BIG TORELLI GROUPS: GENERATION AND COMMENSURATION

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Abstract. For any surface Σ of infinite topological type, we study the *Torelli subgroup* $\mathcal{I}(\Sigma)$ of the mapping class group $\mathrm{MCG}(\Sigma)$, whose elements are those mapping classes that act trivially on the homology of Σ. Our first result asserts that $\mathcal{I}(\Sigma)$ is topologically generated by the subgroup of $\mathrm{MCG}(\Sigma)$ consisting of those elements which have compact support. In particular, using results of Birman [4], Powell [22], and Putman [23] we deduce that $\mathcal{I}(\Sigma)$ is topologically generated by *separating twists* and *bounding pair maps*. Next, we prove the abstract commensurator group of $\mathcal{I}(\Sigma)$ coincides with $\mathrm{MCG}(\Sigma)$. This extends the results for finite-type surfaces [8, 6, 7, 14] to the setting of infinite-type surfaces.

1. Introduction

Let Σ be a connected orientable surface of infinite topological type – that is a surface with fundamental group that is not finitely generated. The *mapping class group* of Σ is the group:

$$\mathrm{MCG}(\Sigma) = \mathrm{Homeo}(\Sigma, \partial\Sigma)/\,\mathrm{Homeo}_0(\Sigma, \partial\Sigma),$$

where $\mathrm{Homeo}(\Sigma, \partial\Sigma)$ is the group of self-homeomorphisms of Σ which fix $\partial\Sigma$ pointwise. The group $\mathrm{Homeo}(\Sigma, \partial\Sigma)$ is equipped with the compact–open topology, and $\mathrm{Homeo}_0(\Sigma, \partial\Sigma)$ is the connected component of the identity in $\mathrm{Homeo}(\Sigma, \partial\Sigma)$. We equip $\mathrm{MCG}(\Sigma)$ with the quotient topology.

There is a natural homomorphism $\mathrm{MCG}(\Sigma) \to \mathrm{Aut}(H_1(\Sigma, \mathbb{Z}))$, whose kernel is commonly referred to as the *Torelli group* $\mathcal{I}(\Sigma) < \mathrm{MCG}(\Sigma)$. While Torelli groups of finite–type surfaces have been the object of intense study (see for example [3, 4, 10, 13, 15, 18, 20, 22, 24]) not much is known about them in the case of surfaces of infinite type. The present article aims to be a first step in this direction.

Generation. In a recent article, Patel–Vlamis [21] give a (topological) generating set for the *pure* mapping class group $\mathrm{PMCG}(\Sigma)$, namely the subgroup of $\mathrm{MCG}(\Sigma)$ consisting of those mapping classes which fix every *end* of Σ; see Section 2. More concretely, they show that $\mathrm{PMCG}(\Sigma)$ is generated by the subgroup of elements with compact support if Σ has at most one end *accumulated by genus*; otherwise, $\mathrm{PMCG}(\Sigma)$ is generated by the union of the set of compactly–supported elements and the set of *handle–shifts*; see Section 2.

Observe that $\mathcal{I}(\Sigma) < \mathrm{PMCG}(\Sigma)$. Our first result asserts that, for any infinite–type surface Σ, the set of compactly–supported mapping classes suffices to generate the Torelli group:

Theorem 1. *Let Σ be a connected orientable surface of infinite topological type. Every element of $\mathcal{I}(\Sigma)$ is a limit of compactly–supported mapping classes in $\mathcal{I}(\Sigma)$.*

Birman [4] and Powell [22] showed that the Torelli group of a closed finite–type surface is generated by *separating twists* (i.e. Dehn twists about separating curves), plus *bounding pair maps* (that is, products of twists of the form $T_\gamma T_\delta^{-1}$, where γ and δ are non-separating but their union separates). Putman then proved that the same is true for finite–type surfaces with boundary [23]. In light of this, an immediate consequence of Theorem 1 is:

Corollary 2. *Let Σ be a connected orientable surface of infinite topological type. Then $\mathcal{I}(\Sigma)$ is topologically generated by separating twists and bounding-pair maps.*

Theorem 1 implies $\mathcal{I}(\Sigma)$ is a closed subgroup of $\mathrm{MCG}(\Sigma)$. Since $\mathrm{MCG}(\Sigma)$ is a Polish group [1] and closed subgroups of Polish groups are Polish, we have the following corollary.

Corollary 3. *Let Σ be a connected orientable surface of infinite topological type. Then $\mathcal{I}(\Sigma)$ is a Polish group.*

Commensurations. Recall that, given a group G, its abstract commensurator $\mathrm{Comm}(G)$ is the group of equivalence classes of isomorphisms between finite-index subgroups of G; here, two isomorphisms are equivalent if they agree on a finite-index subgroup. Observe that there is a natural homomorphism

$$\mathrm{Aut}(G) \to \mathrm{Comm}(G).$$

We will prove:

Theorem 4. *For any connected orientable surface Σ of infinite topological type we have*

$$\mathrm{Comm}\,\mathcal{I}(\Sigma) \cong \mathrm{Aut}\,\mathcal{I}(\Sigma) \cong \mathrm{MCG}(\Sigma).$$

Historical context and idea of proof. Theorem 4 was previously known to hold for finite–type surfaces. Indeed, Farb–Ivanov [8] proved it for closed surfaces of genus at least five, which was then extended (and generalized to the Johnson Kernel) by Brendle–Margalit to all closed surfaces of genus at least three [6, 7]. Kida extended the result of Brendle–Margalit to all finite–type surfaces of genus at least four [14]. Finally, recent work of Brendle–Margalit and McLeay has further generalized the result to apply to a large class of normal subgroups of finite–type surfaces [5, 19].

In order to prove the theorem, we closely follow Brendle–Margalit's strategy. First, we adapt ideas to Bavard–Dowdall–Rafi [2] to show that every commensuration of the Torelli group respects the property of being a separating twist or a bounding pair map. From this we deduce that every commensuration induces an automorphism of a combinatorial object called the *Torelli complex*. This complex was originally introduced, for closed surfaces, by Brendle–Margalit [6], who proved that its automorphism group coincides with the mapping class group; this was later extended by Kida [14] to finite–type surfaces with punctures. Using this, plus an inductive argument due to Ivanov [12], we will show that every automorphism of the Torelli complex of an infinite–type surface is induced by a surface homeomorphism. At this point, Theorem 4 will follow easily using a well-known argument of Ivanov [12].

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2. Definitions

In this section we introduce the main objects needed for the proofs of our results.

