Math 752

Homology

Let $\mathcal{C}$ be a chain complex (indexed by $\mathbb{Z}$) of abelian gps.

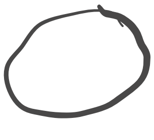
$$\begin{tikzcd}
{} \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial=\partial_{n+1}"] & C_n \arrow[r, "\partial=\partial_n"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] & \cdots \\
 & & \text{``gp of n-chains''} & &
\end{tikzcd}$$

Being ch. cx. means $\partial^2 = \partial\partial = 0$.

Homology of $\mathcal{C}$ is

$$H_n(\mathcal{C}) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$$

$\ker \partial_n =: Z_n$ gp of "n-cycles"



$\operatorname{Im} \partial_{n+1} =: B_n$ gp of "boundaries"

Cohomology of $\mathcal{C}$.

Let G be an abelian gp.
"gp of coefficients."

We "dualize" $\mathcal{C}$ by "Hom-ing" $\mathcal{C}$ into G.

Replace chain gps C_n w/ "cochain gps"

$$C_n^* = \mathrm{Hom}(C_n, G) = \{\text{homomorphisms } C_n \to G.\}$$

and replace ∂ with their dual maps δ.

$$\begin{tikzcd} {} & C_{n+1}^* \arrow[l] & C_n^* \arrow[l, "\delta = \delta_n"'] & C_{n-1}^* \arrow[l, "\delta = \delta_{n-1}"'] & {} \arrow[l] \end{tikzcd}$$

What's δ (coboundary operator)?

$$\varphi \in C_n^* = \mathrm{Hom}(C_n, G).$$

$$\delta\varphi = \varphi \circ \partial$$

$$\delta\varphi(c) = \varphi(\partial c).$$

δ increases dimension

Complex $\mathcal{C}^*$ is still chain complex,

since $\delta\delta\varphi = \varphi\circ\underbrace{\partial\circ\partial}_{0} = 0$.

Ex $\mathcal{A}$: silly ch. cx.

$$\begin{tikzcd} A_{n+1} \arrow[r, "f_{n+1}"'] \arrow[r, shift left=3] & A_n \arrow[r, "f_n"'] \arrow[r, shift left=3] & A_{n-1} \end{tikzcd}$$

If Exact: $\ker f_n = \text{Image}(f_{n+1})$

homology of $\mathcal{A}$ is 0 in all dimension.

But! Dual complex $\mathcal{A}^*$ given coeffs G is still ch. cx. but not nec. exact. So Cohomology of $\mathcal{A}$ can be nontrivial.

$H^n(\mathcal{C}; G)$ cohomology of $\mathcal{C}$ w/ coefficients in G

$\parallel$

$H_n(\mathcal{C}^*)$.

$$H^n(\mathcal{C};G) = \underbrace{\ker \delta_n}_{\text{cocycles}} / \underbrace{\operatorname{Im} \delta_{n-1}}_{\text{coboundaries}}$$

That's cohomology.

letting $\mathcal{C}$ be chain cx of singular chains on a space X gives us singular cohomology $H^n(X;G)$

letting $\mathcal{C}$ be chain cx of cellular chains on a cell cx X gives us cellular cohomology $H^n(X;G)$

Just dualize chain cx and take homology.

What's relationship between H^n and H_n.

Naive guess is that

NOT RIGHT $\boxed{H^n(\mathcal{C}; G) \cong \mathrm{Hom}(H_n(\mathcal{C}), G)}$

But that's too good to be true.

Think about cocycles and coboundaries for a sec.

φ cocycle $\iff \delta\varphi \equiv 0$

$\iff \varphi \circ \partial \equiv 0$

$\iff \varphi$ vanishes on boundaries

φ coboundary : so $\varphi = \delta\psi = \psi \circ \partial$

In this case φ vanishes on cycles

If $\underbrace{z \text{ cycle}}_{\text{i.e. } \partial z = 0}$ $\quad \varphi(z) = \delta\psi(z)$

$$= \psi(\partial z)$$
$$= \psi(0) = 0.$$

If the homology groups are free, then φ coboundary iff φ vanishes on cycles

$$\begin{tikzcd} C_n \arrow[r, "\varphi"] \arrow[d, "\partial"] & G \\ C_{n-1} \arrow[r, "\psi"'] & G \end{tikzcd}$$

φ vanishes on cycles $\ker \partial$ $\Rightarrow$ φ descends to a map $\bar{\varphi}: C_n/Z_n \to G$.

$$\overset{?}{\cong}$$

$$\boxed{B_{n-1} = \operatorname{Im} \partial}$$

$$0 \to Z_{n-1} \to C_{n-1} \to \overset{\text{free}}{B_{n-2}} \to 0$$

$$\Rightarrow \quad C_{n-1} \cong Z_{n-1} \oplus B_{n-2}$$

if $H_{n-1} = Z_{n-1}/B_{n-1}$ is free, then $Z_{n-1} \cong B_{n-1} \oplus Z_{n-1}/B_{n-1}$

So, we can then project $C_{n-1} \cong B_{n-1} \oplus Z_{n-1}/B_{n-1} \oplus B_{n-2}$ onto $B_{n-1} \cong C_n/Z_n$ and compose with $\bar{\varphi}$ to find ψ.

Difference between H^n & H_n

Example: $\mathcal{C}: 0 \to \underset{C_3}{\underset{\shortparallel}{\mathbb{Z}}} \xrightarrow{0} \underset{C_2}{\underset{\shortparallel}{\mathbb{Z}}} \xrightarrow{2} \underset{C_1}{\underset{\shortparallel}{\mathbb{Z}}} \xrightarrow{0} \underset{C_0}{\underset{\shortparallel}{\mathbb{Z}}} \to 0$

$H_3(\mathcal{C}) = \mathbb{Z}/0 = \mathbb{Z}$.

$H_2(\mathcal{C}) = 0/0 = 0$.

ker/Im.

$H_1(\mathcal{C}) = \mathbb{Z}/2\mathbb{Z}$ → torsion in dim. 1.

$H_0(\mathcal{C}) = \mathbb{Z}/0 = \mathbb{Z}$.

let $G = \mathbb{Z}$. take dual ch. cx $\mathcal{C}^*$.

$$\begin{tikzcd}
0 & \mathbb{Z} \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z} \arrow[l, "0"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[dl] & \mathbb{Z} \arrow[l, "0"'] & 0 \arrow[l] \\
 & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & & & & \\
 & C_3^* & C_2^* & C_1^* & C_0^* &
\end{tikzcd}$$

$\varphi \in \mathrm{Hom}(\mathbb{Z},\mathbb{Z})$.

$$\partial(c) = 2c$$

$$\begin{aligned} \delta \underbrace{\varphi(c)} &= \varphi(\partial c) \\ &= \varphi(2c) \\ &= 2\,\underbrace{\varphi(c)} \end{aligned}$$

dual of 2. is 2.

$H^3(\mathcal{C};\mathbb{Z}) = \mathbb{Z}/0 = \mathbb{Z}$.

$H^2(\mathcal{C};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ → torsion in dimension 2.

$H^1(\mathcal{C};\mathbb{Z}) = 0$

$H^0(\mathcal{C};\mathbb{Z}) = \mathbb{Z}$.

Cohomology gps are the same except the torsion has moved up a dimension.

It turns out, that, for $G = \mathbb{Z}$, and if the C_n are finitely gen. *free* abelian gps, "H^n will be isomorphic to H_n, except each torsion subgp. is moved up a dimension."

"That is

$$H^n(\mathcal{C};\mathbb{Z}) \cong H_n(\mathcal{C})/\text{torsion}(H_n(\mathcal{C})) \oplus \text{torsion}(H_{n-1})$$

"

Why is that?

Chain complex of f.g. free abelian gps can be decomposed (at each term), as a direct sum of finitely many ch. cxs of the form

$$\left[\begin{array}{l} 0 \to \mathbb{Z} \to 0 \\ 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0. \end{array}\right.$$

Exercise 2.2 #45

$$\begin{tikzcd}
C_{n+2} \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & C_{n-2} \arrow[r] & {} \\
 & \cong & \cong & \cong & & \\
 & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 & & \\
 & \oplus & \oplus & \oplus & & \\
 & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 & & \\
 & \oplus & \oplus & \oplus & & \\
 & & \vdots & & & \\
 & \oplus & \oplus & \oplus & & \\
 & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 & & \\
 & & \oplus & & & \\
 & 0 \arrow[r] & \mathbb{Z} \arrow[r, "m_1"] & \mathbb{Z} \arrow[r] & 0 & \\
 & & \oplus & \oplus & & \\
 & 0 \arrow[r] & \mathbb{Z} \arrow[r, "m_2"] & \mathbb{Z} \arrow[r] & 0 & \\
 & & \oplus & \oplus & & \\
 & & \vdots & & &
\end{tikzcd}$$

$$\begin{tikzcd}
& 0 \arrow[r] & \mathbb{Z} \arrow[r, "m_p"] & \mathbb{Z} \arrow[r] & 0 \\
& & \oplus & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "l_1"] & \mathbb{Z} \arrow[r] & 0 & \\
& & \oplus & & \\
& & \vdots & & \\
& & \oplus & \oplus & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "l_r"] & \mathbb{Z} \arrow[r] & 0 &
\end{tikzcd}$$

$$\mathbb{Z} \xrightarrow{2\oplus 0} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z} \quad 0$$

$$\mathbb{Z}\oplus\mathbb{Z} / 2\mathbb{Z}\oplus 0 = \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}.$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r] & 0 \\
& \oplus & \oplus & \oplus \\
& 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

So H^n isn't really the dual of H_n, when $G = \mathbb{Z}$, but almost was.

Description was straightforward.

But for arbitrary G, little more subtle.

In general, $\exists$ a s.e.s.

$$0 \to \underbrace{??}_{\text{solve the mystery.}} \to \underbrace{H^n(\mathcal{C};G) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}),G) \to 0}$$

[from now on: $\mathcal{C}$ ch. cx. of **free** abelian gps.]

Thm $\exists$ natural surjective map

$$h: H^n(\mathcal{C};G) \longrightarrow \mathrm{Hom}(\underline{H_n}(\mathcal{C}),\underline{G}).$$

Let $Z_n = \ker \partial_n \subset C_n$ cycles

$B_n = \operatorname{Im} \partial_{n+1} \subset C_n$ boundaries.

A class ξ in $H^n(\mathcal{C}; G) = \underline{\ker \delta}/\operatorname{Im}\delta$

is represented by $\varphi : C_n \to G$.

that vanishes on B_n

(that's what it means to be a cocycle).

Restrict φ to Z_n:

$$\varphi_0 = \varphi\big|_{Z_n} : Z_n \to G.$$

But φ_0 vanishes on B_n.

so it induces a map on Z_n/B_n

$$\overline{\varphi_0} : \underset{\underset{H_n(\mathcal{C})}{\parallel}}{Z_n/B_n} \to G$$

$\in \mathrm{Hom}(H_n(\mathcal{C}), G)$

$h(\xi) = \overline{\varphi_0}$.

Need to check that this is well defined

Any two reps. of ξ differ by an element of $\mathrm{Im}\,\delta$.

So It suffices to show that if φ is in $\mathrm{Im}\,\delta$, then $\overline{\varphi_0} = 0$. (this guarantees that diff. reps. of ξ have same image.)

$\varphi \in \mathrm{Im}\,\delta$.

$\Rightarrow \varphi$ vanishes on cycles, i.e. φ vanishes on Z_n.

$\varphi_0 = \varphi\big|_{Z_n} \equiv 0$

$$\Rightarrow \overline{\varphi_0} \equiv 0.$$

So "h"$(\varphi) = 0$.

So h well defined.

Next time:

$$\begin{tikzcd} H^n(C;G) \arrow[r, "h"] & \mathrm{Hom}(H_n(C),G) \arrow[l, bend left, dashed] \end{tikzcd}$$

This map has a "section"

$$s : \mathrm{Hom}(H_n(C), G) \to H^n(C, G).$$

$$h \circ s = \mathbb{1} \text{ on Hom}.$$

$$\begin{tikzcd} 0 \arrow[r] & \underline{??} \arrow[r] & \underline{H^n(C;G)} \arrow[r, two heads] & \mathrm{Hom}(H_n(C),G) \arrow[r] \arrow[l, bend left, dashed] & 0 \end{tikzcd}$$

eventually have short exact sequence.

section means it splits.

Cohomology

$$\overbrace{\to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to}^{\mathcal{C} \nearrow \text{ Chain cx of } \underline{\text{free}}\ \underline{\text{abelian gps}}}$$

$$\begin{tikzcd} {} \arrow[d, squiggly] \\ {} \end{tikzcd}$$

$$\overbrace{\leftarrow C^*_{n+1} \leftarrow C^*_n \xleftarrow{\delta} C^*_{n-1} \leftarrow}^{\mathcal{C}^*} \left.\vphantom{\int_a^b}\right\} \begin{matrix} C_n^* \\ \text{not} \\ \text{nec.} \\ \text{free.} \end{matrix}$$

$$C_n^* = \mathrm{Hom}(C_n, G)$$

$$H^n(\mathcal{C};G) = H_n(\mathcal{C}^*).$$

$$= \underbrace{\underbrace{\ker\delta}_{\text{cocycles}}}_{\substack{\text{vanish} \\ \text{on} \\ \partial\text{s}}} \Big/ \underbrace{\underbrace{\operatorname{Im}\delta}_{\text{coboundaries}}}_{\substack{\text{these vanish} \\ \text{on cycles} \\ \text{(but not iff)}}}$$

Constructed a map, a homomorphism:

$$h: H^n(\mathcal{C}) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}), G)$$

We want to analyze the kernel.

<u>Recall:</u> Construction of h:

$\xi \in H^n(\mathcal{C})$, then pick representative

$\varphi : C_n \longrightarrow G$. (cocycle)

restrict to $Z_n \subset C_n$ to get

$\varphi_0 : Z_n \longrightarrow G$

and then pass to quotient

$$\overline{\varphi_0} : \underset{\underset{\displaystyle H_n(\mathcal{C})}{\shortparallel}}{Z_n/B_n} \longrightarrow G$$

, which is allowed since $\varphi_0|_{B_n} = 0$.

$h(\xi) = \overline{\varphi_0}$.

Claim: h surjective.

Pf. (assuming C ch.cx of free abelian gps.)

There is a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, "i"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

This is split, since B_{n-1} is free abelian (cuz it's subgp of a free abelian gp.)

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "\partial"] & \overset{\text{free}}{\overline{B_{n-1}}} \arrow[r] \arrow[l, bend left=60, "s"] & 0 \end{tikzcd}$$

$\partial \circ s = \mathbb{1}_{B_{n-1}}$

s defined by just sending each generator to any of its preimages.

---

b generator $b \overset{s}{\longmapsto}$ anything $\in \partial^{-1}(b)$.

(s is very ununique!)

So we can think of

C_n as $\cong Z_n \oplus B_{n-1}$

$$\begin{tikzcd}1 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] \arrow[l, bend left, "s"] & 1\end{tikzcd}$$

in general only know

$B \cong A \rtimes C$, but we're in an abelian setting.

$$C_n \cong Z_n \oplus B_{n-1} \xrightarrow{p} Z_n.$$

So there's a projection

$$p : C_n \longrightarrow Z_n$$

s.t. $\underbrace{p|_{Z} = \mathbb{1}_{Z_n}}_{\text{"retraction."}}$

again p not unique!

Now, to show surjectivity of

$$h : H^n(C;G) \longrightarrow \mathrm{Hom}(H_n(C), G)$$

we build a section of h.

construct a map

$$S : \mathrm{Hom}(H_n(C), G) \longrightarrow H^n(C;G)$$

s.t. $h \circ S = \mathbb{1}_{\mathrm{Hom}}$.

Say given an elt of $\mathrm{Hom}(H_n(C), G)$.

It's represented by some homo.

$$\varphi_0 : Z_n \longrightarrow G$$

I can extend it to homo.

$$\begin{tikzcd}
C_n \arrow[r, "\varphi"] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circ"] & G \\
Z_n \oplus B_{n-1} \arrow[r, "p"] & Z_n \arrow[u, "\varphi_0"']
\end{tikzcd}$$

$\varphi = \varphi_0 \circ p$.

That's true of any $\varphi_0 \in \mathrm{Hom}(Z_n, G)$.

If $\varphi_0\big|_{B_n} = 0$, then so does φ.

(Since $\varphi_0 \in \mathrm{Hom}(Z_n/B_n, G)$, we have that.)

If φ vanishes on B_n, then

$\varphi \in \ker \delta$.

$\Rightarrow$ φ represents a cocycle.

So we get a map

$$\begin{tikzcd}
\mathrm{Hom}(H_n(C), G) \arrow[rr, "\text{“extend” from } Z_n"] \arrow[ddd, "\mathbb{1}"'] \arrow[ddrr, "S"'] \arrow[rr, phantom, shift right=4, "\circ"] & & \ker \delta \arrow[dd, "\text{quotient}"'] \\
& & \\
& & H^n(C; G) \arrow[ddll, "h"] \\
\mathrm{Hom}(H_n(C), G) & &
\end{tikzcd}$$

send coboundaries to 0.

restricts to Z_n and then passes to quotient Z_n/B_n.

h undoes our extension.

$$\begin{tikzcd}
 & Z_n/B_n \arrow[r] & G \arrow[d, squiggly, "\text{take representative}"] & \\
\text{“}s\text{”} & \varphi_0 : Z_n \arrow[r] & G \arrow[d, squiggly, "\text{extend}"] & \\
 & \varphi : C_n \arrow[r] & G \arrow[d, squiggly, "\text{restrict}"] & \text{coboundary} \arrow[dd, mapsto] \\
h & \varphi_0 : Z_n \arrow[r] & G \arrow[d, squiggly] & \\
 & \overline{\varphi_0} : Z_n/B_n \arrow[r] & G & 0
\end{tikzcd}$$

$\square$

when h restricts φ from C_n to Z_n, we lose information

$C_n \cong Z_n \oplus B_{n-1}$ ← This decomposition is not unique!

SPLIT
EXACT SEQUENCE:

$$\begin{tikzcd} 0 \arrow[r] & \underbrace{\ker h}_{??} \arrow[r] & H^n(\mathcal{C};G) \arrow[r, "h"] & \mathrm{Hom}(H_n(\mathcal{C}),G) \arrow[r] \arrow[l, bend left, "s", "\circ"'] & 0 \end{tikzcd}$$

What is the mystery kernel?

The following may seem circuitous:

Start: Consider $\mathcal{C}$:

$$\begin{tikzcd} & {} \arrow[d] & {} \arrow[d, "\partial"] & {} \arrow[d] & \\ 0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_n \arrow[r, "j"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d, "\partial"] & B_{n-1} \arrow[r] \arrow[d] & 0 \\ & \underbrace{\qquad}_{\text{ch. cx.}} & \underbrace{\qquad}_{\mathcal{C}} & \underbrace{\qquad}_{\text{ch. cx.}} & \end{tikzcd}$$

So this is a SES of ch. cxs.

Rows are split!

Fact: If $0 \to A \to B \to C \to 0$

is a split short exact seq.

of abelian gps,

then $\underbrace{0 \leftarrow A^* \leftarrow B^* \leftarrow C^*}_{\text{Hom into } G \text{ ab.}} \leftarrow 0$

is also a SES.

Splitness is important!

In general, if $A \to B \to C \to 0$

is exact, then it's always

true that $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$

But!

$0 \leftarrow A^* \leftarrow B^* \leftarrow C^*$

Doesn't need be exact!

EX

consider "quotient" SES.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[l, dashed, bend left, "\text{no section.}"] & 0 \end{tikzcd}$$

let $G=\mathbb{Z}$:

dual sequence is

$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$$

$$\begin{tikzcd} 0 & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "2"'] & 0 \arrow[l, "0"'] & 0 \arrow[l] \end{tikzcd}$$

Not exact!

Taking dual of the SES of ch. cxs:

we get SES of chain cxs:

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 & Z^*_{n+1} \arrow[l] \arrow[u] & C^*_{n+1} \arrow[l, "j^*"'] \arrow[u] & B^*_n \arrow[l, "\delta"'] \arrow[u] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"'] & C^*_n \arrow[l, "j^*"'] \arrow[u, "\delta"'] & B^*_{n-1} \arrow[l] \arrow[u, "0"'] & 0 \arrow[l] \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

Any SES of chain complexes gives you a long exact sequence in homology:

induced by maps in diagram

$$\begin{tikzcd}
{} & B_n^* \arrow[l] & Z_n^* \arrow[l, "\partial"'] \arrow[d] & H^n(\mathcal{C};G) \arrow[l] & B_{n-1}^* \arrow[l] & {} \arrow[l] \\
 & & \text{Hom of ch.cx. at left\textbackslash right is given by terms c/Z maps are 0.} & & &
\end{tikzcd}$$

The "boundary map" $\partial : Z_n^* \to B_n^*$ is obtained by pull a $\varphi_0 \in Z_n^*$ back to C_n^*, apply δ, and pull back to B_n^*.

Pulling φ_0 back to C_n^* extends φ_0 to $\varphi : C_n \to G$,

The second step precomposes w/ $\partial \rightsquigarrow \varphi \circ \partial$,

last step undoes that composition

and restricts to B_n.

The total effect of this is just restrict to B_n.

So $\partial : Z_n^* \to B_n^*$ is just restriction to B_n, that's dual i_n^* of inclusion $i_n : B_n \to Z_n$.

"Shave off" the rest of long exact sequence to get

$$0 \leftarrow \ker i_n^* \xleftarrow{h} H^n(C;G) \leftarrow \operatorname{Coker} i_{n-1}^* \leftarrow 0$$

$\operatorname{coker}(A \to B)$
$= B/\operatorname{Im}(A)$.

$\downarrow$

$$\begin{aligned}\ker i_n^* &= \{\text{homs on } Z_n \text{ that vanish on } B_n\} \\ &= \{Z_n/B_n \to G\} \\ &= \mathrm{Hom}(H_n(C), G)\end{aligned}$$

and $j^* = h$.

So this funny sequence is <u>our</u> sequence, and $\ker h = \operatorname{coker} i_{n-1}^*$.

<u>Where we going?</u> →

$$0 \to \underline{??} \longrightarrow H^n(C;G) \to \underline{\mathrm{Hom}(H_n, G)} \to 0$$

only depends on H_n and G?

What is coker i_{n-1}^*?

What is i_{n-1}?

SES (just def^n of Homology)

$$\boxed{0 \to \underbrace{B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1}}_{\text{free}} \to H_{n-1}(\mathcal{C}) \to 0}$$

this writes $H_{n-1}(\mathcal{C})$ as a quotient of fr. abelian Z_{n-1} with kernel B_{n-1}.

This is example of <u>free resolution</u>.

If H abelian gp, then there is always a <u>free resolution</u> of H: an exact sequence

$$\cdots \to F_2 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where F_i are free for all i.

Given a free resolution $\mathcal{F}_\bullet$ of H, we can dualize by homing into G to get

$$\begin{tikzcd}
\cdots & F_2^* \arrow[l] & F_1^* \arrow[l, "f_2^*"'] & F_0^* \arrow[l, "f_1^*"'] & H^* \arrow[l, "f_0^*"'] & 0 \arrow[l]
\end{tikzcd}$$

(not nec. exact), it is a chain cx.

So $H^n(\mathcal{F}_\bullet; G) = \ker f_{n+1}^* / \operatorname{Im} f_n^*$

In our case:

$$\begin{tikzcd}
0 \arrow[r] & B_{n-1} \arrow[r, "i_{n-1}"] & Z_{n-1} \arrow[r] \arrow[d, phantom, "?"] & H_{n-1}(C) \arrow[r] & 0 \\
0 & B_{n-1}^* \arrow[l] \arrow[d, equal] & Z_{n-1}^* \arrow[l, "i_{n-1}^*"'] \arrow[d, equal] & H_{n-1}(C)^* \arrow[l] & 0 \arrow[l] \\
 & F_1^* & F_0^* & &
\end{tikzcd}$$

$\boxed{\begin{array}{l} H^0(\mathcal{F}_\bullet; G) = 0 \\ \text{since exact at } F_0^* \end{array}}$

$$H^1(\mathcal{F}_\bullet; G) \cong \operatorname{coker} i_{n-1}^* .$$

__Lemma:__ a) If $\mathcal{F}$ and $\mathcal{F}'$ are free resolutions of H and H', then every homo $\alpha : H \to H'$ induces ch. map:

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\arrow[r] & F_1' \arrow[r] & F_0' \arrow[r, "f_0'"] & H' \arrow[r] & 0
\end{tikzcd}$$

and any two such chain maps are chain homotopic.

b) $\forall$ two resolutions $\mathcal{F}$ and $\mathcal{F}'$ of H $\exists$ canonical isos

$$\begin{array}{c} H^n(\mathcal{F}'; G) \cong H^n(\mathcal{F}; G) \\ \| \\ \mathrm{Ext}^n(H, G) \end{array}.$$

i.e. cohomology groups $H^n(\mathcal{F}; G)$ only depend on H & G.

$$\mathrm{Ext}(H, G) = H^1(\mathcal{F}; G).$$

In our case, $\ker h \cong \operatorname{coker} i_{n-1}^*$

$$= \operatorname{Ext}(H_{n-1}(C), G)$$

<u>Thm</u> Universal coefficient Theorem for Cohomology

∃ split s.e.s:

integral hom

$$0 \to \operatorname{Ext}(H_{n-1}(C), G) \to H^n(C; G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0$$

FACTS: (easy to check)

1) $\operatorname{Ext}(H \oplus H', G) = \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G)$

(take direct sum of resolutions)

2) $\operatorname{Ext}(H, G) = 0$ if H free.

$0 \to H \to H \to 0$

is resolution.

3) $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$.

$\mathrm{Ext}(B, C)$

$\shortparallel$

equivalence classes of extensions of B by C.

$A \qquad B$

$$\boxed{\begin{tikzcd} 1 \arrow[r] & \underset{\sqcup}{C} \arrow[r] & A \arrow[r] & \underset{\sqcup}{B} \arrow[r] \arrow[l, bend left] & 1 \end{tikzcd}}$$

$$\begin{tikzcd} 1 \arrow[r] & \boxed{A} \arrow[r] & \overset{\text{fte}}{G} \arrow[r] & B \arrow[r] \arrow[l, bend left] & 1 \end{tikzcd}$$

order

$m \cdot n$

$\boxed{|B| = n}$

$(m, n) = 1$

$$\boxed{G \cong A \rtimes B.}$$

LAST TIME:

Universal Coefficient Thm for Cohomology.

Split SES:

$$0\to \underbrace{\mathrm{Ext}(H_{n-1}(\mathcal{C}),G)}_{} \to H^n(\mathcal{C};G) \xrightarrow{h} \mathrm{Hom}(H_n(\mathcal{C}),G)\to 0$$

1st Cohomology of Free resolution of $H_{n-1}(\mathcal{C})$:

$$\cdots\cdots \to \overbrace{F_1 \to F_0 \to}^{\mathcal{F}} H_{n-1}(\mathcal{C}) \to 0$$

Hom into G:

$$\underbrace{\cdots\cdots \leftarrow \overbrace{F_1^* \leftarrow F_0^* \leftarrow}^{\mathcal{F}^*} H_{n-1}(\mathcal{C})^*}_{\text{1st homology of } \mathcal{F}_3 \cdot 3} \leftarrow 0$$

$\mathrm{Ext}(H_{n-1}(\mathcal{C}),G).$

---

to compute Ext, use following properties:

1) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.

Pf: Take direct sum of the resolutions.

$$\begin{tikzcd}
\cdots \arrow[r] & F_0 \arrow[r] & H \arrow[r] & 0 \\
\cdots & \oplus & \oplus & \\
\cdots \arrow[r] & F_0 \arrow[r] & H' \arrow[r] & 0
\end{tikzcd}$$

□

2) $\mathrm{Ext}(H, G) = 0$ if H free.

Pf. $0 \to 0 \to H \xrightarrow{\mathrm{Id}} H \to 0$

is a free resolution of H.

Dual cx is

$$\begin{tikzcd}
0 & 0 \arrow[l] \arrow[d, equal] & H^* \arrow[l] & H^* \arrow[l, "\mathrm{Id}"'] & 0 \arrow[l] \\
 & F_1^* & & &
\end{tikzcd}$$

□

3) $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

Dualize f.r.

$$0 \to \underleftarrow{\mathbb{Z} \xrightarrow{n} \mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

get

$$\begin{tikzcd}
0 & \mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},G) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z},G) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z},G) \arrow[l, "n"'] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},G) \arrow[l] \\
 & G/nG & G \arrow[l] & G \arrow[l, "n"'] &
\end{tikzcd}$$

$$0 \leftarrow \mathbb{Z}^* \xleftarrow{f_1^*} \mathbb{Z}^* \leftarrow \mathbb{Z}/n\mathbb{Z}^* \leftarrow 0$$

Ext

$= \mathbb{Z}^*/\text{Image previous } \mathbb{Z}^*.$

$= \mathbb{Z}^*/f_1^*$

Cor. If $H_n(C)$ and $H_{n-1}(C)$ are f.g. with the torsion subgp of $H_n(C)$ denoted T_n, then

$$H^n(C;\mathbb{Z}) \cong H_n(C)/T_n \oplus T_{n-1}.$$

Cor. If X space, then

$$H^1(X;\mathbb{Z}) := H^1(C(X);\mathbb{Z})$$

has no torsion, since $H_0(X;\mathbb{Z})$ is free. $\square$

UCT is "natural" in that

if $\alpha : \mathcal{C} \to \mathcal{C}'$ is a chain map, then α induces a map between the short exact sequences.

<u>Cor.</u> If a ch. map induces isomorphisms on homology, then it induces isomorphisms on cohomology.

We'll see soon that H^* is "more" natural in a sense.

And, in cohomology, there is a natural multiplication s.t. if

$$H^*(\mathcal{C};G) = \bigoplus_i H^i(\mathcal{C};G)$$

Then this object is a graded ring.

Geometric intuition. Let X be a space.

Cohomology of X will mean cohom. of $C(X)$ = singular ch. cx

or = cell. ch. cx

or = simplicial ch cx.

or = CW ch. cx.

$$H^0(X; \mathbb{Z}) = \mathrm{Hom}(H_0(X), \mathbb{Z})$$

$$H_0(X) = \bigoplus_{\substack{\#\text{cmpts} \\ \text{of } X}} \mathbb{Z} .$$

$$H^0(X; \mathbb{Z}) = \prod_{\substack{\#\text{cmpts} \\ \text{of } X}} \mathbb{Z} .$$

$H^1(X; \mathbb{Z})$. Assume X connected.
(so $H_0(X) \cong \mathbb{Z}$)

Since H_0 has no torsion,

$H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X;\mathbb{Z}), \mathbb{Z})$.

(supress coeffs. from now on.)

Consider: $\mathrm{Hom}(H_1(X), \mathbb{Z})$.

$H_1(X)$ is $\pi_1(X,*)^{ab}$

Every homomorphism $\pi_1(X,*) \to \mathbb{Z}$ "factors through" $H_1(X)$, meaning that if $\varphi : \pi_1(X,*) \to \mathbb{Z}$ there's a commutative diagram

$$\begin{tikzcd}
\pi_1(X,*) \arrow[r, "\varphi"] \arrow[d, "q"'] & \mathbb{Z} \\
H_1(X) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

quotient ← q

$\pi_1(X) \to \pi_1(X)^{ab}$

for some ψ.

Why?

$$\underline{\varphi : \pi_1(X,*) \longrightarrow \mathbb{Z}}.$$

$\mathbb{Z}$ abelian. So if $[x,y]$ commutator in $\pi_1(X,*)$, then $\varphi([x,y]) =$ identity in $\mathbb{Z}$.

Since the commutator subgp $[\pi_1(X), \pi_1(X)]$ dies under φ, we get an induced map $\bar{\varphi} : \pi_1(X,*)/[\pi_1(X), \pi_1(X)] \to \mathbb{Z}$.

So $\mathrm{Hom}(H_1(X), \mathbb{Z})$

$= \mathrm{Hom}(\pi_1(X,*), \mathbb{Z}).$

$$\begin{tikzcd} \bar{\varphi} & \psi \arrow[d, mapsto] \\ \varphi \arrow[u, mapsto] & \hat{\psi} \end{tikzcd}$$

If $\psi : H_1(X) \to \mathbb{Z}$

then $\hat{\psi} = \left(\pi_1(X,*) \xrightarrow{\varepsilon} H_1(X) \xrightarrow{\psi} \mathbb{Z}\right)$.

So $H^1(X) = \mathrm{Hom}(\pi_1(X, *), \mathbb{Z})$.

Assume X is cell cx.

By collapsing a maximal tree in $X^{(1)}$ (1-skeleton), we can assume that $X^{(0)} = \{*\}$. (This changes X, but not the homotopy type of X.)

__Claim.__ Every homomorphism $\pi_1(X, *) \to \mathbb{Z}$ is realized by a map $X \to S^1$, meaning that, identifying $\mathbb{Z}$ with $\pi_1(S^1, *)$, and given $\varphi : \pi_1(X, *) \to \mathbb{Z}$, $\exists$ cont. map $f : (X, *) \to (S^1, *)$ s.t. $f_* : \pi_1(X, *) \to \pi_1(S^1, *) \cong \mathbb{Z}$ is equal to φ. ($f_* = \varphi$).

Pf. Define f inductively on skeleton.

Step one:

$$f|_{X^{(0)}} : (X^{(0)}, *) \to (S^1, *)$$

$$f(*) = * .$$

On $X^{(1)}$ what do we do?

Cell by cell.

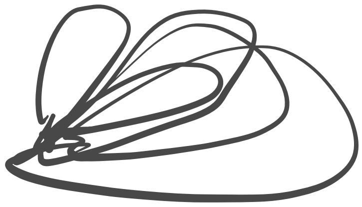
Let e be a 1-cell.

$\pi_1(X, *)$ is generated by the image of $\pi_1(X^{(1)}, *)$ in $\pi_1(X, *)$.

i.e. the map $i_* : \pi_1(X^{(1)}, *) \to \pi_1(X, *)$ induced by inclusion is surjective.

e is a loop. (since $X^{(0)} = *$.)

These loops given by the 1-cells, generate $\pi_1(X,*)$, because they generate $\pi_1(X^{(1)},*)$. $(X^{(1)} = \vee S^1_s)$



So e represents some ele $[e]$ of $\pi_1(X,*)$.

Given our $\varphi : \pi_1(X,*) \longrightarrow \mathbb{Z} \cong \pi_1(S^1,*)$

we want f_* to take $[e]$ to $\varphi([e])$.

$\varphi([e]) \in \mathbb{Z}$.

Define f on e by sending $(e,*) \to (S^1,*)$ by any degree $\varphi([e])$ map.

(if $\varphi[e]=3$)

Do that for every e in $X^{(1)}$.

So now we have a map

$$f:(X^{(1)},*) \to (S^1,*).$$

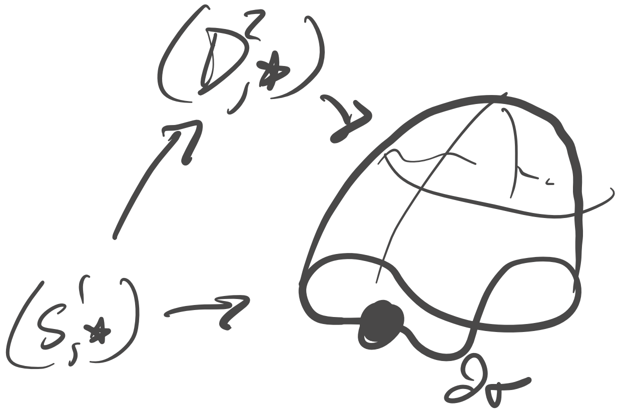
2-skeleton:

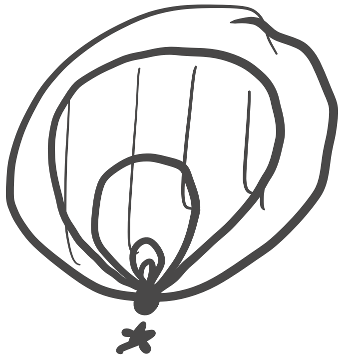
let σ be a 2-cell in $X^{(2)}$.

$\partial\sigma$ is a loop in $X^{(1)}$.

Notice: Since $\partial\sigma$ bounds a 2-cell in X,

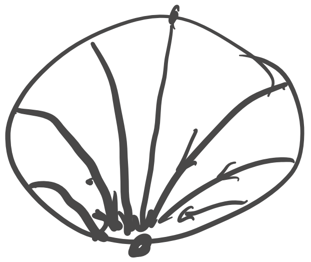
$(\partial\sigma,*)$ is a nullhomotopic loop in $(X,*)$





D^2.

$(\partial D^2, *) \xrightarrow{\text{udpc}} *$



So $[\partial\sigma] = 1 \in \pi_1(X, *)$.

Since ϕ is a homomorphism,

$$\phi[\partial\sigma] = 1 \in \pi_1(S^1, *) = \mathbb{Z} \ (\mathbb{Z} \text{ mult.})$$

$$f|_{\partial\sigma} : (\partial\sigma, *) \longrightarrow (S^1, *)$$

is null-homotopic.

($[\partial\sigma]$ = product of edge loops.)

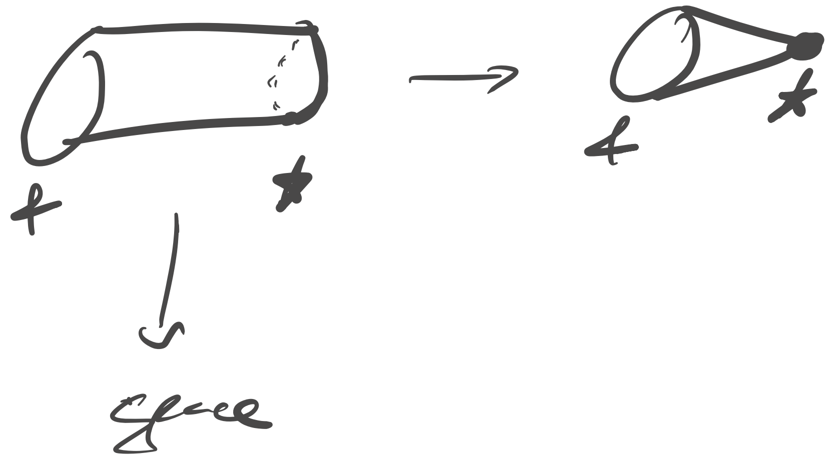
So, since $f|_{\partial\sigma} : (\partial\sigma, *) \to (S^1, *)$

is null-homotopic,

we can extend $f|_{\partial\sigma}$ to

a map $f : (\sigma, *) \to (S^1, *)$.

(fact: map sphere $\simeq *$ iff $\exists$ extension to a ball.)



So we've extended f to σ.

Do that for all 2-cells.

Continue for higher skeleta in same way. have $f : (X^{(2)}, *) \to (S^1, *)$

If σ 3-cell. $\partial\sigma \to S^1$ is null htpc.

$S^2 \longrightarrow S^1$

$S^2 \to S^1$ lifts to univ. cover

$S^2 \to \mathbb{R} = \tilde{S}^1$. so it's null htpc.

So we can extends $\partial\sigma \to S^1$

to $\sigma \to S^1$.

get $f: X^{(3)} \to S^1$.

and so on for all higher skeleton.

$\Rightarrow$ map on entire space

$$f: (X,*) \to (S^1,*)$$

(and it's cont. since it's cont. on all skeleton

(topology on cell cx is "weak" top.))

Why is $f_* = \varphi$?

let $w \in \pi_1(X,*)$.

write w as $[\underbrace{e_1 \cdot e_2 \cdots e_n}_{\text{concatenation of edge loops.}}]$

i.e. $\mathcal{F}$(1-cells in $X^{(1)}$).

$$w=[e_1\cdots e_n] = [e_1]\cdots[e_n]$$

$$f_*(w) = f_*([e_1\cdots e_n]) = f_*[e_1]f_*[e_2]\cdots f_*[e_n]$$

$$= \varphi[e_1]\varphi[e_2]\cdots\varphi[e_n]$$

by construction

$$= \varphi[e_1\cdots e_n]$$

φ homom.

$$= \varphi(w).$$

□

Have:

$$H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X),\mathbb{Z})$$

$$\cong \mathrm{Hom}(\pi_1(X,*),\mathbb{Z})$$

$$\cong \underbrace{\langle (X,*),(S^1,*)\rangle}_{\substack{\text{htpy classes}\\ \text{of maps}\\ (X,*)\to(S^1,*).}}$$

All of this works for

$H^1(X;G)$ if $\exists$ a space

$K(G,1)$ s.t. $\pi_1(K(G,1)) \cong G$

and the universal cover of

$K(G,1)$ is contractible.

$$\mathrm{Hom}(\pi_1(X), G)$$

$$= \langle (X,*), (K(G,1),*) \rangle.$$

homotopy classes of maps.

Pf of Corollary.

$$H^n(X) = H_n(X)/T_n \oplus T_{n-1}$$

$$\mathrm{Hom}\Big(\underbrace{\Big(\bigoplus_{i=1}^{n} \mathbb{Z}\Big) \oplus T_n}_{H_n}, \mathbb{Z}\Big)$$

$$\cong \bigoplus_{i=1}^{n} \mathbb{Z}^* \cong \bigoplus_{i=1}^{n} \mathbb{Z}$$

$$\mathrm{Ext}(H_{n-1}, \mathbb{Z})$$

$$= \mathrm{Ext}\Big(\Big(\bigoplus_{j=1}^{l} \mathbb{Z}\Big) \oplus T_{n-1}, \mathbb{Z}\Big)$$

$$= \mathrm{Ext}\Big(\underbrace{\bigoplus_{j=1}^{l} \mathbb{Z}}_{\text{free}}, \mathbb{Z}\Big) \oplus \mathrm{Ext}(T_{n-1}, \mathbb{Z})$$

$$= \quad 0 \quad \oplus \mathrm{Ext}(T_{n-1}, \mathbb{Z})$$

$$T_{n-1} = \bigoplus_{k=1}^{p} \mathbb{Z}/n_k\mathbb{Z}$$

$$\mathrm{Ext}(T_{n-1}, \mathbb{Z}) \cong \mathrm{Ext}\Big(\bigoplus_{k=1}^{p} \mathbb{Z}/n_k\mathbb{Z}, \mathbb{Z}\Big)$$

$$= \bigoplus_{k=1}^{p} \underbrace{\mathrm{Ext}(\mathbb{Z}/n_k\mathbb{Z}, \mathbb{Z})}_{\mathbb{Z}/n_k\mathbb{Z}}$$

Last time

$$\left.\begin{aligned} H^1(X;\mathbb{Z}) &= \mathrm{Hom}(H_1(X),\mathbb{Z}) \\ &= \mathrm{Hom}(\pi_1(X),\mathbb{Z}) \end{aligned}\right\}$$

$$= \langle (X,*),(S^1,*) \rangle.$$

$$X^{(1)} \subset X$$

$$\pi_1(X^{(1)}) = F \text{ free.}$$

$$\begin{tikzcd} \pi_1(X^{(1)}) \arrow[r, two heads] \arrow[rr, bend right] & \pi_1(X) \arrow[r, "\varphi"] & \mathbb{Z} \end{tikzcd}$$

We built map $(X,*) \to (S^1,*)$ inductively by sending edge loops of $X^{(1)}$ (assuming $X^{(1)} = \bigvee_{\alpha \in \Lambda} S^1_\alpha$) to where they should go according to φ.

Continued building map on higher skeleta using that φ hom.

and that univ. cover of S^1
is contractible.

Only things we used here:

$$\mathrm{Hom}(H_1 X, \mathbb{Z}) = \mathrm{Hom}(\pi_1 X, \mathbb{Z})$$

just uses $\mathbb{Z}$ abelian

$$\mathrm{Hom}(H_1 X, G) = \mathrm{Hom}(\pi_1 X, G)$$

when G abelian.

and: $\pi_1 S^1 = \mathbb{Z}$

and $\widetilde{S^1} \simeq *$.

If G abelian &

If $K(G,1)$ is a space with $\pi_1 = G$

and $\widetilde{K(G,1)} \simeq *$,

entire argument works.

So modulo existence of $K(G,1)$,

we have $H^1(X; G) \cong \langle (X,*), (K(G,1),*) \rangle$

$$\underbrace{\hspace{4cm}}_{\text{htpy classes of } (\mathbb{X},*)\to(K,*)}$$

$K(G,1)$ always exists.

$\downarrow$

Eilenberg–MacLane space.

We'll see later (way at the end) that $\forall$ G abelian and $n \in \mathbb{N}$, $\exists$ space $K(G,n)$ s.t.

$$H^n(\mathbb{X};G) = \langle (\mathbb{X},*),(K(G,n),*)\rangle$$

Need universal contractible to define map on $\mathbb{X}^{(n)}$ $n > 2$.

$$(B^3\times S^1,*) \xrightarrow{?} (S^2\times S^1,*)$$

realizes "$\mathbb{1}$" $: \pi_1(B^3\times S^1,*) \to \pi_1(S^2\times S^1,*)$

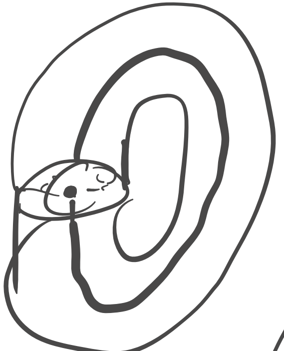
could do $B^3\times S^1 \xrightarrow{\pi} S^1 \longrightarrow * \times S^1$.

