_1(\alpha')$ and $h_1(f(\alpha'))$ defines a flow-compressing disk, as required. $\square$

From the two lemmas above, we deduce the following.

Proposition 3.4. *If $Y \subset S$ is a minimal core for a strongly irreducible end-periodic homeomorphism and $Y_1 \subset \overline{M}_f$ is the Y–viscera, then Y is boundary incompressible.*

Proof. If Y_1 were boundary compressible, Lemma 3.3 would providuce a flow-compressing disk, then Lemma 3.2 would produce a new core $Y' \subset Y$ with $\chi(Y') > \chi(Y)$, which contradicts minimality of Y. $\square$

4. PLEATED SURFACES

For the remainder of this section, we assume that $f\colon S \to S$ is a strongly irreducible end-periodic homeomorphism. Throughout, $Y \subset S$ will denote a core for f.

A *pants–lamination* on a hyperbolic surface X is a geodesic lamination whose leaves are the curves of a pants decomposition together with isolated leaves such that each complementary component is an ideal triangle spiraling into all three cuffs of its pair of pants, see Figure 8. We also make a technical assumption that for each pants curve, the leaves that spiral in towards that curve do so "in the same direction" on both sides of the curve. This is necessary to ensure that in Section 5.2 we can find simplicial hyperbolic surfaces limiting to our choice of pleated surface without adding many additional vertices. See Figure 9 for an illustration of this behavior.

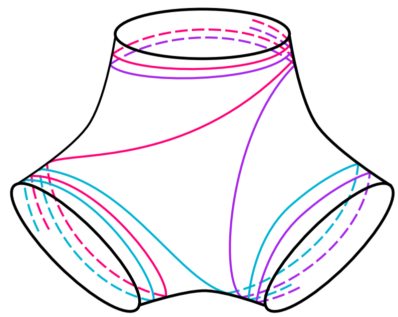

FIGURE 8. An ideal triangulation of a pair of pants. The three biinfinite geodesics shown are examples of the isolated leaves found in a pants–lamination.

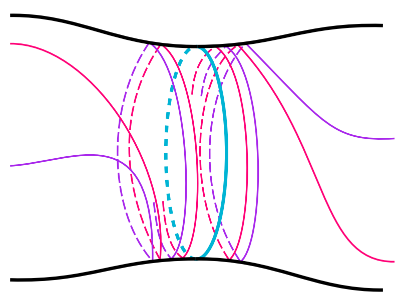

FIGURE 9. The blue pants curve in the middle has leaves of the pants-lamination spiraling in towards it "in the same direction" on both sides, *i.e.* they both turn to the right before begining to spiral.

Let M be a hyperbolic 3-manifold. A *pleated surface* is an arc-length preserving map $\varphi : (X, \sigma) \to M$ from a surface X with a complete hyperbolic metric σ such that: (a) φ maps leaves of some σ–geodesic lamination $\lambda \subset X$ to geodesics, and (b) φ is totally geodesic in the complement of λ. We say that λ is the *pleating locus* of φ if it is the smallest lamination satisfying conditions (a) and (b). See [CEG87, Section 5] for more details.

We introduce the following definition as an adaption of pleated surfaces to our infinite-type setting. A *well-pleated surface adapted to the core* Y in $\overline{M}_f$ is a pleated surface $\varphi\colon (S, \sigma) \to \overline{M}_f$, homotopic to the inclusion of S into $\overline{M}_f$, such that

(1) the pleating locus is contained in a pants–lamination, and
(2) $\varphi(S - Y) \subset \partial \overline{M}_f$.

Observe that "the inclusion" of S into $\overline{M}_f$ is really only well-defined up to pre-composing with powers of f. We will later avoid this issue by passing to the cover $\widetilde{M}_\infty \subset S \times [-\infty, \infty]$, where this ambiguity disappears. We also note that the metric σ on S is compatible with Y and induced by the metric on $\widetilde{M}_\infty$.

One may follow [CEG87, Theorem 5.3.6] to construct a well-pleated surface representing the inclusion, adapted to any given core Y. For this, construct a homotopy $h_t\colon S \to \overline{M}_f$ as in Section 3.2. We replace h_1 with a homotopic map $\varphi'\colon S \to \overline{M}_f$ so that φ' sends each component of ∂Y to a closed geodesic in $\partial\overline{M}_f$. Since h_1 was a local homeomorphism from the neighborhoods of the ends $S - Y$ onto $\partial\overline{M}_f$, we can assume that φ' is as well. We choose a hyperbolic structure on $S - Y$ so that $\varphi'|_{S-Y}$ is a local isometry. Now we choose any pants lamination λ containing ∂Y, and observe that we are now reduced to constructing a pleated surface $Y \to \overline{M}_f$ homotopic to $\varphi'|_{\partial Y}$, rel ∂Y, realizing the finite lamination $\lambda \cap Y$. We can view this restriction $\varphi'|_Y\colon (Y, \partial Y) \to (\overline{M}_f, \partial\overline{M}_f)$ as a pleated surface representative of the Y–viscera, $(Y_1, \partial Y_1) \subset (\overline{M}_f, \partial\overline{M}_f)$, realizing the finite lamination $h_1(\lambda \cap Y) \subset Y_1$.

We let

$$\Omega(f, Y) = \{\varphi\colon (S, \sigma) \to \overline{M}_f \mid \varphi \text{ is well-pleated adapted to } Y\}$$

denote the set of all well-pleated surfaces adapted to Y. A well pleated surface adapted to *some* core will simply be called a well-pleated surface and we denote the set of all such by $\Omega(f)$.

In the proof of the following lemma, we will make use of the theorem of Basmajian [Bas94, Theorem 1.1] that there is a constant $\beta > 0$, depending only on $\xi(f)$, so that the β–neighborhood of $\partial\overline{M}_f$ is a product, $\partial\overline{M}_f \times [0, \beta]$.

Lemma 4.1 (Bounding the boundary)**.** *Suppose that Y is a minimal core of the strongly irreducible, end-periodic homeomorphism $f\colon S \to S$ and suppose $(\varphi\colon (S, \sigma) \to \overline{M}_f) \in \Omega(f, Y)$, then the total σ–length of the boundary of ∂Y satisfies $\ell_\sigma(\partial Y) \le \frac{2\pi|\chi(Y)|}{\beta}$.*

Proof. Take $\epsilon > 0$ maximal such that the open ϵ–neighborhood (with respect to the metric σ) of ∂Y are annuli. The boundary of this neighborhood meets itself at some point, producing an essential arc of length 2ϵ in Y. The area of the ϵ–neighborhood is no more than the total area of (Y, σ), which is $-2\pi\chi(Y) = 2\pi|\chi(Y)|$ by the Gauss–Bonnet formula. On the other hand, the area of this ϵ–neighborhood is $\sinh(\epsilon)\ell_\sigma(\partial Y) > \epsilon\ell_\sigma(\partial Y)$, and so

$$\epsilon\ell_\sigma(\partial Y) < 2\pi|\chi(Y)|.$$

If $\ell_\sigma(\partial Y) > \frac{2\pi|\chi(Y)|}{\beta}$, then $\epsilon < \beta$. In this case, the closure of the ϵ–neighborhood of ∂Y would map into the β–neighborhood of $\partial\overline{M}_f$. It follows that α is mapped into the collar neighborhood of $\partial\overline{M}_f$, and is hence properly homotopic into $\partial\overline{M}_f$. But $\varphi|_Y$ is a pleated surface representative of Y_1, and thus [AR93, Theorem 2.2] implies that Y_1 is boundary-compressible. This contradicts Proposition 3.4, since Y was a minimal core. Therefore, $\ell_\sigma(\partial Y) \le \frac{2\pi|\chi(Y)|}{\beta}$, as required. □

A *minimally well-pleated surface* is any well-pleated surface $\varphi \in \Omega(f, Y)$ for which Y is a minimal core.

Corollary 4.2. *If $\varphi \in \Omega(f)$ is minimally well-pleated and $\beta(\xi(f)) > 0$ is Basmajian's constant, then*

$$\ell_\sigma(\partial\Sigma_\varphi) \leq \frac{2\pi|\chi(f)|}{\beta(\xi(f))}.$$

5. Interpolation and pants

In this section, we describe how to produce paths of pants decompositions that we will use to prove the required lower bounds on volume. In what follows, we typically assume Y is a minimal core for a strongly irreducible end-periodic f, so that $\chi(Y) = \chi(f)$.