2.1. **Surfaces.** Throughout, by a *surface* we mean a connected, orientable, second-countable topological surface. We say that Σ has *finite type* if its fundamental group is finitely generated; otherwise, we say that Σ has *infinite type*. In the finite type case, we will sometimes use the notation $\Sigma = \Sigma_{g,p}^b$, where g, p, and b are, respectively, the genus, the number of punctures, and the number of boundary components of Σ. In this case, we define the *complexity* of Σ to be the integer $3g - 3 + p + b$.

A *subsurface* of Σ is a subset for which the inclusion map is a proper, π_1-injective embedding.

The *space of ends* of Σ is the set

$$\mathrm{Ends}(\Sigma) = \varprojlim \pi_0(\Sigma \setminus K),$$

where the inverse limit is taken over the set of compact subsurfaces $K \subset \Sigma$, directed with respect to inclusion. Here, the topology on $\mathrm{Ends}(\Sigma)$ is given by the limit topology obtained by equipping each $\pi_0(\Sigma \setminus K)$ with the discrete topology. See [25] for further details.

We say that e in $\mathrm{Ends}(\Sigma)$ is *accumulated by genus* if every neighborhood of e has infinite genus; otherwise, we say that e is *planar*. We denote by $\mathrm{Ends}_g(\Sigma)$ the subset of $\mathrm{Ends}(\Sigma)$ consisting of ends accumulated by genus. It is a classical theorem (see [25] for a discussion and proof) that the homeomorphism type of Σ is determined by the tuple

$$\big(g(\Sigma), b(\Sigma), \mathrm{Ends}(\Sigma), \mathrm{Ends}_g(\Sigma)\big),$$

where $g(\Sigma)$ and $b(\Sigma)$ denote the genus and the number of boundary components of Σ.

2.2. **Curves.** By a *curve* on Σ we mean the free homotopy class of a simple closed curve that does not bound a disk or a disk containing a single planar end of Σ. Abusing notation, we will not make any distinction between a curve and any of its representatives.

We say that a curve γ is *separating* if $\Sigma \setminus \gamma$ has two connected components; otherwise, we say that γ is *non-separating*. We say that two curves are *disjoint* if they have disjoint representatives in Σ. A *multicurve* is a set of pairwise disjoint curves.

2.3. **Pure mapping class groups.** The *pure mapping class group* $\mathrm{PMCG}(\Sigma)$ is the subgroup of $\mathrm{MCG}(\Sigma)$ whose elements fix every end of Σ.

The *compactly–supported* mapping class group $\mathrm{PMCG}_c(\Sigma)$ is the group whose elements have compact support, that is, they are represented by a homeomorphism that is the identity outside a compact subsurface of Σ. A classical result due to Dehn and Lickorish (see [9, Section 4], for instance) implies that $\mathrm{PMCG}_c(\Sigma)$ is generated by *Dehn twists*.

2.4. **Handle–Shifts.** For any subgroup $\Gamma < \mathrm{MCG}(\Sigma)$, we denote by $\overline{\Gamma}$ its topological closure in $\mathrm{MCG}(\Sigma)$.

Patel–Vlamis introduced handle–shifts and showed that handle–shifts and Dehn twists topologically generate $\mathrm{PMCG}(\Sigma)$ [21]. Subsequently, in [1] it was shown that $\mathrm{PMCG}(\Sigma) = \overline{\mathrm{PMCG}_c(\Sigma)} \rtimes H$ where H is a particular subgroup generated by pairwise commuting *handle–shifts*, whose definition we now recall.

Let Λ be the surface obtained from $\mathbb{R} \times [-1, 1]$ by removing disks of radius $\frac{1}{4}$ centered at $(t, 0)$ for t in $\mathbb{Z}$ and gluing in a torus with one boundary component, identifying the boundary of the torus with the boundary of the removed disk. Let $\sigma : \Lambda \to \Lambda$ be the homeomorphism that shifts the handle at $(t, 0)$ to the handle at $(t + 1, 0)$, and is the identity on $\mathbb{R} \times \{-1, 1\}$ (see [1] or [21] for an image of such a homeomorphism). The isotopy class of σ is called a handle shift of Λ.

An element h in $\mathrm{MCG}(\Sigma)$ is a *handle–shift* if there exists a proper embedding $\iota : \Lambda \to \Sigma$ which induces an injective map on ends, and such that

$[h] = [\delta]$ where $\delta\,|_{\iota(\Lambda)}= \sigma$ and δ is the identity outside $\iota(\Lambda)$. As a consequence of our definition, we must have $|\mathrm{Ends}_g(\Sigma)| \geq 2$; also, for each handle–shift there is an attracting end ϵ_+ and a repelling end ϵ_- in $\mathrm{Ends}_g(\Sigma)$, and they are distinct.

We say a handle–shift h with attracting end ϵ_+ and repelling end ϵ_- is *dual* to a separating curve γ if each component of $\Sigma \setminus \gamma$ contains exactly one of ϵ_+ and ϵ_-.

2.5. **Principal exhaustions.** We now introduce a minor modification of the the notion of *principal exhaustion* from [1, 2]:

Definition. A *principal exhaustion* of Σ is an infinite sequence of connected subsurfaces $\{P_1, P_2, \dots\}$ such that, for every $i \geq 1$, one has:

1. P_i has finite type, and each component of $\Sigma \setminus P_i$ has infinite type,
2. $P_i \subset P_{i+1}$,
3. every component of ∂P_i is separating
4. no component of ∂P_i is isotopic to a component of ∂P_{i+1}, and
5. $\Sigma = \bigcup P_i$.

Lemma 5. *Let Σ be a connected infinite–type surface and let $\{P_i\}$ be a principal exhaustion of Σ. Then for all i, we have:*

- *for all $j > i$, $H_1(P_j) \cong H_1(P_i) \oplus M$ for some $M < H_1(P_j \setminus P_i)$*
- *$H_1(\Sigma) \cong H_1(P_i) \oplus M'$ for some $M' < H_1(\Sigma \setminus P_i)$*

Proof of lemma. We will let W be either P_j or Σ to prove both cases simultaneously.

Let $\partial_1 P_i, \dots \partial_m P_i$ be the boundary components of P_i.

Since every component of $\Sigma - P_i$ is of infinite type, every component of $\overline{W - P_i}$ either contains an end of Σ or a boundary component of W. So there is a collection of pairwise disjoint rays and arcs $\gamma_1, \dots, \gamma_m$ properly embedded in $\overline{W - P_i}$ such that $\gamma_k \cap \partial_k P_i$ is a single point for all k.

By the Regular Neighborhood Theorem, we may deformation retract W along the γ_k, fixing P_i throughout, to obtain a subsurface Δ homotopy equivalent to W that contains P_i and such that $P_i \cap \overline{\Delta - P_i}$ is a disjoint union of arcs $\alpha_1, \dots, \alpha_m$.