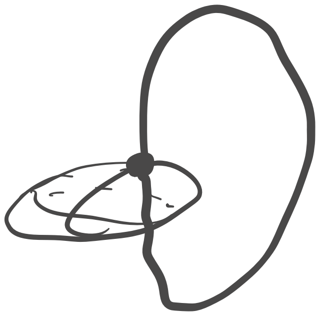
$$\partial B^3 \times * \longrightarrow S^2 \times *$$

if you have something like this at some stage, you can't extend

$$\underline{\partial B^3} \vee S^1 \xrightarrow{\text{obvious}} S^2 \vee S^1$$

has no extension to $B^3 \times S^1$





$$\begin{tikzcd}
B^3 \arrow[dr] & \\
\partial B^3 \arrow[u, "i"] \arrow[r] \arrow[r, bend left=80, "\Rightarrow \text{ it is } \simeq *."] & S^2 \times S^1
\end{tikzcd}$$

not null-htpc.

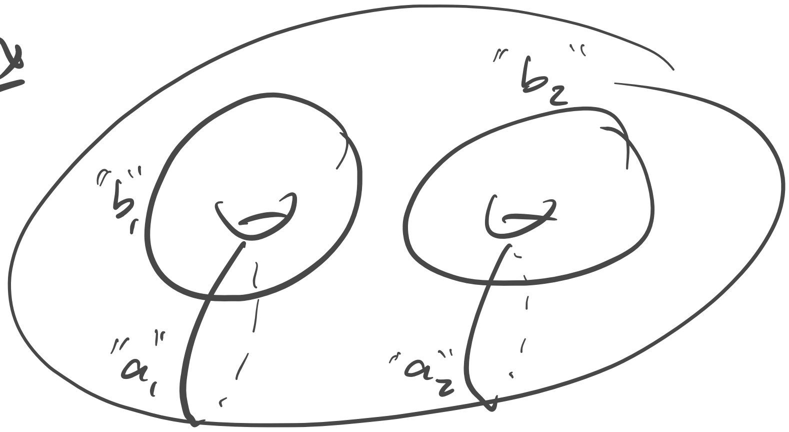
$$\begin{tikzcd}
& \mathbb{R} \arrow[d] \\
S^2 \arrow[ur] \arrow[r] \arrow[ur, phantom, "\circ", shift left=8] & S^1 \\
\pi_1 = 1 & \bullet
\end{tikzcd}$$

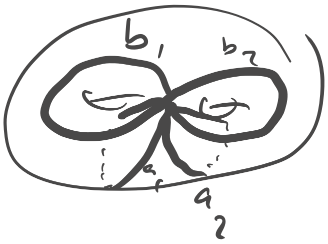
$\Rightarrow S^2 \to S^1$ null-htpc.

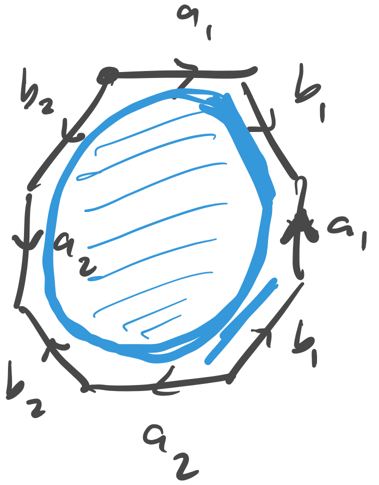
---

③

$X =$







$$\pi_1(X) \cong \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1][a_2,b_2] \rangle$$

$\phi : \pi_1 \to \mathbb{Z}$ defined by

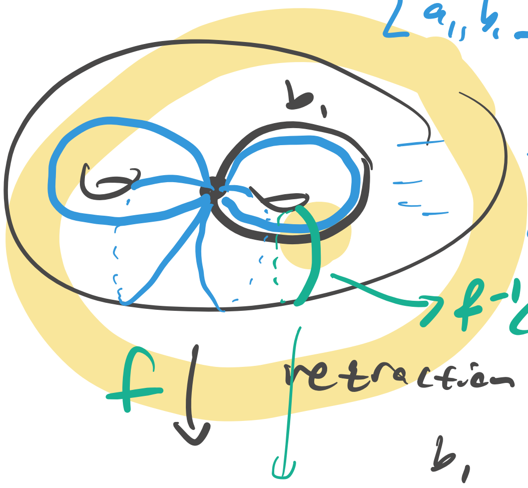
$$\phi(a_1) = 1 = \phi(b_1) = \phi(a_2)$$

$$\phi(b_2) = z, \text{ generator of } \mathbb{Z}.$$

$$\mathbb{Z} = \langle z \rangle = \{ z^i \mid i \in \mathbb{Z} \}.$$

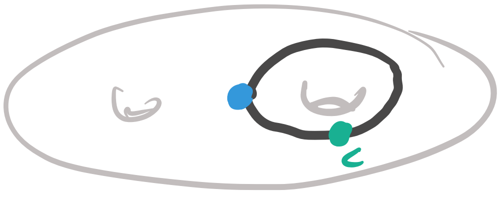
$$[a_1,b_1][a_2,b_2] \mapsto \underset{\text{Id.}}{\overset{}{1}} \in \mathbb{Z}$$

$\Rightarrow$ extend to the disk.



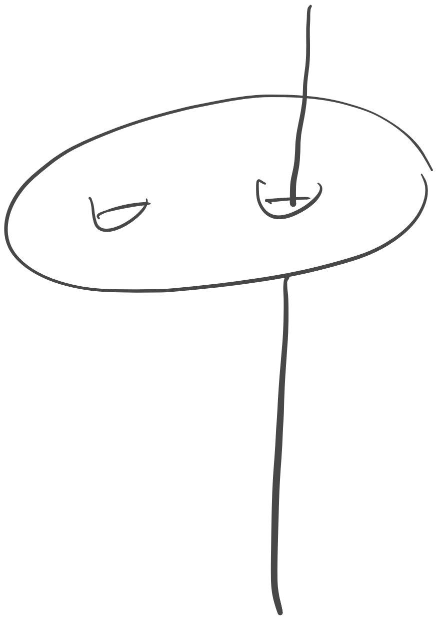
$f^{-1}(c)$ = circle that intersects b_1 once.

retraction onto the loop representing b_1



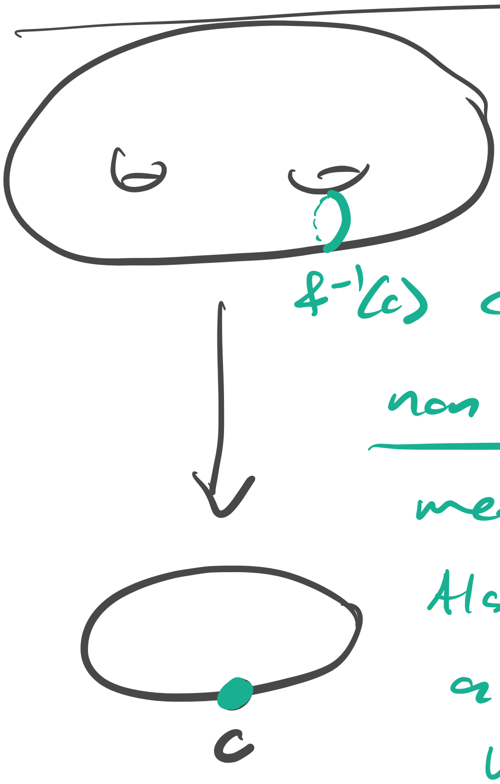
assume our map is transverse to c.

Derivative is just "projection"



$\mathbb{R}^3$ — vertical axis in $\mathbb{R}^3$

$\longrightarrow$ proj onto a circle.



Map $f : \Sigma \to S^1$ representing a cohomology class. "Dual" to it is a homology class $f^{-1}(c)$.

$f^{-1}(c)$ curve (1-submanifold).

non separating.

means $\Sigma - f^{-1}(c)$ connected.

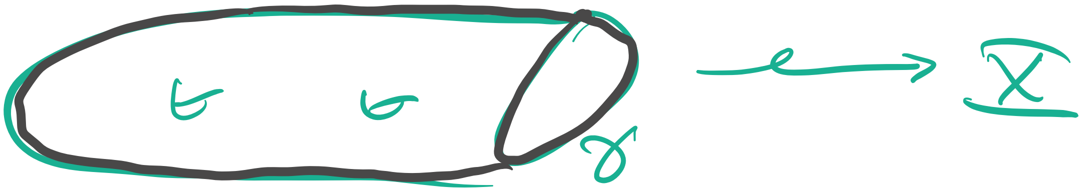
Also means that it is a nontrivial 1-dim^{l} homology class.

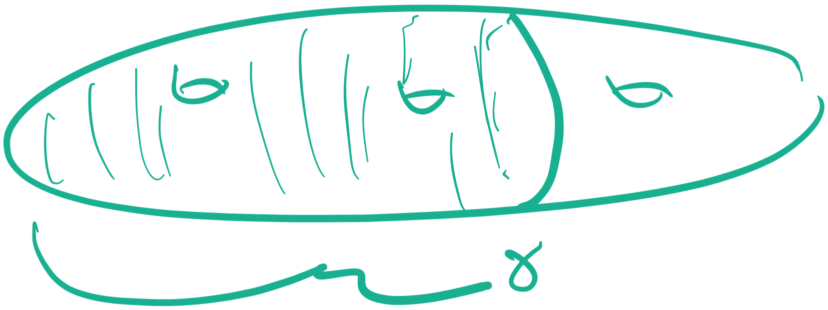
Every 1-dim^{l} homology class in a surface is represented by a 1-dim^{l} submfld.

If 1-submanifold is connected

it is $\neq 0$ in H,

iff nonseparating.





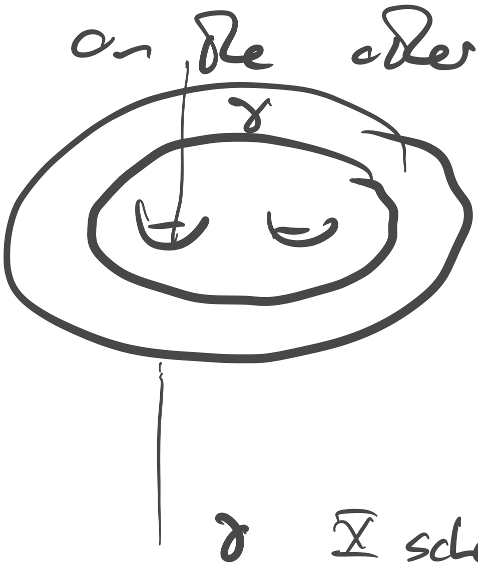
null homology

On the other hand, if you give me

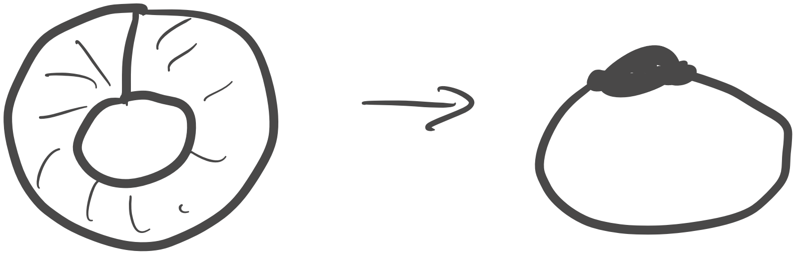
a nonseparating $\gamma \subset \Sigma$

$\exists$ map $g: \Sigma \to S^1$

s.t. $g^{-1}(*) = \gamma$.

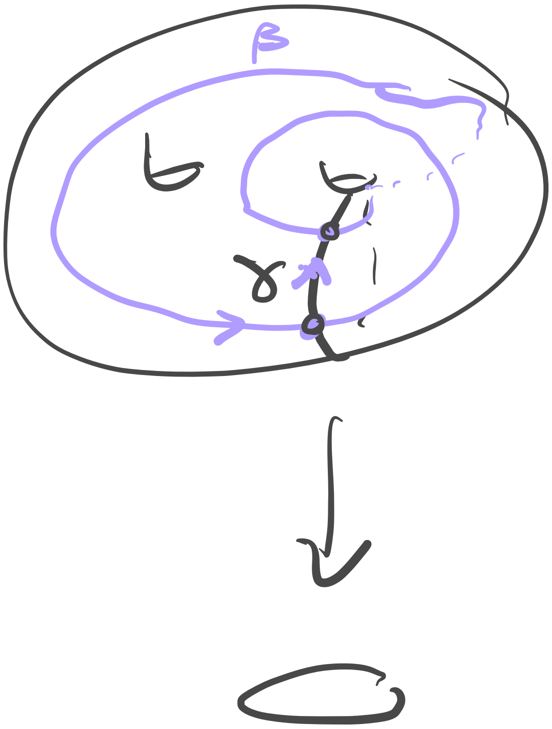


γ Σ schematic.



So, There's a kind of duality here between $\underset{\varphi}{H^1(X)}$ and $\underset{\gamma}{H_1(X)}$.

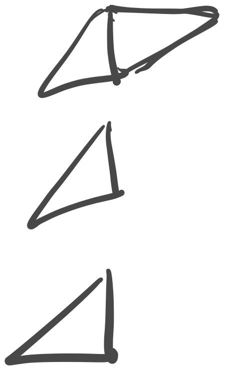
The homomorphism φ representing a class in H^1 can be thought of as the intersection # w/ γ.



$\varphi(a_1)=\varphi(b_1)=\varphi(a_2)=1$

$\varphi(b_1)=z$.

Exercise $\varphi(\beta)=2z$.



$\bigcirc = \partial$ 2-chain

realize 2-chain as a surface w/ $\partial =$ cycle.

Thm (Thom) If M smooth mfld, then every n-dim'l homology class is represented by a map

$$N^n \longrightarrow M$$

as long as $n \leq 7$.

Can always realize an n-chain into any space X.

by a map $K^n \to X$

for some n-cx K.

---

special case: EXERCISE:

If $S^1 \to X$ is a map

and as a singular cycle,

$\alpha : S^1 \to X$ null homologous

then $\exists$ a sfce Σ w/ $\partial\Sigma \cong S^1$

s.t. $\alpha : S^1 \to X$ extends

to a map $\Sigma \to X$.

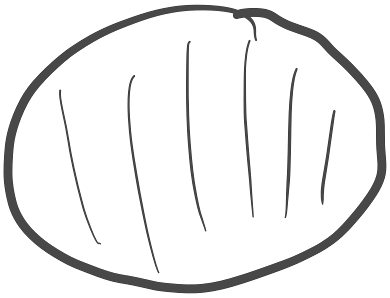
Hint: Build Σ directly from a 2-chain C s.t. $\partial C = \alpha$.

Fun Fact from 3-mfld topology.

Let $S^1 \to S^3$ be a smooth embedding.

Show $\exists$ an embedded surface Σ

in S^3 s.t. $\partial\Sigma = \text{Image}(S^1)$.



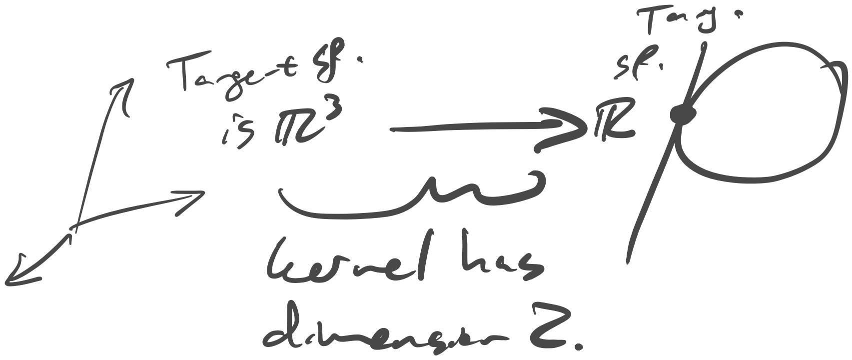
where's Σ?

What if Σ were like a 3-mfld instead of sfce?

M^3 be a 3-mfld.

$f: M^3 \longrightarrow S^1$ smooth.

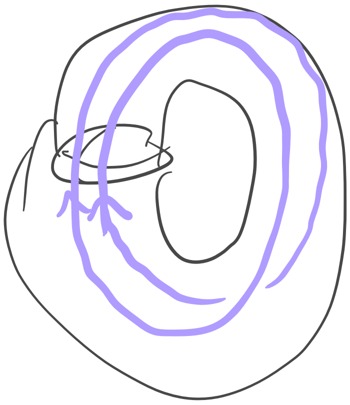
assume f transverse to $c \in S^1$.



Pullback $f^{-1}(c)$ is a 2manifold.

$S^2 \times S^1 \xrightarrow{\text{proj.}} S^1$ realizes $\varphi : \mathbb{Z} \to \mathbb{Z}$.

$f^{-1}(c) = S^2 \times c \longrightarrow c$



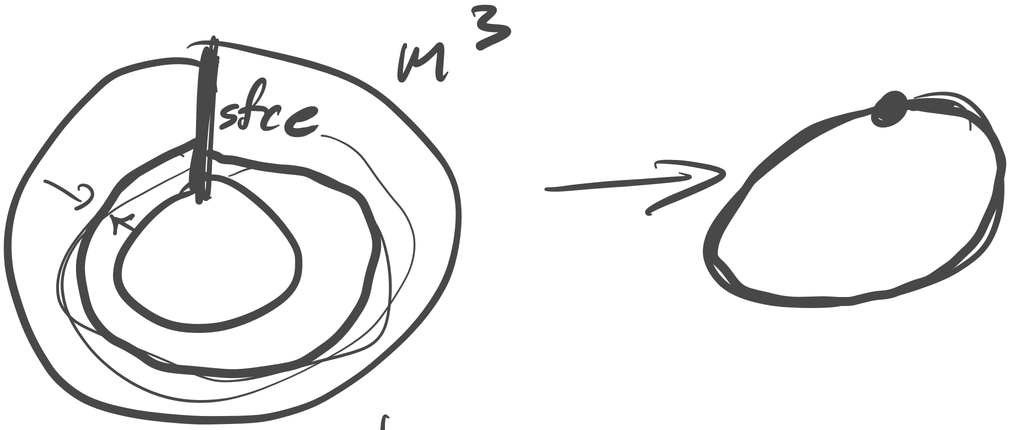
2-mfld "dual" to f.
subsurface $\gamma = f^{-1}(c)$

$\varphi(z)$ measures intersection of a curve with surface γ.

γ represents an elt of $H_2(X)$.

cuz it's a nonseparating surface.

and like before, given a nonseparating surface embedded in M^3 get a map from $M^3 \to S^1$.



an arc in a mfld w/ ∂ is a retract of the manifold.

prop. emb.

elements of $H^1(M^3)$ give me elements of $H_2(M^3)$.

If M is an n-mfld.

elements of $H^1(M^n)$

$\rightsquigarrow$ eles of $H_{n-1}(M^n)$.

(orientable stuff I'm hiding.]

This will actually give an isomorphism

$$\begin{tikzcd} H^1(M^n) \arrow[r] \arrow[d] & H_{n-1}(M^n). \\ \text{(orientable) mfld} & \end{tikzcd}$$

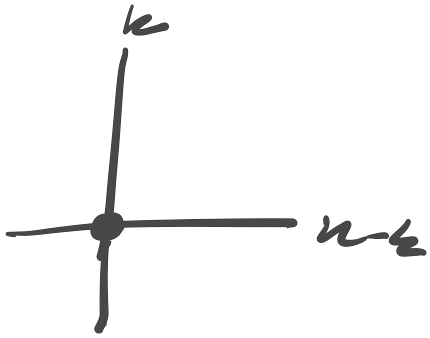
This is a special case of very important theorem

Poincaré Duality Theorem:

If M^n is a closed (no ∂ and compact) orientable n-mfld, then there is a natural isomorphism

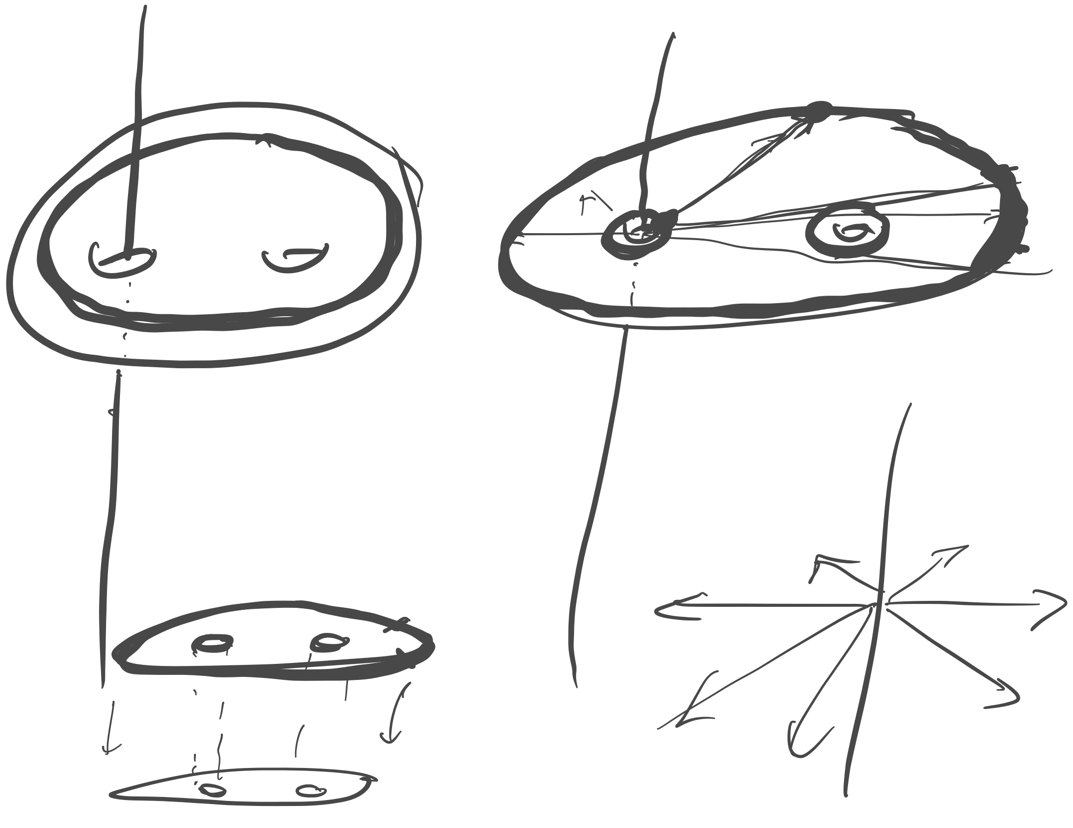
$$H^k(M^n) \cong H_{n-k}(M^n)$$

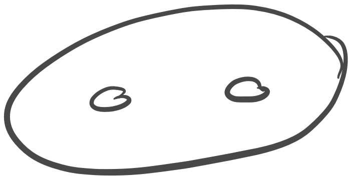
$\forall k$.

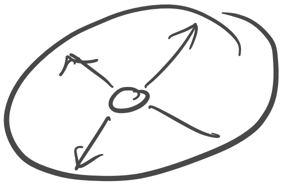


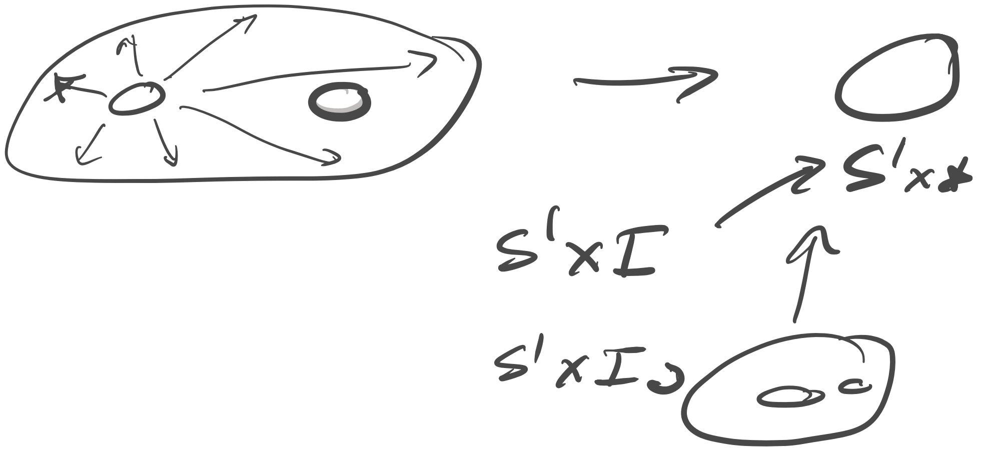
$$B^k \times B^{n-k} \cong \underline{B^n}$$

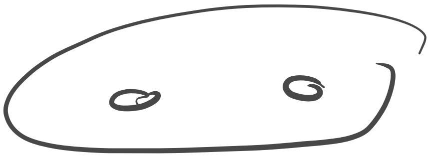
<u>Next time</u>: Some tidying up. Go through a bunch of stuff from Homology & how it shows up in cohomology.

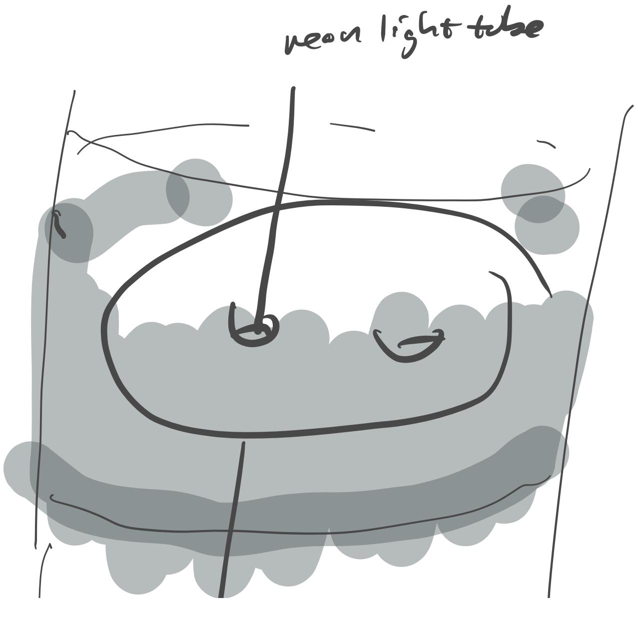







S¹×*
S¹×I
S¹×I ⊃




neon light tube

Cohomology.

Graph. Cell cx of dimension 1. Γ

H^0.

Cellular (simplicial) ch. complex

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

Dualize:

$$0 \leftarrow C_1^* \xleftarrow{\delta} C_0^* \leftarrow 0$$

$$H^0(\Gamma) = \ker(\delta : C_0^* \to C_1^*)$$

If $\varphi \in C_0^*$, what does $\delta\varphi = 0$ mean?

$$\delta\varphi(c) = \varphi(\partial(c)) . = 0$$

σ 1-sx

$\shortparallel$

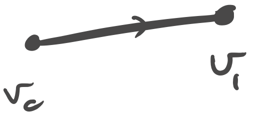
$[v_0, v_1]$

in sing. ch. cx $\shortparallel$

$\sigma : [v_0\, v_1] \to \Gamma$

$\shortparallel$

$\sigma|_{[v_0, v_1]}$



$$\partial\sigma = \partial([v_0, v_1])$$

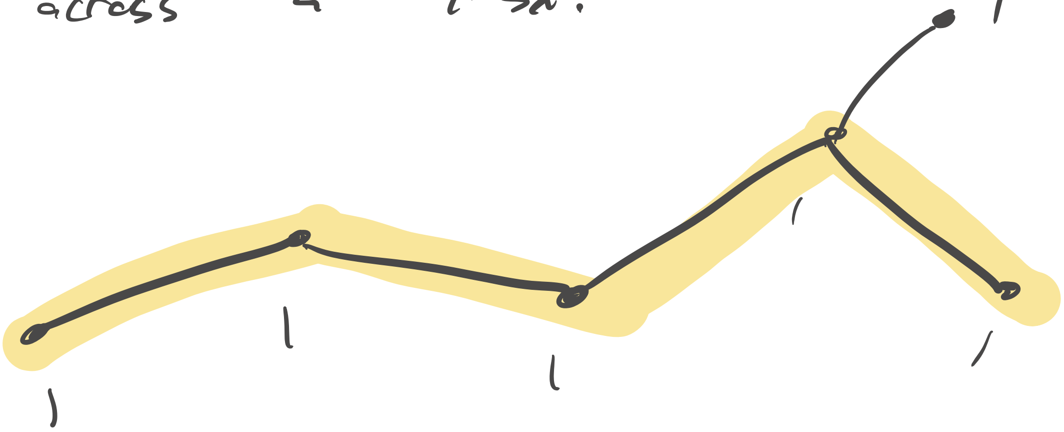
$$= [v_1] - [v_0]$$

$$\varphi(\partial\sigma) = 0 \implies$$

$$\varphi([v_1]) - \varphi([v_0]) = 0$$

Δ

So $\delta\varphi \equiv 0$ means that value of φ doesn't change across a 1-sx.



0-coCycles are the 0-cochains φ such that when restricted to 0-cells in a path component of Γ, φ is constant.

$H^0(\Gamma) \cong \{$ locally constant functions on $\Gamma \}$
$= \{$ functions from the set of cmpts of Γ to $G \}$.

If X space, $\pi_0 X = \{$path components of $X\}$

$$H^0(\Gamma; G) = \{\pi_0 X \to G\} \left(= \langle X, G_{\text{discrete}} \rangle \right)$$

$$= \prod_{\pi_0 X} G \quad .$$

$$H^1(\Gamma; G) \qquad 0 \leftarrow C_1^* \xleftarrow{\delta} C_0^* \leftarrow 0$$

$$\parallel$$

$C_1^*/\text{Im}(\delta) \leftarrow$ what are the coboundaries?

If $\varphi \in C_1^*$, when is there a ψ

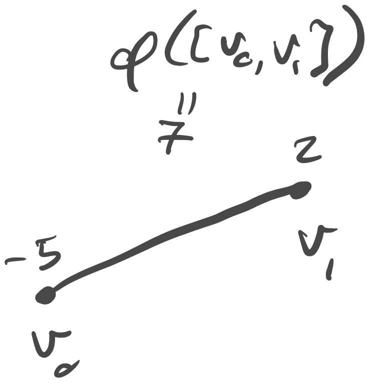
s.t. $\delta\psi = \varphi$? $\;\varphi \to$ 1-cochain

$$\begin{tikzcd} \delta\psi \arrow[d, phantom, "\parallel"] \\ \psi \circ \partial \end{tikzcd}$$

ψ = 0-cochain.

Q is:

Is φ realized as the difference between the values of some 0-cochain ψ across 1-cells?



Does

$$\varphi([v_0,v_1]) = \psi(\partial[v_0,v_1]) = \underbrace{\psi([v_1]) - \psi([v_0])}\ ?$$

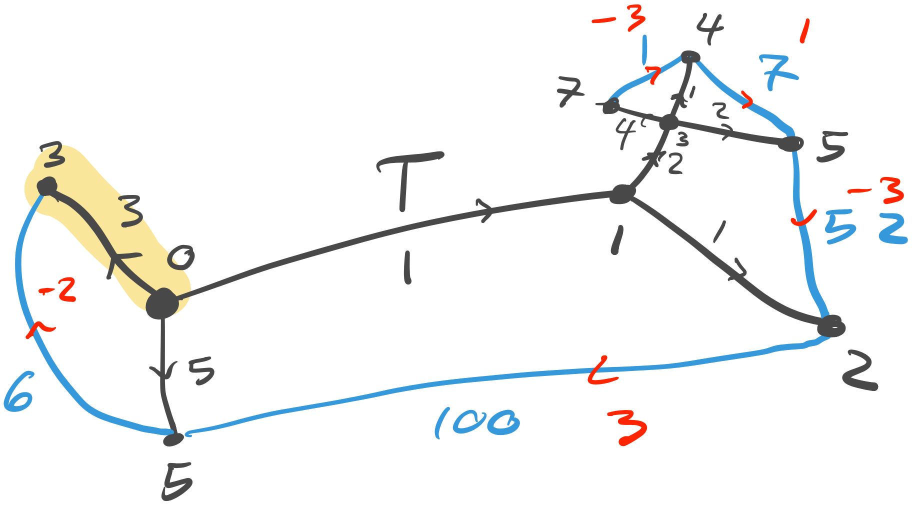
on each cell, sure!

Let's assume Γ is connected.

By Zorn's lemma, $\exists$ maximal tree $T \subset \Gamma$.

So $T \simeq *$ and $\Gamma - T = \sqcup$ intervals.

$\Phi = \delta\psi$



Γ

Differences are wrong across blue edges!

1-cochain Φ chose in nontrivial cohomology class. $\phi = \delta\psi$.

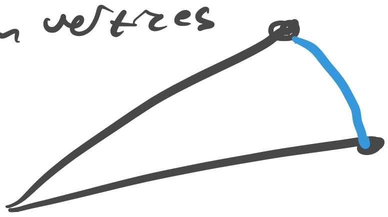
On maximal tree, you can always build ψ s.t. $\phi|_T = \delta\psi|_T$.

start at basepoint: $*$

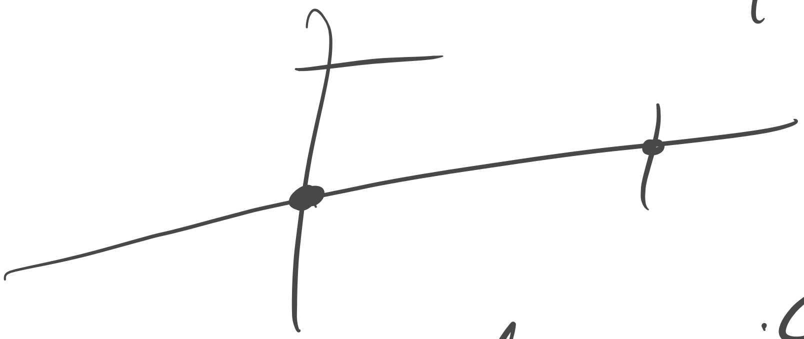
define $\psi(*) = $ anything. 0, say.

Not ambiguous cuz $\exists!$ edge path in T between any two vertices.

But on edges outside of T, the value on edge may not equal difference on vertices



$\varphi = \delta\psi$.



$\psi : C_0 \to G$.

Given φ, can build a ψ s.t. on tree, $\delta\psi = \varphi$.

but, then, $\exists!$ $\underline{\Phi}$ s.t. $\Phi = \delta\psi$

So result is that

$$H^1(\Gamma; G) = \prod_{\lambda \in \Lambda} G$$

where Λ is the set of edges outside of T.

We've shown:

if $\varphi \in C_1^*$, then there is a coboundary Φ that agrees with φ on edges of max. tree T. So $[\varphi] \in H^1$ is represented by $\underbrace{\varphi - \Phi}$.

is 0 on all edges in tree.

and so φ always represented by cochain supported on $\Gamma - T$.

$$H^1(\Gamma) \cong \boxed{\prod_{\lambda \in \Lambda} G}, \text{ where } \Lambda = \{\text{edges not in } T\}.$$

(Γ, T)

$$\underline{\Gamma/T'} \cong \Gamma \cong \underline{\Gamma/T}$$

$$\cong \underbrace{\bigvee_{\lambda \in \Lambda} S^1}$$

Go up a dimension.

Let X be a space.

Let $G = \mathbb{Z}/2\mathbb{Z}$. (supress in notation).

$$0 \leftarrow C_2^* \leftarrow C_1^* \leftarrow C_0^* \leftarrow 0$$

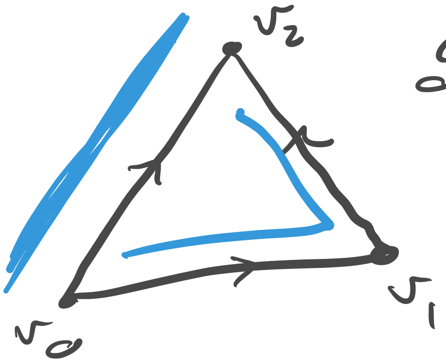
$$H^1(X) = H^1(X; \mathbb{Z}/2\mathbb{Z}).$$

What does it mean to be a cocycle?

$\varphi \in C_1^*$, then φ cocycle

if $$\underset{\substack{\| \\ \varphi \circ \partial}}{\delta \varphi} = 0$$

Consider a 2-cell $\sigma = [v_0, v_1, v_2]$



$$\partial \sigma = \underline{[v_1, v_2]} - \underline{[v_0, v_2]} + \underline{[v_0, v_1]}$$

$$= \sum_{i=0}^{2} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_1]$$

$\delta\varphi(\sigma) = \varphi(\partial\sigma) = 0$

$\Rightarrow \quad 0 = \varphi[v_1, v_2] - \varphi[v_0, v_2] + \varphi[v_0, v_1].$

cocycle cond. true for all σ

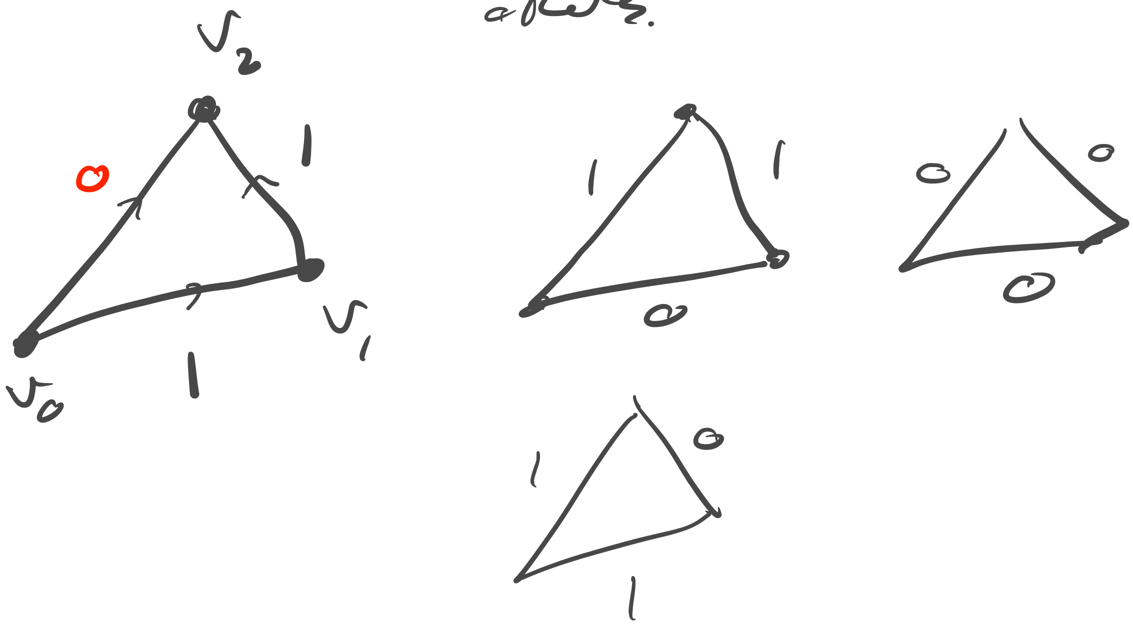
Cocycle condition.

i.e. φ is "additive":

$$\varphi([v_0, v_2]) = \varphi([v_0, v_1]) + \varphi([v_1, v_2])$$

→ value here determined by others.

Mod 2,

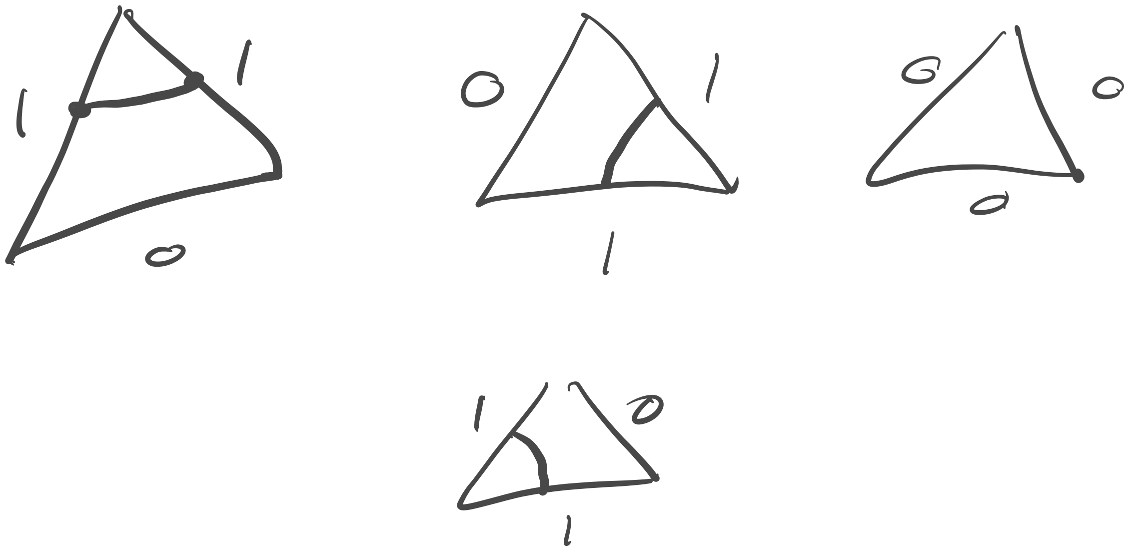


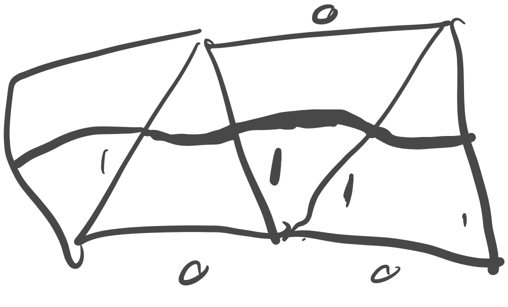
In all cases, the additivity is exactly the condition that exactly 0 or 2 values of φ on $\partial\sigma$ are 1.

(important that $G = \mathbb{Z}/2\mathbb{Z}$.)

Geometric picture:

use φ to build a 1-dim^l submfld.





since X sfce,
only ever 2 2-cells
incident to edge.
get 1-mfld.

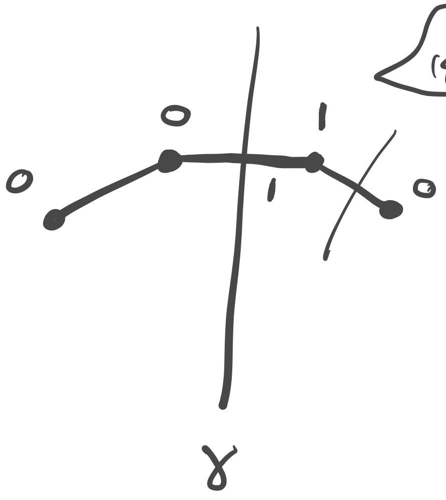
1-cocycle gives you a 1-mfld in X. (cohomologous cocycles $\to$ homologous curves).

What if $\varphi = \delta\psi$.

So ψ 0-cochain and value of φ across edge is difference of the values of ψ.

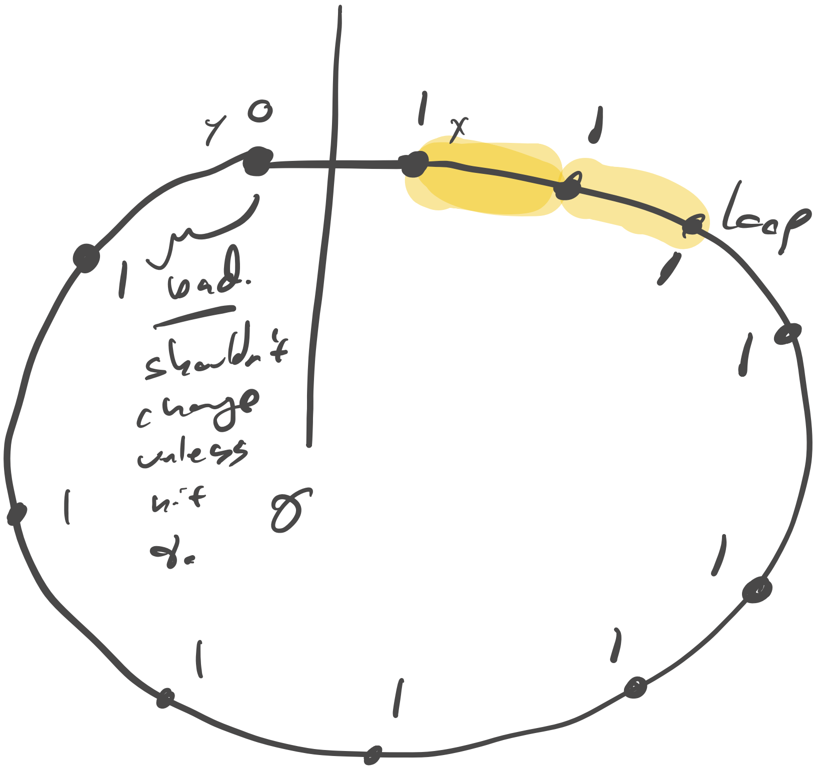
Claim: 1-mfld γ associated to φ is separating cuts the surface into two pieces, on one piece, $\psi = 1$, on the other, $\psi = 0$.