5.1. **Pants and cores.** Suppose $(\varphi\colon (S,\sigma) \to \overline{M}_f) \in \Omega(f,Y)$ is any minimally well-pleated surface (*i.e.* Y is a minimal core), and let $U_+ \cup U_- = S - Y$ be the associated tight nesting neighborhoods. Applying an isotopy if necessary, we assume that for any component α of $\partial_- Y$, $f^{-1}(\alpha)$ is either an essential curve of U_- or else a different component of $\partial_- Y$.

We also set

$$\Delta_+ = \overline{U_+ - f(U_+)},$$

as in Section 2.3, which is a fundamental domain for the action of f on $\mathcal{U}_+$. Note that Δ_+ is the subsurface bounded by (some components of) $\partial_+ Y$ and $f(\partial_+ Y)$, by Lemma 2.5, since Y is balanced by minimality.

Now we define

$$\Sigma = Y \cup \Delta_+,$$

(see Figure 10) and observe that from the discussion in Section 2.3 we have

$$\xi(\Delta_+) \leq \tfrac{3}{2}|\chi(\Delta_+)| = \tfrac{3}{4}|\chi(\partial\overline{M}_f)| = \tfrac{1}{2}\xi(f). \tag{5.1}$$

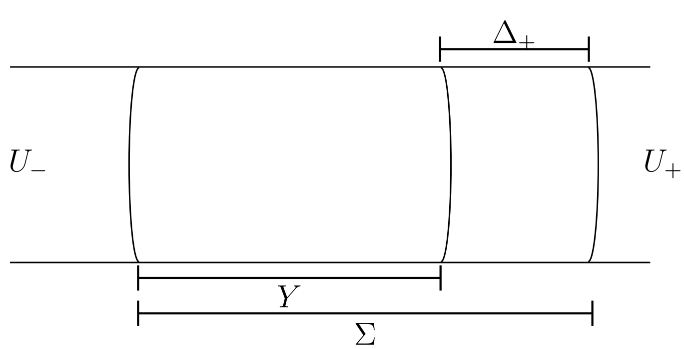

Figure 10. A schematic of the surface S with Y, Δ_+, and Σ labelled. Here Δ_+ is a fundamental domain for the action of f on $\mathcal{U}_+$, and Σ is the union of the minimal core, Y, with Δ_+.

We are now ready to construct a bounded-length, f-asymptotic pants decomposition of S.

Lemma 5.1. *There exists a constant $L = L(\chi(f), \xi(f))$ so that for any minimally well-pleated surface $(\varphi\colon (S,\sigma) \to \overline{M}_f) \in \Omega(f, Y)$, there exists a pants decomposition P of S such that*

1. *$\partial\Sigma \subset P$;*
2. *P and $f(P)$ differ only on Σ; and*
3. *each curve in P has σ–length at most L.*

Proof. Fix $(\varphi\colon (S,\sigma) \to \overline{M}_f) \in \Omega(f, Y)$ minimally well-pleated and adapted to a core Y, and continue with the notation as above, so that $U_\pm$ are the tight nesting neighborhoods defined by Y. Since $\varphi(U_\pm)$ is entirely contained in $\partial\overline{M}_f$, it follows that, with respect to the metric σ, after an isotopy we can assume that f isometrically maps U_+ into itself and f^{-1} isometrically maps U_- into itself.

We now construct the desired pants decomposition P of S. We start with the curves ∂Y. Note that $|\chi(Y)| = |\chi(f)|$ and

$$\ell_\sigma(\partial Y) \leq \frac{2\pi|\chi(Y)|}{\beta(\xi(f))} \leq \frac{2\pi|\chi(f)|}{\beta(\xi(f))},$$

by Corollary 4.2. Since $\xi(Y) \leq \frac{3}{2}|\chi(Y)|$, Theorem 2.9 then guarantees that we can choose a pants decomposition of Y so that each pants curve has length bounded by some $L_0 = L_0(\chi(f), \xi(f))$. We require P to contain this pants decomposition of Y (including the boundary curves ∂Y).

From Equation (5.1), $\xi(\Delta_-) \leq \frac{1}{2}\xi(f)$. Since $\ell_\sigma(\partial Y)$ is uniformly bounded and since f isometrically maps U_+ into itself, we see that there is also a uniform bound on the length of $\partial\Delta_+$ in terms of $\chi(f)$ and $\xi(f)$. So again applying Theorem 2.9, there is a pants decomposition P_+ of Δ_+ (including the boundary curves, $\partial\Delta_+$) such that each pants curve has length bounded by some $L_1 = L_1(\chi(f), \xi(f))$. Then, since

$$U_+ = \bigcup_{j=0}^{\infty} f^j(\Delta_+)$$

with any two distinct translates $f^j\Delta_+$ and $f^{j'}\Delta_+$ intersecting at most in their boundary curves, we can extend P over U_+ so that

$$U_+ \cap P = \bigcup_{j=0}^{\infty} f^j(P_+),$$

We similarly construct P on U_- with each curve's length bounded by some constant $L_2 = L_2(\chi(f), \xi(f))$.

Now set $L = \max\{L_0, L_1, L_2\}$. The lemma follows by construction: Item 3 is by definition, and Item 1 and Item 2 are a consequence of the fact that $\Sigma = Y \cup \Delta_+$. $\square$

Convention 5.2. For the remainder of this section, fix $(\varphi\colon (S,\sigma) \to \overline{M}_f) \in \Omega(f, Y)$ with $\chi(Y) = \chi(f)$ and $L = L(\chi(f), \xi(f))$, and P as in Lemma 5.1. Note that P contains ∂Y, and so we may change our pleated surface φ to assume it realizes a pants lamination containing P. The conclusions of the lemma are still satisfied for this well-pleated surface for the same P.

Next, fix the lift $\widetilde{\varphi}\colon S \to \widetilde{M}_\infty \subset S \times [-\infty, \infty]$ of φ so that $\widetilde{\varphi}$ is homotopic to the identity after projecting onto the first factor.

Composing with the covering transformation F on $\widetilde{M}_\infty$, we get $F \circ \widetilde{\varphi}\colon S \to \widetilde{M}_\infty$, which is another lift of φ, but rather than being homotopic to the identity on S (after projecting onto the first factor) it is homotopic to f, since $F(x,t) = (f(x), t-1)$. Therefore,

$$F \circ \widetilde{\varphi} \circ f^{-1}\colon S \to \widetilde{M}_\infty$$

is homotopic to the identity (again, after projecting to the first factor).

Pulling back the metric by f^{-1} gives a new (lift of a) well-pleated surface

$$F \circ \widetilde{\varphi} \circ f^{-1}\colon (S, (f^{-1})^*\sigma) \to \widetilde{M}_\infty.$$

Since P has length at most L with respect to σ, $f(P)$ has length at most L with respect to $(f^{-1})^*\sigma$.

By Lemma 5.1, P and $f(P)$ differ only in Σ, and we define

$$P_\alpha = P \cap \Sigma \quad \text{and} \quad P_\omega = f(P) \cap \Sigma.$$

These are pants decompositions of Σ such that

$$d_{\mathcal{P}(S)}(P, f(P)) \leq d_{\mathcal{P}(\Sigma)}(P_\alpha, P_\omega), \tag{5.2}$$

where

$$|\chi(\Sigma)| = |\chi(\Delta_-)| + |\chi(Y)| = \tfrac{1}{3}\xi(f) + |\chi(f)| < \xi(f) + |\chi(f)|, \tag{5.3}$$

and each curve of $\partial\Sigma$ has length at most L.