Consideration of the Mayer–Vietoris sequence gives us an exact sequence

$$0 \to H_1(P_i) \oplus H_1(\Delta - P_i) \longrightarrow H_1(W) \xrightarrow{\partial} H_0(\alpha_1 \sqcup \cdots \sqcup \alpha_m).$$

This gives us the direct sum decomposition of $H_1(W)$. Since $\partial_\ell P_i$ are separating, then so are the α_ℓ. This implies that the boundary map ∂ is zero, and since $H_1(\Delta - P_i)$ is naturally a subgroup of $H_1(W - P_i)$, the proof is complete. $\square$

3. Compactly generating the Torelli group

Let Σ be an infinite–type surface. We define the *compactly supported Torelli group*

$$\mathcal{I}_c(\Sigma) := \{f \in \mathcal{I}(\Sigma) \mid f \text{ has compact support}\}.$$

The aim of this section is to prove the first main result of the introduction, whose statement we now recall:

Theorem 1. *For any connected orientable surface Σ of infinite type, we have $\mathcal{I}(\Sigma) = \overline{\mathcal{I}_c(\Sigma)}$.*

We will need to know that certain, possibly infinite, products of handle shifts are inaccessible by compactly supported mapping classes. For a general product of handle–shifts, this is too much to hope for. For example, in a surface with two ends, the product of two commuting handle shifts with opposite dynamics is a limit of compactly supported classes.

More generally, there are products of infinitely many commuting handle shift that are limits of compactly supported classes. For example, there is the "boundary leaf shift," which we now explain.

Example (Boundary leaf shift)**.** Start with an infinite regular tree $\mathcal{T}$ properly embedded in the hyperbolic plane $\mathbb{H}^2$ with boundary a Cantor set in $\partial\mathbb{H}^2$. Orient $\partial\mathbb{H}^2$ counterclockwise. Build a surface by taking the boundary of a regular neighborhood of $\mathcal{T}$ in $\mathbb{H}^2 \times \mathbb{R}$ and attach handles periodically (in the hyperbolic metric) along each side of $\mathcal{T}$, see Figure 1. The orientation on $\partial\mathbb{H}^2$ defines a product $\mathcal{H}$ of handle–shifts by shifting the handles in each region of $\mathbb{H}^2 - \mathcal{T}$ in the clockwise direction.

To see the boundary leaf shift is in $\overline{\mathrm{PMCG}_c(\Sigma)}$, pick a basepoint $*$ in $\mathcal{T}$ and consider the n–neighborhood $B(n)$ of $*$ in $\mathcal{T}$. Then we may move the handles incident to $B(n)$ around in a counterclockwise fashion to get a compactly supported class f_n in $\mathrm{PMCG}_c(\Sigma)$. The sequence $\{f_n\}$ converges to the boundary leaf shift.

Let γ be a separating curve in Σ whose complementary components are both noncompact. Let Σ_- and Σ_+ be the closures of the two components of $\Sigma - \gamma$. By the same argument as in Lemma 5, Σ deformation retracts to a subspace homeomorphic to $X \vee \gamma \vee Y$, where X and Y are subspaces of Σ_- and Σ_+, respectively. It follows that $H_1(\Sigma)$ splits as $A \oplus \langle\gamma\rangle \oplus B$, where $A = H_1(X)$ and $B = H_1(Y)$.

Similarly, if h is a handle–shift dual to γ, then $H_1(\Sigma) \cong L \oplus \langle\gamma\rangle \oplus H_1(\mathrm{supp}(h)) \oplus R$, where L and R are subgroups of A and B.

Definition (Pseudo-handle–shift)**.** We say that a mapping class $\mathcal{H}$ is a *pseudo-handle–shift dual to a separating curve γ with associated handle–shift h* if the following hold:

1. h is a handle–shift dual to γ
2. $\mathcal{H}_*$ agrees with h_* on $H_1(\mathrm{supp}(h))$

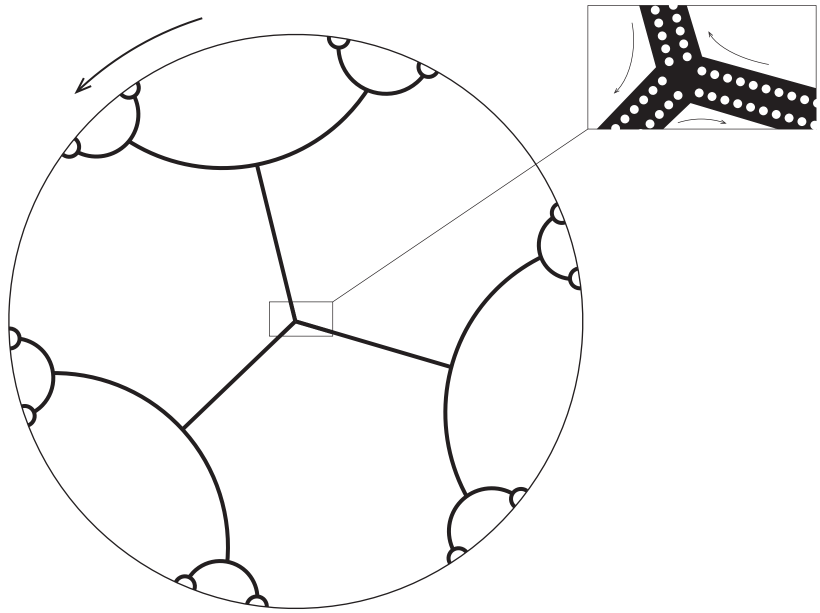

FIGURE 1. The boundary leaf shift.

(3) $\mathcal{H}_*(\gamma) = \gamma$
(4) $\mathcal{H}_*(L) < A$
(5) $\mathcal{H}_*(R) < B$

In what follows, we always assume that the repelling end of h is on the "A–side." We have:

Theorem 6 (Pseudo-handle–shifts are unapproachable). *A pseudo-handle–shift $\mathcal{H}$ dual to a separating curve γ is not a limit of compactly supported mapping classes.*

Proof. Let h be the associated handle–shift dual to γ. Let ϵ_- and ϵ_+ be the ends of Σ corresponding to the repelling and attracting ends of h, respectively, and let Σ_- and Σ_+ be the complementary components of $\Sigma - \gamma$ containing ϵ_- and ϵ_+, respectively. Choose some principal exhaustion $\{P_i\}$ of Σ, and let $\Sigma_-^i = (\Sigma - P_i) \cap \Sigma_-$ and $\Sigma_+^i = (\Sigma - P_i) \cap \Sigma_+$.