$\boxed{\text{if } \varphi = \delta\psi}$



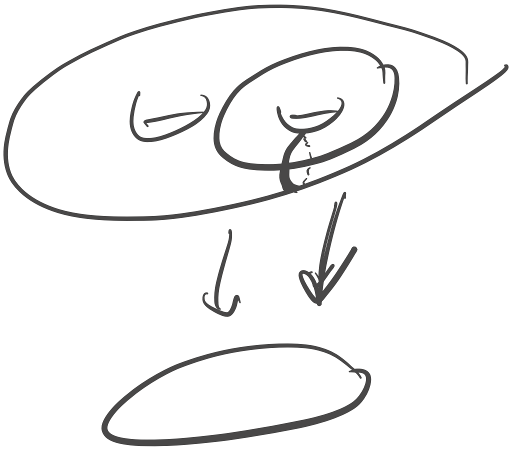
γ hits edges where $\varphi = 1$.
doesn't hit $\varphi = 0$ edges

Each time we cross γ, value on vertices changes.

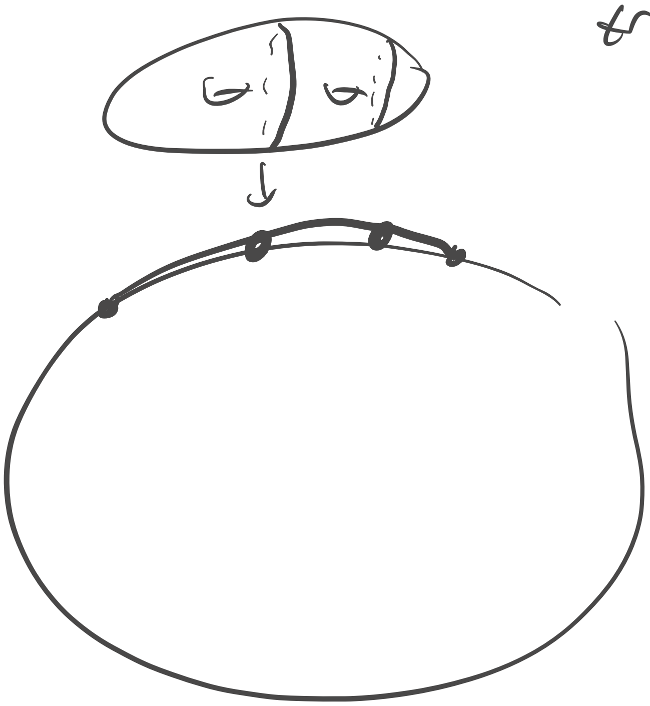


Suppose this loop only intersects γ in a ~~that~~ single point.

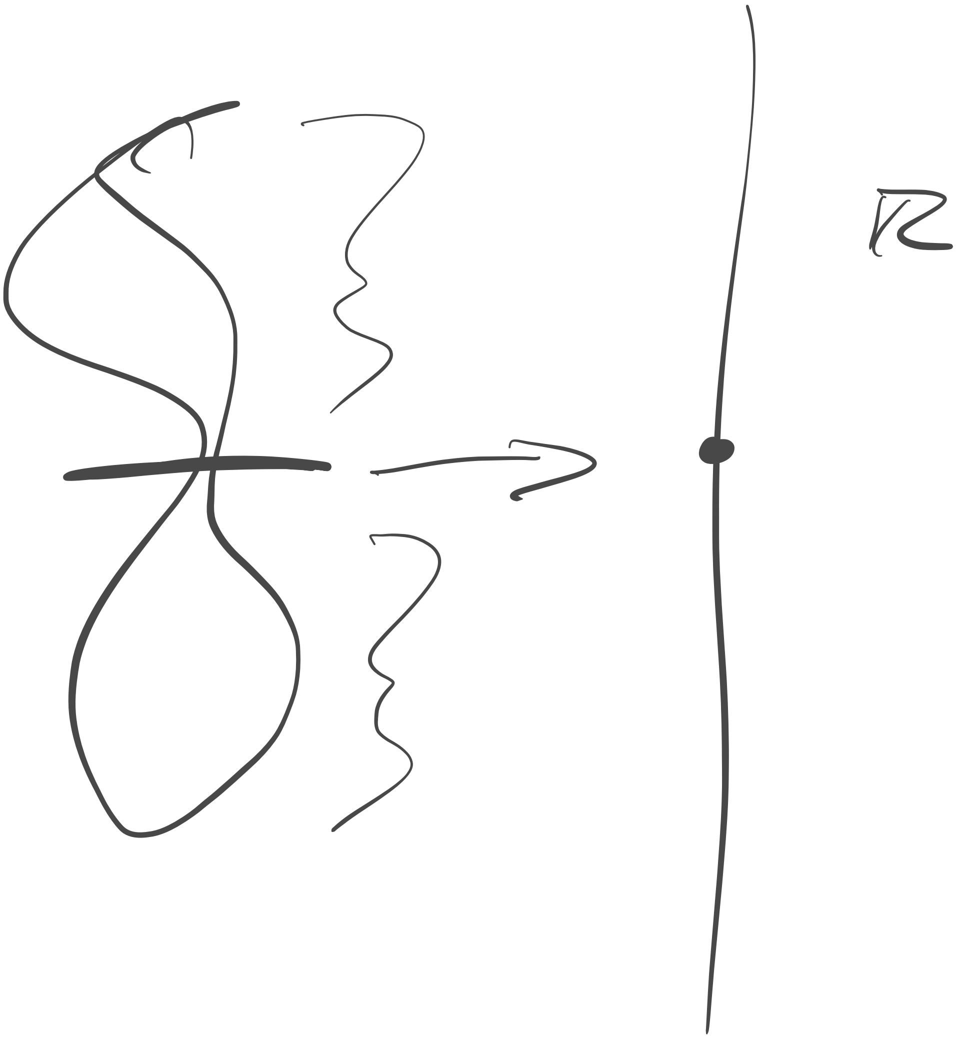
So conclude γ separating.



This is like building the dual curve given by $X \to S^1$ that induces φ.

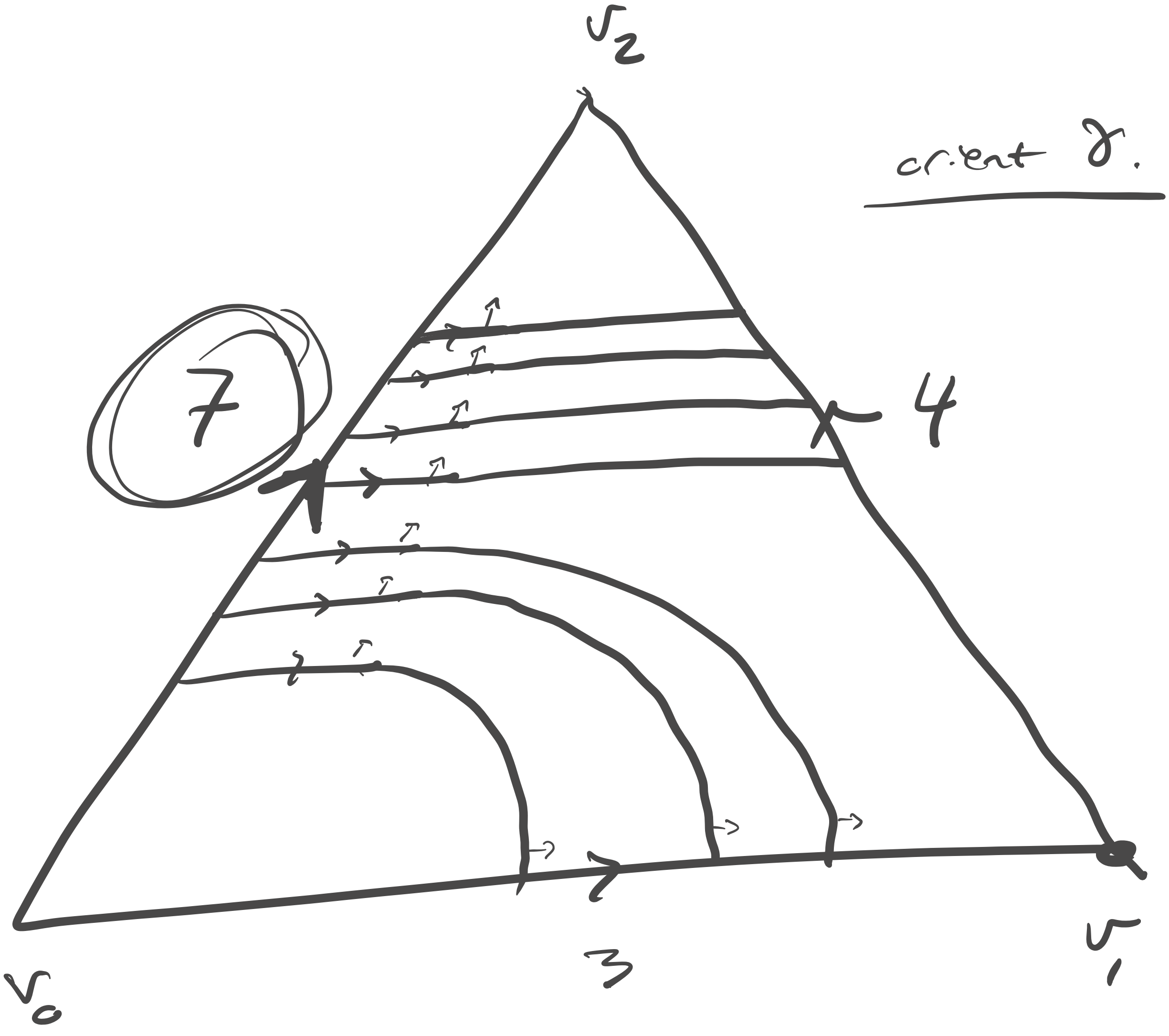


trivial homo. $\pi_1 X \to \mathbb{Z}$,
take nullhtpc map.
pullback of reg. value
will be separating.



What if we're brave?
let $G = \mathbb{Z}$.

orient γ.



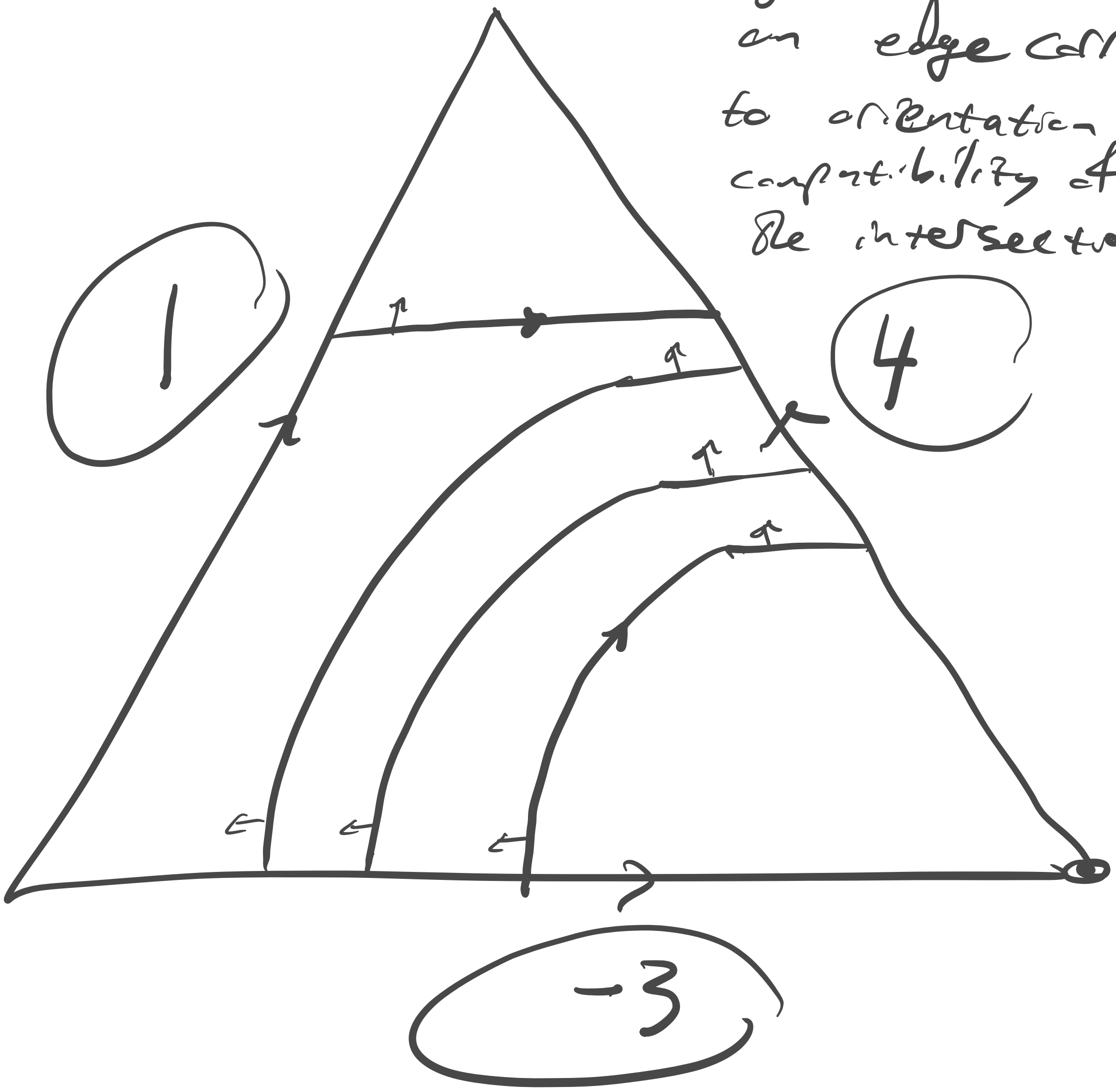
$$\varphi[v_0, v_2] = \varphi[v_0, v_1] + \varphi[v_1, v_2]$$

COCYCLE!

Build a dual curve!

also could pick orientation on γ $\longrightarrow$

sign of value on edge corresp. to orientation compatibility of the intersection.



Value ~~of~~ of ϕ on edge is

the oriented intersection # of γ with

that edge. $\phi(\sigma) = i(\gamma, \sigma)$.

$\varphi(\alpha) = i(\gamma, \alpha).$

Also: If φ coboundary, then γ separating.

---

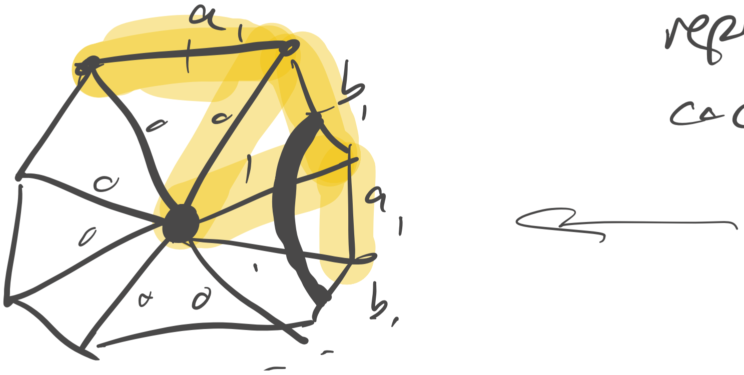
and each complementary component corresponds to a value of ψ if $\varphi = \delta\psi$

When we look at cup product, we will use this interpretation to build a cochain representing a cohomology class.

Given

$\longrightarrow \mathbb{Z}.$

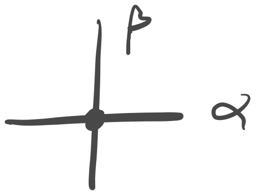
how do I build representative cochain?



for oriented curves α, β on a surface.

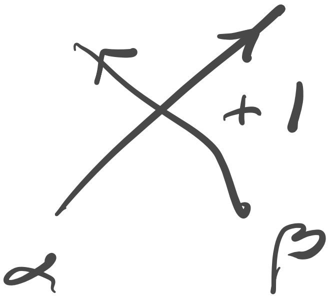
define $i(\alpha, \beta)$ as follows:

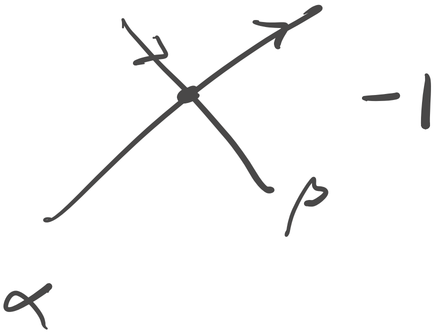
(htpe α, β so that every intersection point is transverse:



)

to compute $i(\underline{\alpha}, \beta)$.

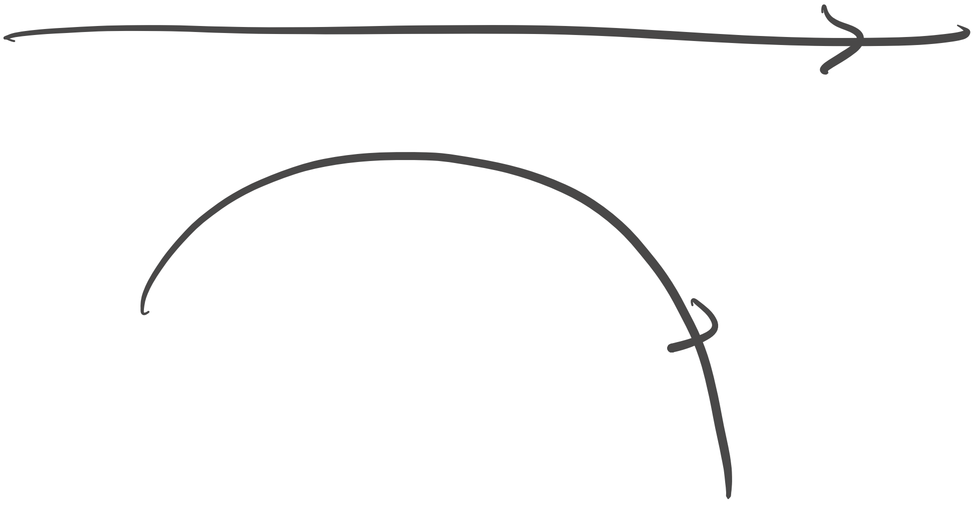


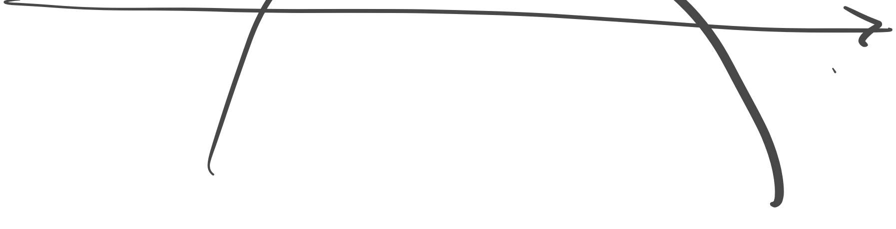


At an intersection point, ±1 contributed to $i(\alpha, \beta)$ according to right hand rule.

local intersection # $i_x(\alpha,\beta)$ at intersection pt. x is ± 1 depending on picture.

$$i(\alpha,\beta) = \sum_{x\in\alpha\cap\beta} i_x(\alpha,\beta)$$





Cohomology of Spaces, and carrying over stuff from Homology. G ab. gp.

X space. $C_n(X)$ singular n-chains.

$$C^n(X;G) = \mathrm{Hom}(C_n(X), G) = C_n^*.$$

$$\Delta^{n+1} = [v_0, v_1, \dots, v_{n+1}]$$

Singular $(n+1)$-simplex is cont. map

$$\sigma : \Delta^{n+1} \longrightarrow X.$$

If $\varphi \in C^n(X;G)$, then

$$\begin{aligned}\delta\varphi(\sigma) &= \varphi(\partial\sigma) \\ &= \varphi\Big(\sum_{i=0}^{n+1} (-1)^i \sigma\Big|_{\underbrace{[v_0, \dots, \hat{v}_i, \dots, v_{n+1}]}_{\text{forget } i\text{th vx.}}}\Big)\end{aligned}$$

$$= \sum_{i=0}^{n+1} (-1)^i \varphi\left(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{n+1}]}\right)$$

$\delta\varphi$ defined on C_n by ext. linearly.

Reduced Cohomology.

Reduced singular chain cx $\tilde{\mathcal{C}}$ is

$$\xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

ε augmentation map

sends chain to sum of its coeffs.

i.e. $$\varepsilon\left(\sum_{i=1}^{k} c_i \sigma_i\right) = \sum_{i=1}^{k} c_i .$$

$\sigma_i : \Delta^0 \to X$

Reduced cohomology is homology of $\tilde{C}^*$. So we hom $\tilde{C}$ into G and take homology.

$\tilde{H}^n(X;G) = H^n(X;G)$ when $n > 0$

UCT tells us that

$$\tilde{H}^0(X;G) = \mathrm{Hom}(\tilde{H}_0(X), G)$$

Saw:

$H^0(X;G)$ is the gp of functions $X \to G$ constant on path components, i.e

$$\underbrace{\mathrm{Hom}(\pi_0 X, G)}_{\text{functions from set of cmp'ts to } G}.$$

What's $\tilde{H}^0(X;G)$?

Think about augmentation. If σ sx.

$$\varepsilon(\sigma) = 1$$

$$\varepsilon^* : \mathrm{Hom}(\mathbb{Z}, G) \to \mathrm{Hom}(C_0, G).$$

If $\varphi \in \mathrm{Hom}(\mathbb{Z}, G)$

ε^* takes φ to the composition

$$\begin{tikzcd} C_0(X) \arrow[r, "\varepsilon"] \arrow[rr, bend right, "\varepsilon^*(\varphi)"'] & \mathbb{Z} \arrow[r, "\varphi"] & G \end{tikzcd}$$

$\varepsilon^*(\varphi)$ take every single sing. sx σ to $\varphi(1)$.

So $\varepsilon^*(\varphi)$ is a constant function on $X = \{$singular 0-simplices$\}$.

$\varepsilon^*(\mathbb{Z})$ is set of constant functions $X \to G$.

So $\tilde{H}^0(X;G) = \ker\delta / \operatorname{Im}\varepsilon^*$

$$= \underset{\substack{\text{locally const} \\ \text{functions} \\ X \to G}}{\underset{\shortparallel}{H^0(X;G)}} \Big/ \text{globally constant functions.}$$

---

reduced gps useful in computations (usually when you're computing cohomology using some LES.)

Relative Gps.

Pair (X, A).

Want rel. gps $H^n(X,A;G)$.

To do that:

Dualize the SES

$$0 \longrightarrow \underbrace{C_n(A)}_{\substack{\shortparallel \\ \{\text{lin. combos} \\ \text{of } \sigma:\Delta \to A\}}} \xrightarrow{i} \underline{C_n(X)} \xrightarrow{j} \underbrace{C_n(X,A)}_{C_n(X)/C_n(A)} \to 0$$

get

$$\begin{tikzcd}
 & {} & & {} & & {} & \\
0 & \underline{C^n(A;G)} \arrow[l] \arrow[u, "\delta"'] & & \underline{C^n(X;G)} \arrow[ll, "i^*"'] \arrow[u, "\delta"'] & & C^n(X,A;G) \arrow[ll, "j^*"'] \arrow[u, "\delta"'] & 0 \arrow[l] \\
 & {} \arrow[u, "\delta"'] & & {} \arrow[u, "\delta"'] & & \left(C_n(X)/C_n(A)\right)^* \arrow[u, equal] & \\
 & & & & & {} \arrow[u, "\delta"'] &
\end{tikzcd}$$

Remember! not immediate that the dual sequence is exact. Exactness may fail at left.

We only <u>need</u> to check exactness at left. We look at right a little too for clarity.

Dual map i^* is dual of inclusion

so i^* just restricts a cochain to A.

Why is i^* surjective?

take $\varphi : C_n(A) \to G$

can extend to $\Phi : C_n(X) \to G$,

by just defining $\Phi(\sigma) = 0$ if

$\sigma : \Delta^n \to X$ doesn't take Δ^n into A.

So $\Rightarrow$ i^* surjective. ✓

The kernel of i^* $\big($= restriction to $C_n(A)\big)$

If you're in $\ker(i^*)$ then you vanish on all simplices that land in A.

But! the homomorphisms $C_n(X) \to G$ that vanish on $C_n(A)$ are in 1-1 correspondence w/ homomorphisms

$$C_n(X)/C_n(A) \to G.$$

$$\ker(i^*) = C^n(X; A; G)$$

thought of as a subset of $C^n(X)$:

the subset of cochains vanishing on $C_n(A)$.

Relative Cohomology.

$$\begin{tikzcd} \delta : C^n(X, A; G) \arrow[r] \arrow[d, phantom, sloped, "\subset"] & C^{n+1}(X, A; G) \arrow[d, phantom, sloped, "\subset"] \\ C^n(X) \arrow[r, "\delta"] & C^{n+1}(X) \end{tikzcd}$$

restriction of

Define $H^n(X, A; G)$ using this chain cx.

Now, i, j commute w/ ∂,

so i^*, j^* " " δ.

The SES above is part of SES of Chain complexes.

So $\exists$ LES in CoHomology:

$$\to H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \to$$

(and a reduced version.)

$$\hat{\varphi}\circ\partial \leftarrow (j^*)^{-1}(\hat{\varphi}\circ\partial)$$

$$\begin{tikzcd}
0 & C^{n+1}(A;G) \arrow[l] & C^{n+1}(X;G) \arrow[l, "i^*"'] & C^{n+1}(X,A) \arrow[l] & 0 \arrow[l] \\
0 & C^{n}(A;G) \arrow[l] \arrow[u, "\delta"'] & C^{n}(X;G) \arrow[l, "i^*"'] \arrow[u, "\delta"'] & C^{n}(X,A;G) \arrow[l, "j^*"'] \arrow[u, "\delta"'] & 0 \arrow[l] \\
 & \varphi & \hat{\varphi} & &
\end{tikzcd}$$

$\hat{\varphi}\circ\partial = \widehat{\underline{\varphi}}\circ j$ since φ cocycle.

So $\hat{\varphi}\circ\partial \in C^{n+1}(X,A)$, i.e. vanishes on A.

h restricts to relative cycles $Z^{n+1}(X,A)$.
= cycles whose ∂ is in A.

So on $Z^{n+1}(X,A)$, $\hat{\varphi}\circ\partial = \varphi\circ\partial = \partial^*\circ h(\varphi)$

$$\begin{tikzcd}
H^n(A) \arrow[r, "\delta"] \arrow[d, "h"'] \arrow[dr, phantom, "\circ"] & H^{n+1}(X,A) \arrow[d, "h"] \\
\mathrm{Hom}(H_n A, G) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_{n+1}(X,A), G)
\end{tikzcd}$$

Induced homomorphisms:

$f : X \to Y$ map.

$f_\# : C_n(X) \longrightarrow C_n(Y)$.

σ $\qquad f_\#(\sigma) = f \circ \sigma$.

$\sigma : \Delta^n \to X$ $\qquad f \circ \sigma : \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$.

Dual map:

$f^\# : C^n(Y; G) \longrightarrow C^n(X; G)$

pull back functions using f.

$f_\# \partial = \partial f_\# \implies \delta f^\# = f^\# \delta$

(chain map!)

So we get a induced map on

$f^* : H^n(Y; G) \longrightarrow H^n(X; G)$.

relative:

$f: (X, A) \to (Y, B)$ map of pairs

$\rightsquigarrow f^*: H^n(Y, B; G) \to H^n(X, A; G)$

and furthermore induce map between LESs of pairs.

Functoriality.

$(fg)^\# = g^\# f^\#$

$\mathbb{1}^\# = \mathbb{1}$

Dualizing

$(fg)_\# = f_\# g_\#$

$\mathbb{1}_\# = \mathbb{1}$.

$\Rightarrow (fg)^* = g^* f^*$ and $\mathbb{1}^* = \mathbb{1}$.

UCT Holds for relative grps.

and $f:(X,A)\to(Y,B)$

then $\exists$ a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X,A),G) \arrow[r] & H^n(X,A;G) \arrow[r] & \mathrm{Hom}(H_n(X,A),G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(Y,B),G) \arrow[r] \arrow[u, "(f_*)^*"] & H^n(Y,B;G) \arrow[r] \arrow[u, "f^*"] & \mathrm{Hom}(H_n(Y,B),G) \arrow[r] \arrow[u, "(f_*)^*"] & 0
\end{tikzcd}$$

Htpy invariance.

Proof for homology dualizes.

$$\underbrace{f \simeq g}_{\text{htpc as maps of pairs}} : (X,A) \to (Y,B)$$

then $f^* = g^*$.

Excision. $Z \subset A \subset X$

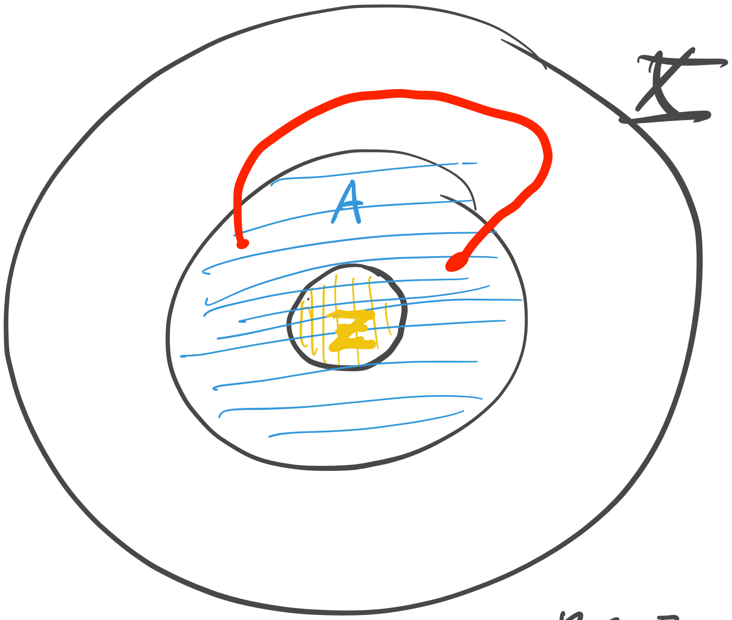
$\overline{Z} \subset \mathring{A}$,

then

$i : (X - Z, A - Z) \to (X, A)$ induces

isomorphisms

$$\begin{tikzcd}
H^n(X, A; G) \arrow[d, "i^*"] \\
H^n(X - Z, A - Z; G)
\end{tikzcd}$$

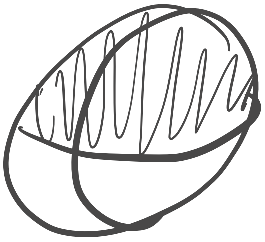


$\forall n$.

eg. $H^n(S^2, \text{Northern Hemisphere})$

$\cong$

$H^n(D^2, \partial D^2)$.



we had this for homology but how to we move it over to cohomology?

Pf: Excision for Homology

+ naturality of UCT

+ 5-Lemma.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X,A),G) \arrow[r] & H^n(X,A;G) \arrow[r] & \mathrm{Hom}(H_n(X,A),G) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & \mathrm{Ext}(H_{n-1}(X-Z,A-Z),G) \arrow[r] \arrow[u, "\cong"', "(i_*)^*"] & H^n(X-Z,A-Z;G) \arrow[r] \arrow[u, "i^*"'] & \mathrm{Hom}(H_n(X-Z,A-Z),G) \arrow[r] \arrow[u, "\cong"', "(i_*)^*"] & 0 \arrow[u]
\end{tikzcd}$$

By excision for homology, the i_* maps are isomorphisms. So their duals are.

5-lemma tells us (since vert map on left is epimorphism & vert. map on right monomorphism) that i^* is $\cong$.

Cohomology has axioms

(we'll use them later.)

Mayer-Vietoris

$X = \mathring{A} \cup \mathring{B}$

then ∃ LES

$$\to H^n(X;G) \xrightarrow{\Psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\Phi} H^n(A \cap B;G) \to H^{n+1}(X;G) \searrow$$

(also all the other ch. cxs.

cellular cohomology

simplicial cohomology

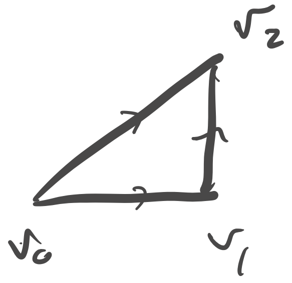
CW cohomology)

<u>Next</u>: New stuff:

Cup product. (Cohomology is a ring!)

If G were a ring, then

we can "multiply cochains pointwise"



I don't have a 2-cochain.

But I do have some 1-cochains, φ, ψ, say.

$\varphi, \psi \in C^1$.

$\rightsquigarrow \Phi \in C^2$?

$[v_0, v_1, v_2]$

[redacted]

$$\Phi([v_0,v_1,v_2]) = \underbrace{\varphi[v_0,v_1]\cdot\psi[v_1,v_2]}_{\text{multiplying in my } \underline{\text{ring}} \ G.}$$

$\parallel$

$\varphi \smile \psi$.

Cup product. $\rightsquigarrow H^*(X; \underset{\substack{\uparrow \\ \text{ring}}}{R}) = \bigoplus H^n$ into a ring.

$$\varphi \smile \psi = \pm \psi \smile \varphi.$$

Cup Product.

Assume that our coefficient gp is a ring R. (usually a nice R).

(There are important situations where R is non-commutative. Most important example is $R = \mathbb{Z}\Gamma$ for some nonabelian gp Γ

$$\mathbb{Z}\Gamma = \{\text{formal lin. combos of eles of } \Gamma\}.$$

$$H^n(M; \mathbb{Z}\pi_1 M).\ )$$

Let X be a space w/ singular ch. complex.

If $\varphi \in C^k(X;R)$ $\swarrow$ k cochains into R

and $\psi \in C^\ell(X;R)$,

the *cup product* $\varphi \cup \psi$ of φ and ψ is a $(k+\ell)$-cochain whose value on

$$\sigma : \Delta^{k+\ell} = [v_0, \dots, v_{k+\ell}] \to X$$

is

$$\varphi \cup \psi(\sigma) = \underbrace{\varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})}_{\text{only multiplication in } R}.$$

Give us a product map $\cup$

$$\smile : C^k(X;R) \times C^\ell(X;R) \to C^{k+\ell}(X;R).$$

This induces a map on cohomology by the following lemma.

Lemma

$$\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^k \varphi \smile (\delta\psi)$$

whenever $\varphi \in C^k$ and $\psi \in C^\ell$.

Pf: let $\sigma : \Delta^{k+\ell+1} \to X$,

a) $(\delta\varphi) \smile \psi(\sigma)$

$$= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

b) $(-1)^k \varphi \smile \underbrace{(\delta \psi)}(\sigma)$

$$= \sum_{i=k}^{k+l+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+l+1}]})$$

If we add these together, the last term of a)

(namely

$$(-1)^{k+1} \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}))$$

and first term of b):

$$(-1)^k \varphi(\sigma|_{[v_0,\dots,v_k]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

cancel.

So we're left with

$$\sum_{i=0}^{k} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\, \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

$$+\sum_{i=k+1}^{k+\ell+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]})$$

$$=(\varphi\smile\psi)(\partial\sigma) = \delta(\varphi\smile\psi). \quad \square$$

---

if φ and ψ are cocycles,

$$\delta\varphi=0, \ \delta\psi=0$$

then

$$\delta(\varphi\smile\psi)= \delta\varphi\smile\psi+(-1)^k\varphi\smile\delta\psi$$
$$= 0\smile\psi+(-1)^k\varphi\smile 0$$
$$=0.$$

so $\varphi\smile\psi$ also a cocycle.

Also, if either of them is a coboundary, then cup product is a coboundary.

To see that. If $\delta\varphi = 0$ (φ cocycle)
ψ any cochain

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi \pm \varphi \cup \delta\psi$$

$$= \underbrace{0 \cup \psi}_{0} \pm \varphi \cup \delta\psi$$

$$= \pm\, \varphi \cup \underbrace{\delta\psi}_{\text{arbitrary coboundary.}}$$

ψ arbitrary.

For any cocycle φ and any cochain ψ
have $\delta(\varphi \cup \psi) = \pm \varphi \cup \underbrace{\delta\psi}_{\text{arbitrary coboundary.}}$

So if φ cocycle and Ψ is coboundary

$$\varphi \cup \underset{\delta\psi}{\underset{\shortparallel}{\Psi}} = \delta(\Phi) = \pm\delta(\varphi \cup \psi)$$

$$= \delta(\pm \varphi \cup \psi).$$

So! $\smile$ descends to a map

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R)$$

Associative and Distributive since it's
" " " at cochain level, since

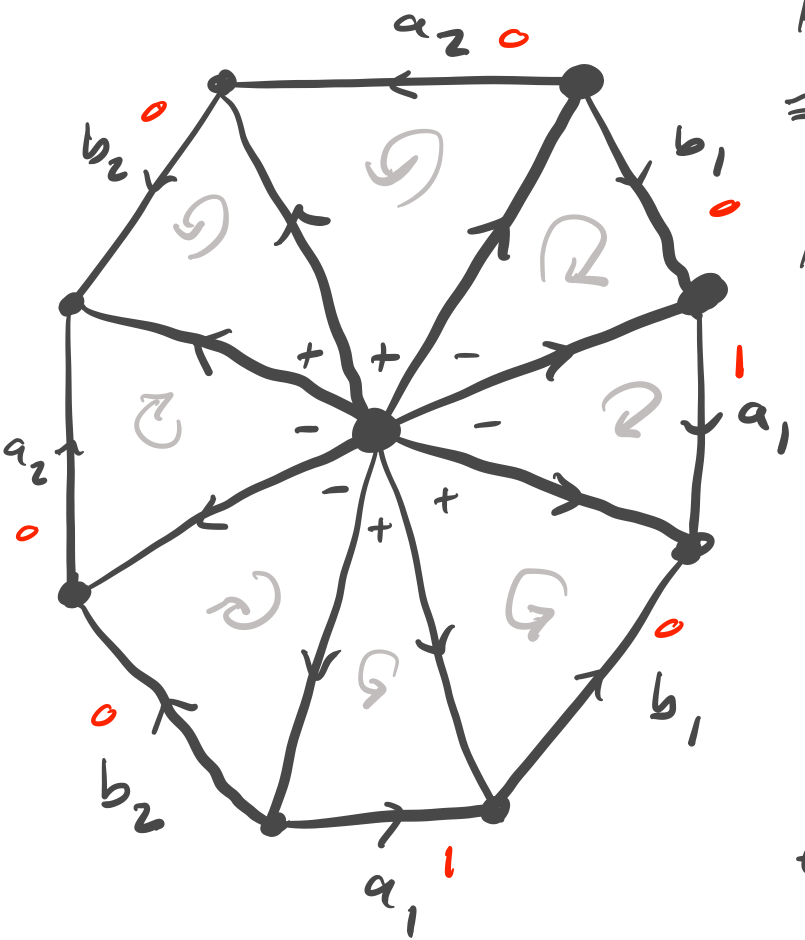
R is assoc. & distrib.

$R = 2\mathbb{Z}$. no 1. rngs.

If R has a 1, then $\smile$ has an identity, namely $1 \in H^0(X;R)$, the cocycle that takes every 0-simplex to 1.

Fact: Canonical isos between simplicial and singular cohomology respect cup product.

Example. M closed orientable surface of genus 2.



$H_1(M)$

$\cong \mathbb{Z}a_1 \oplus \mathbb{Z}b_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}b_2$.

$H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbb{Z})$

$\cong \mathbb{Z}_{a_1}^* \times \mathbb{Z}_{b_1}^* \times \mathbb{Z}_{a_2}^* \times \mathbb{Z}_{b_2}^*$

$\Vert$	$\Vert$
gen by α_1 takes $a_1 \mapsto 1$ and other gens to 0	gen. by β_1 takes $b_1 \mapsto 1$ others ~~gen.~~ to 0.

$\boxed{\Delta\text{-cx.}}$ $\boxed{\text{only 2 vxs}}$ $\alpha_i(a_i) = 1$ $\alpha_i(a_{3-i}) = 0$

$$\alpha_i(b_j) = 0$$

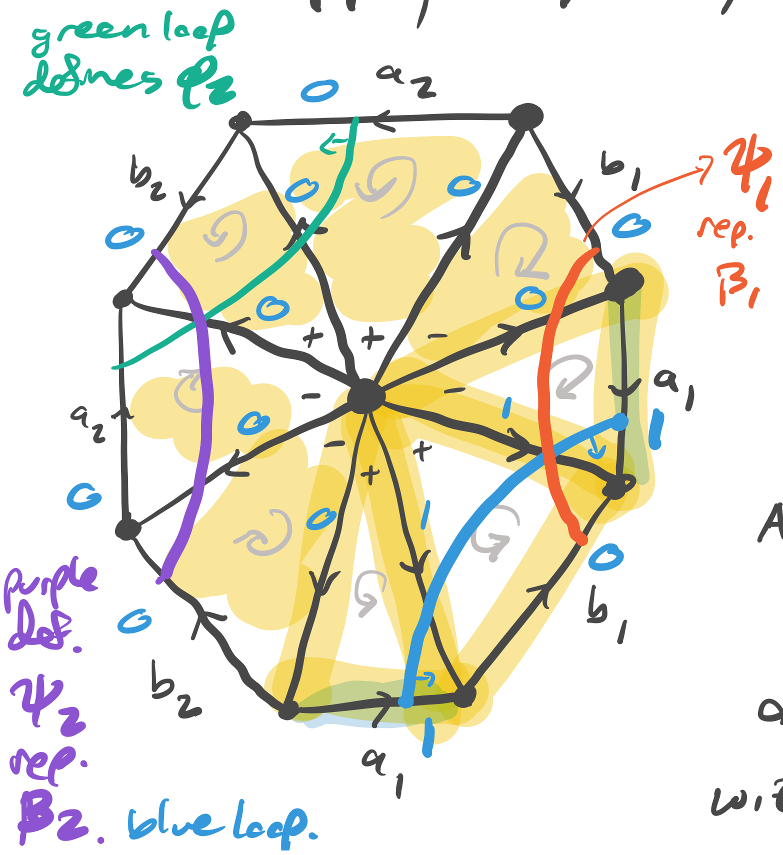
$$\beta_i(b_i) = \quad \beta_i(b_{3-i}) = 0$$

$$\beta_i(a_j) = 0.$$

to compute cup product, we want to represent α_i, β_i w/ 1-cochains.

Build cochain that's cocycle representing these homomorphisms.

Want φ_1, cocycle, representing α_1.



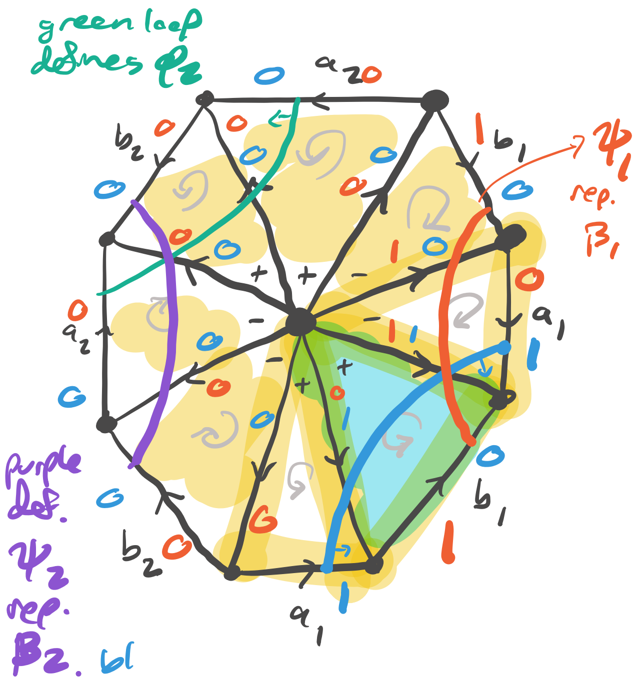
Remember: being a cocycle means that it's additive on edges of the oriented 2-simplices.

Also remember, we have a geometric interpretation of α_1 as intersection # with a 1-mfld.

Assign values of φ_1 according to intersection #s with blue loop.

That defines φ_1 and it's a cocycle.

green loop defines φ_2



$$\varphi_1 \smile \psi_1([v_0, v_1, v_2])$$

$$= \boxed{\varphi_1([v_0, v_1])}\boxed{\psi_1([v_1, v_2])}$$

If $[v_0, v_1, v_2]$ SE corner.

$\varphi_1([v_0, v_1]) = 1$ } edge int. blue loop.

$\psi_1([v_1, v_2]) = 1$ } edge int. orange loop

So for SE corner

$$\varphi_1 \smile \psi_1([v_0, v_1, v_2]) = 1$$

on eastern sx, $\varphi_1([v_0, v_1]) = 0$ $\psi_1([v_1, v_2]) = 0$

$$\varphi_1 \smile \psi_1([v_0, v_1, v_2]) = 0.$$

$$\varphi_1 \smile \psi_1([v_0, v_1, v_2]) = \begin{cases} 1 & \text{if } [v_0, v_1, v_2] \text{ SE sx} \\ 0 & \text{otherwise.} \end{cases}$$

If C is the 2-chain that's just sum of sxs in the picture,

$\partial C = 0$. So C 2-cycle.