To simplify the notation, we write $\phi_\alpha\colon \Sigma \to \widetilde{M}_\infty$ to denote the restriction of $\widetilde{\varphi}$ to Σ and $\phi_\omega\colon \Sigma \to \widetilde{M}_\infty$ to denote the restriction of $F \circ \widetilde{\varphi} \circ f^{-1}$ to Σ. As the notation suggests, ϕ_α maps the curves of P_α to geodesics of length at most L and ϕ_ω maps the curves of P_ω to geodesics of length at most L. We write σ_α and σ_ω to denote the hyperbolic structures so that $\phi_\alpha\colon (\Sigma, \sigma_\alpha) \to \widetilde{M}_\infty$ and $\phi_\omega\colon (\Sigma, \sigma_\omega) \to \widetilde{M}_\infty$ are pleated surfaces (representing the Σ–viscera). Since both ϕ_α and ϕ_ω map $\partial\Sigma$ to geodesics, by precomposing one of these with an isotopy of the identity, we assume (as we may) ϕ_α and ϕ_ω agree on the boundary and that they are homotopic by a homotopy that is stationary on $\partial\Sigma$.

5.2. **Simplicial hyperbolic surfaces.** To produce continuous families of "good" representatives of a homotopy class of $\phi_\alpha\colon \Sigma \to \widetilde{M}_\infty$ (and hence of ϕ_ω) we use simplicial hyperbolic surfaces, following [Can96].

We fix once and for all $k_0 = |\partial\Sigma|$ points, one on each component of $\partial\Sigma$. Let $k \geq k_0$. A *k–simplicial pre-hyperbolic surface in $\widetilde{M}_\infty$* is a map $\eta\colon \Sigma \to \widetilde{M}_\infty$ that satisfies the following:

- There is a triangulation[1] $\mathcal{T}$ of Σ with k vertices, exactly one on each boundary component at the fixed k_0 points, such that η takes each triangle to a non-degenerate totally geodesic triangle in $\widetilde{M}_\infty$.

[1]This is not a triangulation in the classical sense, but rather a Δ–complex structure in the sense of [Hat02]

- The restriction of η to each component of $\partial\Sigma$ is a closed geodesic.
- The map η is homotopic to ϕ_α through maps that are stationary on $\partial\Sigma$.

Note that, for such a surface, each component of $\partial\Sigma$ is parameterized by a single edge of the triangulation under η. Furthermore, the hyperbolic metrics on the triangles induce a singular hyperbolic metric σ_η on Σ. In this metric, the boundary is a smooth geodesic, except possibly at the vertices. The k–simplicial pre-hyperbolic surface is a k–simplicial ***hyperbolic*** surface if the cone angles in the interior are all at least 2π, and those on the boundary are at least π. The set of all k–simplicial hyperbolic surfaces is denoted $\mathcal{SH}_k$, and the set of all simplicial hyperbolic surfaces by $\mathcal{SH} = \cup_k \mathcal{SH}_k$, and we equip both with the compact–open topology. We will be primarily interested in k–simplicial hyperbolic surfaces when $k = k_0$ or $k = k_0 + 1$.

The universal cover $\widetilde{M}$ of $\widetilde{M}_\infty$ may be identified with a convex subset of $\mathbb{H}^3$ whose frontier is a union of totally geodesic hyperbolic planes. This allows the identification of $\overline{M}_f$ as the quotient of such a subset. We choose such an identification, as well as an equivariant lift $\widetilde{\Sigma} \to \widetilde{M}$ of ϕ_α, which also gives us an equivariant lift $\widetilde{\eta}$ of any k–simplicial hyperbolic surface η by lifting the homotopy to ϕ_α.

We let $\mathcal{SH}(\mathcal{T}) \subset \mathcal{SH}$ be the subspace consisting of all simplicial hyperbolic surfaces whose underlying triangulation is $\mathcal{T}$.

Lemma 5.3. *For any triangulation $\mathcal{T}$ of Σ with k_0 vertices, one vertex at each of the fixed points on $\partial\Sigma$, the space $\mathcal{SH}(\mathcal{T})$ is non-empty and any two elements of $\mathcal{SH}(\mathcal{T})$ differ by precomposing with a homeomorphism isotopic to the identity by an isotopy that is stationary on the boundary.*

Proof. We will first show that $\mathcal{SH}(\mathcal{T})$ is nonempty by constructing an element $\eta : \Sigma \to \widetilde{M}_\infty$ of $\mathcal{SH}(\mathcal{T})$ inductively over the skeleta.

We begin by declaring that η agrees with ϕ_α on the boundary of Σ, and hence the vertices of $\mathcal{T}$, which is required for any element of $\mathcal{SH}(\mathcal{T})$. Let e be an edge of $\mathcal{T}$. If the endpoints of e are distinct, then, by lifting to $\mathbb{H}^3$ and using a straight–line homotopy, we may homotope $\phi_\alpha\big|_e$ relative to its endpoints to a locally geodesic segment. If this is the case, we define η to send e to this local geodesic segment which necessarily has positive length: the endpoints are either on distinct boundary components of $\partial\widetilde{M}_\infty$, or on disjoint closed geodesics in a single component of $\partial\widetilde{M}_\infty$. If e is a loop, then it is homotopically nontrivial, and hence it is carried by ϕ_α to an essential loop, and we homotope ϕ_α by a straight–line homotopy again to η mapping e to the geodesic representative of the based homotopy class of loops (if the loop is a boundary component, it is already geodesic and η and ϕ_α agree there). The boundary ∂T of any triangle T in $\mathcal{T}$ is null–homotopic in Σ, and so it also lifts to the universal cover where we can extend our straight line homotopy to a homotopy of $\phi_\alpha|_T$ to $\eta|_T$ mapping T to a (possibly degenerate) geodesic triangle immersed in $\widetilde{M}_\infty$. Having done this for every triangle T, we have defined $\eta : \Sigma \to \widetilde{M}_\infty$ and the homotopy from ϕ_α.

We claim that every geodesic triangle $\eta|_T$ is in fact non-degenerate. This means that the lift $\widetilde{\eta}|_T$ to $\mathbb{H}^3$ is an embedding of a geodesic triangle. Suppose that this is not the case. First, observe that every edge of T in Σ either connects distinct vertices or is a non-null homotopic loop, so the image of each edge is a non-degenerate segment. Thus, the only degeneracy that may occur in $\widetilde{\eta}|_T$ is that all three vertices lie on a single geodesic segment. All three of these points lie on the boundary of $\widetilde{M}$, which we recall is a convex subset of $\mathbb{H}^3$ bounded by hyperbolic planes. Therefore, the entire segment must lie in the boundary.

Now, projecting the segment back to $\widetilde{M}_\infty$, we obtain a segment in $\partial\widetilde{M}_\infty$ passing through three vertices in $\eta(\partial\Sigma)$. Note that the segment cannot be entirely contained in $\eta(\partial\Sigma)$ since then all three edges of T are mapped to the homotopy class of the boundary loop, which is not allowed in the triangulation. Therefore, the segment defines a geodesic path in a component of $\partial\widetilde{M}_\infty$, and hence in either $\mathcal{U}_+ \times \{\infty\}$ or $\mathcal{U}_- \times \{-\infty\}$. Moreover, this segment meets the geodesic boundary $\eta(\partial\Sigma)$ transversely. A subsegment of the path between two of the vertices enters one of $U_+ \times \{\infty\}$ or $U_- \times \{-\infty\}$ from the vertex. Projecting $\widetilde{M}_\infty$ onto the first factor S, this subsegment projects to an essential path in S which cannot be homotoped entirely contained in Σ. This is impossible because the path is homotopic to an edge of the triangulation, and hence an essential arc in Σ. Therefore, the triangle T is non-degenerate.

The link of a vertex v in $\mathcal{T}$ defines a path in the sphere $T^1_{\widetilde{\eta}(v)}(\mathbb{H}^3)$ joining antipodal points—namely, the two tangent vectors to the boundary geodesic—and thus has length at least π, proving that the cone angle at the vertices is at least π (*c.f.* the NLSC [not locally strictly convex] property and [Can96, Lemma 4.2]). Therefore, after reparameterizing if necessary, we conclude that η is the desired k_0–simplicial hyperbolic surface and so, $\mathcal{SH}(\mathcal{T})$ is non-empty.