Picking i large enough, we may assume that the term P_i in our principal exhaustion contains γ and satisfies $\mathcal{H}(\Sigma_-^i) \cap \Sigma_+ = \emptyset$ and $\mathcal{H}(\Sigma_+^i) \cap \Sigma_- = \emptyset$.

The handle–shift h is supported on a strip $\mathcal{S}$ with equally spaced handles and standard basis $\{\alpha_p, \beta_p\}_{k\in\mathbb{Z}}$ of $H_1(\mathcal{S})$ so that $h_*(\alpha_p) = \alpha_{p+1}$ and $h_*(\beta_p) = \beta_{p+1}$. We choose once and for all curves in $\mathcal{S}$ representing these classes. After reindexing the α_p and β_p by translating p, we assume that α_1 and β_1 lie in Σ_-^i. Since α_p and β_p tend to ϵ_+, there is some $j > 1$ such that α_j and β_j lie in Σ_+^i.

Suppose that $\mathcal{H}$ is a limit of compactly supported $\mathcal{H}_n$.

Pick n large enough so that $\mathcal{H}_n$ agrees with $\mathcal{H}$ on P_i and so that $\mathcal{H}_{n*}$ agrees with h_* on both $H_1(P_i)$ and $\langle \alpha_1, \beta_1, \dots, \alpha_j, \beta_j \rangle$. Let P_k be some term in the exhaustion with $k \geq i$ that contains the support of $\mathcal{H}_n$.

We have a direct sum decomposition

$$H_1(P_k) \cong \mathbb{Z}^\ell \oplus \mathbb{Z}^{2j} \oplus \mathbb{Z}^r$$

where $\mathbb{Z}^\ell$ is a subgroup of $H_1(\Sigma_-) \oplus \langle \gamma \rangle$, $\mathbb{Z}^{2j} = \langle \alpha_1, \beta_1, \dots, \alpha_j, \beta_j \rangle$, and $\mathbb{Z}^r$ is a subgroup of $H_1(\Sigma_+)$. Picking a basis $\langle x_1, \dots, x_\ell, \alpha_1, \beta_1, \dots, \alpha_j, \beta_j, y_1, \dots, y_r \rangle$ for $H_1(P_k)$ compatible with this decomposition, we see that $\mathcal{H}_{n*}$ has a block decomposition:

$$\mathcal{H}_{n*} = \begin{array}{c} \\ \ell \\ 2 \\ 2j-2 \\ r \end{array} \begin{array}{c} \begin{array}{cccc} \ell & 2j-2 & 2 & r \end{array} \\ \begin{pmatrix} * & \mathbf{0} & \mathbf{0} & Y \\ * & \mathbf{0} & \mathbf{0} & Z \\ * & I & \mathbf{0} & * \\ X & \mathbf{0} & A & B \end{pmatrix} \end{array}$$

By properties (4) and (5) of a pseudo-handle–shift, and our choice of i, the blocks X, Y, and Z are all zero. So the matrix is:

$$\mathcal{H}_{n*} = \begin{array}{c} \\ \ell \\ 2 \\ 2j-2 \\ r \end{array} \begin{array}{c} \begin{array}{cccc} \ell & 2j-2 & 2 & r \end{array} \\ \begin{pmatrix} * & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & I & \mathbf{0} & * \\ \mathbf{0} & \mathbf{0} & A & B \end{pmatrix} \end{array}$$

This matrix is column equivalent to:

$$\mathcal{H}_{n*} = \begin{array}{c} \\ \ell \\ 2 \\ 2j-2 \\ r \end{array} \begin{array}{c} \begin{array}{cccc} \ell & 2j-2 & 2 & r \end{array} \\ \begin{pmatrix} * & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & B \end{pmatrix} \end{array}$$

But the matrix $[A\ B]$ is an $r \times (r+2)$ matrix, and so its Jordan form cannot have pivot in every column. So the matrix for $\mathcal{H}_{n*}$ is equivalent to a matrix with a zero column. But $\mathcal{H}_{n*}$ is an isomorphism, and this contradiction completes the proof. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. We will first show that $\mathcal{I}(\Sigma) < \overline{\mathrm{PMCG}_c(\Sigma)}$. By [21, Theorem 1], we only need to consider the case when Σ has at least two ends accumulated by genus. We observe that $\mathcal{I}(\Sigma) < \mathrm{PMCG}(\Sigma)$. Let g be in $\mathrm{PMCG}(\Sigma)$ so that g is not a limit of compactly supported mapping classes. We show that g is not in $\mathcal{I}(\Sigma)$.

By Theorem 3 and Corollary 4 from [1], g can be written $g = fk^{-1}$ where f is a limit of compactly supported classes and k is a product of pairwise commuting handle–shifts h_i. The handle–shifts h_i have the property that the support of h_i is disjoint from the dual curve γ_j for h_j whenever $i \neq j$. Such a g cannot be in the Torelli group, for then f would be a pseudo-handle–shift dual to a separating curve that is a limit of compactly supported classes, violating Theorem 6. Therefore

$$\mathcal{I}(\Sigma) < \overline{\mathrm{PMCG}_c(\Sigma)}.$$

If ϕ_n is a sequence in $\mathcal{I}_c(\Sigma)$ that converges to ϕ, then ϕ lies in $\mathcal{I}(\Sigma)$, since $\phi_n(\alpha)$ eventually agrees with $\phi(\alpha)$ for any given simple closed curve α. So

$$\overline{\mathcal{I}_c(\Sigma)} < \mathcal{I}(\Sigma).$$

For the other containment, let ϕ be an element of $\mathcal{I}(\Sigma)$ and let $\{\psi_n\}$ be a sequence in $\mathrm{PMCG}_c(\Sigma)$ converging to ϕ. We would like to convert ψ_n into a sequence of compactly supported ϕ_n in $\mathcal{I}(\Sigma)$ converging to ϕ. The idea is that the homology classes affected by ψ_n must move further and further away from a given basepoint, and so we can precompose the ψ_n with a mapping class supported far from the basepoint to produce the desired ϕ_n.

Fix a principal exhaustion $\{P_i\}$ of Σ. For each i, pick a $j > i$ such that P_j contains $\phi^{-1}(P_i)$. Pick an N large enough so that, for all $n \geq N$, the map ψ_n has a representative that agrees with a fixed representative of ϕ on P_j. Note that ψ_{n*} agrees with ϕ_* on $H_1(P_j)$.

By Lemma 5, we have $H_1(P_k) \cong H_1(P_i) \oplus Q \oplus R$ for some Q a subgroup of $H_1(P_j - P_i)$ and R a subgroup of $H_1(P_k - P_j)$. Let α be element of $H_1(P_k)$ and write $\alpha = \gamma + \mu + \nu$ where γ, μ, and ν are in $H_1(P_i)$, Q, and R, respectively. So $\psi_{n*}(\alpha) = \gamma + \mu + \psi_{n*}(\nu)$.