In fact it generates $H_2(M)$:

$$\underbrace{(\varphi_1 \smile \psi_1)(C) = 1 \text{ generates } \mathbb{Z}.}_{\Rightarrow\ C \text{ generates } H_2(M) = \mathbb{Z}.}$$

$$\underbrace{\text{So } \varphi_1 \smile \psi_1 \text{ generator of } H^2(M;\mathbb{Z})}_{\text{(foreshadowing Poincaré Duality)}}$$

Similarly: can compute

$$\alpha_i \smile \beta_j = 0 \quad \text{if } i \neq j.$$

$$\underbrace{\alpha_i \smile \beta_j = -\beta_j \smile \alpha_i}_{\smile \text{ not commutative.}}$$

$$\underbrace{\alpha_i \cup \alpha_j = 0 \qquad \beta_i \cup \beta_j = 0}_{\text{Dual curves are disjoint!}}$$

The α_i, β_i generate H^1 so this determines all cup products. $\square$

$\cup$ not commutative

says $\underbrace{\alpha \cup \beta = \pm \beta \cup \alpha}_{\text{this is the worst noncomm. can be (later.)}}$

Also see $\underbrace{\alpha^2 = 0. \quad \beta^2 = 0.}_{\text{next time: always true on } H^1.}$

Cup Products.

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R)$$

Relative: (suppress coeffs R)

$$H^k(X) \times H^l(X,A) \xrightarrow{\smile} H^{k+l}(X,A)$$

$$H^k(X,A) \times H^l(X) \xrightarrow{\smile} H^{k+l}(X,A)$$

$$H^k(X,A) \times H^l(X,A) \xrightarrow{\smile} H^{k+l}(X,A).$$

since: if φ or ψ vanishes on $C_n(A)$

then so does $\varphi \smile \psi$.

$$H^k(X,A) \times H^l(X,B) \xrightarrow{\smile} H^{k+l}(X,A\cup B).$$

Prop. If $f: X \to Y$ then

$$f^*: H^n(Y;R) \to H^n(X;R)$$

satisfies $\quad f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta.$

Pf $\quad f^\# \varphi \smile f^\# \psi = f^\# \varphi \smile \psi$ since

$$(f^\# \varphi \smile f^\# \psi)(\sigma)$$

$$= f^\# \varphi(\sigma|_{[v_0, \dots, v_k]}) \cdot f^\# \psi(\sigma|_{[v_k, \dots, v_{k+l}]})$$

$$= \varphi(f \circ \sigma|_{[v_0, \dots, v_k]}) \psi(f \circ \sigma|_{[v_k, \dots, v_{k+l}]})$$

$$= (\varphi \smile \psi)(f \circ \sigma) = f^\#(\varphi \smile \psi)(\sigma). \quad \square$$

Consequence:

Recall: $\quad H^1(X; \mathbb{Z}) = \text{Hom}(\pi_1 X, \mathbb{Z})$

$$= \langle X, S^1 \rangle.$$

So every element of $H^1(X; \mathbb{Z})$ realizes by $f : X \to S^1$

$$\pi_1(X) \to \pi_1(S^1) = \mathbb{Z} = H^1(S^1; \mathbb{Z}).$$

$$H^1(S^1;\mathbb{Z}) = \langle S^1, S^1 \rangle$$

$$\shortparallel$$

$$\mathrm{Hom}(\pi_1(S^1), \pi_1(S^1))$$

$$\shortparallel$$

$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$. $\Rightarrow$ generated by $\mathbb{1}$.

fundamental class $\alpha \in H^1(S^1;\mathbb{Z})$ rep. $\mathbb{1} : H_1 \to \mathbb{Z}$.

So if $\varphi \in H^1(X;\mathbb{Z})$ is realized

by map $f : X \to S^1$

then $\boxed{\varphi = f^*\alpha.}$

Homom. $\varphi : \pi_1 X \to \mathbb{Z}$

$$\begin{tikzcd}
 & \pi_1(S^1) \\
\pi_1(X) \arrow[r, "f_*"] & \pi_1(S^1) \arrow[u, "\alpha"]
\end{tikzcd}$$

$\Big\}$ not doing $\mathbb{1}$.

So <u>in short.</u> every elt of $H^1(X;\mathbb{Z})$ is the pullback $f^*\alpha$ of fundamental class α in $H^1(S^1;\mathbb{Z})$ under a map

f.

So let φ be a class in H^1,

Then consider $\varphi \smile \varphi$.

$$
\begin{aligned}
&= f^*\alpha \smile f^*\alpha \qquad \alpha \text{ gen. } H^1(S^1;\mathbb{Z}) \\
&= f^*(\alpha \smile \alpha) \\
&= f^*(0) \quad \text{since } H^2(S^1;\mathbb{Z}) = 0. \\
&= 0.
\end{aligned}
$$

So in $H^1(X;\mathbb{Z})$, $\varphi \smile \varphi = 0$.

We mentioned that

$$H^n(X;G) \cong \overbrace{\langle X, K(G,n) \rangle}^{\text{htpy classes of maps}}.$$

For the mystery space $K(G,n)$

$H^n(K(G,n))$ has a fundamental class α and every

$\varphi \in H^n(X;G)$ looks like $f^*\alpha$

where $f \in \langle X, K(G,n)\rangle$

represents φ.

$\boxed{\text{Thm } \alpha\cup\beta = (-1)^{kl}\beta\cup\alpha. \text{ Don't prove. } \alpha\in H^1 \Rightarrow 2\alpha^2=0.}$ for comm. R.

Idea: $\varphi\cup\psi$ and $\psi\cup\varphi$

differ by permutation of

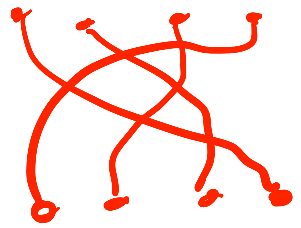
vertices of Δ^{k+l}. Nice permutation:

$$[v_0, \dots, v_n] \to [v_n, \dots, v_0].$$

$$\sigma \qquad\qquad \bar{\sigma}$$

product of $n + (n-1) + \dots + 1 = \frac{n(n+1)}{2}$

transpositions:



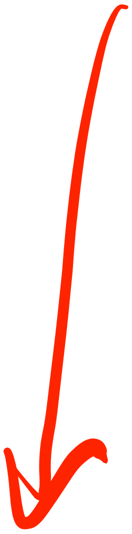
each transposition reverses the order of the n sx.

So we expect a sign change of $\varepsilon_n = (-1)^{n(n+1)/2}$

define $\rho : C_n(\overline{X}) \to C_n(\overline{X})$

by $\rho(\sigma) = \varepsilon_n \overline{\sigma}$.

proof shows that $\rho : C_n \to C_n$ is a chain map chain htpc to identity.



Computation :

$$\rho^*\varphi \smile \rho^*\psi(\sigma)$$

$$= \varphi(\varepsilon_k[v_k,\dots,v_0])\,\psi(\varepsilon_\ell[v_{k+\ell},\dots,v_k])$$

$$\rho^*(\psi\smile\varphi)(\sigma)$$

$$= \varepsilon_{k+\ell}\,\psi([v_{k+\ell},\dots,v_k])\,\varphi([v_k,\dots,v_0])$$

$$= \varepsilon_{k+\ell}\,\varphi([v_k,\dots,v_0])\,\psi([v_{k+\ell},\dots,v_k]$$

since R commutative

So

$$\boxed{\begin{aligned}&\varepsilon_k\varepsilon_\ell(\rho^*\varphi\smile\rho^*\psi)\\ &\quad= \varepsilon_{k+\ell}\,\rho^*(\psi\smile\varphi) \text{ when } R \text{ comm.}\end{aligned}}$$

$$\varepsilon_{k+\ell} = (-1)^{k\ell}\varepsilon_k\varepsilon_\ell$$

$$\Rightarrow \quad \rho^*\varphi\smile\rho^*\psi = (-1)^{k\ell}\rho^*(\psi\smile\varphi).$$

ρ ch. $\sim$ to $\mathbb{1}$

$$\Rightarrow \quad [\varphi]\smile[\psi] = (-1)^{k\ell}[\psi]\smile[\varphi].$$

① $\partial \rho = \rho \partial$

$$\partial \rho(\sigma) = \varepsilon_n \sum_i (-1)^i \sigma|_{[v_n, \dots, \hat{v}_{n-i}, \dots v_0]}$$

$$\rho \partial \sigma = \rho \left(\sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right)$$

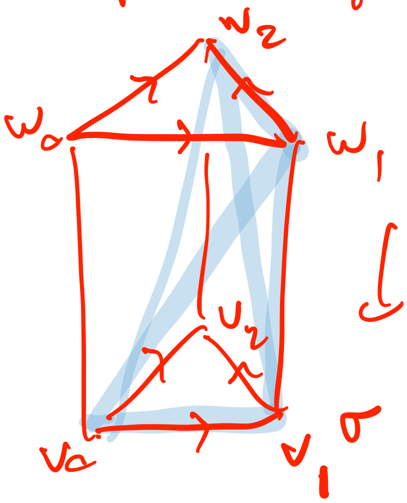
$$= \varepsilon_{n-1} \sum_i (-1)^{n-i} \sigma|_{[v_n, \dots, \hat{v}_{n-i}, \dots, v_0]}$$

$$= \underbrace{\varepsilon_n \quad (-1)^n}_{\varepsilon_{n-1}.} \sum (-1)^{n-i} \underline{\qquad\qquad}$$

Ch. htpy $\quad P : C_n \to C_{n+1}$

$\underline{\text{s.t.} \quad \partial P + P\partial = \rho - \mathbb{1}.}$

Prism operator:



$[\pi$

$$\boxed{P(\sigma) = \sum_i (-1)^i \varepsilon_{n-i} (\sigma\pi)|_{[v_0, \dots, v_i, w_n, \dots w_i]}}$$

Computation:

For simplicity, leave out σ and $\sigma\pi$'s for readability.

$$\partial P = \sum_{\substack{i,j \\ j\le i}} (-1)^i (-1)^j \varepsilon_{n-i}[v_0,\dots,\hat{v}_j,\dots,v_i, w_n,\dots,w_i]$$

$$+ \sum_{\substack{i,j \\ j\ge i}} (-1)^i (-1)^j \varepsilon_{n-i}[\underbrace{\overbrace{v_0,\dots,v_i}^{i+1}, \overbrace{w_n,\dots,\hat{w}_k}^{n-k+1}}_{j+1},\dots,w_i]$$

$$n-k+1 + i+1 = j+1$$

$$\Rightarrow \boxed{k = n+i-j+1}$$

$$= \sum_{\substack{i,j \\ j\le i}} (-1)^i (-1)^j \varepsilon_{n-i}[v_0,\dots,\hat{v}_j,\dots,v_i, w_n,\dots,w_i]$$

$$+ \sum_{\substack{i,k \\ k\ge i}} (-1)^i (-1)^{n+i-k+1} \varepsilon_{n-i}[v_0,\dots,v_i, w_n,\dots,\hat{w}_k,\dots,w_i]$$

$$= \sum_{\substack{i,j \\ j\le i}} (-1)^i (-1)^j \, \varepsilon_{n-i}\left[v_0, \dots, \hat{v}_j, \dots, v_i, w_n, \dots, w_i\right] \quad (\dagger)$$

$$+ \sum_{\substack{i,j \\ j\ge i}} (-1)^i (-1)^{n+i-j+1} \, \varepsilon_{n-i}\left[v_0, \dots, v_i, w_n, \dots, \hat{w}_j, \dots, w_i\right] \quad (*)$$

$j=i$ terms $\rightsquigarrow$

$$\underbrace{\varepsilon_n[w_n, \dots, w_0]}_{i=0} + \sum_{i>0} \varepsilon_{n-i}\left[v_0, \dots, v_{i-1}, w_n, \dots, w_i\right]$$

$$+ \sum_{i<n} (-1)^{n+i+1} \varepsilon_{n-i}\left[v_0, \dots, v_i, w_n, \dots, w_{i+1}\right]$$

$$- \left[v_0, \dots, v_n\right].$$

$$= \mathcal{E}_n[w_n,\dots,w_0]$$

$$+ \sum_{i>0} \mathcal{E}_{n-i}[v_0,\dots,v_{i-1},w_n,\dots,w_i]$$

$$+ \sum_{i>0} (-1)^{n+(i-1)+1}\mathcal{E}_{n-i+1}[v_0,\dots,v_{i-1},w_n,\dots,w_i]$$

$$-[v_0,\dots,v_n]$$

$$\left.\begin{aligned} &= \mathcal{E}_n[w_n,\dots,w_0] \\ &\quad - [v_0,\dots,v_n] \end{aligned}\right\} \nearrow \mathcal{E}_n(\sigma\pi)([w_n,\dots,w_0])$$

$$\left\{\ \boxed{\begin{aligned} &\text{since} \\ &(-1)^{n+i}\mathcal{E}_{n-i+1} \\ &\quad = -\mathcal{E}_{n-i} \end{aligned}}\right.$$

$$(-1)^{n+i}(-1)^{\sum^{n-i+1} k} = (-1)(-1)^{\sum^{n-i} k}$$

$$\sum_{k=n-i}^{n-i+1} k = n-i+1$$

$$(-1)^{n+i}(-1)^{n-i+1} = -1$$

$\Leftrightarrow (-1)^{2n+1} = -1$. ✓

The leftovers represent $\rho(\sigma)-\sigma$. So $i=j$ terms are $\rho(\sigma)-\sigma$.

We want $\partial P + P\partial = \rho - \mathbb{1}$. So we need $i \neq j$ terms to be $-P\partial$. But

$$P\partial\sigma = P\Big(\sum_j (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_n]\Big)$$

$$= \sum_j (-1)^j P([\underbrace{v_0, \dots, \tilde{v}_j, \dots, v_n}_{\underline{n-1 \text{ sx.}}}])$$

$$= \sum_{i<j} (-1)^i (-1)^j \varepsilon_{n-i-1} [v_0, \dots, v_i, w_n, \dots, \hat{w}_j, \dots, w_i]$$

off by 1. Good.

$$+ \sum_{i>j} (-1)^{\boxed{(i-1)}} (-1)^j \varepsilon_{n-i} [v_0, \dots, \hat{v}_j, \dots, v_i, w_n, \dots, w_i].$$

But $\boxed{\varepsilon_{n-i} = (-1)^{n-i}\varepsilon_{n-i-1}}$ plug into $(*)$, and

so $\partial P + P\partial = \rho - \mathbb{1}$.

This proves it when $A = \phi$.

also works with $A \neq \phi$. $\square$

next:

Cohomology ring. $\Downarrow$

$$\varepsilon_{n-i-1} = (-1)^{\sum^{n-i-1} k}$$

$$\varepsilon_{i-i+1} = (-1)^{\sum^{n-i+1} k}$$

$$\varepsilon_{n-i-1} \cdot \varepsilon_{n-i+1} = (-1)^{n-i+n-i+1}$$

$$= (-1)$$

Cohomology Ring

Define $$H^*(X;R) := \bigoplus_{n=0}^{\infty} H^n(X;R)$$

formal l.n. combos $\sum_{i=1}^{k} \alpha_i$ $\alpha_i \in H^i$.

w/ product:

$$\left(\sum_{i=1}^{k} \alpha_i\right) \cdot \left(\sum_{j=1}^{\ell} \beta_j\right) = \sum_{i,j}^{k,\ell} \alpha_i \cup \beta_j .$$

So this makes $H^*(X;R)$ into a ring.

This has 1 in H^0 if R has 1.

$H^*(X)$ is **Graded Ring**. w called the grading.

"pure class"

$|\alpha| = k$ if $\alpha \in H^k$

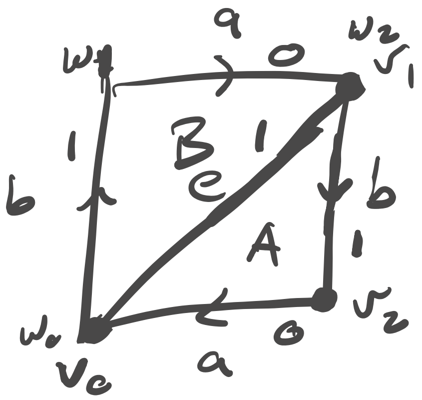
Ex. $H^*(\mathbb{R}P^2; \mathbb{F}_2)$

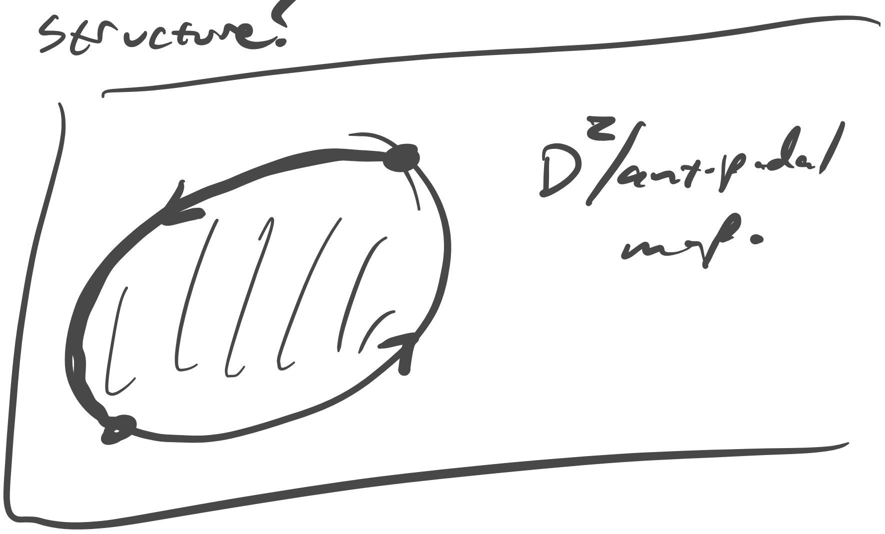
$H^0(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2$

$H^1(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2$

$H^2(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2$

what's ring structure?





$\varphi \in H^1(\mathbb{R}P^2; \mathbb{F}_2)$ as in labeled picture.

$$\varphi \smile \varphi(A) = \varphi \smile \varphi(\widetilde{[v_0, v_1}, v_2])$$
$$= \varphi([v_0, v_1]) \cdot \varphi([v_1, v_2])$$
$$= 1 \cdot 1$$

$$\varphi \smile \varphi(B) = \varphi \smile \varphi([w_0, w_1, w_2])$$

$$= \varphi([w_0, w_1]) \cdot \varphi([w_1, w_2])$$

$$= 1 \cdot 0$$

So:

$\varphi \cup \varphi(A+B) = 1$ so

$A+B$ generates $H_2(\mathbb{R}P^2; \mathbb{F}_2)$

and $\varphi \cup \varphi$ generates $H^2(\mathbb{R}P^2; \mathbb{F}_2)$.

So $H^*(\mathbb{R}P^2; \mathbb{F}_2)$

$\cong \underbrace{\mathbb{F}_2}_{H^0}\underbrace{[\alpha]}_{\alpha \text{ gen } H^1.}/(\alpha^3)$. $\underline{\alpha^2}$ gen $H^2(\mathbb{R}P^2; \mathbb{F}_2)$

but $\alpha^3 = 0$. cuz no H^3.

ring is truncated polynomial ring.

Thm $H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/(\alpha^{n+1})$

$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]$

where $|\alpha| = 1$.

Also, $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$

$H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]$.

$|\alpha| = 2$.

"Need more machinery"

Foreshadowing:

Recall: saw

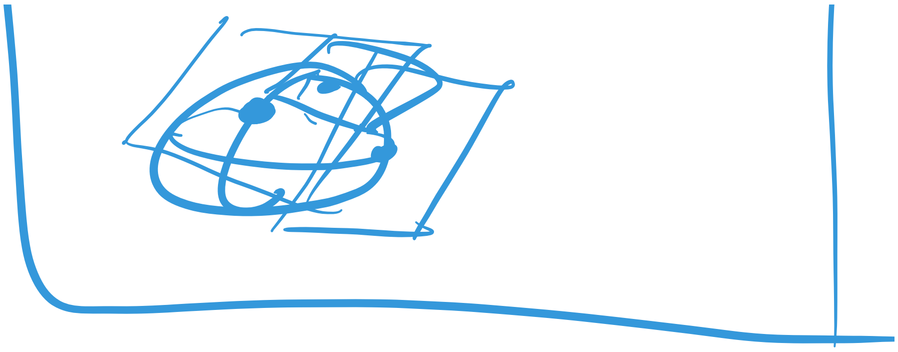
$H^n(X, G) \cong \langle X, K(G,n)\rangle$

$\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$.

So $\underbrace{H^2(X, G) \cong \langle X, \mathbb{C}P^\infty\rangle}_{}$.

$\mathbb{C}P^n$ = lines in $\mathbb{C}^{n+1}$

family of lines that sit on $\mathbb{C}P^n$, called a line bdle.



Another Thm quaternionic projective spaces. $\mathbb{HP}^n$, $\mathbb{HP}^\infty$, where

$$\mathbb{H} = \{ a + bi + cj + dk \}$$

Hamilton's quaternions

w/ multiplication given by relations: $i^2 = j^2 = k^2 = -1$

$$ijk = -1$$

$\mathbb{H}^{n+1}/$scaling.

$$H^*(\mathbb{HP}^n) = \mathbb{Z}[\alpha]/(\alpha^{n+1}) \qquad |\alpha| = 4$$

$$H^*(\mathbb{HP}^\infty) = \mathbb{Z}[\alpha]$$

Ex $H^*(\bigsqcup_{\alpha\in\Lambda} X_\alpha) \longrightarrow \prod_\alpha H^*(X_\alpha)$

map whose coords. are induced by inclusions $i_\alpha : X_\alpha \to \bigsqcup_{\alpha\in\Lambda} X_\alpha$

is a ring isomorphism, since each coord. function is.

Same wedge products:

$$\tilde{H}^*(\bigvee_{\alpha\in\Lambda} X_\alpha) \xrightarrow{\cong} \prod_\alpha \tilde{H}^*(X_\alpha)$$

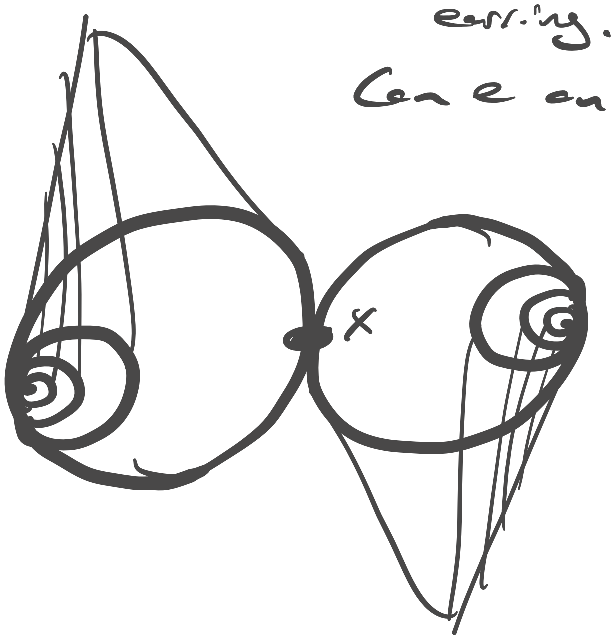
(reduced cohom. is cohomology rel basepoint.)

and should assume that wedge points are deformation retracts of small nbhds of wedge points.

(avoid weird things like X contractible and $X \vee X \not\simeq *$.

earring. E.

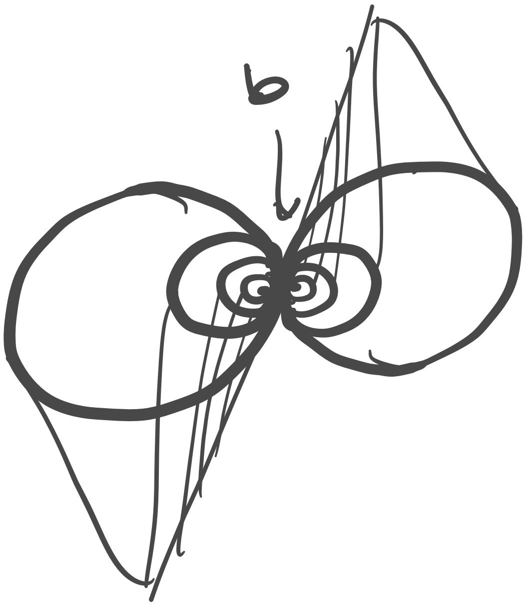
Cone on E, CE.



CE, is contractible

cuz it's a cone.

$CE \vee_x CE \simeq *$.



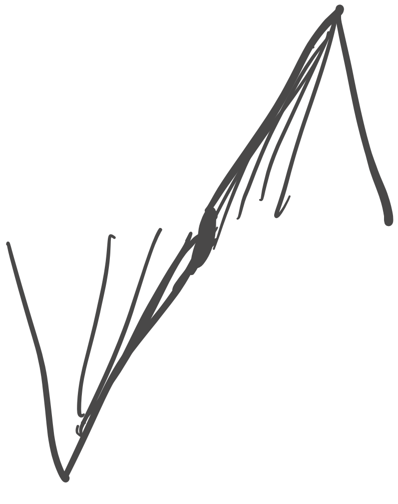
$CE \vee_b CE$

is not contractible.

I think

$\pi_1(CE \vee_b CE, b)$

is not finitely generated.



cone$(\{0\} \cup \{\frac{1}{2}\})$

wedge
not contractible.

Ex $H^*(\bigsqcup_{\alpha\in\Lambda} X_\alpha) \longrightarrow \prod_\alpha H^*(X_\alpha)$

map whose coords. are induced by inclusions $i_\alpha : X_\alpha \to \bigsqcup_{\alpha\in\Lambda} X_\alpha$

is a ring isomorphism, since each coord. function is.

Same wedge products:

$$\tilde{H}^*(\bigvee_{\alpha\in\Lambda} X_\alpha) \xrightarrow{\cong} \prod_\alpha \tilde{H}^*(X_\alpha)$$

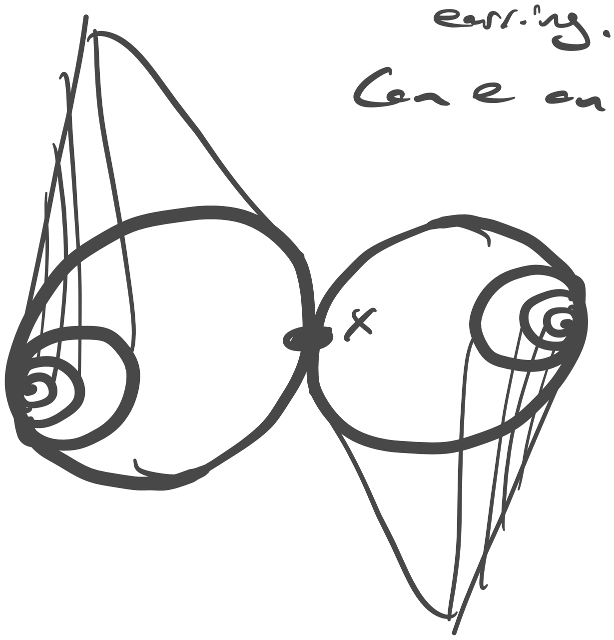
(reduced cohom. is cohomology rel basepoint.)

and should assume that wedge points are deformation retracts of small nbhds of wedge points.

(avoid weird things like X contractible and $X \vee X \not\simeq *$.

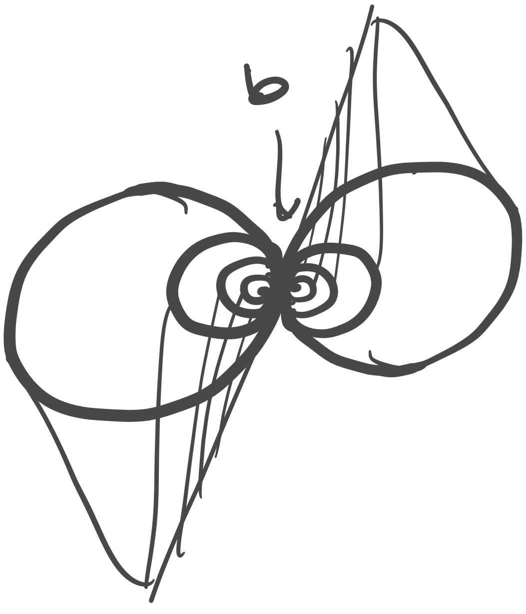
earring. E.

Cone on E, CE.



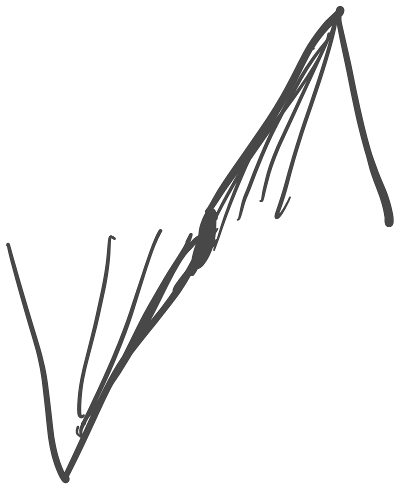
CE is contractible cuz it's a cone.

$CE \vee_x CE \simeq *$.



$CE \vee_b CE$ is not contractible.

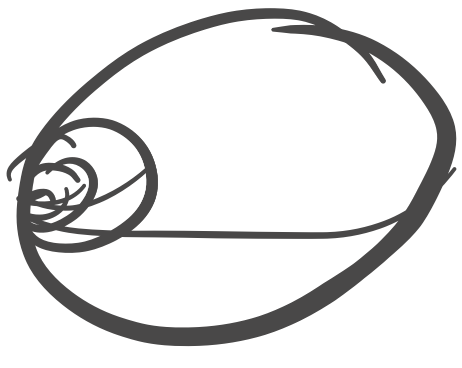
I think $\pi_1(CE \vee_b CE, b)$ is not finitely generated.



cone$(\{0\} \cup \{\frac{1}{n}\})$

wedge
not contractible.

Xmas tree ornament.



Cohomology is skew-comm. graded ring.

$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha$ $|\alpha| = k$
$|\beta| = l$

Reminiscent of Exterior algebra:

Ex. $\Lambda_R[\alpha_1, \dots, \alpha_n]$

R comm Ring.

Free R-module w/ basis

$\alpha_{i_1} \cdots \alpha_{i_k}$ $i_1 < \dots < i_k$

w/ relations $\alpha_i \alpha_j = -\alpha_j \alpha_i$
and $\alpha_i^2 = 0$.

The empt. ed. is the 1.

$$H^*(T^2;R) \cong \Lambda_R[\alpha_1, \alpha_2]$$

We will prove soon that

$$H^*(T^n;R) \cong \Lambda_R[\alpha_1, \dots, \alpha_n]$$

If R a field, this has dim^n $\binom{n}{k}$.

To do this computation, and the computation of $H^*(\mathbb{P}^n;R)$, we need to understand H^* and products.

Products.

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y . \end{tikzcd}$$

There is a **cross product map**

$$H^*(X;R) \times H^*(Y;R) \to H^*(X \times Y;R)$$

$$a \times b = p_1^* a \smile p_2^*(b)$$

cup product is distributive, and so $\times$ is **bilinear**, i.e. linear in each factor individually.


: bilinear maps aren't homomorphisms.

usually

e.g.

$\varphi : A \times B \to C$ bilinear

$$\varphi((a,b)+(a',b'))$$

$$= \varphi((a+a', b+b'))$$

$$= \varphi(a,b) + \varphi(a',b') + \underbrace{\varphi(a,b') + \varphi(a',b]}_{\text{error.}}$$

There is a magic trick to fix this, called the tensor product!

$\boxed{\otimes}$ A, B abelian groups.

$A \otimes B$ is the abelian gp with generators $a \otimes b$ for $a \in A$, $b \in B$ w/ added relations

$$(a+a') \otimes b = a \otimes b + a' \otimes b$$

$$a \otimes (b+b') = a \otimes b + a \otimes b'$$

zero elt of $A \otimes B$ is $0 \otimes 0$

$$= 0 \otimes b = a \otimes 0$$

and $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$

Properties:

1) $A \otimes B \cong B \otimes A$

2) $\left(\bigoplus_i A_i\right) \otimes B \cong \bigoplus_i (A_i \otimes B)$

3) $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

4) $\mathbb{Z} \otimes A \cong A$

5) $\mathbb{Z}/n\mathbb{Z} \otimes A \cong A/nA$

6) homomorphisms $f: A \to A'$
$g: B \to B'$

there's an induced homomorphism

$$f \otimes g : A \otimes B \longrightarrow A' \otimes B'$$

$$(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$$

7) if $\varphi : A \times B \to C$ is bilinear, then there is a

homomorphism:

$\Phi : A \otimes B \to C$

$\Phi(a \otimes b) = \varphi(a, b)$

$\otimes$ defined for modules over commutative ring R.

$A \otimes_R B$ is quotient of $A \otimes B$

where

$$(ra) \otimes b = a \otimes rb$$

for $r \in R$, $a \in A$, $b \in B$.

This makes $A \otimes_R B$ into an R-module.

If R not commutative, then you assume A right R-module

B left R-module

and the relation is $ar \otimes b = a \otimes rb$.

in that case $A \otimes_R B$ only an abelian gp.

$A \otimes_R B = A \otimes B$ when $R = \mathbb{Z}/m\mathbb{Z}$ or $\mathbb{Q}$.

but not same in general:

Example: $R = \mathbb{Q}(\sqrt{2})$ 2-dim^l v.s.

$R \otimes_R R = R$ but $\underline{R \otimes R} = R \otimes_{\mathbb{Z}} R$

is four dimensional vector space

$\sqrt{2}_1 \otimes 1 \neq 1 \otimes \sqrt{2}_2$.

---

(1)-(3), (6), (7) still true.

4) becomes $R \otimes_R A \cong A$

$r \otimes a \mapsto ra$.

Since cross product $\times$ is bilinear, we have a cross product

$$\begin{array}{ccc} H^*(X;R) \otimes_R H^*(Y;R) & \xrightarrow{\times} & H^*(X \times Y;R) \\ a \otimes b & \longmapsto & a \times b \end{array}$$

This tensor product on left can be made into a ring by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd.$$

cross product is a ring homom.

↙

$$\times((a\otimes b)(c\otimes d))$$

$$= (-1)^{|b||c|}\, ac \times bd$$

$$= (-1)^{|b||c|}\, p_1^*(a\smile c)\smile p_2^*(b\smile d)$$

$$= (-1)^{|b||c|}\, p_1^*(a)\smile \underbrace{p_1^*(c)\smile p_2^*(b)}\smile p_2^*(d)$$

$$= (-1)^{|b||c|}(-1)^{|b||c|}\, p_1^*(a)\smile p_2^*(b)\smile p_1^*(c)\smile p_2^*(d)$$

$$= p_1^*(a)\smile p_2^*(b)\smile p_1^*(c)\smile p_2^*(d)$$

$$= (a\times b)\cdot(c\times d)$$

$$= \times(a\otimes b)\cdot\times(c\otimes d). \qquad \square$$

Thm (Künneth Formula)

Let X, Y CW cxs (this hypothesis not nec.)

and suppose that

$H^k(Y;R)$ is a finitely generated free R-module $\forall k$.

Then

$$H^*(X;R)\otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R)$$

is an isomorphism of rings.

Proof of Künneth formula is **beautiful**

Use axioms for cohomology.

Reduced Cohomology Theory is a sequence of contravariant functors $\tilde{h}^n$ from CW cxs to category of abelian groups (R-modules) w/ natural coboundary maps

$$\delta : \tilde{h}^n(A) \to \tilde{h}^n(X/A)$$

for CW pairs (X,A) s.t.

1) if $f \simeq g : X \to Y$ then

$$f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)$$

2) $\forall$ CW pairs (X,A), $\exists$ LES

$$\xrightarrow{\delta} \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \to$$

where i is inclusion $A \to X$

$q : X \to X/A$ quotient map.

"naturality" of δ means maps btw LES have commutative squares.

3) for $X = \bigvee_\alpha X_\alpha$ w/ $i_\alpha : X_\alpha \hookrightarrow X$

then map

$$\prod_\alpha i_\alpha^* : \tilde{h}^n(X) \to \prod_\alpha \tilde{h}^n(X_\alpha)$$

is an isomorphism.

Unreduced theory.

contra. functor from CW pairs to ab. gps (or R modules)

$h^n(X,A)$. $\qquad h^n(X,\emptyset) =: h^n(X)$.

Replace axiom 1) w/ rel. version.

2) is split into two axioms

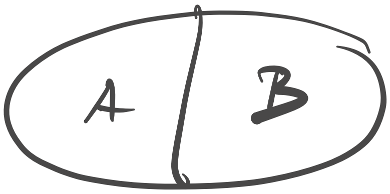
2)' : LES of pairs w/ rel. gps

2)'' : Excision:

$$h^n(X,A) \cong h^n(X/A, A/A)$$

(equiv. $h^n(X,A) \cong h^n(B, A\cap B)$

whenever $A \cup B = X$



)

3) same but w/ disjoint union instead of wedge product.

Don't assume that $\tilde{h}^*(*) = 0$.

$\boxed{\tilde{h}^*(*)=0 \text{ "Dimension axiom"}}$

$h^n(*)$ are "coefficients of the theory."

For formula: Fix Y

$$h^n(X,A) = \bigoplus_i H^i(X,A;R) \otimes H^{n-i}(Y;R)$$

$$k^n(X,A) = \underbrace{H^n(X \times Y, A \times Y ; R)}_{\text{on RHS of Künneth formula}} .$$

h^n LHS.

Show that h^n and k^n are unred. cohomology theories and show that $h^*(*) = k^*(*)$.

$\Rightarrow$ $h^n = k^n$.

$$H^*(X;R) \times H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R)$$

cross product:

$$p_1 : X \times Y \to X$$
$$p_2 : X \times Y \to Y$$

$$(\alpha, \beta) \longmapsto p_1^*\alpha \smile p_2^*\beta .$$

$\times$ is a Bilinear map.

So this is not usually linear.

But $\rightsquigarrow$ linear map

$$H^*(X;R) \otimes H^*(Y;R) \xrightarrow{P} H^*(X \times Y;R)$$
$$a \otimes b \longmapsto a \times b .$$

Tensor product on left is a ring when you define multiplication by $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd.$

with this mult.

Cross product is a ring homomorphism:

$$P(a\otimes b)(c\otimes d) = (-1)^{|b||c|}\, ac \times bd$$

$$= (-1)^{|b||c|}\, p_1^*(a\smile c) \smile p_2^*(b\smile d)$$

$$= (-1)^{|b||c|}\, p_1^*(a)\smile \underleftrightarrow{p_1^*(c) \smile p_2^*(b)} \smile p_2^*(d)$$

$$= \underbrace{(-1)^{|b||c|}(-1)^{|b||c|}}_{\overset{\shortparallel}{1}}\, p_1^*(a)\smile p_2^*(b) \smile p_1^*(c)\smile p_2^*(d)$$

$$= (a\times b)\cdot(c\times d)$$

$$= P(a\otimes b)\cdot P(c\otimes d).$$

Thm (Künneth formula)
If X, Y are CW cxs and $H^k(Y;R)$ is f.g. free R-module for all k, then

$$H^*(X;R)\otimes H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R)$$

is an isomorphism.

(CW hypothesis is unnecessary.)
(Freeness of $H^k(Y)$ can't be relaxed.)

$$\boxed{\begin{array}{l} H^*(S^2)\otimes H^*(S^2) \\ \neq \text{ exterior algebra.} \\ \qquad \alpha\otimes\beta = \beta\otimes\alpha. \end{array}}$$

<u>**Proof**</u> uses axioms of cohomology.

Unreduced cohomology theory h^n

contravariant funct. CW → Ab.

relative "cohomology gps"

$h^n(X,A)$, $h^n(X) := h^n(X,\emptyset)$

when (X,A) CW pair.

satisfying axioms:

1) $f \simeq g \Rightarrow f^* = g^*$

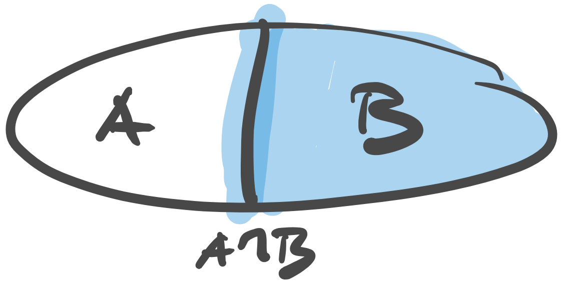
2) LES of pairs

3) Excision: $h^n(X,A) \cong h^n(X/A, A/A)$.

(equivalently:

$h^n(X,A) \cong h^n(B, A\cap B)$

when $X = A \cup B$



)

4) $X = \coprod_{\alpha\in\Lambda} X_\alpha$ w/ $i_\alpha : X_\alpha \hookrightarrow X$

then $\prod_{\alpha\in\Lambda} i_\alpha^* : h^n(X) \longrightarrow \prod_{\alpha\in\Lambda} h^n(X_\alpha)$

is isomorphism.

Don't require that $h^n(*)=0$ when $n\neq 0$.

That would be the "Dimension Axiom"

$h^*(*)$ = "coefficients of the homology theory"

Fix Y. (as in the Künneth formula)

Define functors

$$h^n(X, A) = \bigoplus_i \left(H^i(X, A; R) \otimes H^{n-i}(Y; R) \right)$$

$$k^n(X, A) = H^n(X \times Y, A \times Y; R)$$

X product ⇝ map

$$\mu : h^n(X, A) \to k^n(X, A).$$

We want μ to be isomorphism when X CW cx and $A = \emptyset$.

To do: 1) h^* and k^* are cohomology theories on category of CW pairs

2) μ is a natural transformation i.e. it commutes w/ induced homomorphisms and coboundary maps in LESs of pairs.

✓3) Observe $\mu: h^n(X) \to k^n(X)$ is an isomorphism if $X = *$, since μ is just scalar multiplication

$$R \otimes_R H^n(X;R) \to H^n(X;R).$$

4) The following proposition:

Prop. If a natural transf. μ between unreduced cohomology theories on CW pairs is an iso when $(X,A) = (*, \emptyset)$, then μ is isomorphism for all pairs (X,A) !!!

Pf of prop. Let $\mu: h^*(X,A) \to k^*(X,A)$ be the natural transformation. By 5-Lemma applied to

$$\mu: \text{LES} \to \text{LES},$$

it's enough to prove prop. in

absolute case.

Finite dimensional case (Infinite dimensional case uses telescoping argument similar to proof of 2.34 in Hatcher.)

Induct on dimension.

Base case:

X 0-dimensional.

Theorem holds by assumption and Disjoint union axiom.

Induction Step.

Consider pair $(\overset{n\text{-skel.}}{X^n}, \overset{n-1\text{ skel.}}{X^{n-1}})$

h gives us a map btw LES w/ commuting squares

5-Lemma

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] & B \arrow[r] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, "\varphi"] & D \arrow[r] \arrow[d, "\cong"] & E \arrow[d, hook] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd} \Rightarrow \varphi \text{ iso}$$

$$\begin{tikzcd}
h^{l-1}(X^{n-1}) \arrow[r] \arrow[d, "\mu"', "\cong"] & h^{l}(X^{n}, X^{n-1}) \arrow[r] \arrow[d, "\mu"] & h^{l}(X^{n}) \arrow[r] \arrow[d, "\mu"] & h^{l}(X^{n-1}) \arrow[r] \arrow[d, "\mu"', "\cong"] & h^{l+1}(X^{n}, X^{n-1}) \arrow[d, "\mu"] \\
k^{l-1}(X^{n-1}) \arrow[r] & k^{l}(X^{n}, X^{n-1}) \arrow[r] & k^{l}(X^{n}) \arrow[r] & k^{l}(X^{n-1}) \arrow[r] & k^{l+1}(X^{n}, X^{n-1})
\end{tikzcd}$$

Blue arrows are isos by induction.

So by 5-lemma, we'll have

$$\mu : h^{n}(X^{n}) \to k^{n}(X^{n}) \text{ iso}$$

if

$$h^{l}(X^{n}, X^{n-1}) \xrightarrow{\mu} k^{l}(X^{n}, X^{n-1})$$

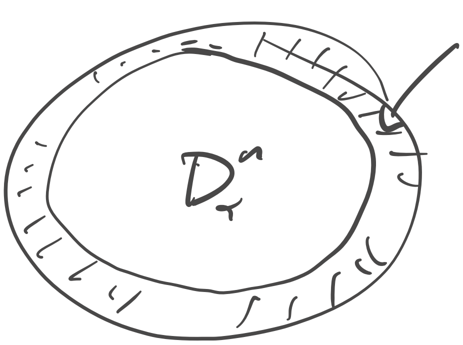
is isomorphism $\forall\, l$.

Let $$\Phi : \underbrace{\coprod_{\alpha} (D^{n}_{\alpha}, \partial D^{n}_{\alpha})}_{\text{Disjoint union of all of the } n\text{-cells.}} \to (X^{n}, X^{n-1})$$

Φ on ∂D^{n}_{α} is just attaching map

Φ on $\overset{\circ}{D}{}^n_\alpha$ is just inclusion.

By excision axiom, Φ^* is isomorphism for both h^* and k^*



write $X = X^n$

As $\left(\coprod_\alpha \overset{\circ}{D}{}^n_\alpha\right) \cup \text{nbhd}(X^{n-1})$.
$\quad B \qquad\qquad A$

$B \cap A \simeq \coprod \partial D^n_\alpha$

So excision says

$h^*(X, A) \cong h^*(B, B\cap A)$

same for k.

So by naturality,

only need to check the case

of $\coprod_\alpha (D^n_\alpha, \partial D^n_\alpha)$.

By disjoint union axiom, only need μ iso for $(D_\alpha^n, \partial D_\alpha^n)$.