Now, given any $\eta, \eta' \in \mathcal{SH}(\mathcal{T})$, the two maps are homotopic to ϕ_α by a homotopy that is stationary on $\partial\Sigma$, they are also homotopic to each other by such a homotopy. Since each edge e of $\mathcal{T}$ must be sent to the geodesic in the relative homotopy class of $\phi_\alpha\big|_e$, it follows that η and η' differ on each edge by a reparameterization constant on the endpoints. Therefore, η and η' differ by reparameterization that maps simplicies to simplices. Since each triangle maps to a non-degenerate triangle, the reparameterization is necessarily a homeomorphism preserving the triangulation which is the identity on the vertices. It follows that this reparameterizing homeomorphism is isotopic to the identity rel the vertices, and thus η and η' differ by precomposing by a homeomorphism isotopic to the identity rel the vertices. □

For each component $\gamma \subset \partial\Sigma$, $\phi_\alpha(\gamma) = \phi_\omega(\gamma)$ is a closed geodesic in the totally geodesic boundary $\partial\widetilde{M}_\infty$ of length at most L. Consider the annular cover of the component of $\partial\widetilde{M}_\infty$ to which this curve lifts. This annulus is divided by (the lift of the image of) γ into two half-open annuli with boundary γ, and we let $\breve{\Sigma} \supset \Sigma$ be obtained by gluing these half-open annuli to each boundary component of Σ. For any $\eta\colon (\Sigma, \mathcal{T}) \to \widetilde{M}_\infty \in \mathcal{SH}$, we have $\eta|_\gamma = \phi_\alpha|_\gamma$, and so we can extend η to a map

$$\breve{\eta}\colon \breve{\Sigma} \to \widetilde{M}_\infty,$$

whose restriction to the added half-open annuli is the restriction of the covering map to $\partial\widetilde{M}_\infty$. The singular hyperbolic metric σ_η extends to a singular hyperbolic metric of the same name on $\breve{\Sigma}$, so that $\breve{\eta}$ is a local isometry on each half-open annulus. Then let $\sigma_\eta^{\mathcal{H}}$ be the hyperbolic uniformization of the conformal structure on $(\breve{\Sigma}, \sigma_\eta)$, considered as a point in the Teichmüller space $\mathcal{T}(\breve{\Sigma})$. We write $\ell_{\sigma_\eta^{\mathcal{H}}}(\gamma)$ for the length of the $\sigma_\eta^{\mathcal{H}}$–geodesic representative of any essential closed curve γ in $\breve{\Sigma}$.

Theorem 5.4. *For any $\eta\colon (\Sigma, \mathcal{T}) \to \widetilde{M}_\infty$ in $\mathcal{SH}$, the identity map*

$$\mathrm{id}\colon (\breve{\Sigma}, \sigma_\eta^{\mathcal{H}}) \to (\breve{\Sigma}, \sigma_\eta)$$

is 1–Lipschitz. Furthermore, for each component $\gamma \subset \partial\Sigma$, the $\ell_{\sigma_\eta^{\mathcal{H}}}(\gamma) \leq 2L$.

Proof. The first statement is a consequence of the Ahlfors–Schwartz–Pick Theorem [Ahl38]. For the second statement, we will use a modulus argument (see [EM06, Theorem 2.16.1] for a precise statement of the correspondence between modulus of an annulus and the length of its core curve). Note that the length of $\phi_\alpha(\gamma) = \eta(\gamma)$ in $\widetilde{M}_\infty$ is at most L, and so the annular cover of the component of $\partial\widetilde{M}_\infty$ containing this curve has modulus at least $\frac{\pi}{L}$. Consequently, the interior of the half-open annulus (which is half of this annular cover) has modulus at least $\frac{\pi}{2L}$. But this annulus lifts to the annular cover of $\breve{\Sigma}$ to which γ lifts, and hence this cover has modulus at least $\frac{\pi}{2L}$ by monotonicity of moduli of annuli. This in turn implies that $\ell_{\sigma_\eta^{\mathcal{H}}}(\gamma) \leq 2L$, as required. □

5.3. **Interpolation and pants paths.** Two triangulations $\mathcal{T}$ and $\mathcal{T}'$ differ by a *flip move* if there are edges e of $\mathcal{T}$ and e' of $\mathcal{T}'$ so that e and e' intersect transversely in a single point and their complements in the 1–skeleta agree:

$$\mathcal{T}^{(1)} - e = \mathcal{T}'^{(1)} - e'.$$

In this case, let $\mathcal{T} * \mathcal{T}'$ be the triangulation obtained from $\mathcal{T}$ by adding a vertex at $e \cap e'$, subdividing each of e and e' and adding the subdivided e' to the 1–skeleton. This is the "minimal common subdivision" of $\mathcal{T}$ and $\mathcal{T}'$ which is illustrated in Figure 11.

Given $\eta\colon (\Sigma, \mathcal{T}) \to \widetilde{M}_\infty$ in $\mathcal{SH}_{k_0}$ and a triangulation $\mathcal{T}'$ differing from $\mathcal{T}$ by a flip, we may reparameterize η by precomposing with a homeomorphism isotopic to the identity so that it is also a (k_0+1)–simplicial hyperbolic surface

$$\eta\colon (\Sigma, \mathcal{T} * \mathcal{T}') \to \widetilde{M}_\infty.$$

The proof of [Can96, Lemma 5.3] can be applied to prove the following.

Lemma 5.5. *Suppose $\mathcal{T}$ and $\mathcal{T}'$ differ by a flip and that $\eta\colon (\Sigma, \mathcal{T}) \to \widetilde{M}_\infty$ and $\eta'\colon (\Sigma, \mathcal{T}') \to \widetilde{M}_\infty$ lie in $\mathcal{SH}_{k_0}$. Then there is a 1–parameter family*

$$\{\eta_t\colon (\Sigma, \mathcal{T} * \mathcal{T}') \to \widetilde{M}_\infty \mid t \in [0,1]\} \subset \mathcal{SH}_{k_0+1}$$

such that $\eta_0 = \eta$ and $\eta_1 = \eta'$, up to reparameterization by homeomorphisms isotopic to the identity by an isotopy that is stationary on the boundary. □

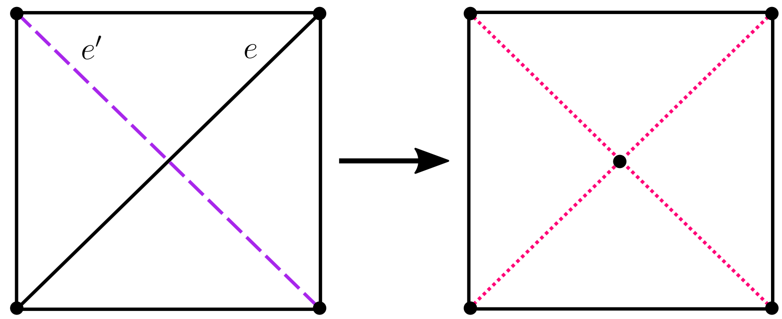

FIGURE 11. The minimal common subdivision $\mathcal{T} * \mathcal{T}'$ (shown on the right) of $\mathcal{T}$ and $\mathcal{T}'$ (shown on the left) where $e \subset \mathcal{T}$ is represented by the solid black diagonal edge and $e' \subset \mathcal{T}'$ is represented by the dashed purple diagonal edge. The four pink dotted edges in $\mathcal{T} * \mathcal{T}'$ are the new edges introduced by subdividing e and e'

Rather than repeat all the details of the proof as in [Can96], we explain the basic idea. By further reparameterization if necessary, the two maps η and η' will agree outside the "square" with diagonals e and e', and the interpolation takes place entirely within this square. Lifting the restrictions of η and η' to the square, the two original triangles of $\mathcal{T}$ in this square, together with the two triangles of $\mathcal{T}'$ define a tetrahedron in $\mathbb{H}^3$. Connecting the image of e via the lift of η to the image of e' via the lift of η' by a geodesic segment, the one-parameter family is essentially obtained by "sliding" the new vertex along this geodesic, and then projecting back to $\widetilde{M}_\infty$.

We will also need the following result; see e.g. Hatcher [Hat91].

Lemma 5.6. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulations of Σ with k_0 vertices, one on each component of $\partial\Sigma$. Then there is a sequence $\mathcal{T}_0 = \mathcal{T}, \mathcal{T}_1, \ldots, \mathcal{T}_m$ of triangulations, each differing from the previous one by a flip so that $\mathcal{T}_m = \mathcal{T}'$, up to isotopy which is stationary on $\partial\Sigma$.* □

We will use these two lemmas together with Lemma 5.3 to prove the following corollary.