The class ν is represented by a 1–manifold $\mathcal{N}$ in $P_k - P_j$. By our choice of j and n, the 1–manifold $\psi_n(\mathcal{N})$ is disjoint from P_i. So $\psi_{n*}(\nu)$ is in $Q \oplus R$. Therefore $\psi_{n*} : H_1(P_k) \to H_1(P_k)$ may be represented by a square matrix

$$A = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$$

where I is the identity on $H_1(P_i)$ and B is a square matrix. Since A is the induced map on homology associated to a homeomorphism of P_k, it is invertible and respects the intersection form, and so the same is true of B. The action on homology surjects mapping class groups of compact surfaces onto their symplectic groups. Therefore the matrix B is represented by a homeomorphism $F : \overline{P_k - P_i} \to \overline{P_k - P_i}$ that is the identity on $\partial P_i \cap \overline{P_k - P_i}$. We extend this by the identity to all of Σ and continue to call the extension F.

Now consider the homeomorphism $\phi_n = \psi_n \circ F^{-1}$. By construction of F, this homeomorphism ϕ_n acts trivially on the homology of Σ, and agrees with ψ_n on P_i.

This completes the proof. $\square$

4. Abstract commensurators of the Torelli group

In this section we prove Theorem 4. As mentioned in the introduction, the first step of the argument consists of proving that an element of $\operatorname{Comm}\mathcal{I}(\Sigma)$ induces a simplicial automorphism of a combinatorial object associated to Σ, called the *Torelli complex*, introduced by Brendle–Margalit in [6].

4.1. Torelli complex.

Recall that the *curve complex* of Σ is the (infinite-dimensional) simplicial complex whose vertex set is the set of isotopy classes of curves in Σ, and where a collection of vertices spans a simplex if and only if the corresponding curves are pairwise disjoint. The curve complex was used by Ivanov [12], Korkmaz [16], and Luo [17] to prove that, for all but a few finite–type surfaces Σ,

$$\operatorname{Comm}\operatorname{MCG}(\Sigma) \cong \operatorname{Aut}\operatorname{MCG}(\Sigma) \cong \operatorname{MCG}(\Sigma).$$

Subsequently, Bavard–Dowdall–Rafi [2] established the analogous result for infinite–type surfaces. In a similar fashion, Farb-Ivanov [8], Brendle–Margalit [6, 7, 5], and Kida [14] proved that, for all but a few finite–type surfaces,

$$\operatorname{Comm}\mathcal{I}(\Sigma) \cong \operatorname{Aut}\mathcal{I}(\Sigma) \cong \mathcal{I}(\Sigma).$$

Here, we will adapt the ideas of Brendle–Margalit [6] to the infinite–type setting. Given an infinite–type surface Σ, we define its Torelli complex to be the (infinite-dimensional) simplicial complex whose vertex set is the set of isotopy classes of separating curves and bounding pairs in Σ, and where a collection of vertices spans a simplex if and only if the corresponding curves are pairwise disjoint. In order to relax notation, we will blur the distinction between vertices of $\mathcal{T}(\Sigma)$ and the curves (or multicurves) they represent. We record the following folklore observation as a separate lemma, as we will need to make use of it later:

Lemma 7. *The Torelli complex $\mathcal{T}(\Sigma)$ has infinite diameter if and only if Σ has finite type.*

Proof. If Σ has finite type, a slick limiting argument due to Feng Luo (see the comment after Proposition 4.6 of [18]) shows that the curve complex has infinite diameter. The obvious adaptation of this method to the case of the Torelli complex also implies that $\mathcal{T}(\Sigma)$ has infinite diameter.

For the other direction, suppose Σ has infinite type. Since curves are compact, given multicurves $\gamma, \delta \subset \Sigma$, we can find a separating curve $\eta \subset \Sigma$ which is disjoint from both γ and δ. In particular, $\mathcal{T}(\Sigma)$ has diameter two. □

4.2. Automorphisms of the Torelli complex.

Denote by $\operatorname{Aut}(\mathcal{T}(\Sigma))$ the group of simplicial automorphisms of $\mathcal{T}(\Sigma)$, and observe that there is a natural homomorphism $\operatorname{MCG}(\Sigma) \to \operatorname{Aut}(\mathcal{T}(\Sigma))$. We want to prove:

Theorem 8. *Let Σ be an infinite–type surface. The natural homomorphism $\operatorname{MCG}(\Sigma) \to \operatorname{Aut}\mathcal{T}(\Sigma)$ is an isomorphism.*

As noted above, the finite–type case is due to Brendle–Margalit [6, 7, 5] and Kida [14]. Indeed, the notion of *sides* which is used in this section is adapted from arguments that may be found in Brendle-Margalit [6], and which find their way back to ideas of Ivanov [12].

Sides. Recall that the *link* of a vertex v of a simplicial complex X is the set of all vertices of X that span an edge with v. In particular, v is not an element of its link. For any finite-dimensional simplex σ let $\mathrm{Link}(\sigma)$ be the intersection of the links of each of the vertices in σ. We say that two vertices α, β in $\mathrm{Link}(\sigma)$ lie on the same *side* of σ if there exists a vertex γ in $\mathrm{Link}(\sigma)$ that fails to span an edge with both α and β, that is, if there exists a curve in $\mathrm{Link}(\sigma)$ that intersects both α and β. Observe that "being on the same side" defines an equivalence relation $\sim_\sigma$ on $\mathrm{Link}(\sigma)$, that is, the *sides* of σ are the equivalence classes of $\sim_\sigma$ in $\mathrm{Link}(\sigma)$.

In particular, we may consider the sides of a vertex of $\mathcal{T}(\Sigma)$. We say that γ in $\mathcal{T}(\Sigma)$ is k-sided if there are k equivalence classes with respect to $\sim_\gamma$. As we shall see, k is in $\{1, 2\}$.

For any vertex γ of $\mathcal{T}(\Sigma)$ there exist two subsurfaces $R, L \subset \Sigma$ obtained by cutting Σ along γ such that γ is isotopic to the boundary components of both R and L. Suppose R is of finite type. We call γ a *pants curve* if γ is a separating curve and $R \cong \Sigma^1_{0,2}$, a sphere with two punctures and one boundary component. We call γ a *genus curve* if γ is a separating curve and $R \cong \Sigma^1_{1,0}$, a torus with one boundary component. If γ is any other type of separating curve then we say it is type S.