<u>For that:</u>

$$\begin{tikzcd}
h^{\ell-1}(\partial D^n) \arrow[r] \arrow[d, "\mu"] \arrow[d, phantom, swap, "\overset{\text{ind}}{\cong}"] & h^\ell(D^n, \partial D^n) \arrow[r] \arrow[d, "\mu"] & h^\ell(D^n) \arrow[r] \arrow[d, "\mu"] \arrow[d, phantom, swap, "\cong"] & h^\ell(\partial D^n) \arrow[r] \arrow[d, "\mu"] \arrow[d, phantom, swap, "\overset{\text{ind}}{\cong}"] & h^{\ell+1}(D^n, \partial D^n) \arrow[d, "\mu"] \\
k^{\ell-1}(\partial D^n) \arrow[r] & k^\ell(D^n, \partial D^n) \arrow[r] & k^\ell(D^n) \arrow[r] & k^\ell(\partial D^n) \arrow[r] & k^{\ell+1}(D^n, \partial D^n)
\end{tikzcd}$$

this iso
since $D^n \simeq *$
and theories
agree on a $*$.

5-lemma tells us that the rest are isomorphisms. $\square$

(and ∞-dim'l case.)

Now need to check that they're coh. theories.

Need to check axioms for our theories h^* and k^*:

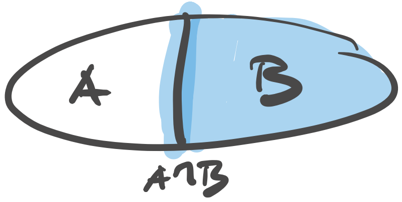
$$h^n(X,A) = \bigoplus_i \left(H^i(X,A;R)\otimes H^{n-i}(I;R)\right)$$

$$k^n(X,A) = H^n(X\times I, A\times I; R).$$

Axioms:

1) htpy invariance $f\simeq g \Rightarrow f^* = g^*$

clear for each.



2) Excision:

need $h^*(X,A) \cong h^*(B, A\cap B)$

for A, B subcxs & $X = A\cup B$.

follows from excision & $H^*(X,A)$

also for k^* by fact that

$$A\times I \cup B\times I = (A\cup B)\times I$$

and $(A\times I)\cap(B\times I) = (A\cap B)\times I$

and excision for usual cohomology.

3) **LES.** Trivial for k^*.

for h^*:

tensor LES of ordinary cohomology with **free** **R-module** $H^k(\Phi)$, for fixed k.

This produces a LES since

$$H^k(\Phi; R) = \oplus R$$

and so tensoring gives a direct sum of copies of the original LES and so still exact.

Now let k vary, take direct sum w/ k^{th} sequence shifted by k. ✓

4) Disjoint union axiom.

Clear for k^*.

For h^*:

$\exists$ Canonical isomorphism

$$\left(\prod_\alpha M_\alpha\right) \otimes_R N \to \prod_\alpha (M_\alpha \otimes N)$$

for R modules M_α and f.g. free R-mod N.

$$N = \bigoplus_{\beta \in \Lambda} R_\beta \qquad R_\beta \cong R$$

$$|\Lambda| < \infty$$

$$\Rightarrow \quad M_\alpha \otimes_R N \cong \bigoplus_\beta M_{\alpha\beta} = \bigoplus_\beta M_\alpha \otimes_R R_\beta$$

then what we want is

$$\prod_\beta \prod_\alpha M_{\alpha\beta} \cong \prod_\alpha \prod_\beta M_{\alpha\beta}.$$

Naturality of μ.

cup product is natural w.r.t. maps and so μ natural w.r.t. maps.

Naturality w.r.t. δ:

We want

$$\begin{tikzcd}
\arrow[r] & h^k(A) \arrow[r, "\delta"] \arrow[d, "\mu"] & h^{k+1}(X,A) \arrow[r] \arrow[d, "\mu"] & {} \\
\arrow[r] & k^k(A) \arrow[r, "\delta"] & k^{k+1}(X,A) \arrow[r] & {}
\end{tikzcd}$$

to commute.

This is true if

$$\begin{tikzcd}
H^k(A) \times H^l(Y) \arrow[r, "\delta \times 1"] \arrow[d, "\times"] & H^{k+1}(X,A) \times H^l(Y) \arrow[d, "\times"] \\
H^{k+l}(A \times Y) \arrow[r, "\delta"] & H^{k+l+1}(X \times Y, A \times Y)
\end{tikzcd}$$

commutes.

(φ, ψ) representing something in upper left.

Recall construction of $\delta : H^k(A) \to H^{k+1}(X, A)$

$$\begin{tikzcd}
 & & \hat{\varphi}\circ\partial & (j^*)^{-1}(\hat{\varphi}\circ\partial) \arrow[l] & \\
0 & C^{n+1}(A;G) \arrow[l] & C^{n+1}(X;G) \arrow[l, "i^*"'] & C^{n+1}(X,A) \arrow[l] & 0 \arrow[l] \\
0 & C^{n}(A;G) \arrow[l] \arrow[u, "\delta"'] & C^{n}(X;G) \arrow[l, "i^*"'] \arrow[u, "\delta"'] & C^{n}(X,A;G) \arrow[l, "j^*"'] \arrow[u, "\delta"'] & 0 \arrow[l] \\
 & \varphi & \hat{\varphi} & &
\end{tikzcd}$$

extend φ to $\hat{\varphi}$

$$\underbrace{(\varphi, \psi) \longmapsto (\hat{\varphi}, \psi) \longmapsto (\delta\hat{\varphi}, \psi)}_{\delta \times \mathbb{1}.}$$

cross product:

$$p_1^{\#}(\delta\hat{\varphi}) \smile p_2^{\#}\psi.$$

other way:

$$(\varphi, \psi) \xrightarrow{\text{cross prod.}} p_1^{\#}\varphi \smile p_2^{\#}\psi$$

Then to $\delta(p_1^{\#}\hat{\varphi} \smile p_2^{\#}\psi)$

since $p_1^{\#}\hat{\varphi} \smile p_2^{\#}\psi$ extends

$p_1^{\#}\varphi \smile p_2^{\#}\psi$ over $C^n(X \times I)$

from $C^n(A \times I)$.

finally

$$\delta(p_1^{\#}\hat{\varphi} \smile p_2^{\#}\psi)$$

$$= (\delta p_1^{\#}\hat{\varphi}) \smile p_2^{\#}\psi + (-1)^{\varepsilon} p_1^{\#}\hat{\varphi} \smile \delta p_2^{\#}\psi$$

$$= \delta p_1^{\#}\hat{\varphi} \smile p_2^{\#}\psi + 0$$

$$= p_1^{\#}\delta\hat{\varphi} \smile p_2^{\#}\psi \quad . \; \blacksquare$$

Relative Künneth

for (X,A), (Y,B), the cross prod. gen

$$H^*(X,A) \otimes_R H^*(Y,B) \to$$

$$H^*(X \times Y, A \times Y \cup X \times B; R)$$

iso of rings if

$H^k(Y,B)$ is f.g. free R-mod.

Next time: examples

Thm $H^*(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/(\alpha^{n+1})$

$H^*(\mathbb{RP}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha]$, $|\alpha| = 1$.

& $H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$

$H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha]$, $|\alpha| = 2$.

Pf. $P^n = \mathbb{RP}^n$, suppress $\mathbb{F}_2$.

$\mathbb{R}^{n+1} - \{0\} / \lambda \underline{x} \sim \underline{x}$, $\lambda \in \mathbb{R}^\times$.

Inclusion $i : P^{n-1} \hookrightarrow P^n$. (There's an incl. for every codim-1 subspace of $\mathbb{R}^{n+1}$)

(alternatively: $P^n = B^n /$ antipodal map on ∂B^n

Inclusion on cells.

induces isomorphism on hom. & cohom i.e. on

cellular cochain gps (1 cell in each dim)

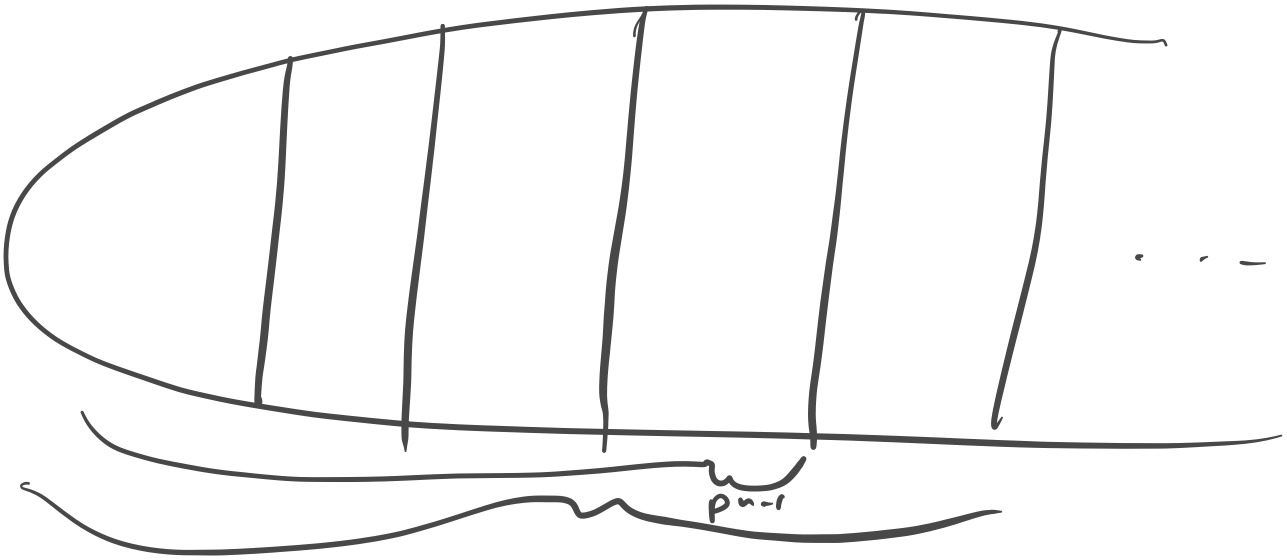
$$\begin{tikzcd}
& \mathbb{F}_2 \arrow[d, equal] & \mathbb{F}_2 \arrow[d, equal] & \mathbb{F}_2 \arrow[d, equal] & \\
{} \arrow[r] & C^{k-1}(P^n) \arrow[r, "{2, 0}"] \arrow[d, "i^\#"] & C^k(P^n) \arrow[r, "{2, 0}"] \arrow[d, "i^\#"] & C^{k+1}(P^n) \arrow[r] \arrow[d, "i^*"] & {} \\
{} \arrow[r] & C^{k-1}(P^n) \arrow[r, "{2, 0}"] & C^k(P^{n-1}) \arrow[r, "{2, 0}"] & C^{k+1}(P^{n-1}) \arrow[r] & {}
\end{tikzcd}$$

groups on top $\mathbb{F}_2$ as long as $k \leq n$

gps " bottom $\mathbb{F}_2$ " " " $k \leq n-1$.

when both domain and codomain of $i^\#$ are $\mathbb{F}_2$, then $i^\#$ is isomorphism. since $\partial = 0$ $\forall$ ∂s we get isos on H^k as long as $k \leq n-1$.

So, To prove Thm, we should show that cup product of generator of $H^{k-1}(P^k)$ and $H^1(P^k)$ generates $H^k(P^k)$. $\forall k$.



P_n

Actually:
Show cup prod. of gens. of H^i & H^{n-i} generates H^n.

Let $j = n - i \Rightarrow i + j = n$.

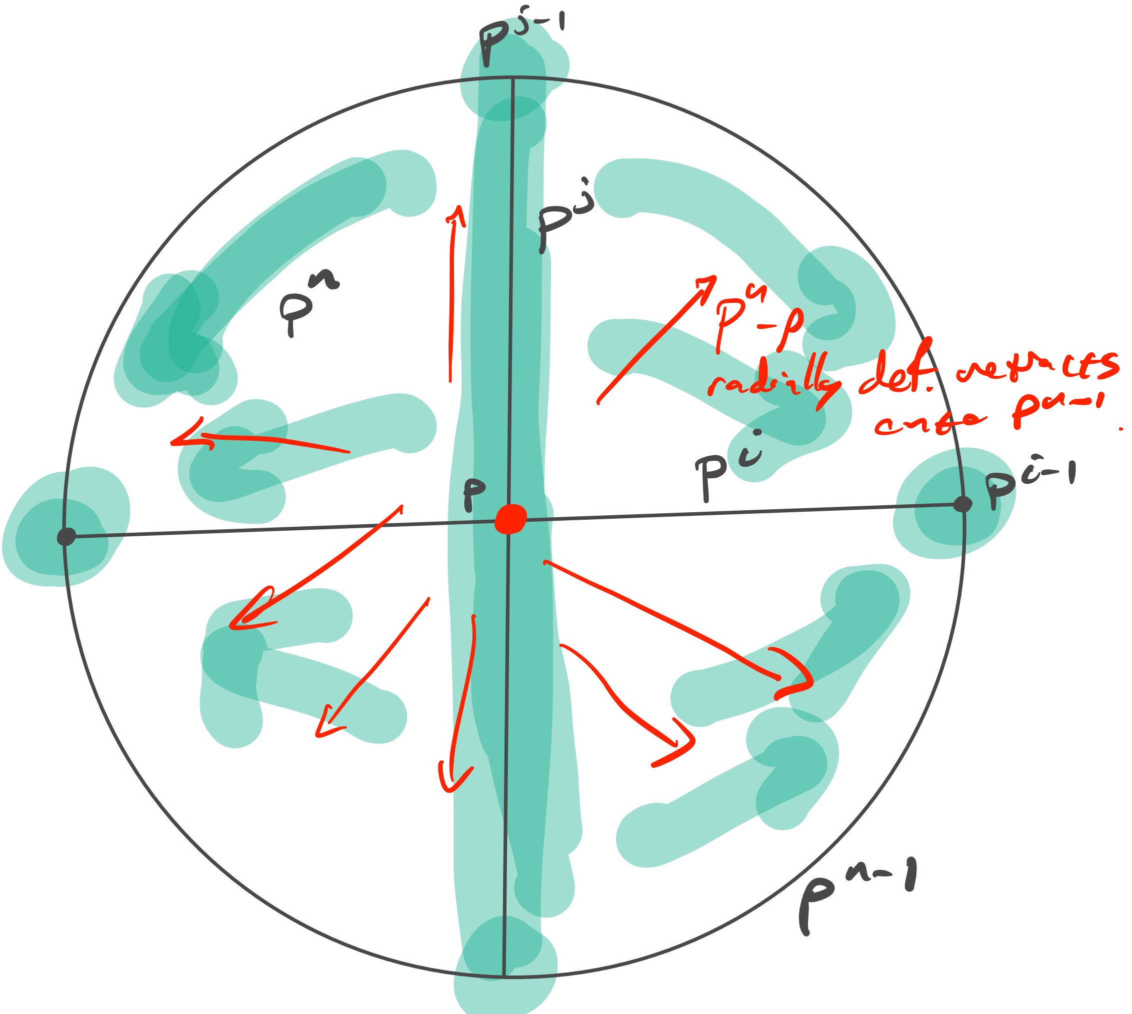
$$P^n = \{(x_0, \overset{\underline{x}}{\overset{\shortparallel}{\dots}}, x_n) \in \mathbb{R}^{n+1}\} / \lambda \underline{x} \sim x \quad \forall \lambda \in \mathbb{R}^\times .$$

P^i sits as image of subspace $\mathbb{R}^{i+1} - \{0\}$ in $\mathbb{R}^{n+1}$

$$P^i = \{(x_0, \dots, x_n) \mid \underline{x_{i+1}, \dots, x_n} = 0\} / \sim$$

$$P^j = \{(x_0,\dots,x_{i-1},\dots,x_n) \mid x_0,\dots,x_{i-1} = 0\}/\sim$$

$$P^i \cap P^j = \{p\} = \{(x_0,\dots,x_i,\dots,x_n) \mid x_k = 0 \text{ if } k \neq i\}/\sim$$



Let $\mathcal{U} = \{(x_0,\dots,x_n) \mid x_i \neq 0\}/\sim$

= everything that has representative

$$(\underbrace{x_0,\dots,x_{i-1}}_{\text{arbitrary}}, 1, \underbrace{x_{i+1},\dots,x_n}_{\text{arbitrary.}})$$

Note: homogeneous coords.

$[x_0 : x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_n]$

$\mathcal{U} \cong \mathbb{R}^n$. $\mathcal{U} \ni p$.

Imagine hence $\mathcal{U} \to \mathbb{R}^n$, $p \mapsto \underline{0}$.

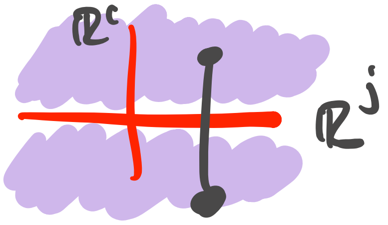
$$\mathbb{R}^n = \underset{\overset{\|}{\{(x_0, \dots, x_{i-1})\}}}{\mathbb{R}^i} \times \mathbb{R}^j = \{(x_{i+1}, \dots, n)\}$$

$$\begin{tikzcd}
H^i(P^n) \times H^j(P^n) \arrow[r, "\smile"] & H^n(P^n) \\
H^i(P^n, P^n - P^j) \times H^j(P^n, P^n - P^i) \arrow[u, "\text{inclusion of relative cochains}"'] \arrow[r, "\smile"] \arrow[d, "i^*"] \arrow[ur, phantom, "\circ"] \arrow[dr, phantom, "\circ"] & H^n(P^n, P^n - p) \arrow[u, "\text{incl. of rel. cochains}"'] \arrow[d, "i^*"] \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \arrow[r, "\smile"] & H^n(\mathbb{R}^n, \mathbb{R}^n - \underline{0}\}
\end{tikzcd}$$

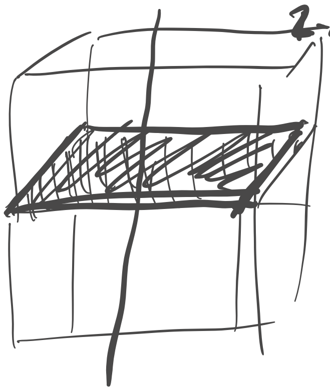
$$\underbrace{\underline{\text{gen}} \times \underline{\text{gen}} \longmapsto \underline{\text{gen}}}_{\text{By Künneth formula.}}$$

$H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j)$

$= H^i(I^i, \partial I^i)$



So we have $H^i(I^i, \partial I^i) \times H^j(I^j, \partial I^j) \xrightarrow{\sim} H^n(I^n, \partial I^n)$



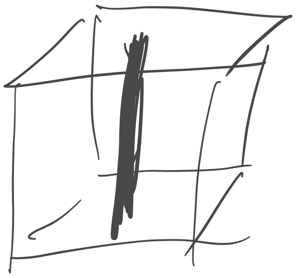
2-dim relative chain in pair $(\mathbb{R}^3, \mathbb{R}^3 - z\text{ axis})$

By excision

in cube.

$(I^2, \partial I^2)$

1-dim relative chain.



Want map at top to take gen × gen ↦ gen.

Want vertical maps to be isos.

SE MAP. Iso by excision.

NE MAP. $H^n(P^n, P^n - p) \to H^n(P^n)$.

$P^n - p \underset{\text{def. retr.}}{\searrow} P^{n-1}$.

$\Rightarrow$ that $H^n(P^n, P^n-\{p\}) \to H^n(P^n, P^{n-1})$ is an iso by 5-lemma applied to the map between LES of Pairs.

But $H^n(P^n, P^{n-1})$ is isomorphic to $H^n(P^n)$ just by computing cellular cohomology.

For Western: Consider commutative diagram:

$$\begin{tikzcd}
H^i(P^n) \arrow[d, "\cong"'] \arrow[dr, phantom, "\text{All maps on this } \square \cong \text{ by cell cohomology.}"] & H^i(P^n, P^{i-1}) \arrow[l, "\cong"'] \arrow[d, "\cong"] \arrow[dr, phantom, "\circ"] & H^i(P^n, P^n - P^j) \arrow[l, "\cong"', "\text{since } P^n-P^j \searrow P^{i-1}"] \arrow[r, "\cong \text{ by known isos}"] \arrow[d, "\cong \text{ by known isos}"] \arrow[dr, phantom, "\circ"] & H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \arrow[d, "\cong \text{ by projecting away } \mathbb{R}^j"] \\
H^i(P^i) & H^i(P^i, P^{i-1}) \arrow[l, "\cong"'] & H^i(P^i, P^i - \{p\}) \arrow[l, "\cong \text{ since } P^i-p \searrow P^{i-1}"] \arrow[r, "\cong \text{ by excision.}"'] & H^i(\mathbb{R}^i, \mathbb{R}^i - 0)
\end{tikzcd}$$

<u>Claim</u>: every map in this diagram is an iso. ✓

If you want explicit $P^n - P^j \searrow P^{i-1}$,

$P^n - P^j = \{ (x_0, \dots, x_n) \mid \text{at least one of } x_0, \dots, x_{i-1} \neq 0 \}$

$f_t(\underline{v}) = (x_0, \dots, x_{i-1}, (1-t)x_i, \dots, (1-t)x_n)$ $\quad t \in [0,1]$.

this is a deformation retraction onto

$$\{(x_0,\dots,x_{i-1},0,0,\dots,0)\}/\sim$$

$$\cong P^{i-1}.$$

($f_t(\underline{v})$ makes sense on $P^n - P^j$ since $f_t(\lambda\underline{v}) = \lambda f_t(\underline{v})$.)

So all vertical maps in our first diagram are isomorphisms and therefore: cup prod of gen. of H^i and gen of H^j generates H^n.

For P^∞, we have

$P^n \hookrightarrow P^\infty \rightsquigarrow$ isos on H^i, $i \leq n$

by cell. cohomology.

$\Rightarrow$ theorem for P^∞.

For $\mathbb{C}P^n$ & $\mathbb{C}P^\infty$, run entire argument replacing $\mathbb{R}$ w/ $\mathbb{C}$ and $\mathbb{F}_2$ w/ $\mathbb{Z}$. $\square$

Example:

$$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]$$

Künneth

$$H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha_1] \otimes \mathbb{F}_2[\alpha_2]$$

$$\cong \mathbb{F}_2[\alpha_1, \alpha_2].$$

$$H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha_1, \alpha_2].$$

Cor: $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{F}_2)$

$\cong H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$

$\cong \mathbb{F}_2[\alpha, \beta]$.

$$H^*\left(\bigtimes_{i=1}^{n} \mathbb{R}P^\infty; \mathbb{F}_2\right) = \mathbb{F}_2[\alpha_1, \dots, \alpha_n].$$

$|\alpha_i| = 1$.

$$H^*\left(\bigtimes_{i=1}^{n} \mathbb{C}P^\infty; \mathbb{Z}\right) \cong \mathbb{Z}[\alpha_1, \dots, \alpha_n]$$

$|\alpha_i| = 2$.

$$H^*\left(\bigtimes_{i=1}^{k} S^{n_i}; \mathbb{Z}\right) = \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_k]$$

if n_i odd $\forall\ i$

$|\alpha_i| = n_i$.

tacking on even dimensional spheres adds $\otimes$ factors like $\underline{\mathbb{Z}[\alpha]/(\alpha^2)}$.

Consequence:

Thm (Hopf) If $\mathbb{R}^n$ has the structure of a division algebra, then n is a power of 2.

(Harder theorem is that $n = 1, 2, 4, 8$.)
Frobenius and Hopf, and Bott-Milnor, Kervaire.
$|ab| = |a||b|$.

Pf. "Algebra" is a ring A with 1 together with ring homom.

$$\mathbb{R} \to A$$
$$1_{\mathbb{R}} \mapsto 1_A$$

so that

$$f(\mathbb{R}) \subset \text{center}(A).$$

(It's an $\mathbb{R}$-module that's also a ring containing $\mathbb{R}$ and the $\mathbb{R}$ is a submodule.)

If $\mathbb{R}^n$ is an algebra over $\mathbb{R}$, then $\exists\, m: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

(write $ab = m(a,b)$) s.t.

$a(b+c) = ab + ac$ and $(a+b)c = ac + bc$

and $\alpha(ab) = (\alpha a)b = a(\alpha b)$ $\forall \alpha \in \mathbb{R}$.

Division algebra if $ax = b$ and $xa = b$ have sol^{ns} element $a \neq 0$.

$x \mapsto ax$ and $x \mapsto xa$ linear isos for $a \neq 0$.

So $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$$\rightsquigarrow \overset{\text{continuous}}{h} : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}.$$

$h|_{\text{factors}}$ is a homeomorphism.

$$h^* : H^*(\mathbb{RP}^{n-1}; \mathbb{F}_2) \to H^*(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2)$$

is a ring homo

$$\mathbb{F}_2[\alpha]/(\alpha^n) \to \mathbb{F}_2[\alpha_1, \alpha_2]/(\alpha_1^n, \alpha_2^n).$$

1 dim^l part of RHS is generated by α_1, α_2. So $h^*(\alpha) = k_1\alpha_1 + k_2\alpha_2$.

Now,

$\alpha^n = 0. \implies h^*(\alpha^n) = 0$

$$\implies 0 = (\alpha_1 + \alpha_2)^n = \underbrace{\sum_k \binom{n}{k} \alpha_1^k \alpha_2^{n-k}}_{\text{using that poly. ring is comm.}}$$

This is e_2^n in

$$\mathbb{F}_2[\alpha_1, \alpha_2]/(\alpha_1^n, \alpha_2^n) \quad \text{so}$$

$\binom{n}{k} \equiv 0$ whenever $0 < k < n$.

<u>Fact</u>. $\binom{n}{k} \equiv_2 0 \quad \forall \ 0 < k < n$ only when n is 2^ℓ.

<u>Equiv. Fact.</u> $(1+x)^n = 1 + x^n$ in $\mathbb{F}_2[x]$ **only** when $n = 2^\ell$.

<u>Proof</u>: Write out binary expansion of n:

$$n = \sum_{j=0}^{m} \varepsilon_j 2^j \quad \text{where} \quad \varepsilon_j \in \{0, 1\}.$$

So in binary, $n = \varepsilon_0 \varepsilon_1 \dots \varepsilon_m$.

The sum of the nonzero ones is

$$\sum_{i=1}^{k} n_i . \qquad n_i \text{ power of } 2.$$

Then $(1+x)^n = (1+x)^{n_1} \cdots (1+x)^{n_k}$

$$= (1+x^{n_1}) \cdots (1+x^{n_k}) \quad \text{Freshman's dream.}$$

Claim: no terms combine on right

Expand:

$$\overbrace{(1+x^{n_1}) \cdots (1+x^{n_{k-1}})}^{2^{k-1} \text{ terms}} \Big\} \Rightarrow \text{largest exponent is } \sum_{i=1}^{k-1} n_i$$

$$+ \underbrace{x^{n_k}(1+x^{n_1}) \cdots (1+x^{n_{k-1}})}_{2^{k-1} \text{ terms}} \Big\} \Rightarrow \text{smallest exponent is } n_k .$$

But,

$$\sum_{i=1}^{k-1} n_i = \sum_{j=0}^{m-1} \varepsilon_j 2^j \le \sum_{j=0}^{m-1} 2^j \le 2^m$$

i.e. $\underbrace{1111111}_{m-1} \leq 1\underbrace{00000000}_{m-1}.$

and so there are no like terms.

So 2^k terms total.

But equals $1+x^n$ iff $k=1$. $\square$

Manifolds.

An <u>n-mfld</u> M^n is a second countable Hausdorff space such that every point x has a nbhd that's homeomorphic to $\mathbb{R}^n$.

A manifold is <u>closed</u> if it's <u>compact</u>.

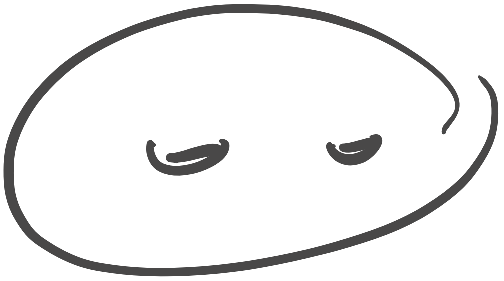
An <u>n-mfld with boundary</u> is a second countable Hausdorff space M^n s.t. every point x has a nbhd homeomorphic to $\mathbb{R}^n$ or $[0,\infty) \times \mathbb{R}^{n-1}$.

$$B^n = \{ \underline{x} \in \mathbb{R}^n \mid \|\underline{x}\| \le 1 \}$$

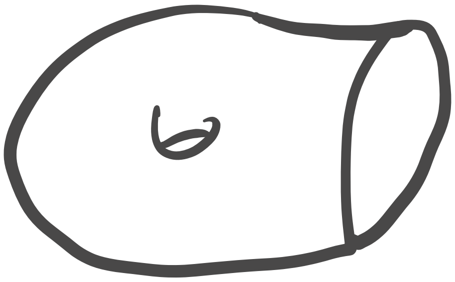
is a mfld w/ ∂.

The set of points x s.t. the nbhd is $\cong [0,\infty) \times \mathbb{R}^{n-1}$

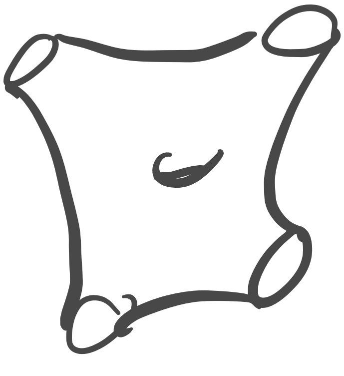
is the boundary of M^n,
it's denoted ∂M^n.
Often "manifold" means manifold w/∂.
A closed mfld is a compact mfld M with $\partial M = \emptyset$.



genus-2 surface is closed 2-mfld.



$T^2 - \mathring{B}^2$ is mfld w/ ∂.



Fun theorem. Define "large mfld" to be same defn but not require 2nd countable.

Thm (?) The # of connected large 2-mflds is $2^{\aleph_1}$.

$\aleph_1$ is the first uncountable cardinal #.

$S^1 \times$ Long Line.

Often mflds are obtained as $\mathbb{R}^n/G$ G group acting on $\mathbb{R}^n$. If action is badly behaved, you might get a non-Hausdorff "mfld" instead of a mfld.

Let M^n be an n-mfld.

If $x \in M^n$ then a local orientation μ_x at x is a choice of a generator of local homology

$$\langle \mu_x \rangle = H_n(M^n, M^n - x; \mathbb{Z})$$

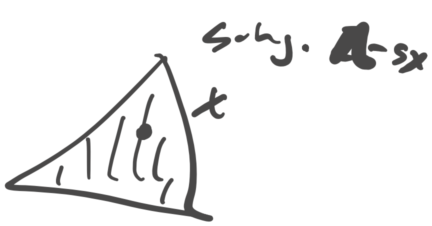
$$\overset{\text{excision}}{\cong} H_n(B^n, B^n - x; \mathbb{Z})$$

$$\cong H_n(B^n, \partial B^n; \mathbb{Z})$$

$$\cong \mathbb{Z}.$$

$B^n \ni x$ a ball

(look at long exact seq. of $(B^n, \partial B^n)$.



An orientation is a consistent choice μ_x of a local orientation.

$\forall z$ $\exists$ $B \ni z$ in M^n an embedded closed ball $\cong B^n$, s.t. if $x, y \in B$,

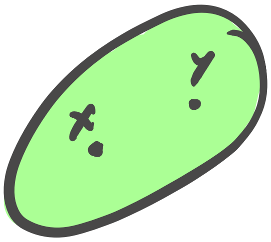
then $\mu_x \in H_n(M, M-x; \mathbb{Z})$

and $\mu_y \in H_n(M, M-y; \mathbb{Z})$

are both the image of the same generator $\mu_B \in H_n(M, M-B; \mathbb{Z})$

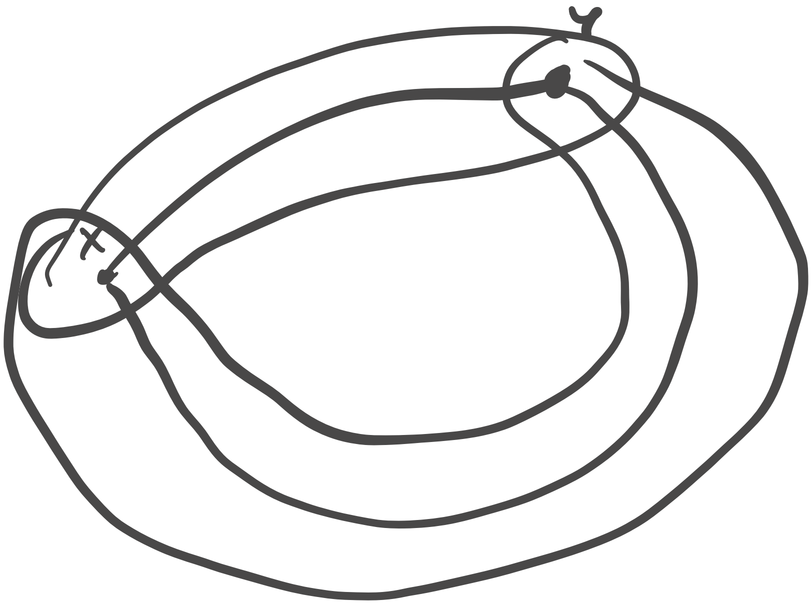
$$M - B \subset M - x$$
$$M - B \subset M - y$$

$$\begin{tikzcd}
\mu_B \arrow[r, mapsto] & \mu_x \\
(M, M-B) \arrow[r, "i_x"] \arrow[dr, "i_y"] & (M, M-x) \\
\mu_B \arrow[dr, mapsto] & (M, M-y) \\
 & \mu_y
\end{tikzcd}$$



$(i_x)_* \cong$

$(i_y)_* \cong$



Say M^n is orientable if it

$\exists$ an orientation of M^n.

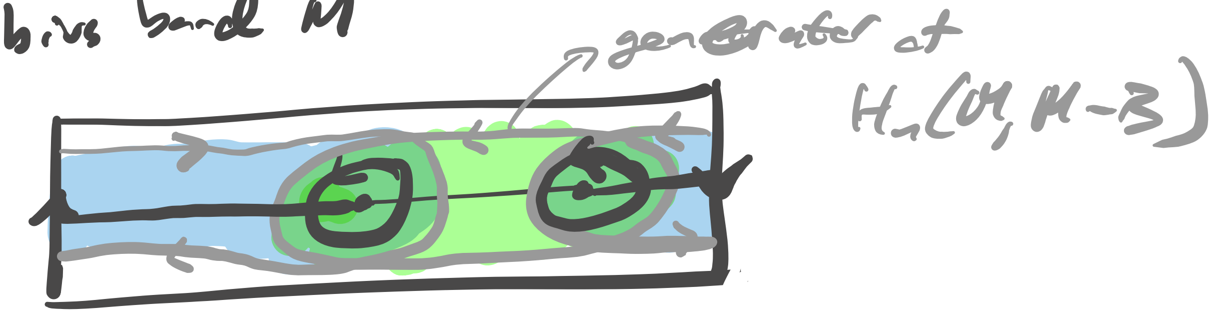
an oriented mfld is a mfld

with an orientation.

orientable.

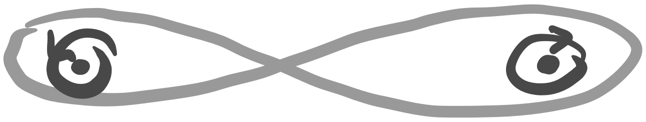
$\mathbb{RP}^2$ not orientable, "nonorientable"

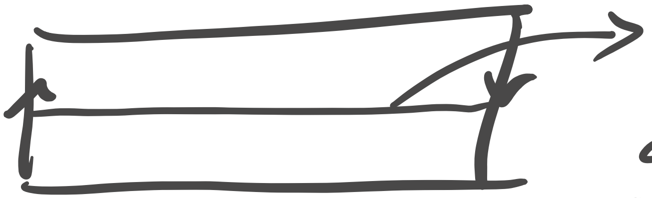
Möbius band M



Making cons. choice in Green ball and the blue ball simultaneously is impossible.

$$H_n(M, M-x)$$
$$\cong H_n(B, \partial M) \cong \underbrace{H_{n-1}(S^{n-1})}_{}$$

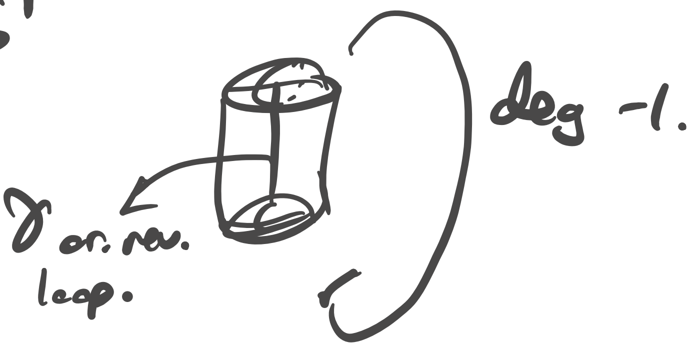




γ is an orientation reversing loop.

$S^2 \times S^1$ orientable mfld.

$S^2 \tilde{\times} S^1$



non-orientable.

loop γ is orientation reversing if nbhd is non orientable, i.e. $\cong S^{n-1} \tilde{\times} S^1$. $S^{n-1} \times I/f$ deg -1.

Orientable $\Longleftrightarrow$ no orientation reversing loops.

Thm Every n-mfld M has a degree 2 covering space that's orientable, called the orientable double cover $\tilde{M}$.

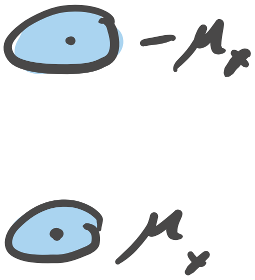
Pf. $\tilde{M} = \{ \mu_x \mid x \in M$ and μ_x is a generator of $H_n(M, M-x; \mathbb{Z}) \}$

Want a topology on $\tilde{M}$ so that $\tilde{M}$ is a mfld.

Given a open ball $\mathring{B}$ around x, take two copies of $\mathring{B}$, one for each choice of μ_x.

Define one copy to be a nbhd of μ_x and the other a copy of $-\mu_x$.

This gives us a basis for a topology on $\tilde{M}$.



$\tilde{M}$ is mfld by constr. and there is an obvious covering map $\tilde{M} \to M$ that sends $\mu_x \mapsto x$.

And $\tilde{M}$ is orientable, because choices are made for you in the construction of $\tilde{M}$.

If $\tilde{M}$ disconnected, then M orientable.

If $\tilde{M}$ disconnected.

It's a 2-fold cover of M

So if it's disconnected, each component is a 1-fold cover of M and so homeo. to M.

So that orients M, cuz $\tilde{M}$ oriented.

Conversely, if you're orientable then this cover will be disconnected.

M non-orientable iff $\tilde{M}$ connected.

<u>Cor.</u> If M non-orientable, then $\pi_1(M) \neq 1$, because M has 2-fold cover and so π_1 has index 2 subgp. So simply conn. mflds are always orientable

<u>Thm</u> $H_n(M) \longrightarrow H_n(M, M-x)$ is isomorphism iff M orientable and $H_n(M) = 0$ otherwise.

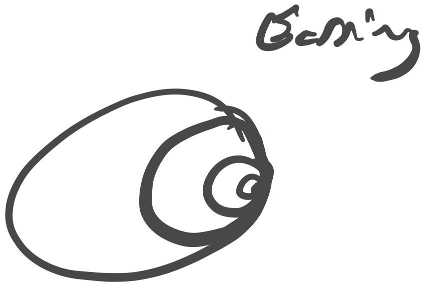
$H_n(M; \mathbb{F}_2) \cong \mathbb{F}_2$ always.

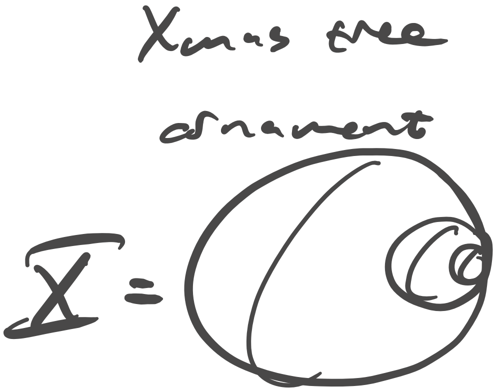
<u>Thm</u> $H_k(M) = 0$ if $k > n$.

For topological mflds this isn't obvious. If M cellular, then clear.

Singular Homology of topological spaces can be deceiving.

Ex. (Milnor)



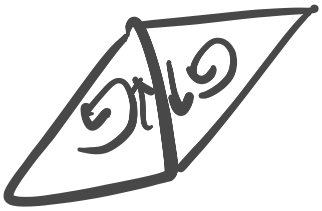


$$H_3(X; \mathbb{Z}) \neq 0.$$

Thm M^n oriented and connected and triangulated. Think of fact that $H_n(M; \mathbb{Z}) = \mathbb{Z}$

→ chain in $C_n(M)$ take 1 of each top dimensional cell. the orientation means that I can

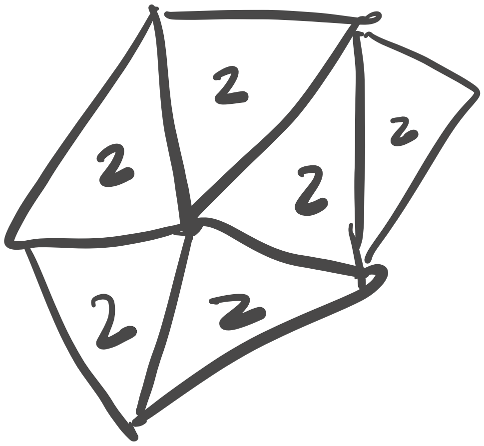
orient the sxs
so that this is
a cycle



Consist. orientation means
$\partial(\text{chain}) = 0$.
cycle.

non-zero cycle $[M]$
in $H_n(M;\mathbb{Z})$.

Geometrically clear that



every cycle in H_n
is a multiple of
$[M]$.

Next time Mflds w/ triangulations.

Clarify orientation:

orientation is a function

$M \longrightarrow$ set of local orientations

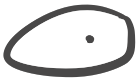
$= \{ (x, \mu_x) \mid x \in M, \mu_x \text{ gen. } H_n(M, M-x) \}$

$x \longmapsto \mu_x$ gen. $H_n(M, M-x)$.

s.t. $\forall x$ $\exists$ a ball nbhd B in which the choice of μ_x is consistent.

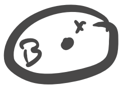
This makes it more obvious that $\tilde{M}$ is orientable.

(x, μ_B)

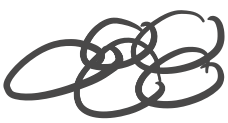


μ_B gen. $H_n(M, M-B)$

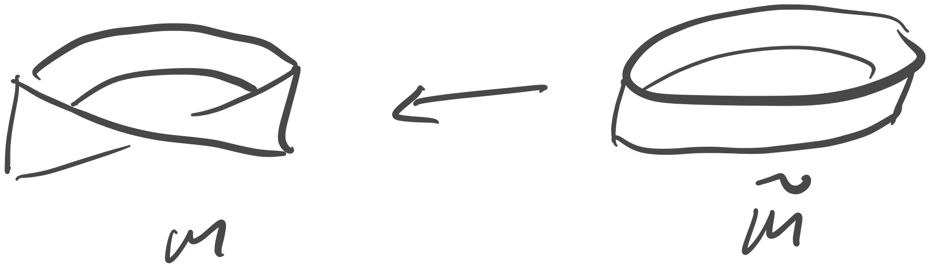
$\tilde{M}$



$(x, -\mu_B)$



cover M w/ emb. closed balls $\cong B^n$.



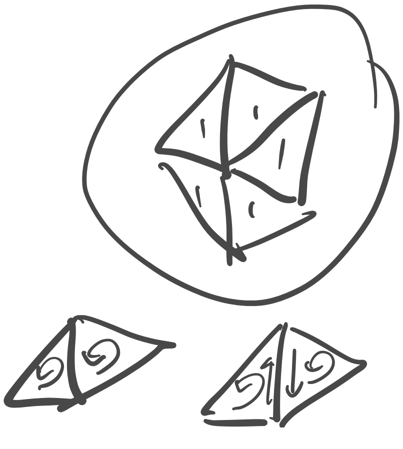
$$\mathbb{RP}^2 \longleftarrow S^2$$

Klein bottle $\longleftarrow T^2$

$\vdots$

Major detour away from Hatcher.

$H_n(M;\mathbb{Z}) = \mathbb{Z}$ when M^n orientable n-manifold.

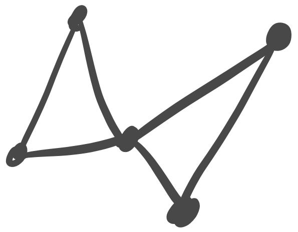


For triangulated mflds get fundamental class in $H_n(M)$ thanks to orientation

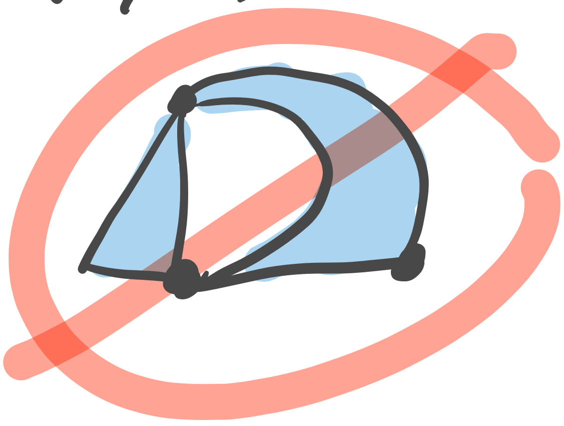
More combinatorial approach.