Corollary 5.7. *Suppose $\eta\colon (\Sigma, \mathcal{T}) \to \widetilde{M}_\infty$ and $\eta'\colon (\Sigma, \mathcal{T}') \to \widetilde{M}_\infty$ are two k_0–simplicial hyperbolic surfaces. Then there is a 1–parameter family*

$$\{\eta_t\colon (\Sigma, \mathcal{T}_t) \to \widetilde{M}_\infty \mid t \in [0, 1]\} \subset \mathcal{SH}$$

such that $\eta_0 = \eta$ and $\eta_1 = \eta'$, up to reparameterization by homeomorphisms isotopic to the identity by an isotopy that is stationary on the boundary.

Proof. Let $\mathcal{T} = \mathcal{T}_0, \ldots, \mathcal{T}_m = \mathcal{T}'$ be the sequence of flips from Lemma 5.6. By Lemma 5.3, there is a sequence of simplicial hyperbolic surfaces

$$\{\eta_j\colon (\Sigma, \mathcal{T}_j) \to \widetilde{M}_\infty\}_{j=0}^m$$

and $\eta_0 = \eta$, $\eta_m = \eta'$, up to reparameterization (isotopic to the identity by an isotopy that is stationary on the boundary). By Lemma 5.5, for each $j = 1, \dots, m$, we can interpolate between η_{j-1} and η_j by a one-parameter family of simplicial hyperbolic surfaces. Concatenating these one-parameter families, and reparameterizing by precomposing by isotopies between these families whenever necessary, produces the required 1-parameter family from η to η', as required. □

We are now able to define a continuous path in Teichmüller space given by the hyperbolic structures obtained from uniformization of the 1-parameter family of simplicial hyperbolic surfaces given to us in Corollary 5.7.

Lemma 5.8. *Given the family $\{\eta_t \colon (\Sigma, \mathcal{T}_t) \to \widetilde{M}_\infty \mid t \in [0,1]\} \subset \mathcal{SH}$ from Corollary 5.7, the map $[0,1] \to \mathrm{Teich}(\breve{\Sigma})$, given by $t \mapsto \sigma^{\mathcal{H}}_{\eta_t}$ defines a (continuous) path in $\mathrm{Teich}(\breve{\Sigma})$.*

Proof. The family in Lemma 5.5 defines a continuous path in $\mathrm{Teich}(\breve{\Sigma})$ since the shapes of the hyperbolic triangles in the interpolation vary continuously; see [Can96]. The terminal point of the path from η_{j-1} to η_j and the initial point of the path from η_j to η_{j+1} differ by reparameterization by a homeomorphism isotopic to the identity. This isotopy defines a constant path in $\mathrm{Teich}(\breve{\Sigma})$ since the cone metrics are all obtained by pulling back the same cone metric by the homeomorphisms throughout the isotopy. Therefore, the paths can be concatenated to produces a path from $\sigma^{\mathcal{H}}_{\eta}$ to $\sigma^{\mathcal{H}}_{\eta'}$, as required. □

Recall that P_α, P_ω are pants decompositions on Σ so that with respect to σ_α and σ_ω, the lengths of each component of P_α and P_ω, respectively are at most $L = L(\chi(f), \xi(f))$, from Lemma 5.1. The next lemma says we can find simplicial hyperbolic surfaces whose cone metrics have hyperbolic uniformizations in which P_α and P_ω are also bounded length.

Lemma 5.9. *There exists $L_1 = L_1(\chi(f), \xi(f))$ and a pair of simplicial hyperbolic surfaces $\eta_\alpha \colon (\Sigma, \mathcal{T}_\alpha) \to \widetilde{M}_\infty$ and $\eta_\omega \colon (\Sigma, \mathcal{T}_\omega) \to \widetilde{M}_\infty$ in $\mathcal{SH}_{k_0}$ such that each component of P_α and P_ω have length at most L_1 in $\sigma^{\mathcal{H}}_{\mathcal{T}_\alpha}$ and $\sigma^{\mathcal{H}}_{\mathcal{T}_\omega}$, respectively.*

Remark. We will ultimately use an interpolation through simplicial hyperbolic surfaces to find a path in the pants graph between P_α and P_ω via Lemma 5.8. In the finite-type case [Bro03b], Brock similarly constructs such a path, though in his situation, the initial and terminal pants decompositions are *defined* from a simplicial hyperbolic surface and its image under a power of the monodromy. In our case, Σ is not invariant by any nontrivial power of f. However, Lemma 5.9 allows us to choose simplicial hyperbolic surfaces that are adapted to our *existing* pants decompositions P_α and P_ω (as opposed to choosing these pants decompositions from the simplicial hyperbolic surfaces themselves). Our construction in Lemma 5.9 is guided by the minimally well-pleated surfaces ϕ_α and ϕ_ω that realize P_α and P_ω, respectively.

Proof of Lemma 5.9. We carry out the proof for P_α with the one for P_ω being identical. Recall that P_α is the pants decomposition in Σ so that the pleating locus of ϕ_α is contained in a pants lamination λ containing P_α. Choose a triangulation $\mathcal{T}$ of Σ so that the arcs of intersection of the 1–skeleton with each pair of pants connect every pair of distinct boundary curves in those pants. To do this, we can first find a collection of pairwise disjoint essential arcs with this property, slide the endpoints to lie on the k_0 vertices, then extend to a triangulation.

For each closed curve δ in $P_\alpha \cup \partial\Sigma$, the non-compact leaves spiraling in toward δ spiral to the left or to the right (from both sides when $\delta \subset P_\alpha$, by assumption; see Section 4). Let $D\colon \Sigma \to \Sigma$ be the multitwist obtained by applying a right-handed Dehn twist around those $\delta \subset P_\alpha \cup \partial\Sigma$ for which the spiraling is to the right, and a left-handed Dehn twist around those δ for which the spiraling is to the left, and set $\mathcal{T}_j = D^j(\mathcal{T})$, for all integers $j \geq 0$. We also let $\eta_j\colon (\Sigma, \mathcal{T}_j) \to \widetilde{M}_\infty$ be the simplicial hyperbolic surface given by Lemma 5.3.

Consider the sequence of triangulations $\mathcal{T}_j$ on Σ, straightened to be geodesic with respect to σ_α. Lift ϕ_α to the universal covers $\tilde{\phi}_\alpha\colon \widetilde{\Sigma} \to \widetilde{M} \subset \mathbb{H}^3$. We write $\widetilde{\mathcal{T}}_j$ for the lifted triangulation.

Claim 5.10. *Given $\epsilon > 0$, there exists $J > 0$ so that for all $j \geq J$ and every edge e of $\widetilde{\mathcal{T}}_j$, $\tilde{\phi}_\alpha\big|_e$ is a $(1+\epsilon, \epsilon)$–quasi-geodesic.*

Proof. First, we observe that as j tends to infinity, we have Hausdorff convergence of the 1–skeleta, $\mathcal{T}_j^{(1)} \to \lambda$. This is because all angles of intersection with $P_\alpha \cup \partial\Sigma$ tend to zero, so the limit is a lamination containing $P_\alpha \cup \partial\Sigma$. See Figure 12. Further, in any pair of pants, our original choice of $\mathcal{T}$ guarantees that there are non-compact leaves that spiral between any two boundary components. Finally, our choice of D ensures that the spiraling toward all the curves is in the correct direction. These conditions uniquely determine the pants lamination λ. If $\tilde{\lambda}$ is the lifted lamination to $\widetilde{\Sigma}$, then we also have $\widetilde{\mathcal{T}}_j^{(1)} \to \tilde{\lambda}$ as $j \to \infty$. Thus, given $c, r > 0$ there exists $J > 0$ so that for all $j > J$, every edge e of $\widetilde{\mathcal{T}}_j$, and every segment $\delta \subset e$ of length at most r and there is a segment δ' of a leaf of λ such that the endpoints of δ' and the endpoints of δ are at most c apart.