If γ is a bounding pair and one of the associated subsurfaces of Σ is homeomorphic to $\Sigma^2_{1,0}$ then we call it a *genus bounding pair.*

Lemma 9. *A vertex γ in $\mathcal{T}(\Sigma)$ is 2-sided if and only if it is type S or it is a genus bounding pair. Otherwise, γ is 1-sided.*

Proof. We first prove that if γ is type S then it has exactly two sides. Let R and L be the two subsurfaces of Σ obtained by cutting along γ. Let α, β lie in $\mathrm{Link}(\gamma)$. If $\alpha \subset R$ and $\beta \subset L$, then any vertex of $\mathcal{T}(\Sigma)$ that intersects both α and β must also intersect γ. This implies that γ has at least two sides. If $\alpha, \beta \subset R$ then there exists an element of the $\mathrm{MCG}(\Sigma)$-orbit of α that intersects both α and β and is contained in R. An identical argument holds for two vertices contained in L and so it follows that γ has exactly two sides.

Now let γ be a genus one separating curve or a pants curve. Define $L, R \subset \Sigma$ as above. Recall that neither $\Sigma^1_{0,2}$ nor $\Sigma^1_{1,0}$ contains any non-peripheral separating curves or bounding pairs. Therefore $\mathrm{Link}(\gamma)$ does not contain any curves in R. As above, all vertices contained in L are on the same side and so γ is 1-sided.

We now move on to the case where γ is a bounding pair. We define R and L as above. Assume that neither R nor L is homeomorphic to $\Sigma^2_{0,1}$ or $\Sigma^2_{1,0}$.

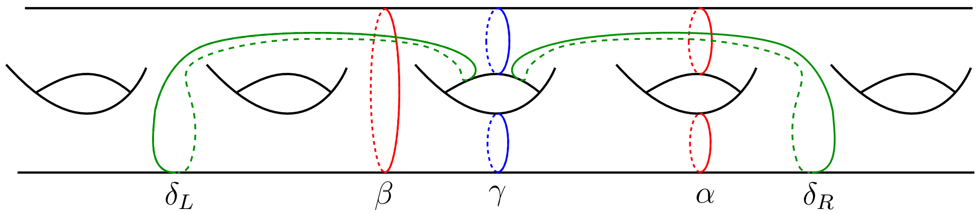

FIGURE 2. A general bounding pair γ is 1-sided. For any two vertices α, β in $\mathcal{T}(\Sigma)$ adjacent to γ, we can find a bounding pair not adjacent to α and β but adjacent to γ. Informally, bounding pairs can "pass through" each other.

Let α, β in $\mathrm{Link}(\gamma)$ be such that $\alpha \subset R$ and $\beta \subset L$. As shown in Figure 2, there exists a bounding pair $\gamma' = \{\delta_R, \delta_L\}$ such that:

- any pair of curves in γ or γ' forms a bounding pair,
- $\delta_R \subset R$ and $\delta_L \subset L$, and
- $\delta_R \cap \alpha \neq \emptyset$ and $\delta_L \cap \beta \neq \emptyset$.

That is, γ' is in $\mathrm{Link}(\gamma)$ and there is no edge between γ' and α or between γ' and β. It follows that γ has exactly one side.

If γ is a genus bounding pair then no such γ' exists. Indeed, every nonseparating curve in R that forms a bounding pair with a curve in γ is also isotopic to a curve in γ. By the same argument as for type S vertices, we conclude that γ is 2-sided.

If R is homeomorphic to $\Sigma^2_{0,1}$ then γ is 1-sided. Indeed, the only vertex of $\mathcal{T}(\Sigma)$ contained in R is γ and so all vertices of $\mathrm{Link}(\gamma)$ are contained in L. This completes the proof. □

Let σ be a finite-dimensional simplex of $\mathcal{T}(\Sigma)$ consisting entirely of curves of type S. Using similar methods to the above proofs it is straightforward to show that the set of sides of σ is in bijective correspondence with the subsurfaces of Σ obtained by cutting Σ along σ.

We are finally in a position to prove Theorem 8:

Proof of Theorem 8. Let

$$\Phi : \mathrm{MCG}(\Sigma) \to \mathrm{Aut}(\mathcal{T}(\Sigma))$$

be the natural homomorphism; that is, for f in $\mathrm{MCG}(\Sigma)$, $\Phi(f)$ is the automorphism of $\mathcal{T}(\Sigma)$ determined by the rule

$$\Phi(f)(\gamma) = f(\gamma)$$

for every separating curve or bounding pair γ.

First, we show that Φ is injective. To this end, suppose $\Phi(f) = \mathrm{Id}$. Then we argue that $f(\gamma) = \gamma$ for every curve γ. Indeed, if γ is separating, then γ is a vertex of $\mathcal{T}(\Sigma)$, so $\Phi(f)(\gamma) = \gamma$ and we are done. If γ is nonseparating, there is some curve γ' such that γ and γ' form a bounding pair. Because

$\Phi(f)$ fixes the vertex corresponding to $\gamma \cup \gamma'$, it must be the case that either $f(\gamma) = \gamma$ and $f(\gamma') = \gamma'$ or $f(\gamma) = \gamma'$ and $f(\gamma') = \gamma$. But there exists a separating curve η that intersects γ but not γ'. Because $f(\eta) = \eta$, it cannot be the case that $f(\gamma) = \gamma'$. Therefore $f(\gamma) = \gamma$ as desired.

By the Alexander method for infinite–type surfaces, due to Hernández–Moralez–Valdez [11], we deduce that f is the identity.

We now show that Φ is surjective. Let $\phi : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ be an automorphism. Fix a principal exhaustion $\{P_1, P_2, \dots\}$ of Σ such that P_1 has complexity at least six. Define σ_i to be the simplex of $\mathcal{T}(\Sigma)$ corresponding to the multicurve ∂P_i. Note that by construction, σ_i contains only type S vertices. Denote by $\mathcal{P}_i$ the subcomplex of $\mathcal{T}(\Sigma)$ spanned by the curves and bounding pairs contained in P_i. By Lemma 7, we know that $\mathcal{P}_i$ is the unique side of σ_i whose diameter is infinite. By construction, $\mathcal{P}_i$ is connected for all i.

Since ϕ is a simplicial automorphism, it induces a bijection between the sides of σ_i and the sides of $\phi(\sigma_i)$. Because all simplicial automorphisms of $\mathcal{T}(\Sigma)$ are isometries, $\phi(\sigma_i)$ has a unique side of infinite diameter. From Lemma 9 we have that every vertex of $\phi(\sigma_i)$ is of type S or it is a genus bounding pair. If $\phi(\sigma_i)$ contains a genus bounding pair, then the unique side of $\phi(\sigma_i)$ with infinite diameter is disconnected. This contradicts the fact that ϕ is an isometry, and so no vertex of $\phi(\sigma_i)$ is a genus bounding pair.