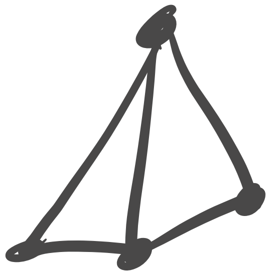
Focus on triangulated manifolds.

Def. A polyhedron is a space that's (a subset of $\mathbb{R}^N$) that is a union of simplices s.t. if two simplices intersect, they intersect in a single face (cx hull of subset of vertices of the sx)



very special CW cx.





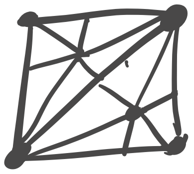
When algebraic topology started.

Polyhedra were first things considered.

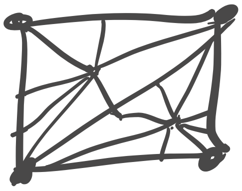
Simplicial homology (defined just like CW homology).

Once Homology is defined, why is it, e.g., a homeomorphism invariant?

T_1



T_2



Why would Homology be the same?

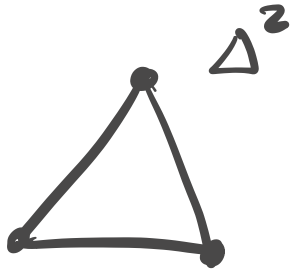
In this example, T_2 is a "subdivision of T_1".

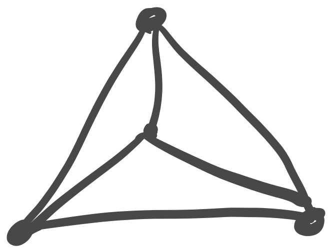
Say X a polyhedron and T_1, T_2 are triangulations

T_2 is a subdivision of T_1

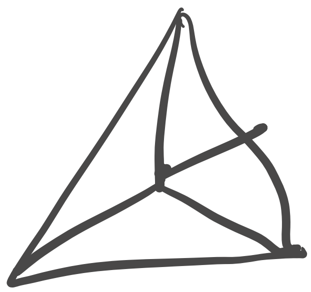
if every sx of T_2 is contained in a sx of T_1.

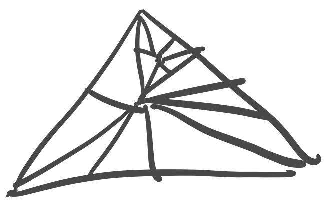
Some subdivisions:





T just 1-2x
3-1xs
3-0xs.

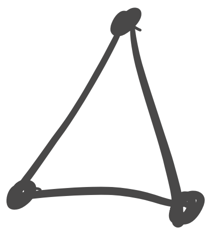




Canonical example of a subdivision:

(first) Barycentric subdivision of T

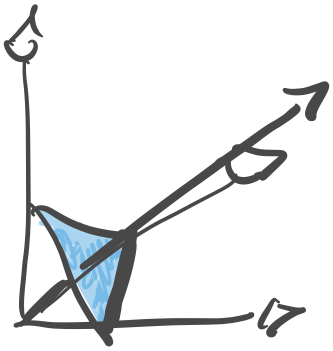
Suffices to define for a sx:



In $\mathbb{R}^{n+1}$, consider the standard basis $e_0, \dots, e_n$.

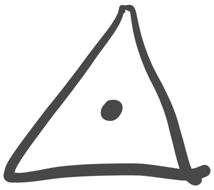
$\Delta^n = \{$cx combos. of $e_0, \dots, e_n\}$

$$= \left\{ \sum_{i=0}^{n} c_i e_i \;\middle|\; \sum_{i=0}^{n} c_i = 1 \text{ and } c_i \geq 0 \right\}$$



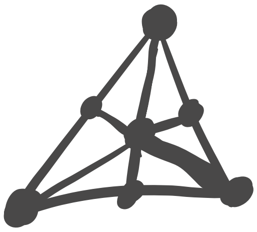
faces of Δ^n obtained by letting some of the c_i $= 0$.

Barycenter of $\Delta^n = \frac{1}{n+1} e_0 + \frac{1}{n+1} e_1 \ldots + \frac{1}{n+1} e_n$.



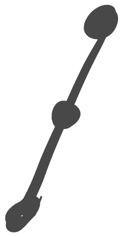
1st barycentric subdivision of Δ^n:

take all the barycenters of all the faces.

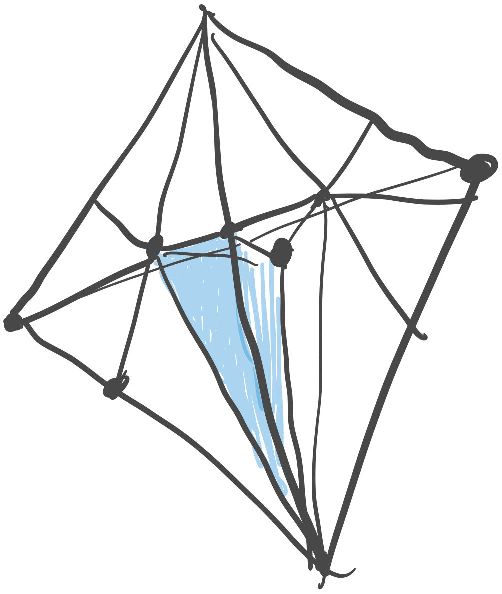


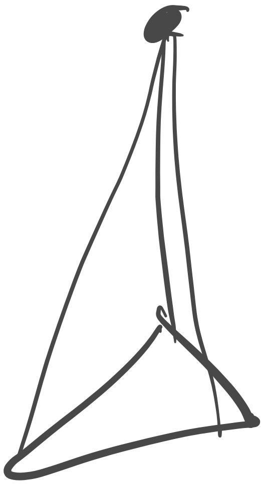
Define bar. subdiv. inductively.

Take barycentric subdivisions of faces.



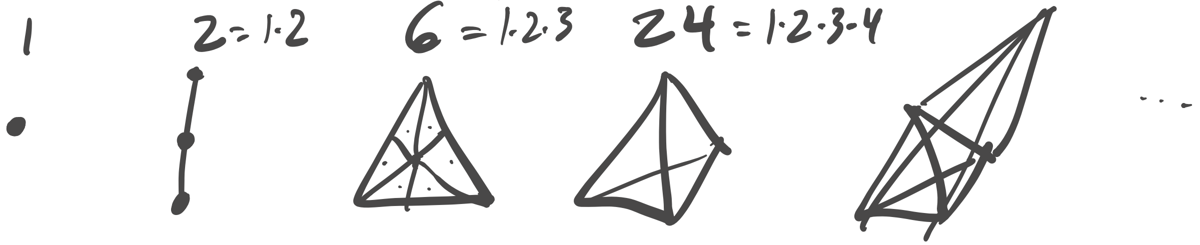
Cone the simplices in subdiv. of faces with barycenter of Δ^n.





How many n-simplices are there in 1st baryc. subdiv. of Δ^n?

1 $2=1\cdot2$ $6=1\cdot2\cdot3$ $24=1\cdot2\cdot3\cdot4$ $1\cdot2\cdot3\cdot4\cdot5$



. . .

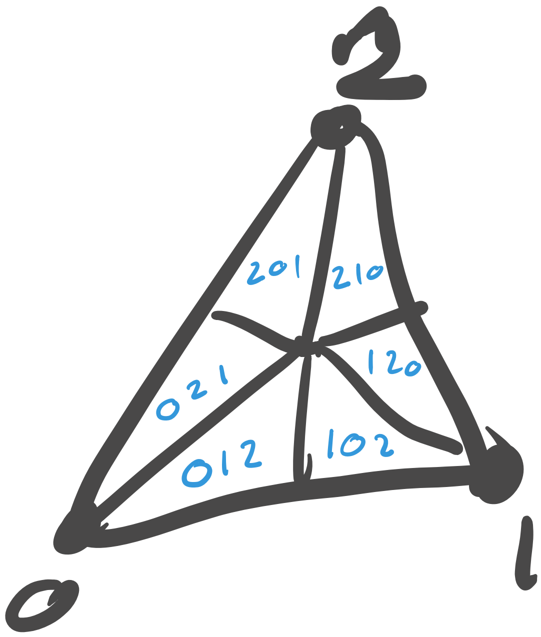
Δ^4

# of n-sxs in 1st barycentric subdiv. of Δ^n is $(n+1)!$

4 2-dim'l faces of Δ^3 each of which has 6 2-sxs in its 1st barycentric subdivision. So coning them off to barycenter of Δ^3, get $6\cdot4$ 3-sxs

Do it again, get 2nd barycentric subdiv.
3rd
4th etc.

n-sxs of 1st b-subdiv. of Δ^n
can be organized using symmetric
group S_{n+1}:



Q. Is simplicial homology homeomorphism invariant?

Strategy Idea: ① Show that a subdivision gives isomorphic homology.

② Show that if K_1 and K_2 are homeomorphic polyhedra, then the two triangulations have a common subdivision. (combinatorially equivalent), if possible.

Q. Hauptvermutung (Main Conjecture)

If K_1 and K_2 are homeomorphic polyhedra, then they have a common subdivision.

Thm (Rado - Kerekjato) (Top sfces are triang. and Hauptvermutung true for surfaces.

Fun *exercise*: Show that any two triangulations of a disk have a common subdivision.

Thm (Moise)

Hauptvermutung true for 3-mflds.
(and 3-mflds have triangulations)

Thm (Papakyriakopolous)

Hauptvermutung is true for 2-dim'l simplicial complexes.

Thm (Milnor)

Hauptvermutung is false.

Milnor's example is not a manifold.

Also false for mflds, forgot reference.

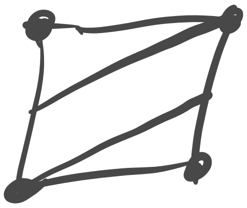
Here's Milnor's example: $(7,q)=1$.

Consider the lens space $L(7,q)$. (a 3-mfld)

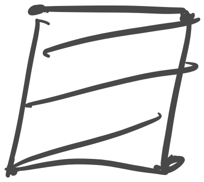
$D^2 \times S'$

$D^2 \times S'$

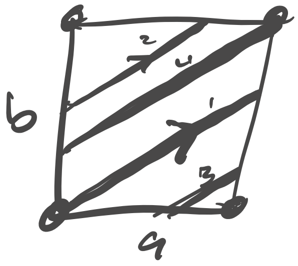
glue these together along boundaries so that β goes to a $(7,q)$-curve on $\partial(D^2 \times S') = T^2$.



$(2,1)$ curve



$(3,1)$



$(3,2)$

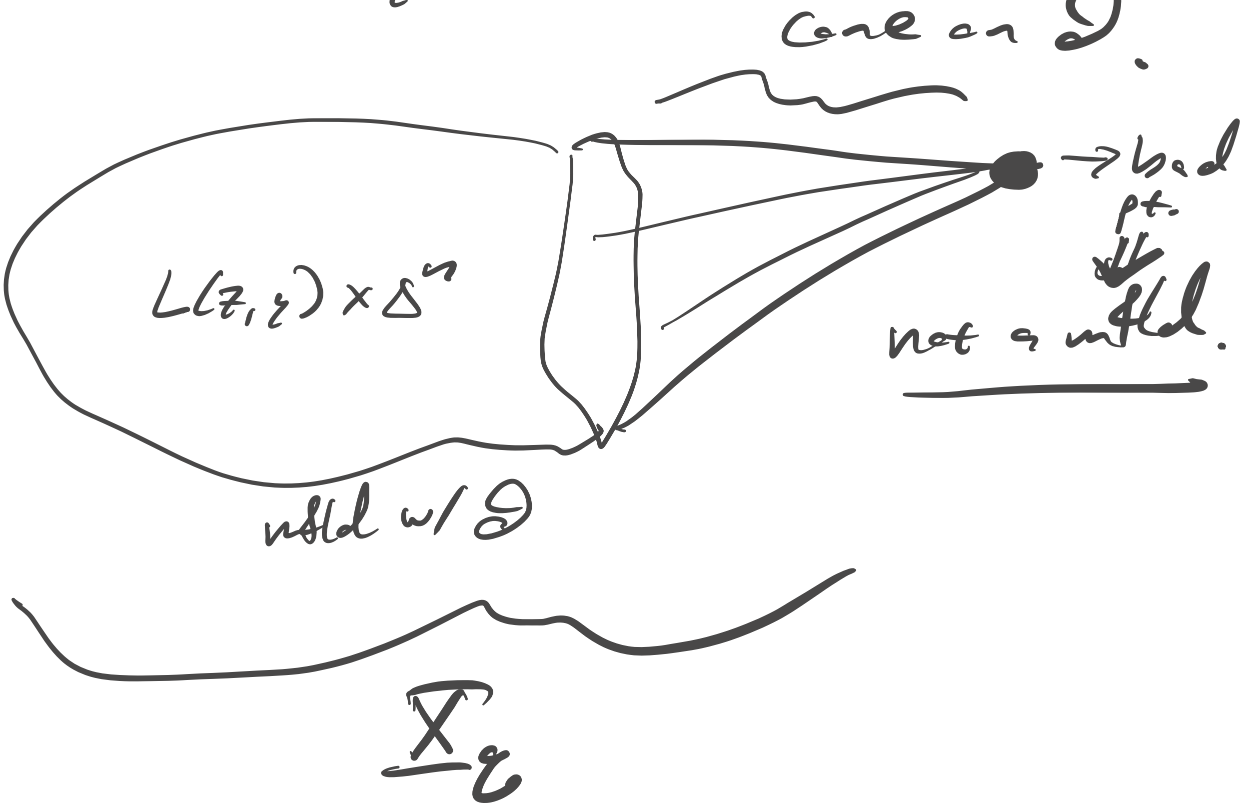
ex.

homology class $pa+qb$ is always represented by an embedded loop when $(p,q)=1$.

$\underbrace{L(7,q)}_{\text{triangulate.}} \times \Delta^n$ triangulate in "obvious" way.

$$\partial(L(7,q)\times\Delta^n) = L(7,q)\times\partial\Delta^n$$

attach the cone $C(\partial(L(7,q)\times\Delta^n))$ to $L(7,q)\times\Delta^n$.



Milnor shows that $X_1 \cong X_2$.

But the triangulations have no common

subdivision as long as $n \geq 3$.

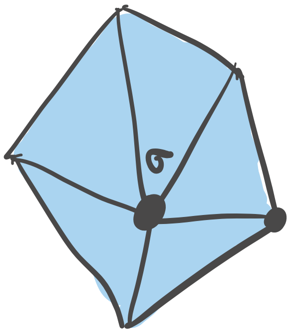
We're going to consider triangulated manifolds.

Terminology. K polyhedron.

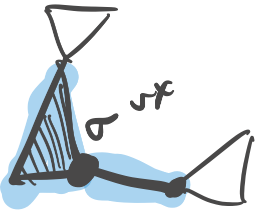
Let σ be a simplex of K.

The closed star of σ is the union of all closed simplices containing σ.

Exs:

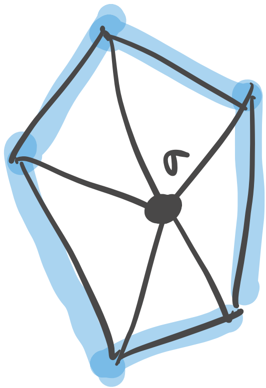


σ vertex in middle



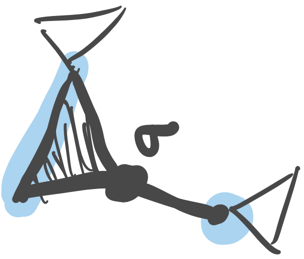
σ 1-sx surrounded by 2-sxs

Link of σ, $Lk(\sigma)$ is the union of the closed sxs in the star(σ) that don't intersect σ at all.

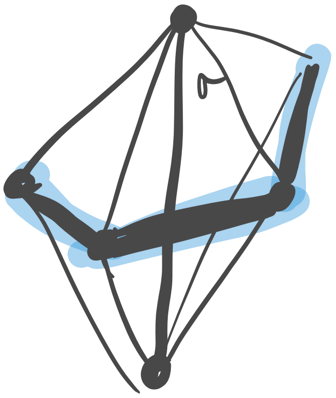


σ vertex

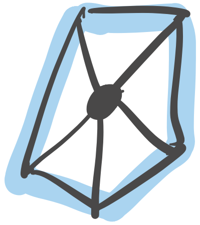
$Lk(\sigma)$ in blue.



$Lk(\sigma)$ in blue.

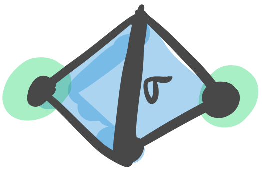


A triangulation of an n-mfld is PL ("piecewise linear") iff the link of every k-sx is an $(n-k-1)$-sphere.

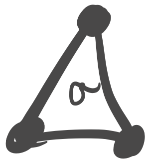


lk(vertex) is a circle
0 sx
$2-0-1=1$ sphere

Lk(σ) in green.



Lk(1-sx) is a $2-1-1=0$ sphere.



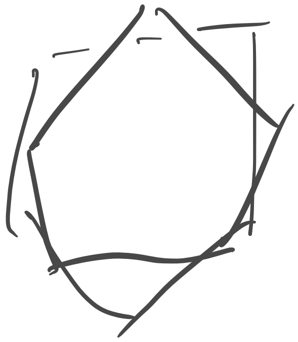
σ 2sx $\mathrm{lk}(\sigma) = \emptyset = -1$ sphere.

Why do we need this notion of PL triangulation?

Poincaré Homology sphere.

M = Dodecahedron / gluing opposite sides by a translation and a minimal clockwise rotation.

3-mfld.



$$\pi_1(M) = \langle r, s, t \mid r^2 = s^3 = t^5 = rst \rangle \neq 1$$

Binary Icosahedral group.

$$\pi_1(M)^{ab} = 1.$$

$$H_1(M) = 1.$$

Can check that $H_2(M) = 0$.

and it's orientable so $H_3(M) = \mathbb{Z}$.

$H_*(M; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$.

$M \not\cong S^3$ since $\pi_1 M \neq 1$.

By Moise, M triangulable,

But can triangulate directly

using dodecahedron.

Originally Poincaré asked:

Q: If N closed 3-mfld and

$$H_*(N) \cong H_*(S^3)$$

Is $N \cong S^3$?

A: (Poincaré) No. Because of PHS.

Q: (Poincaré Conjecture)

If $\pi_1(N) \cong 1$, Is $N \cong S^3$?

A (Perelman) Yes.

Same is true in high dimensions:

If $N \overset{h.e.}{\simeq} S^n$ then $N \cong S^n$

$n \geq 5$ (Smale) $\qquad \pi_1(S^2 \times S^2) = 1.$

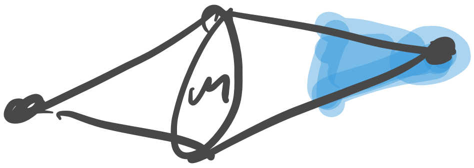
$n = 4$ (Freedman). $\qquad S^2 \times S^2 \ncong S^4$

by cohomology.

Let M be PHS.

$M \rightsquigarrow SM$ suspension.

$$\begin{array}{c} SM \\ \parallel \\ CM \cup_M CM \end{array}$$

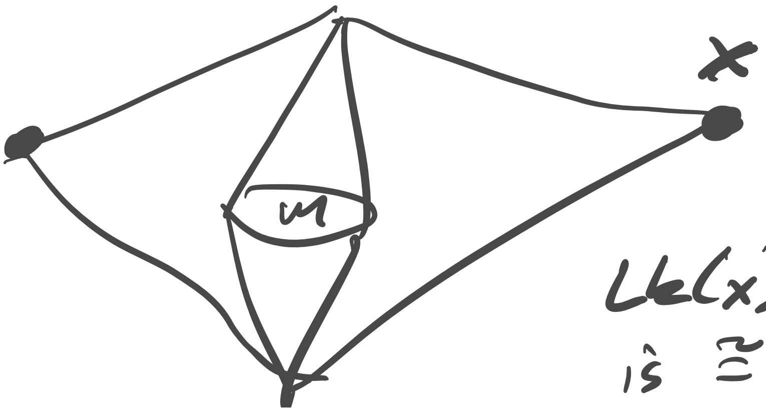


SM not a mfld.

"not locally simply connected."

But!

Thm (Edwards) $SSM \cong S^5$.



$\cong S^5$

$Lk(x)$ is $\cong SM \neq$ mfld.

not a PL triangulation.

Thm (Cannon-Edwards) M homology S^3 $\Rightarrow SSM \cong S^5$.

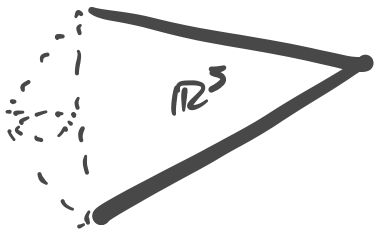
So S^5 has a triangulation that is not PL.

Link of cone point in SSM is not even a manifold!

of course S^5 has a different, nice triangulation.

In example,

$CSM - S1 \times \{0\} \cong \mathbb{R}^5$



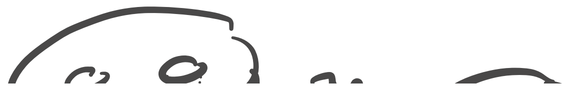
, which is weird.

In high dimensions, weird things happen.

Weird things can happen when cross with $\mathbb{R}$.

$\mathbb{C} - \{0, 1\} = S^2 - 3pts$

$(\mathbb{C} - \{0, 1\}) \times \mathbb{R} \cong (T^2 - \Delta) \times \mathbb{R}$



Nice visual

Exercise.

Whitehead continuum:

$\subset S^3$

$N(K) = \text{nbhd}(K) \cong D^2 \times S^1$.

Pick homeomorphism

$D^2 \times S^1 \longrightarrow N(K)$.

image of K in N(K)

Do that with that smaller solid torus.

over and over.

W = intersection of all of them.

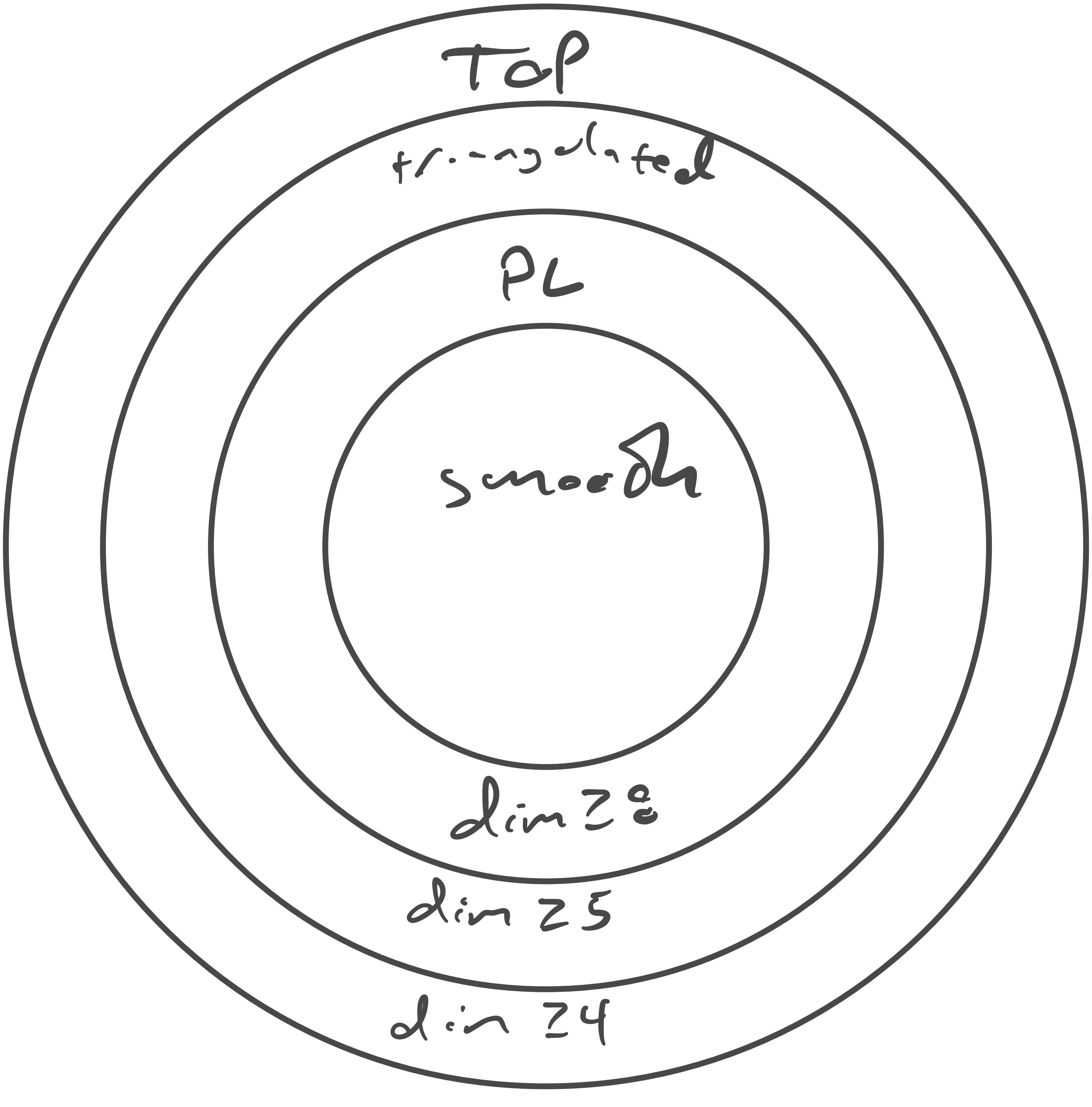
$$S^3/W \times \mathbb{R} \cong \underline{S^3 \times \mathbb{R}}.$$

$\underbrace{S^3/W}$ not a mfld.

Whole subject of wild geometric topology.

Landscape:

Manifolds:



smooth $\Rightarrow$ PL (Cairns, Whitehead)

Top $\not\Rightarrow$ PL (Freedman $n=4$)

(Kirby-Siebenmann $n \geq 5$)

Top $\not\Rightarrow$ Triangulable

(Freedman $n=4$)

(Manolescu $n \geq 5$)

Positive results

If $n \neq 4$ then $M \cong$ CW cx.

(Kirby-Siebenmann $n \geq 6$

Quinn $n = 5$)

Every $M^n \underset{h.e.}{\simeq}$ to an n-dim'l simplicial complex.

[when $n \neq 4$, $M \cong$ CW cx
$\Rightarrow M \cong$ sx cx
by Hatcher 2C.5
when $n=4$, harder.]

Open problem

Is every $M^4 \cong$ cell cx?

Common picture of geometric top:

3 categories:

TOP	top mflds w/ cont. maps
PL	PL mflds w/ PL maps
smooth	smooth mflds w/ smooth maps.

In smooth category:

"Smooth Poincaré Conjecture:"

Q: If M, N smooth mflds $\cong$ to S^n, are M and N diffeomorphic?

A: No. Milnor showed that there are 28 different smooth structures on S^7.

SPC open when $n=4$.

$n=4$ strange:

$\exists$ uncountably many smooth structures on $\mathbb{R}^4$.

Back to Poincaré Duality.

Poincaré Duality

M^n is a closed orientable n-mfld then $H_k(M^n;\mathbb{Z}) \cong H^{n-k}(M^n;\mathbb{Z})$

and $\forall$ closed mflds

$$H_k(M^n;\mathbb{F}_2) \cong H^{n-k}(M^n;\mathbb{F}_2).$$

Poincaré-Lefschetz Duality

M^n is compact, orientable mfld w/ ∂

then $H_k(M^n,\partial M^n;\mathbb{Z}) \cong H^{n-k}(M^n;\mathbb{Z})$

and $H^k(M^n,\partial M^n;\mathbb{Z}) \cong H_{n-k}(M^n;\mathbb{Z})$.

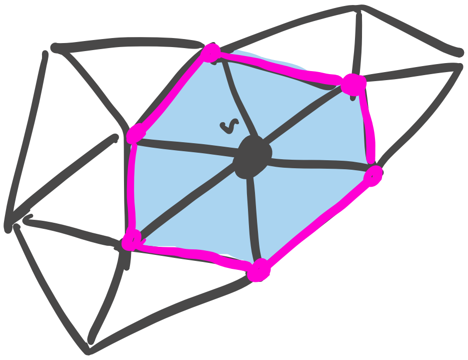
(also for $\mathbb{F}_2$ coeffs even if M^n non-orientable.)

[illegible] proof for PL mflds.

Talk about

PL mflds.

PL n-mfld M is a triangulated n-mfld with the property that the **link** of every k-simplex is an $(n-k-1)$-sphere.

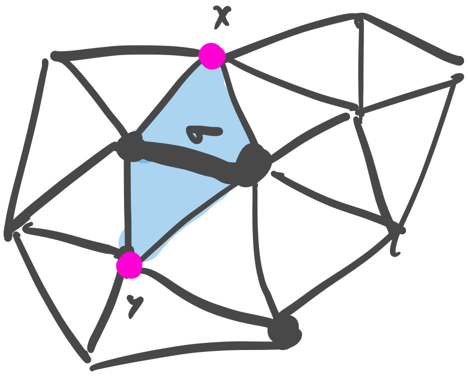


$\bigcup$ All closed sxs containing v

Blue $''$ $= \mathrm{Star}(v)$.

Pink $= \mathrm{Link}(v)$.

$$'' \\ S^1 = S^{2-0-1}$$



$$\mathrm{Star}(\sigma) = \bigcup_{\tau > \sigma} \tau$$

$$\mathrm{Link}(\sigma) = \mathrm{Lk}(\sigma) = x \sqcup y = S^0 = S^{2-1-1}$$

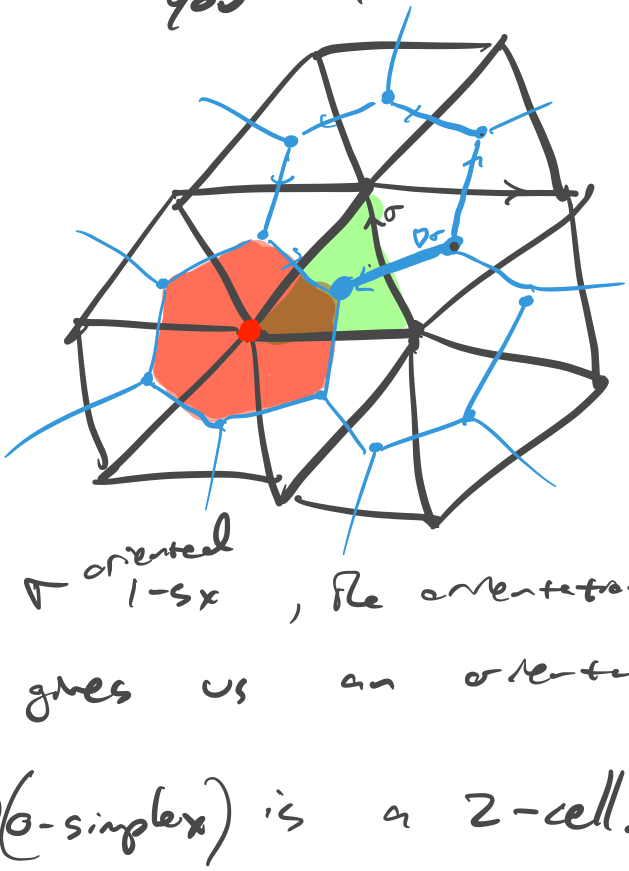
Seems like, at least for vertices, seems like $Lk(v) = S^{n-1}$ as long as M is a mfld.

That's false for arbitrary triangulations as we saw last time.

Idea: PL triangulation gives you a dual cell decomposition.

Use barycenters.

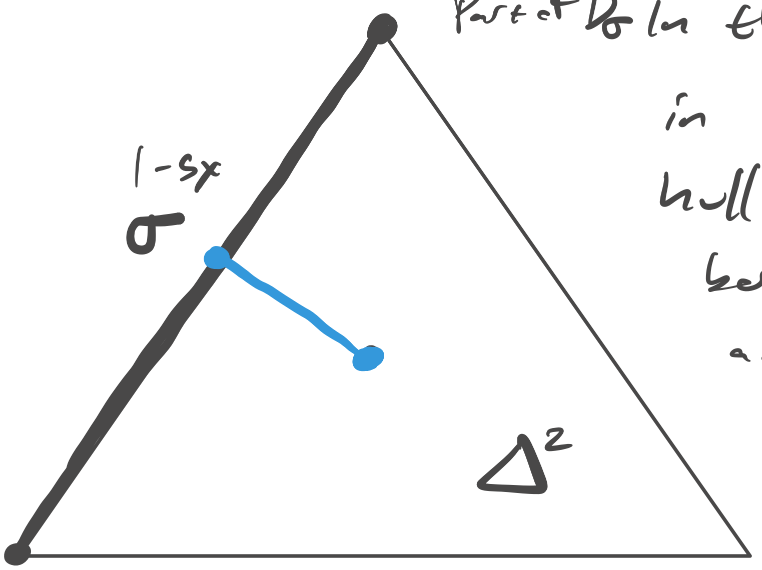
If σ is n-sx. Dual cell $D\sigma$ is the barycenter.



σ oriented 1-sx, the orientation of M gives us an orientation of $D\sigma$

D(0-simplex) is a 2-cell.

How to define these carefully?

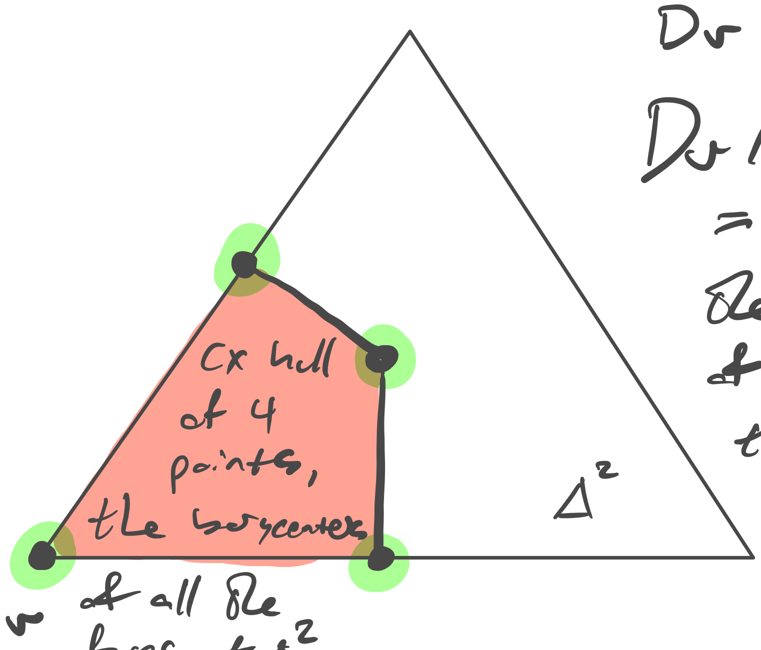


Part of D_σ in this 2sx Δ^2, in the convex hull of the barycenter of σ and the barycenter of Δ^2.

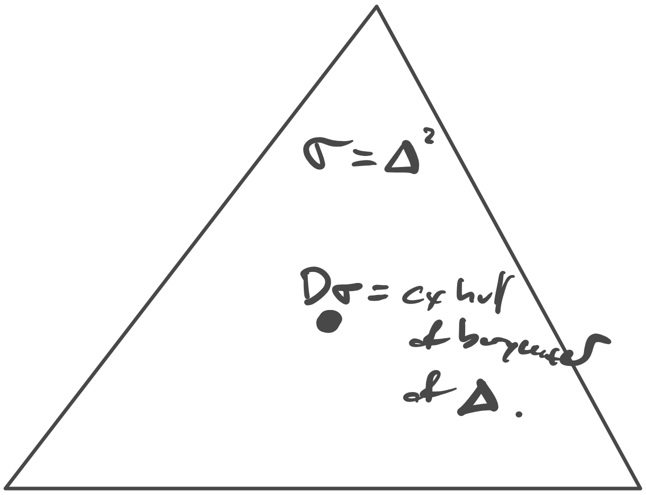
In Δ^2, σ and Δ^2 are the only sxs that contain σ.

D_v in Δ^2

$D_v \cap \Delta^2$ = cx hull of the barycenters of sxs in Δ^2 that contain v.



cx hull of 4 points, the barycenters of all the faces of Δ^2 that contain v.



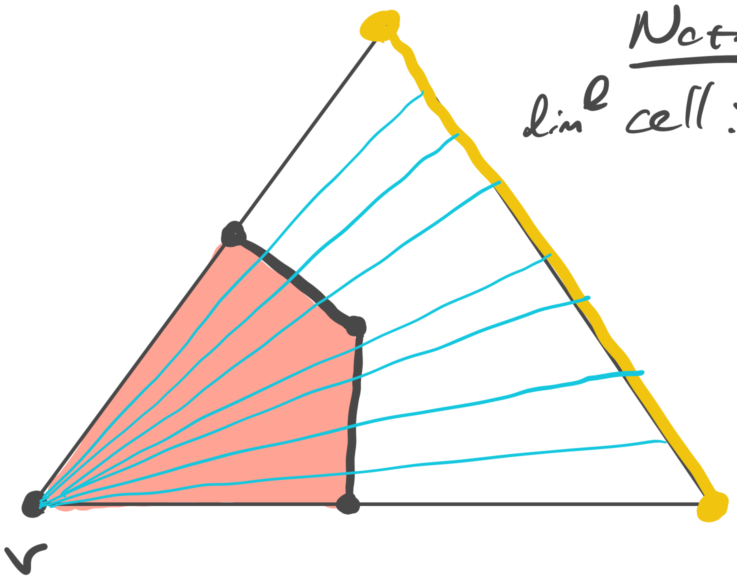
In a PL triangulation of M,

if σ k-simplex,

Then $\underset{\substack{\downarrow \\ \text{top dim}^l \\ \text{simplex}}}{D\sigma \cap \Delta^n}$ = cx hull of barycenters of the faces of Δ^n that contain σ.

Why is $D\sigma$ a cell?

<u>Claim</u>: $D\sigma$ is an $(n-k)$-cell.



Notice: In Δ top dim^l cell: $\underline{Lk(v)}$

$\|$

yellow part.

In this example, $D_v \cong C(Lk(v))$

In an n-simplex Δ

If $\sigma < \Delta$ is a face.

Then $\Delta \cong \text{Join}(\sigma, Lk(\sigma))$.

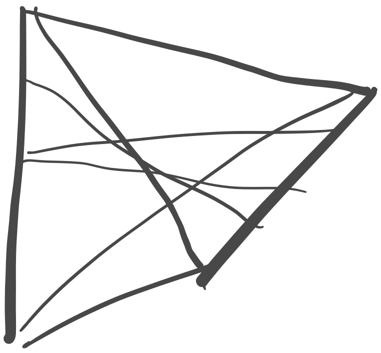
$$\overbrace{[\underbrace{\overbrace{v_0, \dots, v_k}}_{\substack{k\text{-sx} \\ \sigma}}, \underbrace{v_{k+1}, \dots, v_n}]}^{\Delta}$$

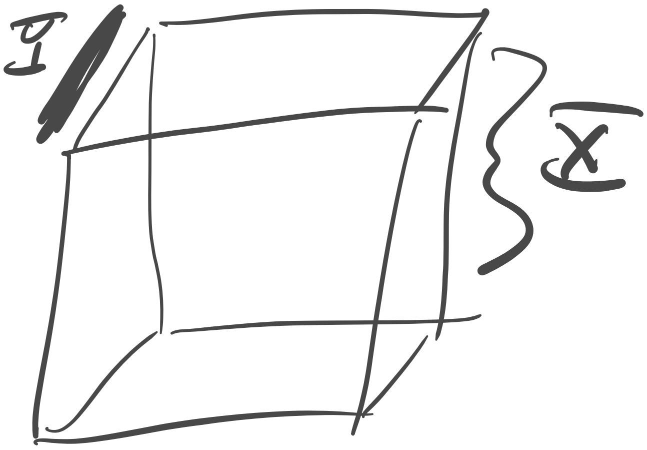
If X, Y spaces, form a new space

$J(X,Y) =$ all lines joining X to Y.

$$\shortparallel$$

$$X \times I \times I / * \times I \times \{0\}, X \times * \times \{1\}$$





Using $\Delta = J_{\Delta}(\sigma, Lk(\sigma))$ in Δ

see $D\sigma \cap \Delta = \underbrace{C}_{\substack{\text{conepoint} \\ = \text{barycenter} \\ \text{of } \sigma.}}(Lk(\sigma))$

Then $D\sigma = C\underbrace{Lk(\sigma)}_{\substack{\text{sphere} \\ \text{since} \\ \text{Triang. is PL}}} = \text{Ball.}$

Poincaré Duality

If M^n is an orientable n-mfld, then $H_k(M^n;\mathbb{Z}) \cong H^{n-k}(M^n;\mathbb{Z})$ and $H_k(M^n;\mathbb{F}_2) \cong H^{n-k}(M^n;\mathbb{F}_2)$ even if M not orientable.

True for all topological manifolds. We'll prove it for PL mflds.

---

i.e.: Triangulation w/ Links homeomorphic to spheres.

Let M^n be a PL n-mfld, orientable.

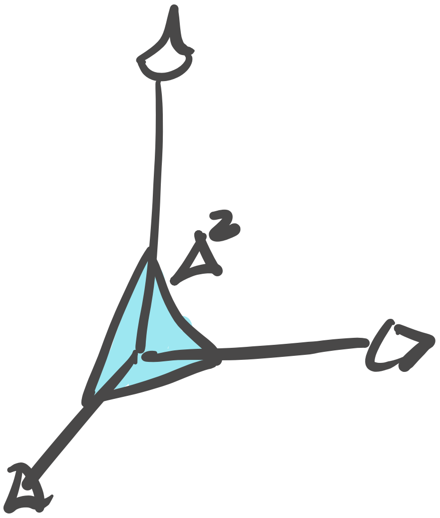
If $\sigma \subset \Delta^n = \{ \sum_{i=0}^{n} c_i e_i \mid c_i > 0 \ \forall i$

$\downarrow$

k-simplex

and $\sum_{i=0}^{n} c_i = 1$

for e_i std basis in $\mathbb{R}^{n+1} \}$



every k-sx $\sigma \subset M$ is in some n sx Δ^n in M.

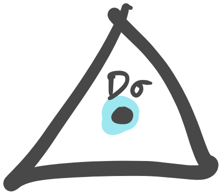
We define the <u>dual cell</u> D_σ of σ geometrically by declaring that

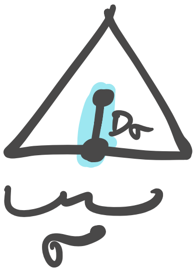
$D_\sigma \cap \Delta^n$ is the convex hull of the barycenters of all the faces of Δ^n containing σ.

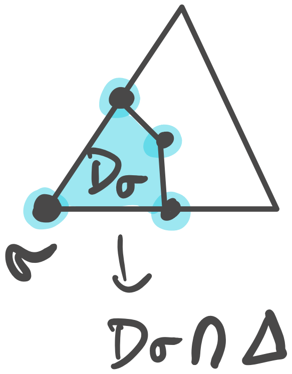
Let $\underline{\sigma = k\text{-sx}}$.

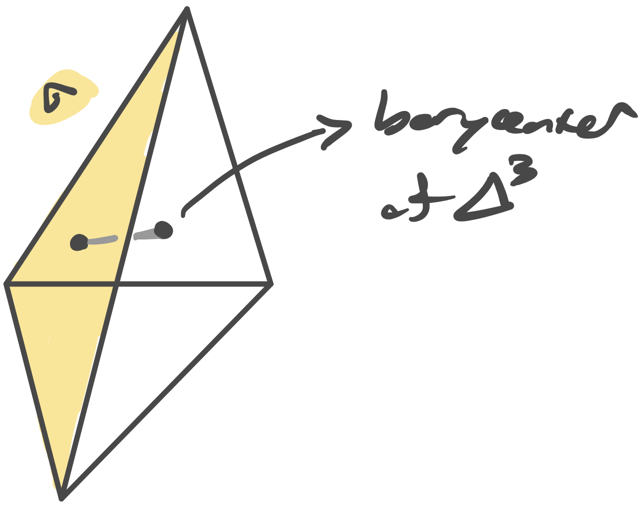
Ex:

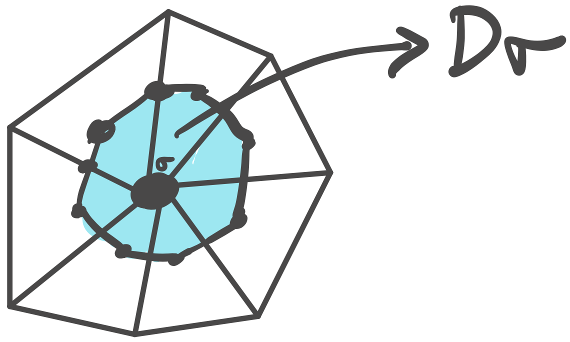
$\sigma = \Delta^n$











Dσ subcomplex of the first barycentric subdivision of M.