Since $\tilde{\phi}_\alpha$ maps each leaf of $\tilde{\lambda}$ to a geodesic and is 1–Lipschitz, it follows that the endpoints of $\tilde{\phi}_\alpha(\delta)$ are within c of the endpoints of the geodesic $\tilde{\phi}_\alpha(\delta')$. Therefore, if x, y are the endpoints of δ, we have

$$d(\tilde{\phi}_\alpha(x), \tilde{\phi}_\alpha(y)) \geq \ell(\tilde{\phi}_\alpha(\delta')) - 2c = \ell(\delta') - 2c \geq \ell(\delta) - 4c = d(x, y) - 4c.$$

Since $\tilde{\phi}_\alpha\big|_e$ is 1–Lipschitz, it follows that this path is an r–local, $(1, 4c)$–quasi-geodesic (*i.e.* every segment of length less than r is a $(1, 4c)$–quasi-geodesic). Using hyperbolic geometry, we can find r sufficiently large and c sufficiently small so that such a path is also $(1+\epsilon, \epsilon)$–quasi-geodesic. Specifically, take $r = 4$, and consider consecutive points $x_0, x_1, \ldots, x_n$ along e with $1 \leq d(x_j, x_{j+1}) \leq 2$. Taking c sufficiently small, the angle at $\tilde{\phi}_\alpha(x_j)$ between geodesic segments $[\tilde{\phi}_\alpha(x_{j-1}), \tilde{\phi}_\alpha(x_j)]$ and $[\tilde{\phi}_\alpha(x_j), \tilde{\phi}_\alpha(x_{j+1})]$ in $\widetilde{M} \subset \mathbb{H}^3$ can be made arbitrarily close to π (depending on c), and then we can apply [CEG87, Theorem 4.2.10] to see that the concatenation of geodesic segments is arbitrarily close to the geodesic. Then $\tilde{\phi}_\alpha\big|_e$ is also as close to a geodesic as we like, and we can promote the local quasi-geodesic to a global quasi-geodesic. Taking J large enough to find such an r and c, completes the proof of the claim. □

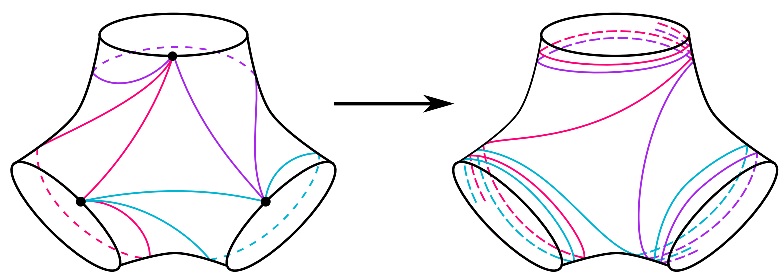

FIGURE 12. By "spinning" the triangulation on the left around each boundary component—more precisely, applying powers of Dehn twists about curves parallel to the boundary—we obtain the pants lamination in the limit, as shown on the right.

Claim 5.11. *Given $\epsilon > 0$, there exists $J > 0$ so that for all $j \geq J$:*

(1) *For every component $\gamma \subset P_\alpha$, we have $\ell_{\sigma_{\eta_j}}(\gamma) \leq \ell_{\sigma_\alpha}(\gamma) + \epsilon$, and*
(2) *all cone angles in the boundary of Σ for $\mathcal{T}_j$ are at most $\pi + \epsilon$.*

Note that because σ_α maps any component $\gamma \subset P_\alpha$ to a geodesic, and since η_j is 1–Lipschitz with respect to σ_{η_j}, we have $\ell(\phi_\alpha(\gamma)) = \ell_{\sigma_\alpha}(\gamma) \leq \ell_{\sigma_{\eta_j}}(\gamma)$, so part (1) in the claim really says that $\ell_{\sigma_\alpha}(\gamma)$ and $\ell_{\sigma_{\eta_j}}(\gamma)$ are nearly the same.

Proof. Let m be the maximum geometric intersection number between (the union of arcs in) $\mathcal{T}^{(1)}$ and any component of P_α. Since $\mathcal{T}_j^{(1)} = D^j(\mathcal{T}^{(1)})$, this maximum, m, also bounds the geometric intersection number between $\mathcal{T}_j^{(1)}$ and any component of P_α.

Now, let $\tilde{\eta}_j \colon \widetilde{\Sigma} \to \widetilde{M} \subset \mathbb{H}^3$ be the lift of the simplicial hyperbolic surface $\eta_j \colon (\Sigma, \mathcal{T}_j) \to \widetilde{M}_\infty$. From Claim 5.10, we deduce that, for j sufficiently large, and for every edge e of $\widetilde{\mathcal{T}}_j$, $\tilde{\eta}_j\big|_e$ and $\tilde{\phi}_\alpha|_e$ can be made arbitrarily close (depending on j).

For any component $\gamma \subset P_\alpha$ pick a lift $\widetilde{\gamma} \subset \widetilde{\Sigma}$ and a segment $\bar{\gamma}$ that serves as a fundamental domain for the action of $\langle g \rangle$, the stabilizer of $\widetilde{\gamma}$ in $\pi_1\Sigma$. Then, for any j consider the set of (at most m) edges of $\widetilde{\mathcal{T}}_j$ that cross $\widetilde{\gamma}$ at a point of $\bar{\gamma}$. For j sufficiently large, the $\tilde{\phi}_\alpha$-image of these edges are arbitrarily close to the geodesic $\tilde{\phi}_\alpha(\widetilde{\gamma})$ on arbitrarily long segments (depending on j), thus the same is true for the $\tilde{\eta}_j$ images of these edges.

Next, pick the edge $e \subset \widetilde{\mathcal{T}}_j$ intersecting $\bar{\gamma}$ as close to the initial point of $\bar{\gamma}$ as possible and assume that j is sufficiently large so that all the edges between e and $g \cdot e$ (ordered by the intersection with $\widetilde{\gamma}$) are mapped within $\frac{\epsilon}{2(m+1)}$ of $\tilde{\phi}_\alpha(\widetilde{\gamma})$ on segments of length at least $\ell_{\sigma_\alpha}(\gamma) + \epsilon$ centered at the points of intersection with $\widetilde{\gamma}$.

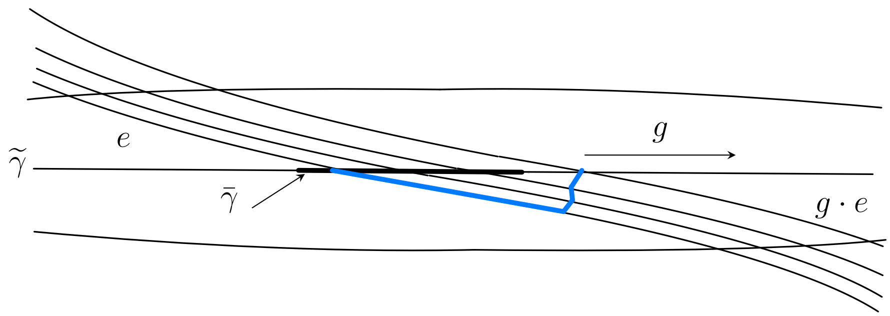

FIGURE 13. The edges of $\widetilde{\mathcal{T}}_j$ through $\widetilde{\gamma}$ between e and $g \cdot e$ are mapped close to $\tilde{\phi}_\alpha(\widetilde{\gamma})$ for a long time by η_j. The blue segment connects $e \cap \widetilde{\gamma}$ to $g \cdot e \cap \widetilde{\gamma}$, and projects to a closed loop in Σ.

From this we can construct a path in $\widetilde{\Sigma}$ from $e \cap \widetilde{\gamma}$ to $g \cdot e \cap \widetilde{\gamma}$ built from a segment of e of length at most $\ell_{\sigma_\alpha}(\gamma) + \frac{\epsilon}{2(m+1)}$ and at most m short segments of length at most $2\frac{\epsilon}{2(m+1)}$ between consecutive edges of $\widetilde{\mathcal{T}}_j$. Thus the σ_{η_j}–length of this path is at most

$$\ell_{\sigma_\alpha}(\gamma) + \frac{\epsilon}{2(m+1)} + m\left(\frac{\epsilon}{(m+1)}\right) < \ell_{\sigma_\alpha}(\gamma) + \epsilon.$$

See Figure 13. This path then projects to a loop in Σ homotopic to γ of σ_{η_j}–length at most $\ell_{\sigma_\alpha}(\gamma) + \epsilon$. This proves part (1) of the claim.