We write $\mathcal{Q}_i \subset \mathrm{Link}(\phi(\sigma_i))$ for the side of $\phi(\sigma_i)$ with infinite diameter, and $Q_i \subset \Sigma$ for the finite–type subsurface which it defines. By construction, $\mathcal{P}_i \cong \mathcal{T}(P_i)$ and $\mathcal{Q}_i \cong \mathcal{T}(Q_i)$ (note that if σ_i contains a bounding pair then this may not be the case). Furthermore, ϕ restricts to an isomorphism

$$\phi_i : \mathcal{P}_i \to \mathcal{Q}_i.$$

Since each P_i is assumed to have complexity at least six, the combination of results of Kida [14] and Korkmaz [16] implies that ϕ_i is induced by a homeomorphism f_i. Moreover, the homeomorphism f_{i+1} restricts to f_i on the subsurface P_i. Since $\Sigma = \bigcup P_i$, we deduce that ϕ is induced by the limit of the f_i, completing the proof. □

4.3. **Algebraic characterization of twists and bounding pair maps.** Before proving Theorem 4 we will need one more ingredient. Notice that the vertices of $\mathcal{T}(\Sigma)$ define supports of elements in $\mathcal{I}(\Sigma)$. We must now show that commensurations of $\mathcal{I}(\Sigma)$ preserve such elements and therefore define a permutation of the vertices of the complex. We will adapt the algebraic characterization of Dehn twists of Bavard–Dowdall–Rafi [2] to our setting.

We first introduce some terminology to facilitate the characterization of twists and bounding pairs. Let $G < \mathrm{MCG}(\Sigma)$. We denote by $\mathcal{F}_G$ the set of elements of G whose conjugacy class (in G) is countable. Bavard–Dowdall–Rafi prove that if G is finite-index in $\mathrm{MCG}(\Sigma)$ then f is in $\mathcal{F}_G$ if and only if it has compact support [2, Proposition 4.2]. Using similar methods, we will show:

Proposition 10. *Let $G < \mathcal{I}(\Sigma)$ be a finite-index subgroup. An element f in G has compact support if and only if f is in $\mathcal{F}_G$.*

Proof. It is clear that compactly-supported mapping classes have countable conjugacy classes. For the opposite direction, the argument in [2, Proposition 4.2] exhibits a infinite sequence of pairwise-disjoint curves a_i such that the Dehn twists about the a_i give rise to uncountably many conjugates of f. Since S has infinite type, the curves a_i may be chosen to be separating, so that the corresponding twists belong to $\mathcal{I}(S)$. Hence the result follows. □

Given a group H and a subgroup H', we denote by $Z(H')$ the center of H' in H. If h is in H, we write $C_H(h)$ for the centralizer of h in H.

Given a finite-index subgroup $G < \mathcal{I}(\Sigma)$ we write $\mathcal{M}_G$ for the set of elements f in G which satisfy the following three conditions:

(1) $f \in \mathcal{F}_G$,
(2) $Z(\mathcal{F}_G \cap C_G(f))$ is an infinite cyclic group, and
(3) $C_G(f) = C_G(f^k)$ for every $k > 0$.

We now prove that, for any G, powers of Dehn twists and bounding pair maps belong to the set $\mathcal{M}_G$.

Lemma 11. *Let $G < \mathcal{I}(\Sigma)$ be a finite-index subgroup. If f in G is a power of a Dehn twist or a bounding pair map then f in $\mathcal{M}_G$.*

Proof. Since f has compact support, we have that f is in $\mathcal{F}_G$. Suppose first f is a power of a Dehn twist about the separating curve γ. We have that

$$C_{\mathcal{I}(\Sigma)}(T_\gamma^k) = \{g \in \mathcal{I}(\Sigma) \mid g(\gamma) = \gamma\},$$

for $k \neq 0$. It follows that all powers of T_γ have equal centralizer in $\mathcal{I}(\Sigma)$ and hence, in any subgroup. A similar argument holds if f is a power of bounding pair map. This implies the third condition in the definition of $\mathcal{M}_G$.

To see that f satisfies the second condition, once again assume first that f is a power of the Dehn twist T_γ about a separating curve. Let g be a nontrivial element of $Z(\mathcal{F}_G \cap C_G(f))$ and assume that g is not a power of T_γ. Then there exists a curve δ disjoint from γ such that $g(\delta) \neq \delta$. If δ is a separating curve then T_δ^k is in $C_G(f)$, for some $k > 0$, and $gT_\delta^k \neq T_\delta^k g$, which is a contradiction. Suppose now that, on the other hand, δ is a nonseparating curve; without loss of generality, we may assume that δ is contained in a connected component of $\Sigma \setminus \gamma$ of infinite topological type. Then there exists a curve $\bar{\delta}$ that is disjoint from γ that forms a bounding pair with δ, and once again we arrive at contradiction in our choice of g.

Using a similar argument, one can show that if f is a power of a bounding pair $T_{\gamma_1} T_{\gamma_2^{-1}}$ then any choice of g as above leads to a contradiction. □

When G is a finite-index subgroup of $\mathrm{MCG}(\Sigma)$, all elements of $\mathcal{M}_G$ are powers of Dehn twists, see [2, Lemma 4.5]. In stark contrast, this is no longer true in our setting, as the set $\mathcal{M}_G$ contains elements which are not powers of Dehn twists or bounding pair maps. Moreover, $\mathcal{M}_G$ may contain elements which are not supported on a disjoint union of annuli: for example, we may take a pure braid on a nonseparating planar subsurface with at least three boundary components.

In other words, we need some further work in order to obtain the desired algebraic characterization of separating twists and bounding pair maps. We will need the following terminology from [2]. Given f in G we set

$$(\mathcal{M}_G)_f = \{g \in \mathcal{M}_G \mid fg = gf\}.$$

Note that if g is in $(\mathcal{M}_G)_f$, then $g(\partial Y) = \partial Y$, as otherwise f and g do not commute. Moreover, since f is in $\mathcal{M}_G$, then ∂Y consists only of nonseparating curves, no two of which form a bounding pair. In particular, no pair of boundary components of Y are homologous to each other and therefore f and g both fix ∂Y pointwise. In other words, we have proved:

Lemma 12. *Let f in $\mathcal{M}_G$ have support Y. Then every g in $(\mathcal{M}_G)_f$ (and in particular f) fixes every connected component of ∂Y. As a consequence, f preserves every connected component of Y, and every finite-type component of $\Sigma \setminus Y$.*

We now define a further subset; if the support of f is Y we define

$$(\mathcal{P}_G)_f = \{g \in (\mathcal{M}_G)_f \mid g \text{ is supported in } \Sigma \setminus Y\}.$$

It follows from Lemma 12 that each element of $(\mathcal{M}_G)_f$ can be written as the product of an element supported on Y with an element supported in a finite-type subsurface of $\Sigma \setminus Y$. The next lemma tells us that the elements supported in Y are precisely those which are central.