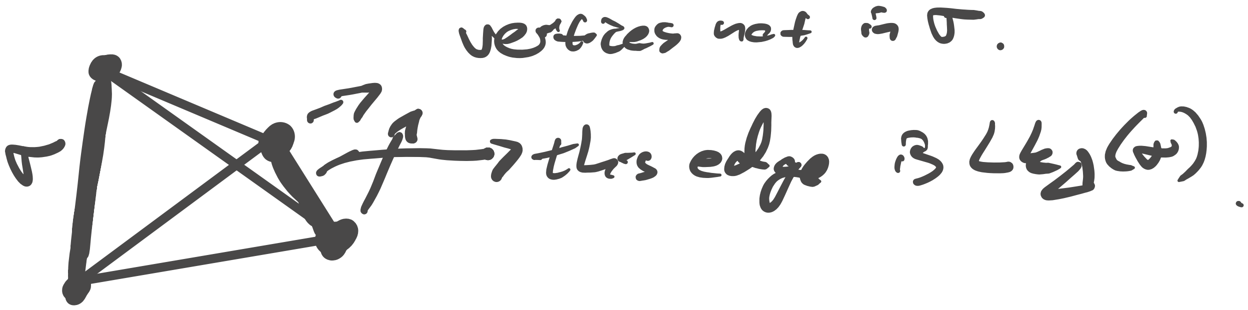
Claim: Dσ is a cell of dimension n-k.

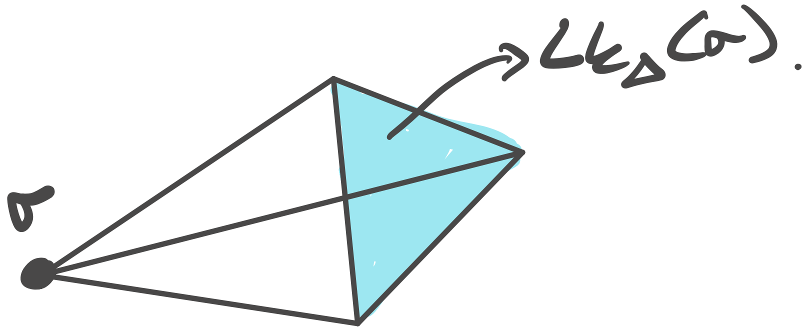
Think of $\Delta^n = \{0, 1, \dots, n\}$

If σ is a k-sx in Δ^n,

Then Link of σ in Δ^n, $Lk_\Delta(\sigma)$

is the convex hull of the vertices of Δ^n that aren't in σ:



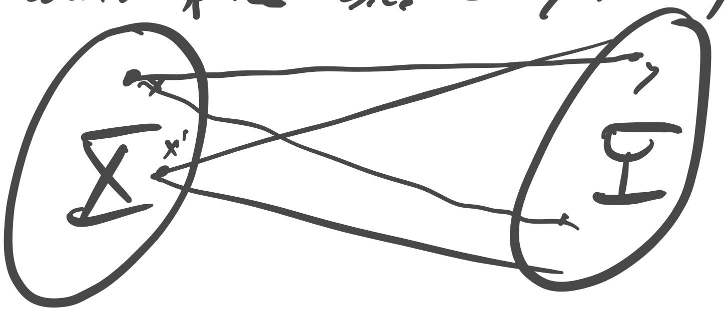


Notice also, that Δ^n is the "join" of σ and $Lk_\Delta(\sigma)$:

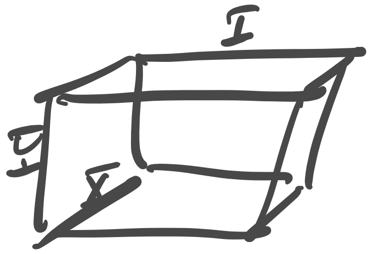
<u>Def</u>. The join of two spaces X and Y

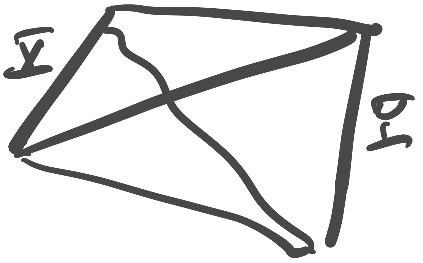
$$X \times Y \times I \Big/ \begin{array}{l} * \times Y \times I = x \\ \qquad\qquad\quad \# \\ X \times * \times I = y \end{array} = J(X, Y)$$

Want space s.t. every $x \in X$, $y \in Y$ joined by ! line segment.



$X = Y = I$





$$\Delta^n \cong J(\sigma, Lk_\Delta(\sigma)) = \sigma \times Lk_\Delta(\sigma) \times I \Big/ {}^{\sigma \times * \times I}_{* \times Lk \times I}$$

$$\|$$

$$\{\textstyle\sum c_i e_i \mid c_i > 0 \text{ and } \|\sum c_i e_i\|_1 = \sum c_i = 1\}$$

$$\|$$

$$\{\underbrace{t\underline{x} + (1-t)\underline{y}} \mid \underline{x} \in \sigma, \ \underline{y} \in Lk_\Delta(\sigma)\}$$

easiest to visualize when $\sigma = \{0, \ldots, k\}$

$$\sum_{i=0}^{n} c_i e_i = \sum_{i=0}^{k} c_i e_i + \sum_{i=k+1}^{n} c_i e_i$$

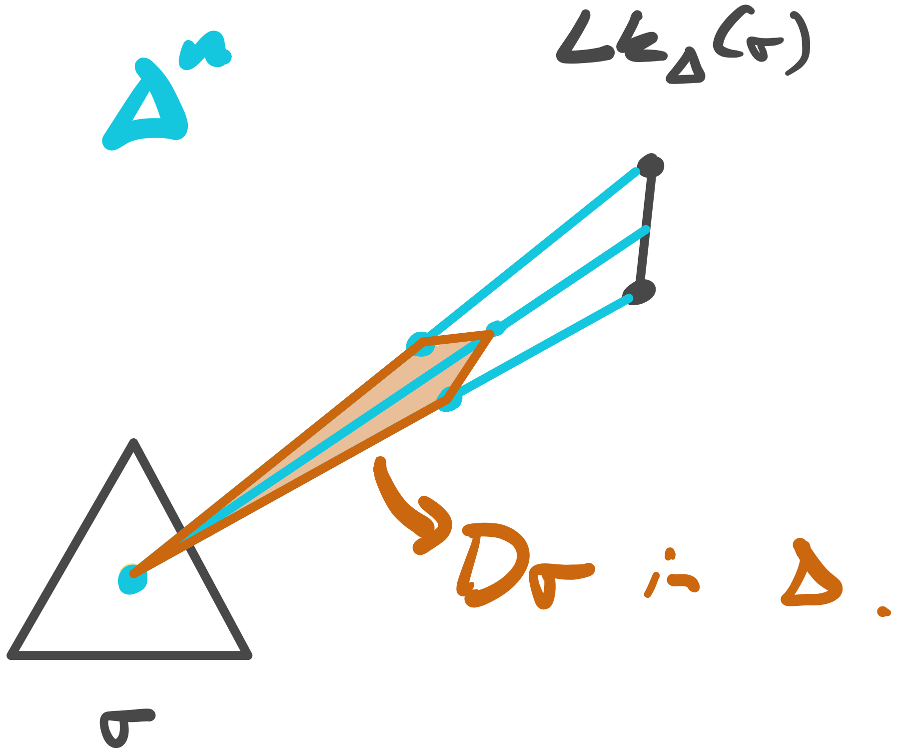
$$= X + Y$$

$$= \|X\|_1 \underbrace{\frac{X}{\|X\|_1}}_{\text{in } \sigma} + \|Y\|_1 \underbrace{\frac{Y}{\|Y\|_1}}_{\text{in } \mathrm{Lk}_\Delta(\sigma)}$$

since $\sum c_i = 1$, $\|Y\|_1 = 1 - \|X\|_1$.

☑

Why talking about joins?



From this picture we see that the cell D_σ in Δ is the cone on $Lk_\Delta(\sigma)$ with conepoint being the barycenter of σ.

$$D_\sigma \cap \Delta \cong C_{z(\sigma)}\{(Lk_\Delta(\sigma))\}.$$

$\Rightarrow$ $D_\sigma \subset M$ is homeomorphic to $C(Lk_M(\sigma))$

Since M is PL, $Lk_M(\sigma) \cong S^{n-k-1}$

and so $C(Lk_M(\sigma)) \cong B^{n-k}$

so $D_\sigma \cong B^{n-k}$, a cell!

The dual cells give us a "dual" cell decomposition.

Note on orientability:

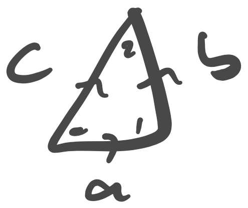
Here we can define an equivalence relation on simplices with orderings on vertices by declaring that a sx Δ with an ordering is equivalent to Δ w/ a different ordering iff orders differ by an even permutation.

With this definition, equivalent Δs have same $\partial\Delta$s. (exercise).

The equiv. class of the ordering is an *orientation* of Δ.

If Δ^{op} is Δ with opp. orientation, then $\Delta + \Delta^{op}$ is a cycle:

$\partial \Delta = a + b - c$ $\quad \partial \Delta^{op} = -a + c - b$

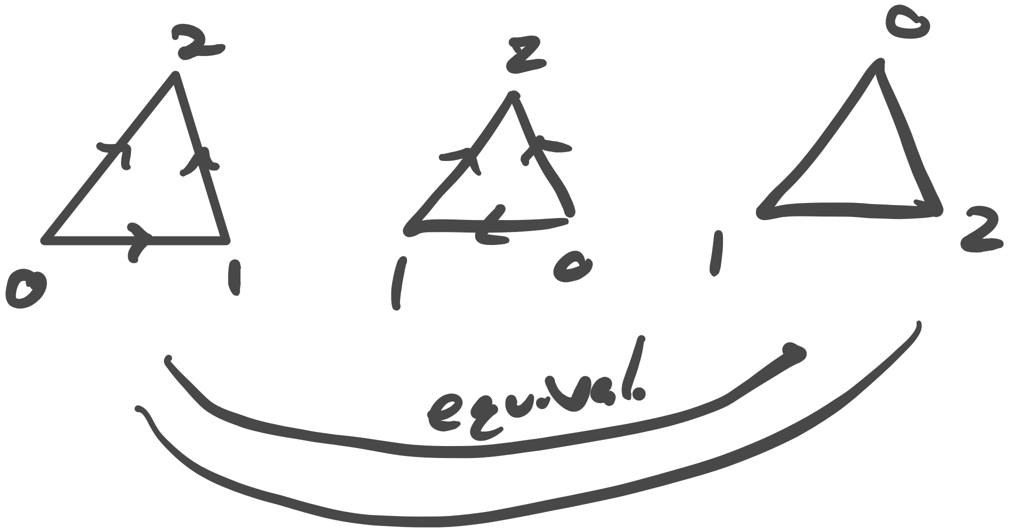


And in fact:

$\Delta + \Delta^{op}$ is a boundary.

Assuming M is orientable, that means we can compatibly assign a preferred orientation to each Δ^n in M.

i.e. ∃ a choice of ordering of the vertices of each Δ^n s.t. ∂ operator makes all the $n-1$ sxs cancel. Combinatorially, an Orientation of Δ^n is an equiv. class of orderings of the vertices two orderings are equivalent if they differ by an even permutation.

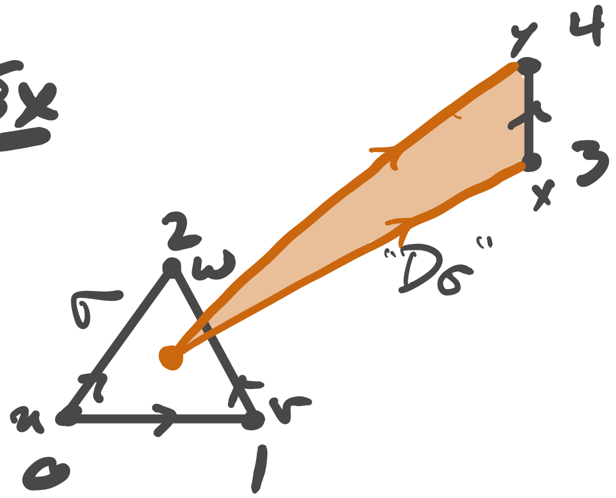


The orientability is helpful!

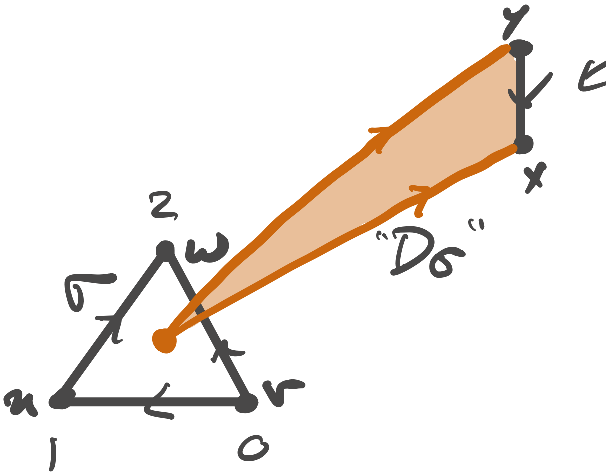
Given an oriented σ, in X, there is a canonical orientation of $D\sigma$, namely the orientation that orients Δ^n according to the orientation on M:

Ex

Δ^4

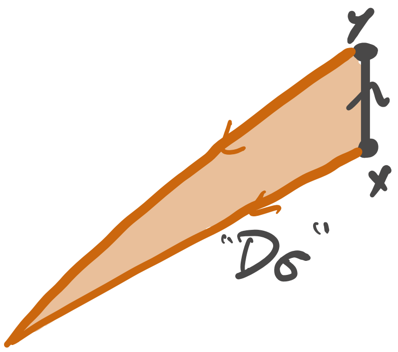


If we change orientation of σ

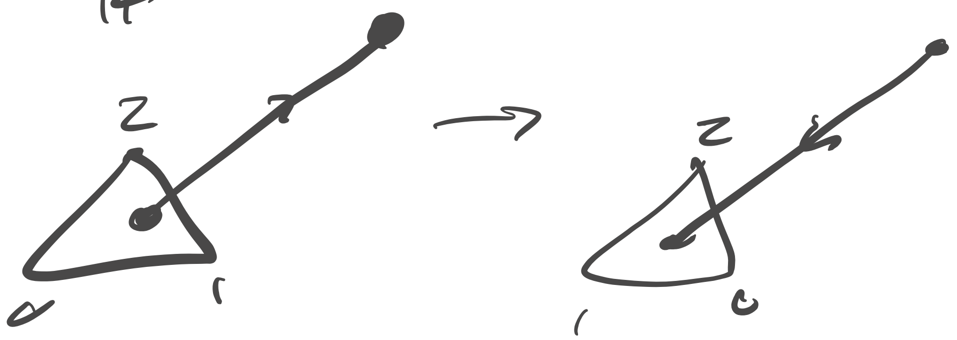


new orientation on $D\sigma$.

Why not the 3?



[If do odd perm to σ, need an odd permutation on $D\sigma \cap \Delta$]



So given oriented σ, we have oriented $D\sigma$.

Now: Compute simplicial homology using the triangulation and the cellular cohomology of M using Dual cell decomposition.

IDea: ∂ becomes δ when we dualize $\chi_{D\sigma}$ and replace cells w/ cochains.

$$\begin{tikzcd} \sigma \arrow[r, mapsto, "\partial"] \arrow[d] \arrow[dr, phantom, "\circ"] & \partial\sigma \arrow[d] \\ \chi_{D\sigma} \arrow[r, mapsto, "\delta"'] & \chi_{\partial\sigma} \end{tikzcd}$$

Given an oriented k-sx σ,

let χ_{D_σ} be the cellular $(n-k)$-cochain

s.t. $\chi_{D_\sigma}(e^{n-k}) = \pm 1$ if $e^{n-k} = \pm D_\sigma$.

and $\chi_{D_\sigma}(e^{n-k}) = 0$ otherwise.

let $\underbrace{\delta \chi_{D_\sigma}(e^{n-k+1})}_{\delta \text{ cell coboundary operator}} =$ degree of ∂e^{n-k+1} passing over D_σ, where ∂ is ∂ of cell.

---

what happens when we look at dual cells of the ∂ of σ?

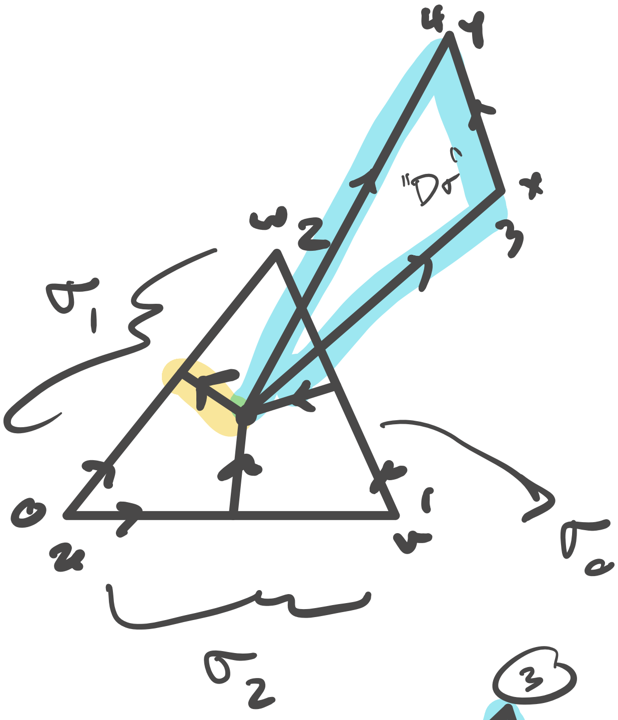
For ease of notation, let

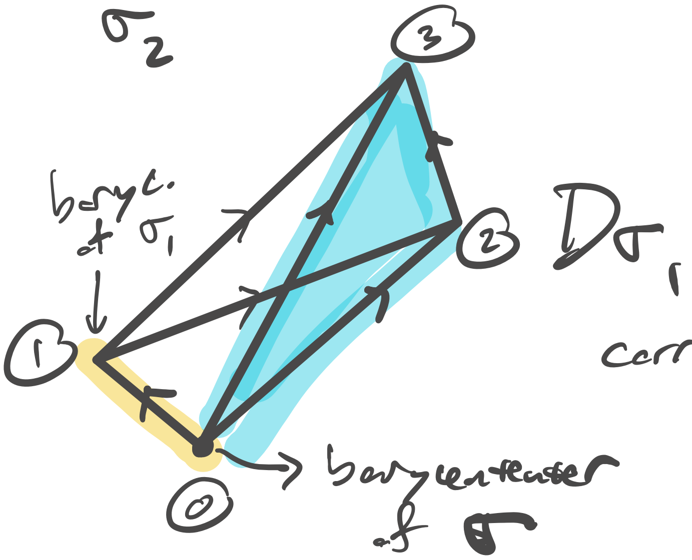
$\sigma_i = \sigma$ w/ ith vertex deleted.

so $\partial\sigma = \sum (-1)^i \sigma_i$.

what are the duals of σ_i?

Let's look at $D\sigma_1$.





$$\partial D\sigma_1 = [\textcircled{1},\textcircled{2},\textcircled{3}] - \overbrace{\underbrace{[\textcircled{0},\textcircled{2},\textcircled{3}]}_{\text{negative sign.}}}^{D\sigma} + [\textcircled{0},\textcircled{1},\textcircled{3}] - [\textcircled{0},\textcircled{1},\textcircled{2}]$$

So, the degree w/ which ∂D_{σ_1} goes over D_σ is -1.

Similarly, ∂D_{σ_0}, ∂D_{σ_2} pass over it with degree $+1$.

In general, ∂D_{σ_i} passes over D_σ with degree $(-1)^i$.

Now! Consider the correspondence

$$\begin{tikzcd}
\sigma \arrow[r, mapsto, "\partial"] \arrow[d, "D"'] & \sum_{i=0}^{k} (-1)^i \sigma_i \arrow[d, "D"] \\
\chi_{D_\sigma} \arrow[r, mapsto] & \underbrace{\sum_{i=0}^{k} (-1)^i \chi_{D_{\sigma_i}}}_{\varphi_\sigma \text{ a cochain.}}
\end{tikzcd}$$

Now, the only $(n-k+1)$-cells in dual cell decomposition that pass over $D\sigma$ are the $D\sigma_i$.

i.e. $\varphi_\sigma(e^{n-k+1}) = 0$ unless

e^{n-k+1} is $\pm D\sigma_i$

Furthermore, by the above,

$\varphi_\sigma(D\sigma_i) = (-1)^i$

But that's exactly what

$\delta\chi_{D\sigma}$ does.

So $\varphi_\sigma = \delta\chi_{D\sigma}$.

$$\begin{array}{l} \quad \| \\ \sum(-1)\chi_{D\sigma_i} = \chi_{D(\partial\sigma)} \end{array}$$

So kth simplicial homology of M is isomorphic to the $(n-k)$th cellular cohomology of M (from dual cell decomp.)

and $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$.

and $H_k(M;\mathbb{F}_2) = H^{n-k}(M;\mathbb{F}_2)$.

The map being our dualization map $\sigma \longmapsto \chi D\sigma$.

☑

Cor. Since $H^*(X;F)$
$$\cong H_*(X;\mathbb{F})^* \cong H_*(X;\mathbb{F})$$
we have $H_k(M^n;\mathbb{F}) \cong H_{n-k}(M^n;\mathbb{F})$

and $H^k(M^n;\mathbb{F}) \cong H^{n-k}(M^n;\mathbb{F})$.

Cor. If M or. odd dim'l mfld, then $\chi(M) = 0$.

$\square$

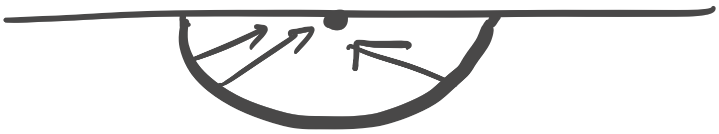
Poincaré-Lefschetz Duality

If M R-or. n-mfld w/ ∂

$$H^k(M;R) \cong H_{n-k}(M,\partial M;R)$$

and $H_k(M;R) \cong H^{n-k}(M,\partial M;R)$.

Geometric proof works here too



Alexander Duality

K is a compact, locally contractible, nonempty proper subspace of S^n, then

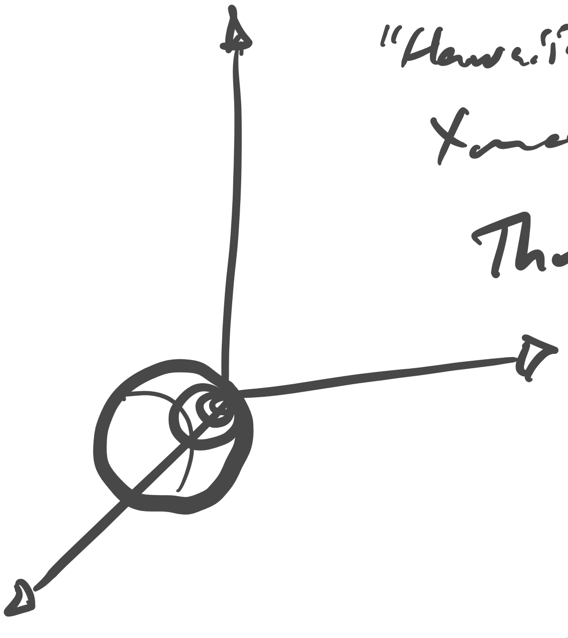
$$\tilde{H}_i(S^n - K;\mathbb{Z}) \cong \tilde{H}^{n-i-1}(K;\mathbb{Z}).$$

Cor. If $K \subset \mathbb{R}^n$ cpct locally contractible then $H_i(K;\mathbb{Z}) = 0$ if $i \geq n$, and torsion-free if $i = n-1$ and $n-2$.

Need the local contractibility.

E.g. Let $K = \bigcup_{n=1}^{\infty} S^2_n$

where S^2_n = sphere of radius $1/n$ centered at $(\frac{1}{2}, 0, 0)$.



2-dim'l "Hawaiian earring," Hawaiian Xmas tree ornament.

Thm (Milnor)

$$H_3(K; \mathbb{Z}) \neq 0.$$

E.g. of Alexander Duality in action:

Let K be a knot in S^3, i.e. $K \cong S^1$ sitting in S^3.

figure 8 knot.

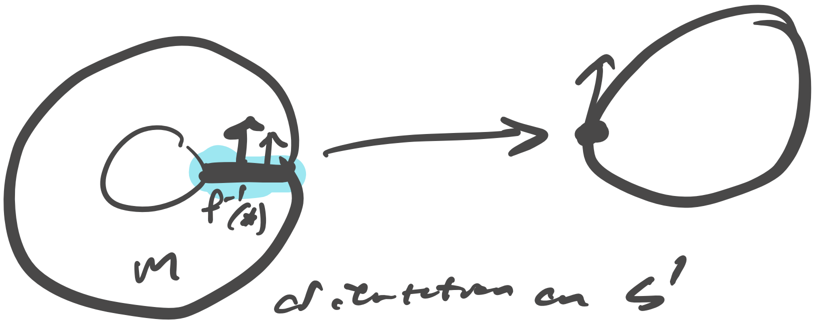
$$\underbrace{\boxed{H_1(S^3-K;\mathbb{Z})}}_{\text{can also use Mayer Vietoris.}} \cong H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$$

Using techniques we talked about earlier to construct a smooth map

$$f:(S^3-\text{nbhd}K) \longrightarrow S^1$$

pull back regular value and get

$f^{-1}(*)$ = orientable surface.



Change map get new sfce.

orientation on S^1

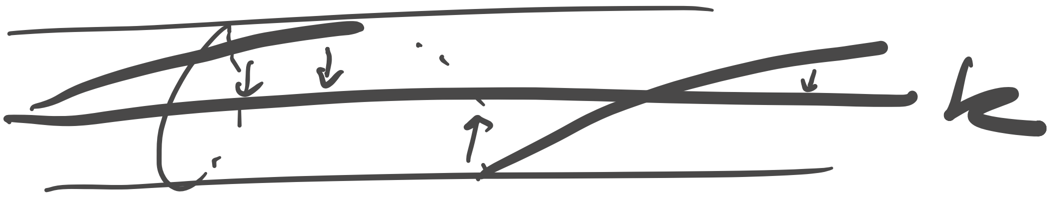
and orientation on M

⇝ orientation of $f^{-1}(*)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad S$

called a seifert surface for K. This a surface whose ∂ is a copy of K



∂S is isotopic (htpc through embeddings) to K.

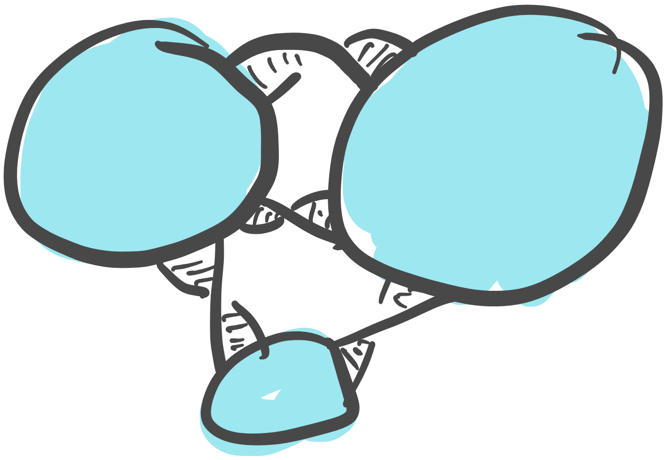
$\chi(F) = -1$

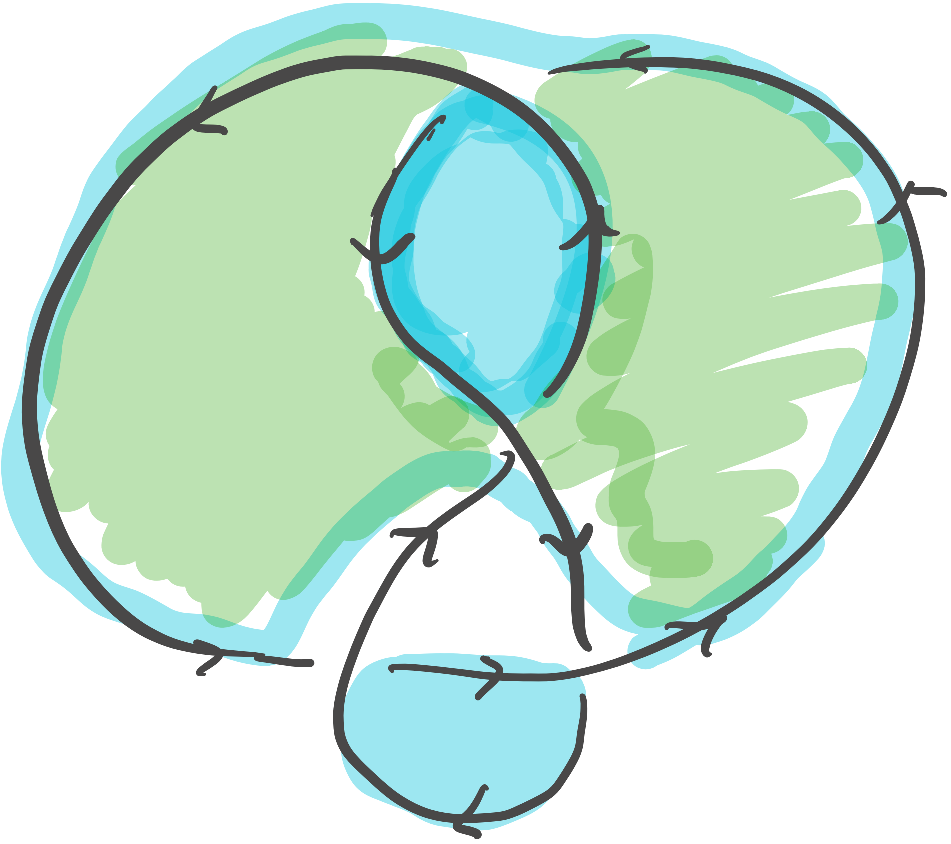
1 boundary component

F = punctured klein bottle.

$K = \partial F$

F is not orientable!





$T^2 - \mathring{B}^2$.

Thm (Hirsch) Every closed orientable 3-mfld embeds smoothly into S^5.

Cor (Rohlin) Every orientable 3-mfld is the boundary of orientable 4-mfld.

(closed nonor. 3-mflds also embed in S^5. and they are also always ∂s! ($\mathbb{RP}^2$ is not a ∂ at all)

Proof of Rohlin

By Hirsch, M^3 closed orientable embeds in S^5.

By Alexander Duality,

$$H_1(S^5 - M^3) \cong H^{5-1-1}(M;\mathbb{Z}) \cong \mathbb{Z}$$

So $H_1(S^5 - \text{nbhd}(M^3)) \cong \mathbb{Z}$.

So build a map

$f: S^5 - \text{nbhd of } M^3 \longrightarrow S^1$

and pull back a regular value to get an orientable 4-mfld $f^{-1}(*)$ whose ∂ is $\cong M^3$. $\square$.

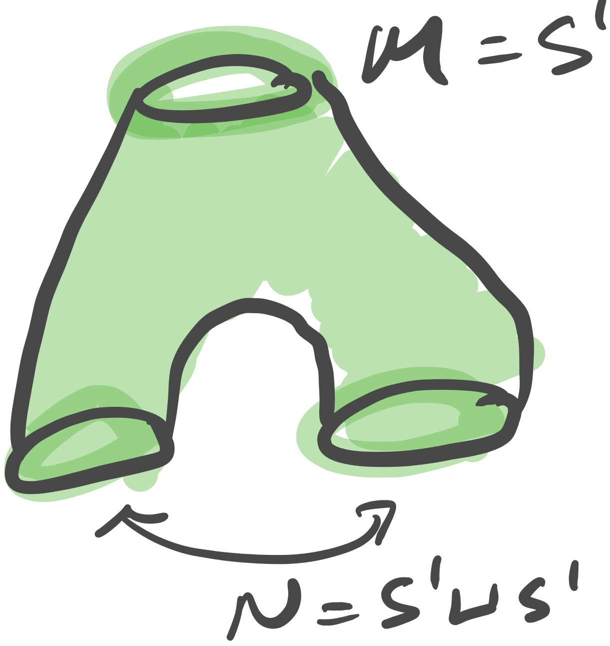
This a theorem in the study of "cobordism."

Basic cobordism stuff.

Look at set of all closed unoriented n-mflds with following equiv^N relation:

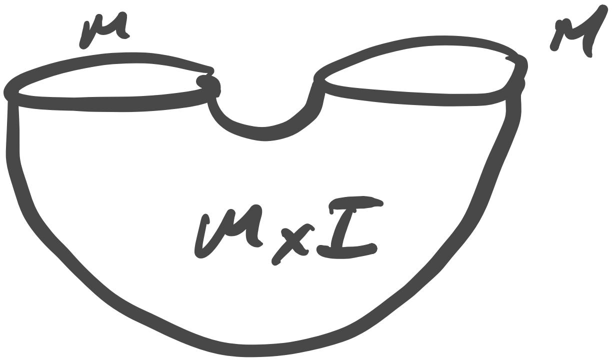
$$M \sim N \text{ if } M \sqcup N \cong \partial W$$

where W is a connected compact $n+1$ mfld with ∂.



<u>Notice</u>:

$M \sqcup M \sim \phi$



So $M \sqcup M \sim S^n$

$\sim \phi$.

$S^n \sim \phi$

$\phi = 0$ then

The set of equivalence classes is a abelian group. $M = M^{-1}$

operation is $\sqcup / \sim$

N_n = unoriented n-dim'l cobordism gp.

2-torsion. $2M = 0$.

$N_0 = \mathbb{Z}/2\mathbb{Z}$

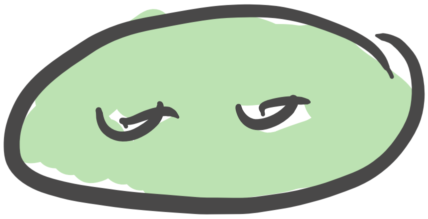
$$\begin{matrix} \mathbb{Z}/2\mathbb{Z} \\ \shortparallel \\ \langle * \rangle \end{matrix}$$

$N_1 = 0$

N_2

$$\boxed{\begin{matrix} \mathbb{RP}^2 \\ \nparallel \\ 0 \end{matrix}}$$

klein bottle



$S_g = \partial H_g$

$H_g = \underbrace{\text{nbhd}(\text{graph in } S^3)}_{\text{handlebody.}}$

$S^1 \times I/_{\substack{x \sim -x \\ \text{on ends}}} = \partial$ twisted solid torus.

$K = \partial(D^2 \times I/_{\substack{x \sim -x \\ \text{on ends}}})$

$M \# N$ = Delete small open balls from both and glue spheres together.

Connect sums of Klein bottles bound and others don't all cobordant to $\mathbb{RP}^2$

$$\mathbb{RP}^2 \# \underbrace{\mathbb{RP}^2 \# \mathbb{RP}^2}_{k} \# \underbrace{\dots}_{} \# \underbrace{\mathbb{RP}^2 \# \mathbb{RP}^2}_{}$$

$N_2 = \mathbb{Z}/2\mathbb{Z} = \langle \mathbb{RP}^2 \rangle.$

$N_3 = 0$ (Rokhlin–Wall–Lickorish)

$N_4 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$

$N_5 = \mathbb{Z}/2\mathbb{Z}.$

<u>oriented version</u>

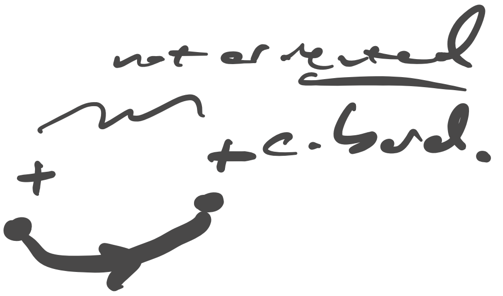
<u>oriented</u> <u>mfld</u>.

$M \qquad M^{-1} = -M$

Now it's not nec. 2-torsion.

∂M might not be null cobordant.

This gp Ω_n.



$\Omega_0 = \mathbb{Z}$

$\Omega_1 = 0$

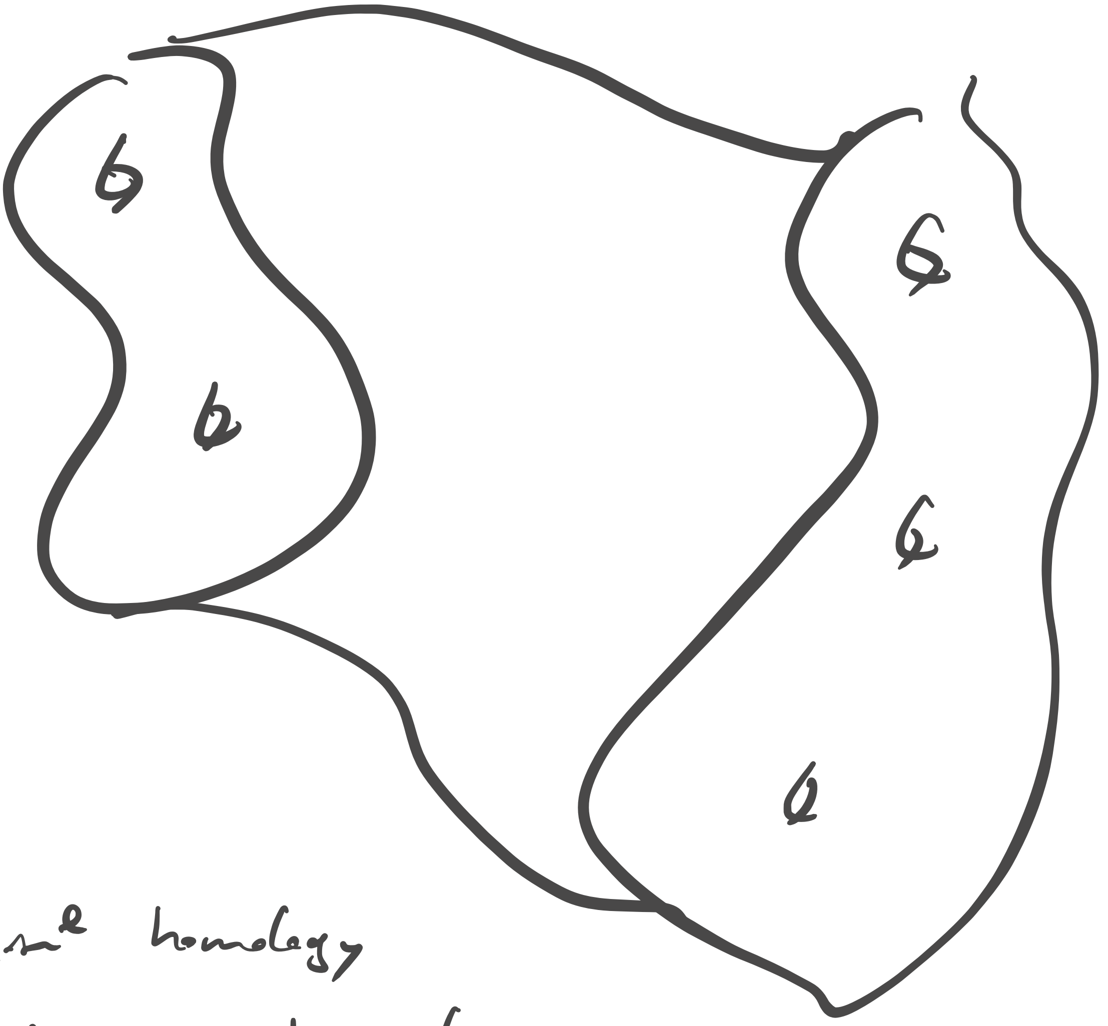
$\Omega_2 = 0$

$\Omega_3 = 0$

$\Omega_4 = \mathbb{Z} = \langle \mathbb{C}P^2 \rangle$

$\Omega_5 = \mathbb{Z}/2\mathbb{Z}$

Originally cobordism came up as attempt to construct homology for mflds using only mflds.



2-dim'l homology classes homologous iff ∃ a map of a cobordism between them into your mfld X whose homology you're studying.

A few things about the PD isomorphism in top. case

Cap Product.

X any top, R

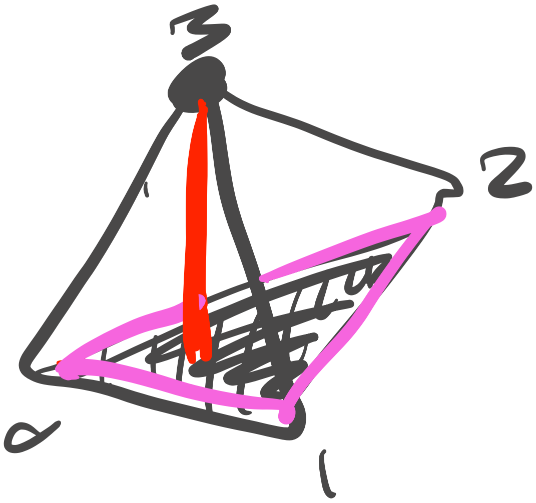
$\exists$ R-linear product

$$\frown : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$$

$$(\underset{\text{sx}}{\sigma}, \varphi)$$

$\sigma|_{[v_0,\dots,v_k]}$

$$\longmapsto \varphi(\sigma|_{[v_0,\dots,v_\ell]})\,\sigma|_{[v_\ell,\dots,v_k]}.$$



φ 2-chain

Check $\partial(\sigma \frown \varphi)$

$$= (-1)^{\ell}(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

so $\partial(\underbrace{\text{cycle} \frown \text{cocycle}}_{\Rightarrow \text{cycle}}) = 0$

cap prod. of a cycle C

and coboundary $\varphi = \delta\Phi$.

then $(-1)^{\ell+1} C \frown \varphi = \partial(C \frown \Phi)$

$\Rightarrow$ cap prod of cycle and cobound. is boundary

also if φ cocycle, $C = \partial D$

then $(-1)^{\ell}\partial D \frown \varphi = \partial(D \frown \varphi)$

So cap of bound. and cocycle is ∂.

$\Longrightarrow$

$$\frown : H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-1}(X;R)$$

R-linear in each variable

Also rel. versions

$$H_k(X,A) \times H^\ell(X) \xrightarrow{\frown} H_{k-\ell}(X,A)$$

$$\underbrace{H_k(X,A) \times H^\ell(X,A) \xrightarrow{\frown} H_{k-1}(X)}$$

$$C_k(X) \times C^\ell(X) \xrightarrow{\frown} C_{k-\ell}(X)$$

restricts to 0 on

$C_k(A) \times C^\ell(X,A)$ so

get $C_k(X,A) \times C^\ell(X,A) \to C_{k-\ell}(X)$

Naturality: $f: X \to Y$

$$\begin{tikzcd}
H_k(X) \arrow[d, "f_*"] & \times & H^l(X) & \arrow[r, "\cap"] & H_{k-l}(X) \\
H_k(X) & \times & H^l(Y) \arrow[u, "f^*"] & \arrow[r] & H_{k-l}(Y)
\end{tikzcd}$$

$$f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^* \varphi).$$

Poincaré Duality.

M R-orientable n-mfld w/ fundamental class $[M] \in H_n(M;R)$, then

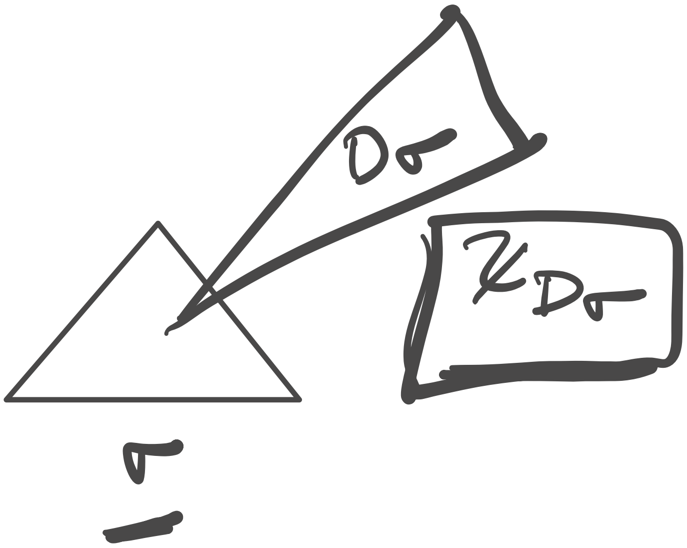
$D: H^k(M;R) \to H_{n-k}(M;R)$

given by $D(\alpha) = [M] \cap \alpha$

is an isomorphism ∀k.

$$\varphi \xmapsto{D} \underbrace{\varphi([0,\dots,k])[v_k,\dots,v_n]}.$$

where $[0,\dots,n]$ chosen usx chosen by $[M]$.



One idea from the proof is Cohomology with compact supports.

∃ lots of noncompact mflds.

Duality for noncompact mflds.

$$H^k_c(M;R) \cong H_{n-k}(M;R)$$

for R-oriented mflds

Cohomology but only allow cochains supported on finitely many cells.

Next time briefly talk about this.

$$H^1_c(\mathbb{R};\mathbb{Z}) = \mathbb{Z}.$$

Back to Poincaré Duality.