For part (2), we similarly observe that for any vertex v, any edge e adjacent to v has η_j–image that stays arbitrarily close to the geodesic image of the boundary component $\eta_j(\gamma)$ for an arbitrarily long time (depending on j). Now the boundary component γ gives two adjacencies to v pointing in opposite directions, and all other edges adjacent to v have η_j–image making arbitrarily small angle with the η_j–image of exactly one of these. It follows that there is one angle equal to at most π, and at most m angles that can be made arbitrarily small, depending on j. It follows that for j sufficiently large, the angle can be made at most $\pi + \epsilon$, proving part (2). $\square$

To complete the proof, we now observe that since the σ_{η_j}–length of each boundary curve is the same as its length with respect to σ_α, it is bounded by L. Further, since the cone angle is less than $\pi + \epsilon$, then provided $\epsilon < \pi$, we can enlarge Σ to an open surface, $\Sigma \subset \Sigma^\circ$, and extend σ_α to a complete *non-singular* hyperbolic metric. The convex core $\bar{\Sigma}$ of Σ° will contain Σ; see Figure 14.

Given any $\epsilon' > 0$, there exists $\epsilon > 0$ so that if $\gamma \subset \partial\Sigma$ is any component and if the cone angle at the vertex on γ has angle less than $\pi + \epsilon$, then $\bar{\Sigma} \subset N_{\epsilon'}(\Sigma)$ on Σ° with respect to σ_{η_j}, for any $j \geq J$ as in Claim 5.11. We note that the choice of ϵ depends not only on ϵ', but also on a lower bound on $\ell_{\sigma_{\eta_j}}(\gamma) = \ell_{\sigma_\alpha}(\gamma)$.

For any $0 < \epsilon < 1$, $j \geq J$ as in Claim 5.11, and any component $\gamma \subset P_\alpha$, $\ell_{\sigma_{\eta_j}}(\gamma) < L + \epsilon < 1$. By the collar lemma, there exists $w = w(L+1)$ such that

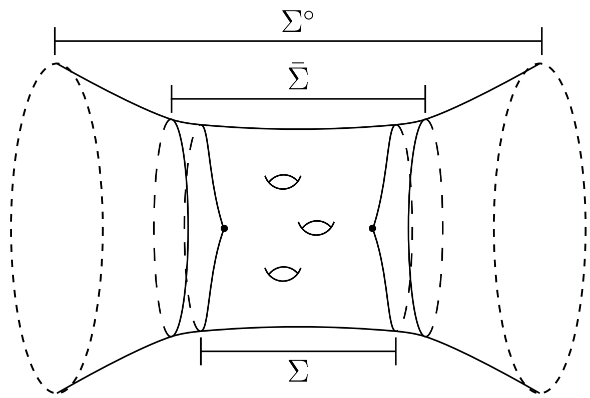

Figure 14. Enlarging Σ to Σ°, with convex core $\bar\Sigma$.

the w–neighborhood of γ is an embedded annulus in $\bar\Sigma$. Thus, if $\epsilon' < \frac{w}{2}$, and ϵ is chosen as in the previous paragraph, then the collar of width $w/2$ is entirely contained in $\Sigma \subset \breve\Sigma$. This gives a lower bound on the σ_{η_j}–moduli of γ, and hence an upper bound L_1 on $\ell_{\sigma^{\mathcal{H}}_{\eta_j}}(\gamma)$ depending on L and w, which in turn both depend only on $\chi(f)$ and $\xi(f)$, as required. □

5.4. **Pants distances and bounded length curves.** Write $\mathcal{T}_{2L}(\breve\Sigma)$ to denote the subspace of $\mathcal{T}(\breve\Sigma)$ consisting of complete hyperbolic structures for which the (geodesic representatives of) components of $\partial\Sigma$ have length at most $2L$. Set

$$\Xi = \Xi(f) = \tfrac{3}{2}\left(\xi(f) + |\chi(f)|\right). \tag{5.4}$$

From Equation (5.3) we have that $\xi(\Sigma) \leq \frac{3}{2}|\chi(\Sigma)| \leq \Xi$. Now set

$$L_2 = L_2(\chi(f), \xi(f)) = \max\{L_1, L_B(\Xi, 2L)\},$$

where L_B is as in Theorem 2.9.

For any pants decomposition P_0 of Σ, let

$$V(P_0, L_2) = \{\sigma \in \mathcal{T}_{2L}(\Sigma) \mid \ell_\sigma(\gamma) < L_2 \text{ for each component } \gamma \subset P_0\}.$$

By Theorem 2.9, the set $\{V(P_0, L_2)\}_{P_0 \in \mathcal{P}(\Sigma)}$ is an open cover of $\mathcal{T}_{2L}(\Sigma)$.

Now let

$$\{(\eta_t \colon (\Sigma, \mathcal{T}_t) \to \widetilde{M}_\infty) \mid t \in [0,1]\} \subset \mathcal{SH}$$

be the path of simplicial hyperbolic surfaces from Lemma 5.8 connecting the k_0–simplicial hyperbolic surfaces $\eta_\alpha \colon (\Sigma, \mathcal{T}_\alpha) \to \widetilde{M}_\infty$ to $\eta_\omega \colon (\Sigma, \mathcal{T}_\omega) \to \widetilde{M}_\infty$ from Lemma 5.9. By compactness (and Lemma 5.9), there is a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

and pants decompositions P_j, for $j = 1, \ldots, n$, so that $P_1 = P_\alpha$, $P_n = P_\omega$, and

$$\sigma_t \in V(P_j, L_2)$$

for all $t \in [t_{j-1}, t_j]$ and $j = 1, \ldots, n$.

Since P_j and P_{j+1} both have length at most L_2 with respect to $\sigma^{\mathcal{H}}_{\eta_{t_j}}$, Lemma 3.3 of [Bro03a] implies that $d_{\mathcal{P}}(P_j, P_{j+1}) \leq \kappa = \kappa(L_2, \Xi)$.

Write $\mathcal{S}(P_1, \ldots, P_n)$ to denote the union of all curves in all pants decompositions $P_1, \ldots, P_n$. The next result is also due to Brock [Bro03a, Lemma 4.3].

Lemma 5.12. *There exists $\mathcal{K}$ depending on $\kappa > 0$ and $\xi(\Sigma)$ with the following property. Let $P_1, \ldots, P_n$ be a sequence of pants decompositions of Σ such that*

$$d_{\mathcal{P}}(P_j, P_{j+1}) \leq \kappa,$$

for all $j = 1, \ldots, n-1$. Then

$$d_{\mathcal{P}}(P_1, P_n) \leq \mathcal{K}|\mathcal{S}(P_1, \ldots, P_n)|.$$

For us, the key application of this lemma is the following.

Corollary 5.13. *Let $P_\alpha = P_1, \ldots, P_n = P_\omega$ be as above and $\mathcal{K}$ as in Lemma 5.12 (which depends only on $\xi(f)$ and $\chi(f)$). Then*

$$d_{\mathcal{P}}(P_\alpha, P_\omega) \leq \mathcal{K}|\mathcal{S}(P_1, \ldots, P_n)|.$$

Finally, from Theorem 5.4 we have the following.

Lemma 5.14. *Let $P_\alpha = P_1, \ldots, P_n = P_\omega$ be as above. Then the geodesic representative in $\widetilde{M}_\infty$ of each curve in $\mathcal{S}(P_1, \ldots, P_n)$ has length bounded above by L_2.* □

6. Bounding volume

We are almost ready to prove the main theorem. We will need one more result, again due to Brock [Bro03a, Lemma 4.8]. Suppose M is compact, convex, hyperbolic 3-manifold, $\mathcal{L} > 0$, and let $\mathcal{G}_{\mathcal{L}}(M)$ denote the set of closed geodesics in M with length less than $\mathcal{L}$.