Lemma 13. *For any element f in $\mathcal{M}_G$ we have that*

$$(\mathcal{M}_G)_f = Z((\mathcal{M}_G)_f) \oplus (\mathcal{P}_G)_f.$$

Proof. Consider g in $(\mathcal{M}_G)_f$. Denoting Y the support of f, we want to show that if g has support in Y, then g is in $Z((\mathcal{M}_G)_f)$. To this end, let h lie in $(\mathcal{M}_G)_f$. If the support of h is contained in $\Sigma \setminus Y$ then the result is clear, so assume this is not the case. By Lemma 12, the mapping classes f, g and h all preserve every component of Y and of ∂Y. Let $Y_1, \dots, Y_s$ be the components of Y. Then there exists $k \geq 1$ such that

$$\begin{aligned} f^k &= f_1 f_2 \dots f_s, \\ g^k &= g_1 g_2 \dots g_s, \\ h^k &= h_1 h_2 \dots h_s, \end{aligned}$$

where f_i (resp. g_i, h_i) denotes the restriction of f^k (resp. g^k, h^k) to Y_i, and is either the identity or a pseudo-Anosov. Note that, in the latter case, each restriction must be a power of *the same* pseudo-Anosov, as f^k commutes with g^k and h^k. It follows that g^k and h^k commute, and therefore so do g and h, by condition (3) in the definition of the set $\mathcal{M}_G$. □

Finally, we can prove the characterization of Dehn twists and bounding pair maps.

Proposition 14. *Let $G < \mathcal{I}(\Sigma)$ be a finite-index subgroup, and let f lie in G. Then f is a power of a Dehn twist or of a bounding pair map if and only f is in $\mathcal{M}_G$, and for all g in $\mathcal{M}_G$ such that $(\mathcal{P}_G)_g = (\mathcal{P}_G)_f$ we have that $f^i = g^j$.*

Proof. The forward direction is clear.

For the other direction, we prove the contrapositive. If f is not a power of a Dehn twist or bounding pair map, we find g in $\mathcal{M}_G$ with the same support as f such that no powers of f and g are equal, but $(\mathcal{P}_G)_f = (\mathcal{P}_G)_g$.

Let Y for the support of f. Since f is in $\mathcal{M}_G$, we may assume that Y has at least one connected component Z on which $f|_Z$ is a pseudo-Anosov. The Torelli group $\mathcal{I}(\Sigma)$ is normal in $\mathrm{MCG}(\Sigma)$. Therefore, for every h in $\mathrm{MCG}(\Sigma)$ that preserves each connected component of Y and ∂Y we have hfh^{-1} in $\mathcal{I}(\Sigma)$ with support contained in Y.

Since Z supports a pseudo-Anosov, we may choose an h in $\mathrm{MCG}(\Sigma)$ that is pseudo-Anosov on Z, agrees with f on the rest of Y, and such that the restriction to Z of f and $g = hfh^{-1}$ are two independent pseudo-Anosovs. In particular, f and g have no power in common and Lemma 13 implies that $(\mathcal{P}_G)_f = (\mathcal{P}_G)_g$, as desired. □

4.4. **Abstract commensurators of the Torelli group.** We can now finally prove Theorem 4. For a bounding pair $\gamma = \{\gamma_1, \gamma_2\}$ we use the shorthand T_γ for the bounding pair map $T_{\gamma_1} T_{\gamma_2}^{-1}$.

Proof of Theorem 4. Let $[\psi]$ be an element of $\mathrm{Comm}\, \mathcal{I}(\Sigma)$ representing the isomorphism of finite index subgroups

$$\psi : G_1 \to G_2.$$

Let γ be a separating curve or a bounding pair and choose n in $\mathbb{N}$ so that T_γ^n is in G_1. By Proposition 14, T_γ^n in $\mathcal{M}_{G_1}$ and for all g in $\mathcal{M}_{G_1}$ such that $(\mathcal{P}_{G_1})_g \subset (\mathcal{P}_{G_1})_{\gamma^n}$, we have that $(\gamma^n)^i = g^j$. Since these conditions are preserved by isomorphism, we have that $\psi(T_\gamma)$ lies in $\mathcal{M}_{G_2}$, Proposition 14 implies there exists some m in $\mathbb{N}$ such that

$$\psi(T_\gamma^n) = T_\delta^m,$$

where δ is a separating curve or a bounding pair.

At this point, and again with respect to the above notation, we obtain that ψ induces a map

$$\psi_* : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma);$$
$$\gamma \mapsto \delta.$$

We observe that ψ_* is a simplicial map, since powers of Dehn twists and bounding pair maps commute if and only if the underlying curves are disjoint. Moreover, the map is also bijective, with inverse the simplicial map associated to the inverse of ψ^{-1}.

By Theorem 8, there exists an f MCG(Σ) such that $\psi_*(\gamma) = f(\gamma)$ for every separating curve or bounding pair γ. Now, for any g in G_1 we have

$$\psi(gT_\gamma^n g^{-1}) = \psi(g)\psi(T_\gamma^n)\psi(g^{-1}) = \psi(g)T_{f(\gamma)}^n\psi(g^{-1}) = T_{\psi(g)f(\gamma)}^n,$$

and therefore

$$T_{\psi(g)f(\gamma)}^n = \psi(gT_\gamma^n g^{-1}) = \psi(T_{g(\gamma)}^n) = T_{fg(\gamma)}^n.$$

Therefore $\psi(g)f(\gamma) = fg(\gamma)$. By use of the Alexander method [11] we conclude that $\psi(g) = fgf^{-1}$. This shows that every abstract commensurator is defined by conjugation by a mapping class, and in particular, so is every automorphism.

On the other hand, suppose there exists an f in MCG(Σ) and a finite-index subgroup $H < \mathcal{I}(\Sigma)$ such that conjugation by f induces the identity map on H. For any separating curve or bounding pair γ, there exists some $m \geq 1$ such that T_γ^m lies in H. Thus

$$T_\gamma^m = fT_\gamma^m f^{-1} = T_{f\gamma}^m.$$

By [2, Lemma 2.5], $f\gamma = \gamma$, and thus f is the identity by Theorem 8. This completes the proof. □

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