Proposition 6.1. *Given $\mathcal{L} > 0$ greater than the Bers constant for closed surfaces of complexity ξ, there is a constant $\mathcal{V} = \mathcal{V}(\xi, \mathcal{L}) > 0$ with the following property. Given a compact hyperbolic 3-manifold M with totally geodesic boundary and $\xi(\partial M) \leq \xi$, then*

$$\mathrm{Vol}(M) \geq \mathcal{V}|\mathcal{G}_{\mathcal{L}}(M)|.$$

Remark. Brock's statement in [Bro03a] involves an additive error as well that depends on $\chi(\partial M)$ (and not $\mathcal{L}$). However, it does not require $\mathcal{L}$ sufficiently large (in our statement, greater than the Bers constant). Because we assume M is acylindrical in this statement, there is a uniform lower bound to the volume (approximately 6.452...) by a result of Kojima and Miyamoto [KM91], and, because we have assumed $\mathcal{L}$ is greater than the Bers constant, $\mathcal{G}_{\mathcal{L}}(M)$ is nonempty. Consequently, we can absorb the additive constant into the multiplicative one, arriving at the version that is most useful for our purposes.

Proof of Theorem 1.1. Let $P_\alpha = P_1, \ldots, P_n = P_\omega$ be the sequence of pants decomposition constructed in §5.4 on Σ and $\mathcal{S}(P_1, \ldots, P_n)$ the set of all curves in all pants decompositions $P_1, \ldots, P_n$. By Lemma 5.14,

$$\mathcal{S}(P_1, \ldots, P_n) \subset \mathcal{G}_{L_2}(\widetilde{M}_\infty).$$

By Lemma 3.1, setting $N = 2\xi(\Sigma) \leq 2\Xi(f)$, we have that Σ and $f^m(\Sigma)$ have no closed curves in common for all $m \geq N$. In particular, no two curves in $\mathcal{S}(P_1, \ldots, P_n)$ differ by an element of $\langle f^N \rangle$. Consequently, no two elements of $\mathcal{S}(P_1, \ldots, P_n)$ project to the same homotopy class in $\overline{M}_{f^N}$. In particular, we have

$$|\mathcal{S}(P_1, \ldots, P_n)| \leq |\mathcal{G}_{L_2}(\overline{M}_{f^N})|. \tag{6.1}$$

Now observe that

$$\begin{aligned} \tau(f) &\leq d_{\mathcal{P}}(P, f(P)) \leq d_{\mathcal{P}}(P_\alpha, P_\omega) \leq \mathcal{K}|\mathcal{S}(P_1, \ldots, P_n)| \\ &\leq \mathcal{K}|\mathcal{G}_{L_2}(\overline{M}_{f^N})| \leq \frac{\mathcal{K}}{\mathcal{V}}\mathrm{Vol}(\overline{M}_{f^N}) = \frac{N\mathcal{K}}{\mathcal{V}}\mathrm{Vol}(\overline{M}_f). \end{aligned}$$

The first inequality is by definition. The second follows from Equation (5.2). The third is by Corollary 5.13. The fourth follows from Equation (6.1). The fifth inequality comes from Proposition 6.1 (note that we must adjust $\mathcal{V}$ because of the power N, but this is also uniform depending on the capacity). The final equality comes from the fact that $\overline{M}_{f^N}$ is an N–fold cover of $\overline{M}_f$. Since $N, \mathcal{K}, \mathcal{V}$ all depend only on $\xi(f)$ and $\chi(f)$, this completes the proof. □

7. Bounded length invariant components

Any f–invariant component $\Omega \subset \mathcal{P}(S)$ determines a pants decomposition P_Ω of $\partial\overline{M}_f$ (see [FKLL23]). More precisely, if $P \in \Omega$ is any pants decomposition representing a vertex in this component then, after identifying $\partial\overline{M}_f$ with the quotient $S_+ \sqcup S_- = (\mathcal{U}_+ \sqcup \mathcal{U}_-)/\langle f \rangle$, the preimage of P_Ω in $\mathcal{U}_+ \sqcup \mathcal{U}_-$ agrees with P on neighborhoods of the attracting and repelling ends of $\mathcal{U}_+$ and $\mathcal{U}_-$, respectively. Given an f–invariant component $\Omega \subset \mathcal{P}(S)$, the pants decomposition P_Ω can be constructed by first observing that for any $P \in \Omega$ there are good nesting neighborhoods $U_\pm$ so that P defines a pants decomposition $P|_{U_\pm}$ of $U_\pm$ (in particular, $\partial U_\pm$ is a union of curves in P) and so that $f^{\pm 1}(P|_{U_\pm}) \subset P|_{U_\pm}$. It follows that

$$\bigcup_{k=0}^{\infty} f^{\mp k}(P|_{U_\pm})$$

is a pants decomposition of $\mathcal{U}_\pm$ which is $\langle f \rangle$–invariant, and hence, descends to a pants decomposition P_Ω on $S_\pm$.

The construction of the pants decomposition P in the proof of Lemma 5.1 defines such an f–invariant component $\Omega_0 \subset \mathcal{P}(S)$, and can be explicitly described as follows. The subsurface $\Delta_+ \subset \overline{U_+} \subset \mathcal{U}_+$ is a fundamental domain for the action of $\langle f \rangle$, and the chosen pants decomposition P_+ from that proof projects to the components of P_{Ω_0} contained in $\partial_+\overline{M}_f$. A similar statement is true for the components of P_{Ω_0} in $\partial_+\overline{M}_f$. We note that the components of P_{Ω_0} have uniformly bounded length, depending only on the capacity by Lemma 5.1.

Every f–invariant component $\Omega \subset \mathcal{P}(S)$ has its own translation length

$$\tau_\Omega(f) = \inf_{P\in\Omega} \lim_{k\to\infty} \frac{d_{\mathcal{P}}(P, f^k(P))}{k}.$$

As was shown in [FKLL23], for any strongly irreducible end-periodic homeomorphisms f, there is always a sequence of f–invariant components Ω_n so that $\tau_{\Omega_n}(f) \to \infty$ as $n \to \infty$. On the other hand, by definition

$$\tau(f) = \inf_{k,\Omega} \frac{\tau_\Omega(f^k)}{k},$$

where the infimum is taken over all $k \geq 1$ and f^k–invariant components Ω.

Thus there are two measures of efficiency for a component $\Omega \subset \mathcal{P}(S)$ with respect to a strongly irreducible end-periodic homeomorphism $f: S \to S$. The first is that P_Ω has bounded length, which is a geometric condition in terms of the hyperbolic geometry of $\overline{M}_f$. The second is purely topological/combinatorial, and is that $\tau_\Omega(f)$ approximates $\tau(f)$. The next result says that these can be achieved simultaneously.

Theorem 1.2 *Given $f: S \to S$, a strongly irreducible end-periodic homeomorphism, there is a component $\Omega \subset \mathcal{P}(S)$ and $E > 0$ (depending on the capacity), so that each curve in $P_\Omega \subset \partial\overline{M}_f$ has length at most E, and so that $\tau_\Omega(f) \leq E\tau(f)$.*

Proof. We claim that the component Ω_0 defined by P from Lemma 5.1 satisfies the conditions of the proposition. Since $U_+ \cup U_-$ projects locally isometrically to $\partial\overline{M}_f$, and since the components of P have length bounded by $L = L(\xi(f), \chi(f))$, the component of P_{Ω_0} are similarly bounded by L.

To see that $\tau_{\Omega_0}(f)$ is bounded by a uniform constant multiple of $\tau(f)$, we first observe that the proof of Theorem 1.1 in fact shows that

$$\tau_{\Omega_0}(f) \leq \frac{1}{C_2}\mathrm{Vol}(\overline{M}_f),$$

where C_2 was explicitly shown to be given by $\frac{\mathcal{V}}{N\mathcal{K}}$. On the other hand, $\mathrm{Vol}(\overline{M}_f) \leq V_{\mathrm{oct}}\tau(f)$ by the main result of [FKLL23]. Therefore,

$$\tau_{\Omega_0}(f) \leq \frac{V_{\mathrm{oct}}}{C_2}\tau(f).$$

Setting $E = \max\left\{L, \frac{V_{\mathrm{oct}}}{C_2}\right\}$ proves the theorem. □

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