Poincaré Duality

M^n is a closed orientable n-mfld then $H_k(M^n;\mathbb{Z}) \cong H^{n-k}(M^n;\mathbb{Z})$

and $\forall$ closed mflds

$$H_k(M^n;\mathbb{F}_2) \cong H^{n-k}(M^n;\mathbb{F}_2).$$

Poincaré-Lefschetz Duality

M^n is compact, orientable mfld w/ ∂ then $H_k(M^n,\partial M^n;\mathbb{Z}) \cong H^{n-k}(M^n;\mathbb{Z})$

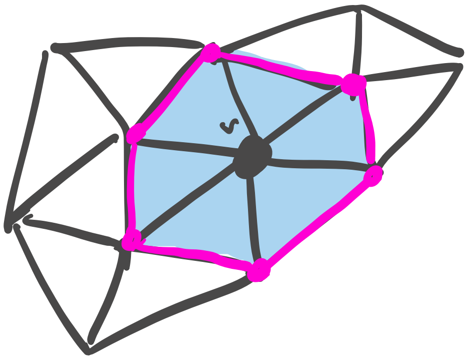
and $H^k(M^n,\partial M^n;\mathbb{Z}) \cong H_{n-k}(M^n;\mathbb{Z})$.

(also for $\mathbb{F}_2$ coeffs even if M^n non-orientable.)

Talk about proof for PL mflds.

PL mflds.

PL n-mfld M is a triangulated n-mfld with the property that the link of every k-simplex is an $(n-k-1)$-sphere.



$\cup$ All closed sxs containing v

Blue ''

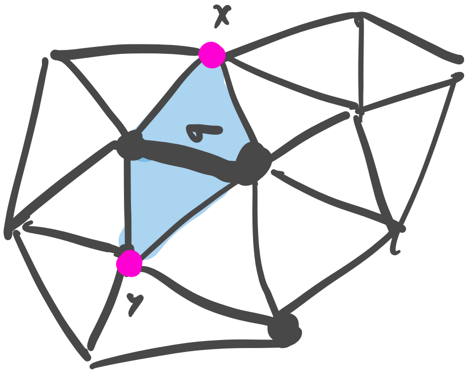
$= \text{Star}(v)$.

Pink

$= \text{Link}(v)$.

''

$S^1 = S^{2-0-1}$



$\text{Star}(\sigma) = \bigcup_{\tau > \sigma} \tau$

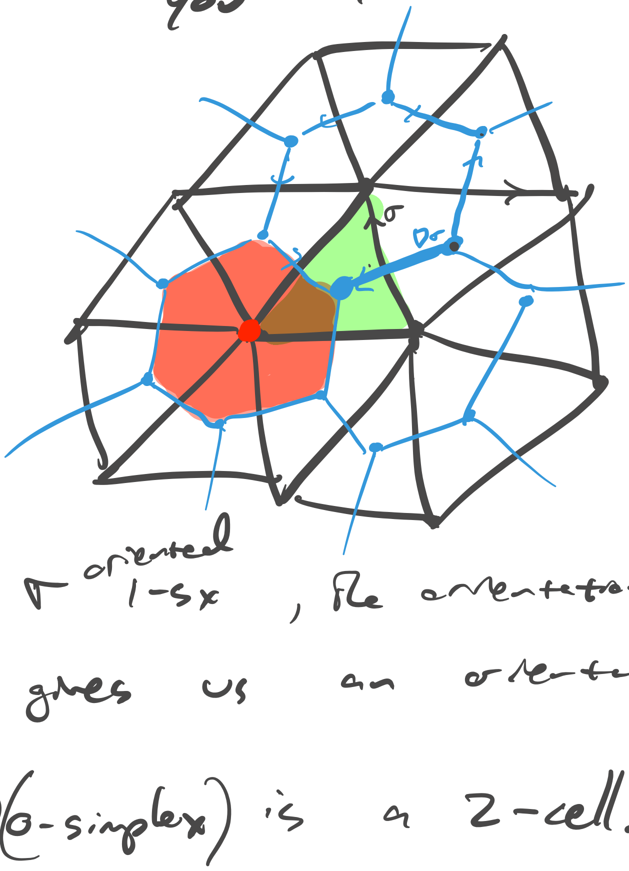
$\text{Link}(\sigma) = \text{Lk}(\sigma)$

$= x \sqcup y = S^0 = S^{2-1-1}$

Seems like, at least for vertices, seems like $Lk(v) = S^{n-1}$ as long as M is a mfld.

That's false for arbitrary triangulations as we saw last time.

Idea: PL triangulation gives you a dual cell decomposition.



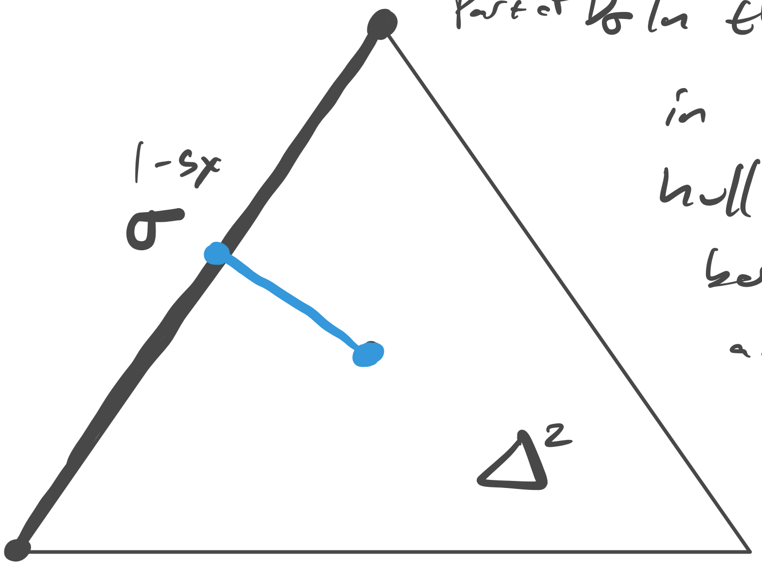
Use barycenters.

If σ is n-sx. Dual cell $D\sigma$ is the barycenter.

σ oriented 1-sx, the orientation of M gives us an orientation of $D\sigma$

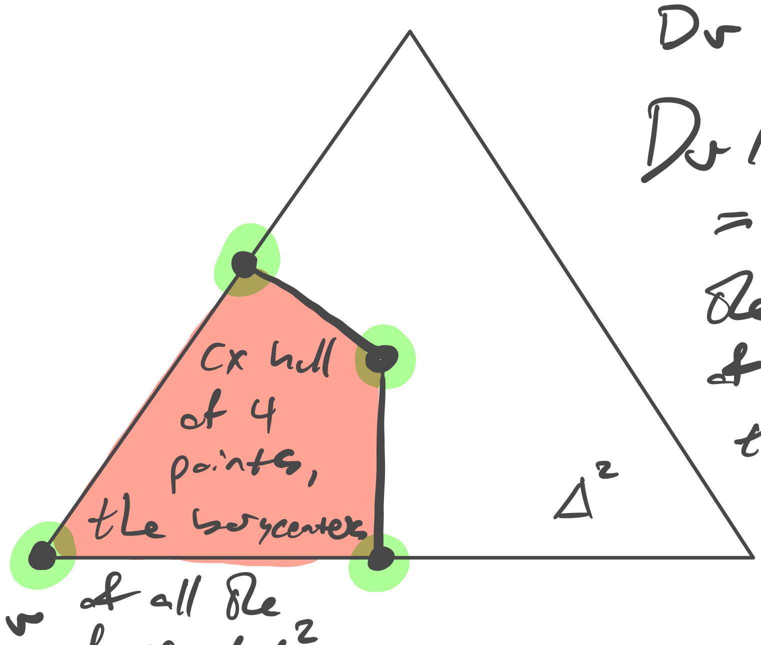
D(0-simplex) is a 2-cell.

How to define these carefully?



Part of D_σ in this 2sx Δ^2, in the convex hull of the barycenter of σ and the barycenter of Δ^2.

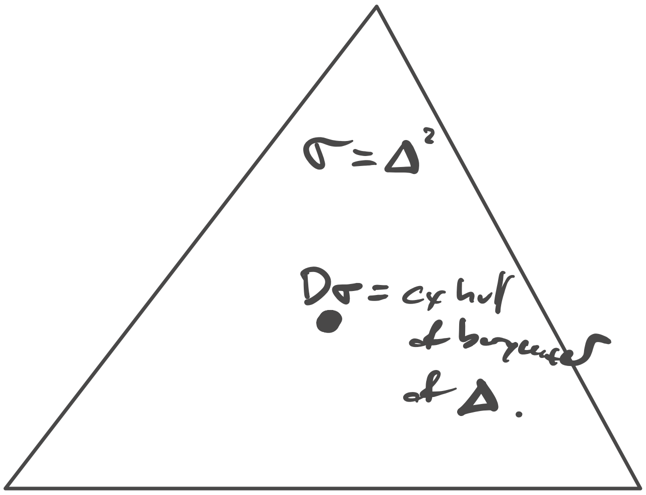
In Δ^2, σ and Δ^2 are the only sxs that contain σ.



cx hull of 4 points, the barycenters of all the faces of Δ^2 that contain v.

D_v in Δ^2

$D_v \cap \Delta^2$ = cx hull of the barycenters of sxs in Δ^2 that contain v.



In a PL triangulation of M,

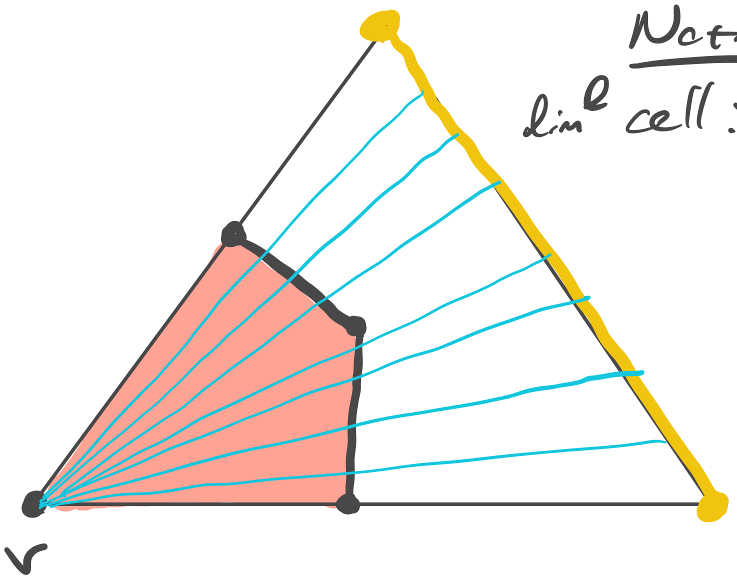
if σ k-simplex,

Then $D\sigma \cap \Delta^n$ = cx hull of barycenters of the faces of Δ^n that contain σ.

Δ^n ↓ top dim^l simplex

Why is $D\sigma$ a cell?

<u>Claim</u>: $D\sigma$ is an $(n-k)$-cell.



Notice: In Δ top dim^l cell: $Lk(v)$

$\|$

yellow part.

In this example, $D_v \cong C(Lk(v))$

In an n-simplex Δ

If $\sigma < \Delta$ is a face.

Then $\Delta \cong \mathrm{Join}(\sigma, Lk(\sigma))$.

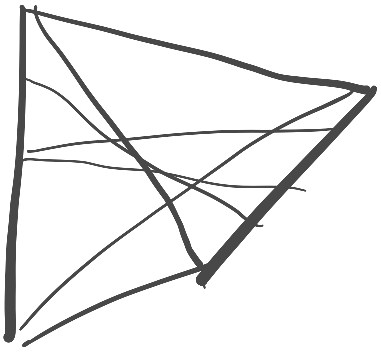
$$\overbrace{[\underbrace{\overbrace{v_0,\dots,v_k}}_{\substack{k\text{-sx}\\ \sigma}},\ \underbrace{v_{k+1},\dots,v_n}]}^{\Delta}$$

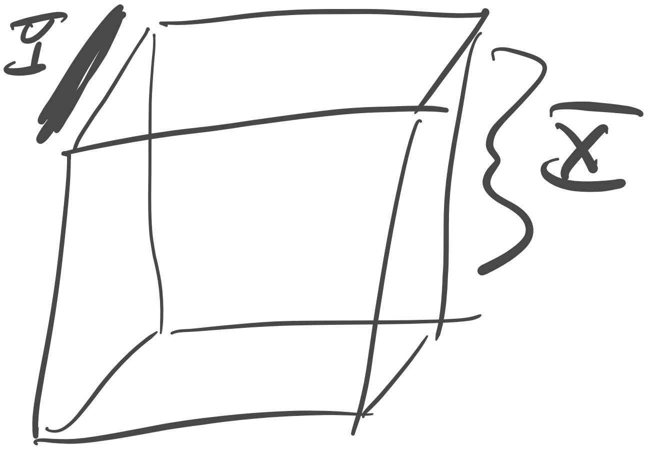
If X, Y spaces, form a new space

$J(X,Y) =$ all lines joining X to Y.

$$\|$$

$$X \times I \times I / * \times I \times \{0\}, X \times * \times \{1\}$$





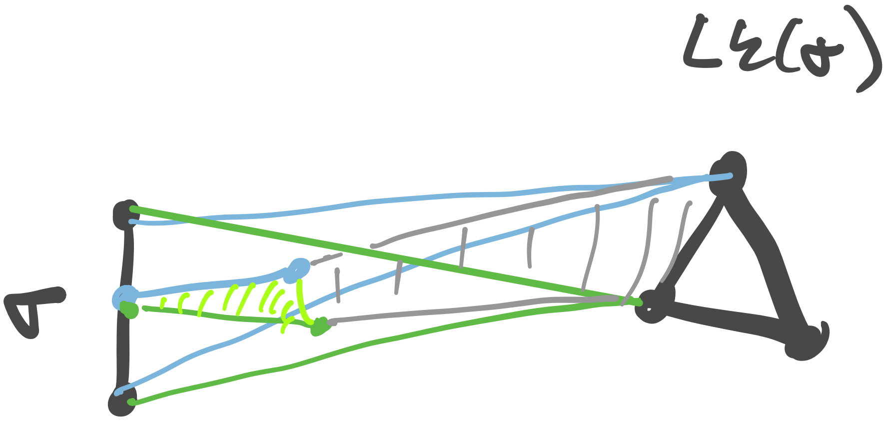
Using $\Delta = J_\Delta(\sigma, Lk(\sigma))$ in Δ

see $D\sigma \cap \Delta \cong \underbrace{C}_{}(Lk(\sigma))$

cone point = barycenter of σ.

Then $D\sigma = \underbrace{C\,Lk(\sigma)}_{\text{sphere}} = $ Ball.

since Triang. is PL


Lk(σ)
σ

Poincaré Duality

If M^n is an orientable n-mfld, then $H_k(M^n;\mathbb{Z}) \cong H^{n-k}(M^n;\mathbb{Z})$ and $H_k(M^n;\mathbb{F}_2) \cong H^{n-k}(M^n;\mathbb{F}_2)$ even if M not orientable.

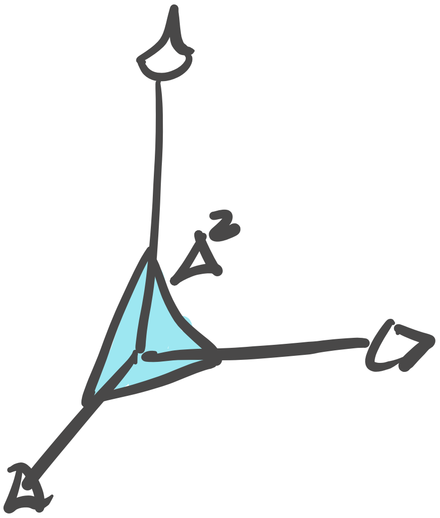
True for all topological manifolds. We'll prove it for PL mflds.

---

i.e.: Triangulation w/ links homeomorphic to spheres.

Let M^n be a PL n-mfld, orientable.

If $\underset{\substack{\downarrow \\ k\text{-simplex}}}{\sigma} \subset \Delta^n = \left\{ \sum_{i=0}^{n} c_i e_i \;\middle|\; \begin{array}{l} c_i > 0 \ \forall i \\ \text{and} \\ \sum_{i=0}^{n} c_i = 1 \\ \text{for } e_i \text{ std} \\ \text{basis in } \mathbb{R}^{n+1} \end{array} \right\}$



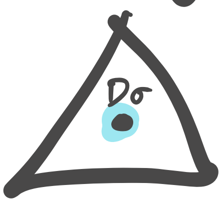
every k-sx $\sigma \subset M$ is in some n sx Δ^n in M.

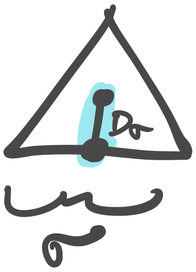
We define the **dual cell** D_σ of σ geometrically by declaring that $D_\sigma \cap \Delta^n$ is the convex hull of the barycenters of all the faces of Δ^n containing σ.

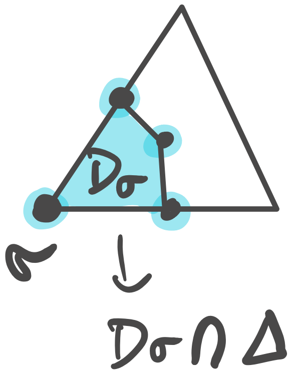
Let $\sigma = k$-sx.

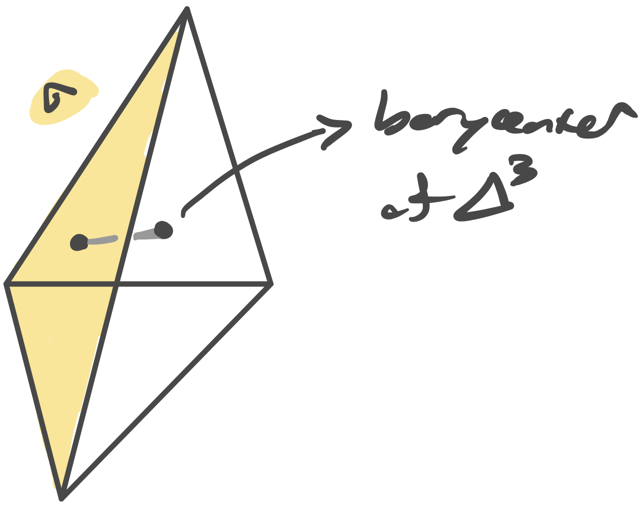
Ex:

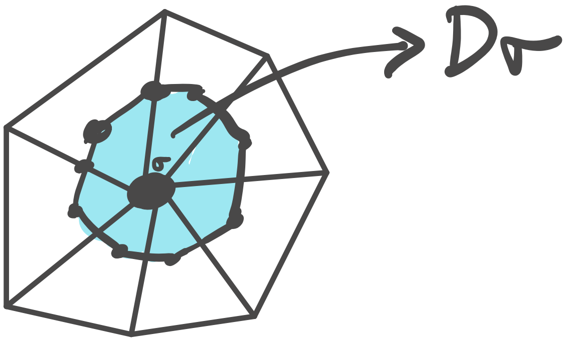
$\sigma = \Delta^n$











Dσ subcomplex of the first barycentric subdivision of M.

Claim: Dσ is a cell of dimension n-k.

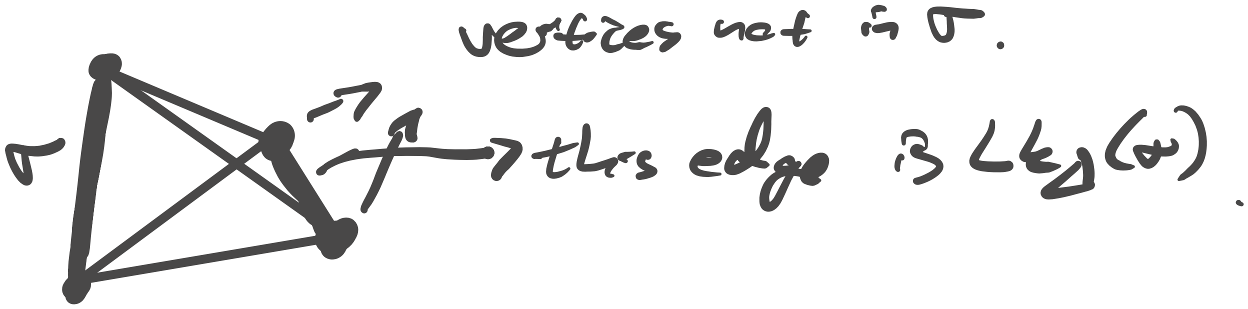
Think of $\Delta^n = \{0, 1, \dots, n\}$

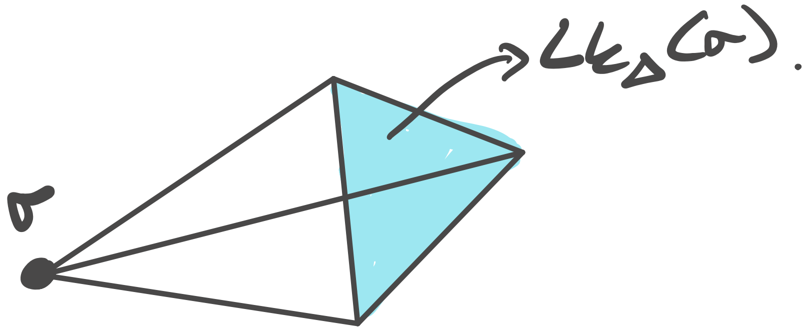
If σ is a k-sx in Δ^n,

Then Link of σ in Δ^n, $Lk_\Delta(\sigma)$

is the convex hull of the vertices

of Δ^n that aren't in σ:





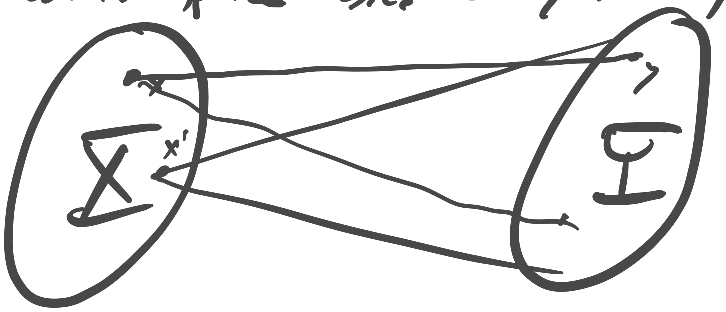
Notice also, that Δ^n is the "join"

of σ and $Lk_\Delta(\sigma)$:

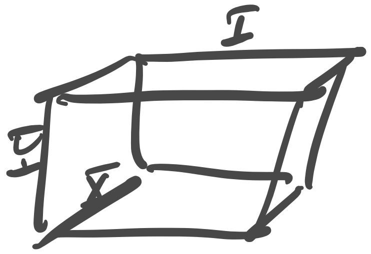
Def. The join of two spaces X and Y

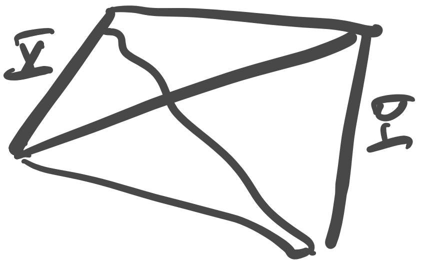
$$X \times Y \times I \Big/ \begin{array}{l} * \times Y \times I = x \\ X \times * \times I = *_y \end{array} = J(X, Y)$$

Want space s.t. every $x \in X$, $y \in Y$ joined by ! line segment.



$X = Y = I$





$$\Delta^n \cong J(\sigma, Lk_\Delta(\sigma)) = \sigma \times Lk_\Delta(\sigma) \times I \Big/ {\sigma \times * \times I \atop * \times Lk \times I}$$

$$\|$$

$$\left\{ \sum c_i e_i \,\middle|\, c_i > 0 \text{ and } \left\|\sum c_i e_i\right\|_1 = \sum c_i = 1 \right\}$$

$$\|$$

$$\left\{ t\underline{x} + (1-t)\underline{y} \,\middle|\, \underline{x} \in \sigma,\ \underline{y} \in Lk_\Delta(\sigma) \right\}$$

easiest to visualize when $\sigma = \{0, \ldots, k\}$

$$\sum_{i=0}^{n} c_i e_i = \sum_{i=0}^{k} c_i e_i + \sum_{i=k+1}^{n} c_i e_i$$

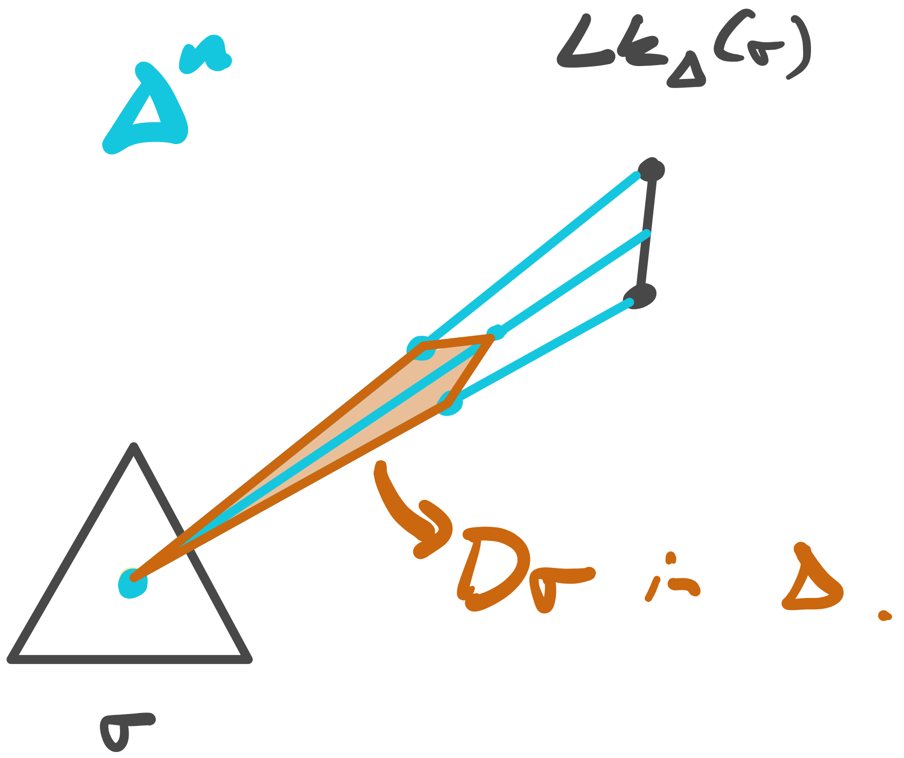
$$= X + Y$$

$$= \|X\|_1 \underbrace{\frac{X}{\|X\|_1}}_{\text{in } \sigma} + \|Y\|_1 \underbrace{\frac{Y}{\|Y\|_1}}_{\text{in } \mathrm{Lk}_\Delta(\sigma)}$$

since $\sum c_i = 1$, $\|Y\|_1 = 1 - \|X\|_1$.

☑

Why talking about Joins?



From this picture we see that the cell D_σ in Δ is the cone on $Lk_\Delta(\sigma)$ with conepoint being the barycenter of σ.

$$D_\sigma \cap \Delta \cong C_{z(\sigma)}((Lk_\Delta(\sigma))).$$

$\Rightarrow D_\sigma \subset M$ is homeomorphic to $C(Lk_M(\sigma))$

Since M is PL, $Lk_M(\sigma) \cong S^{n-k-1}$

and so $C(Lk_M(\sigma)) \cong B^{n-k}$

so $D_\sigma \cong B^{n-k}$, a cell!

The dual cells give us a "dual" cell decomposition.

Note on orientability:

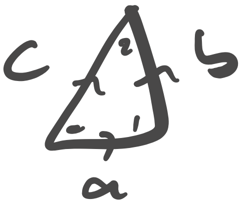
Here we can define an equivalence relation on simplices with orderings on vertices by declaring that a sx Δ with an ordering is equivalent to Δ w/ a different ordering iff orders differ by an even permutation.

With this definition, equivalent Δs have same $\partial\Delta$s. (exercise).

The equiv. class of the ordering is an <u>orientation</u> of Δ.

If $\bar{\Delta}$ is Δ with opp. orientation, then $\Delta + \bar{\Delta}$ is a cycle:

$\partial\Delta = a + b - c$ $\partial\bar{\Delta} = -a + c - b$

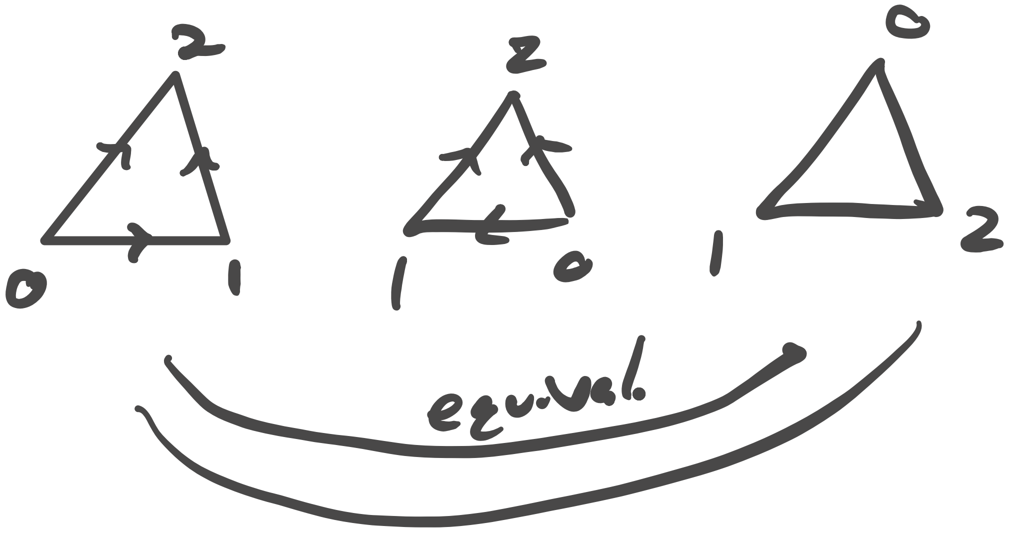


And in fact:

$\Delta + \bar{\Delta}$ is a boundary.

Assuming M is orientable, that means we can compatibly assign a preferred orientation to each Δ^n in M.

i.e. $\exists$ a choice of ordering of the vertices of each Δ^n s.t. ∂ operator makes all the $n-1$ sxs cancel. Combinatorially, an Orientation of Δ^n is an equiv. class of orderings of the vertices two orderings are equivalent if they differ by an even permutation.

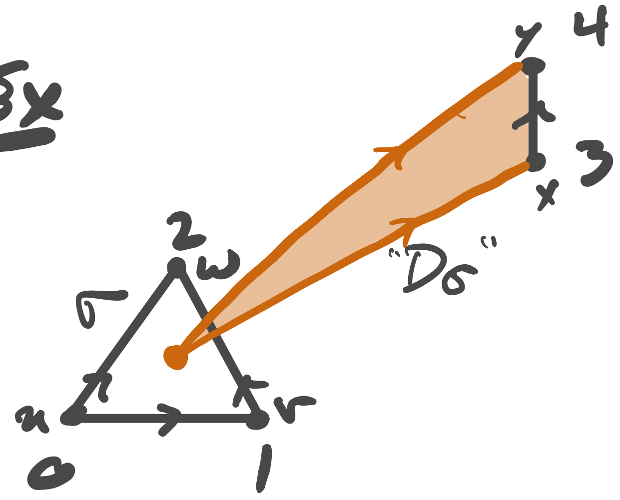


The orientability is helpful!

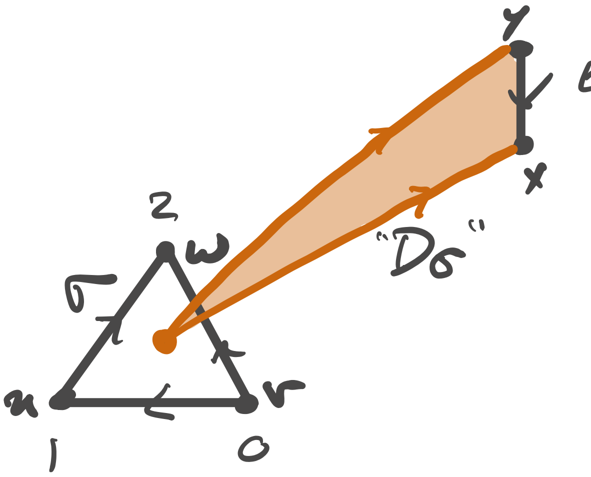
Given an oriented σ, h.s X, there is a canonical orientation of $D\sigma$, namely the orientation that orients Δ^n according to the orientation on M:

Ex

Δ^4

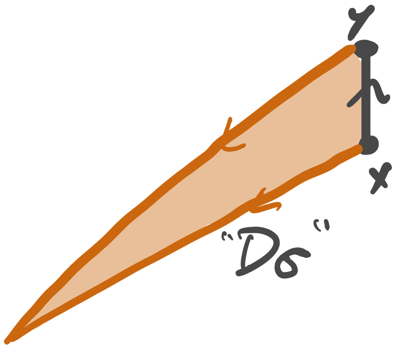


If we change orientation of σ



new orientation on Dσ.

Why not this?



[If do odd perm to σ, need an odd permutation on $D\sigma \cap \Delta$]

14

So given oriented σ, we have oriented D_σ.

Now: Compute simplicial homology using the triangulation and the cellular cohomology of M using Dual cell decomposition.

Idea: ∂ becomes δ when we dualize and replace cells w/ cochains.

$$\begin{tikzcd}
\sigma \arrow[r, mapsto, "\partial"] \arrow[d] \arrow[dr, phantom, "\circ"] & \partial\sigma \arrow[d] \\
\chi_{D_\sigma} \arrow[r, mapsto, "\delta"'] & \chi_{\partial\sigma}
\end{tikzcd}$$

Given an oriented k-sx σ,

let ψ_{D_σ} be the cellular (n-k)-cochain

s.t. $\psi_{D_\sigma}(e^{n-k}) = \pm 1$ if $e^{n-k} = \pm D_\sigma$.

and $\psi_{D_\sigma}(e^{n-k}) = 0$ otherwise.

let $\underbrace{\delta \psi_{D_\sigma}(e^{n-k+1})}_{\delta \text{ cell coboundary operator}} =$ degree of ∂e^{n-k+1} passing over D_σ, where ∂ is ∂ of cell.

---

what happens when we look at dual cells of the ∂ of σ?

For ease of notation, let

$\sigma_i = \sigma$ w/ i^{th} vertex deleted.

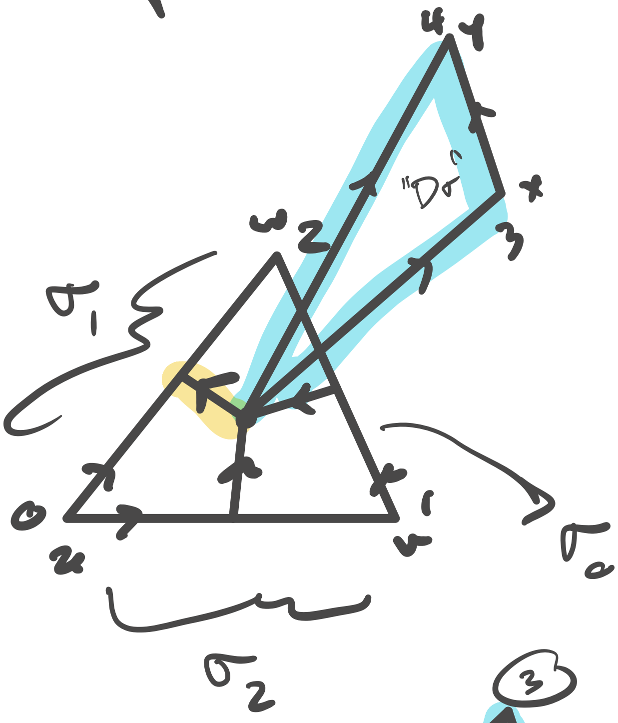
So $\partial \sigma = \sum (-1)^i \sigma_i$.

⌜ ⌝

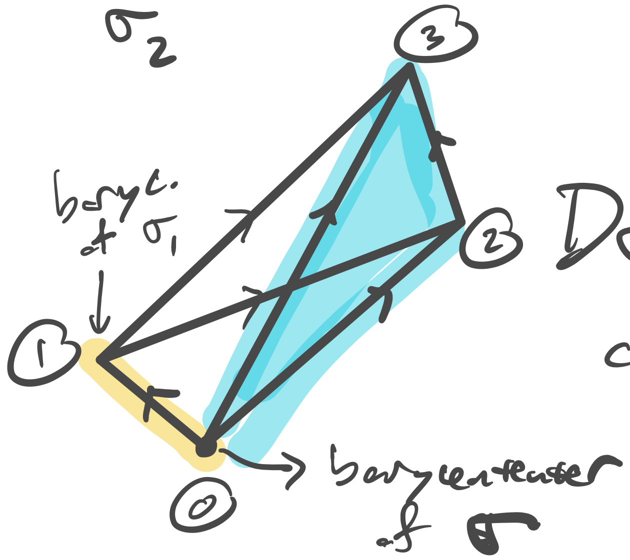
$$\{\underbrace{u,v,w}_{\sigma},x,y\}$$

What are the duals of σ_i?

Let's look at $D\sigma_1$.



σ_1 is a "negative" edge of σ.



$D\sigma_1$

correct orientation

$$\partial D\sigma_1 = [\textcircled{1},\textcircled{2},\textcircled{3}] \underbrace{-}_{\text{negative sign.}} \overbrace{[\textcircled{0},\textcircled{2},\textcircled{3}]}^{D\sigma} + [\textcircled{0},\textcircled{1},\textcircled{3}] - [\textcircled{0},\textcircled{1},\textcircled{2}]$$

So, the degree w/ which $\partial D\sigma_1$ goes over $D\sigma$ is -1.

Similarly, $\partial D\sigma_0$, $\partial D\sigma_2$ pass over it with degree $+1$.

In general, $\partial D\sigma_i$ passes over $D\sigma$ with degree $(-1)^i$.

Now! Consider the correspondence

$$\begin{tikzcd}
\sigma \arrow[r, mapsto, "\partial"] \arrow[d, "D"'] & \displaystyle\sum_{i=0}^{k} (-1)^i \sigma_i \arrow[d, "D"] \\
\chi_{D\sigma} \arrow[r, mapsto] & \underbrace{\displaystyle\sum_{i=0}^{k} (-1)^i \chi_{D\sigma_i}}_{\varphi_\sigma \text{ a cochain.}}
\end{tikzcd}$$

Now, the only $(n-k+1)$-cells in dual cell decomposition that pass over $D\sigma$ are the $D\sigma_i$.

i.e. $\varphi_\sigma(e^{n-k+1}) = 0$ unless

e^{n-k+1} is $\pm D\sigma_i$

Furthermore, by the above,

$$\varphi_\sigma(D\sigma_i) = (-1)^i$$

But that's exactly what

$\delta \chi_{D\sigma}$ does.

So $\varphi_\sigma = \delta \chi_{D\sigma}$.

$$\parallel$$

$$\sum (-1) \chi_{D\sigma_i} = \chi_{D(\partial\sigma)}$$

So kth simplicial homology of M is isomorphic to the $(n-k)$th cellular cohomology of M (from dual cell decomp.)

and $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$.

and $H_k(M;\mathbb{F}_2) = H^{n-k}(M;\mathbb{F}_2)$.

the map being our dualization map $\sigma \longmapsto \pm D\sigma$.

☑

Cor. Since $H^*(X;F)$

$$\cong H_*(X;\mathbb{F})^* \cong H_*(X;\mathbb{F})$$

we have $H_k(M^n;\mathbb{F}) \cong H_{n-k}(M^n;\mathbb{F})$

and $H^k(M^n;\mathbb{F}) \cong H^{n-k}(M^n;\mathbb{F})$.

Cor. If M or. odd dim'l mfld, then $\chi(M) = 0$.

$\square$

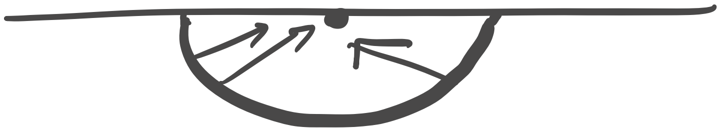
Poincaré-Lefschetz Duality

If M R-or. n-mfld w/ ∂

$$H^k(M;R) \cong H_{n-k}(M,\partial M;R)$$

and $H_k(M;R) \cong H^{n-k}(M,\partial M;R)$.

Geometric proof works here too



Alexander Duality

K is a compact, locally contractible, nonempty proper subspace of S^n, then

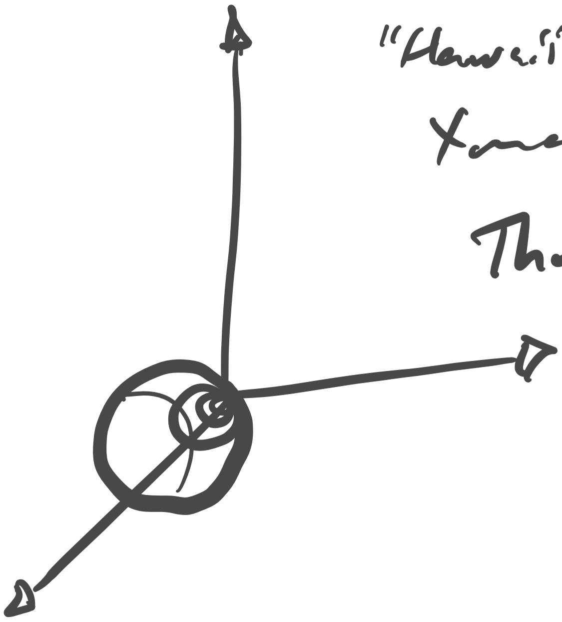
$$\tilde{H}_i(S^n - K;\mathbb{Z}) \cong \tilde{H}^{n-i-1}(K;\mathbb{Z}).$$

Cor. If $K \subset \mathbb{R}^n$ cpct locally contractible then $H_j(K;\mathbb{Z}) = 0$ if $j \geq n$, and torsion-free if $j = n-1$ and $n-2$.

Need the local contractibility.

E.g. Let $K = \bigcup_{n=1}^{\infty} S^2_n$

where S^2_n = sphere of radius $1/n$ centered at $(\frac{1}{2}, 0, 0)$. 2-dim'l "Hawaiian earring," Hawaiian Xmas tree ornament.



Thm (Milnor)

$$H_3(K; \mathbb{Z}) \neq 0.$$

E.g. of Alexander Duality in action:

Let K be a knot in S^3, i.e. $K \cong S^1$ sitting in S^3.

figure 8 knot.

$$\underbrace{\boxed{H_1(S^3-K;\mathbb{Z})}}_{\text{can also use Mayer Vietoris.}} \cong H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$$

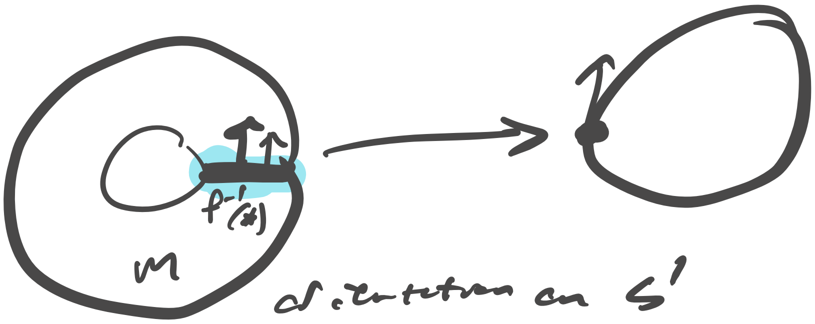
Using techniques we talked about earlier to construct a smooth map

$$f:(S^3-\text{nbhd } K) \longrightarrow S^1$$

pull back regular value and get

$f^{-1}(*) =$ orientable sfce.

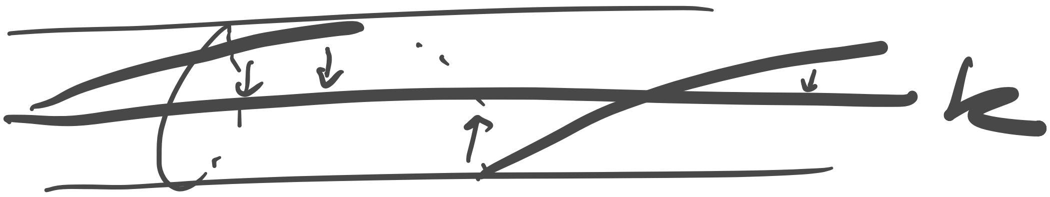
∂f^{-1} could be a bunch of loops.



Change map get new sfce.

orientation on S^1 and orientation on M ⇝ orientation of $f^{-1}(*)$ $= S$

called a seifert surface for K. This a surface whose ∂ is a copy of K



∂S is isotopic (htpc through embeddings) to K.

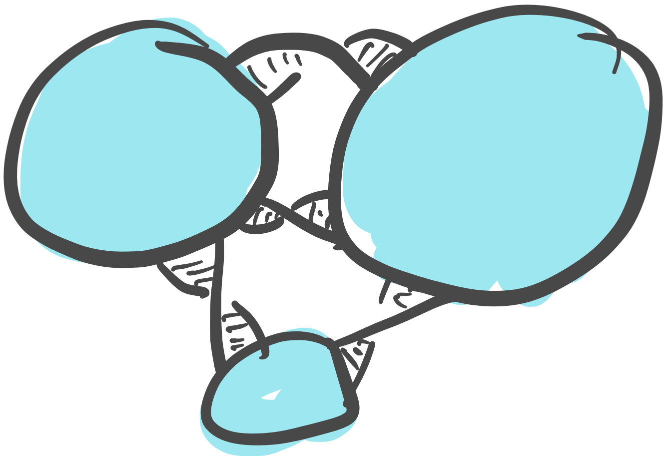
$\chi(F) = -1$

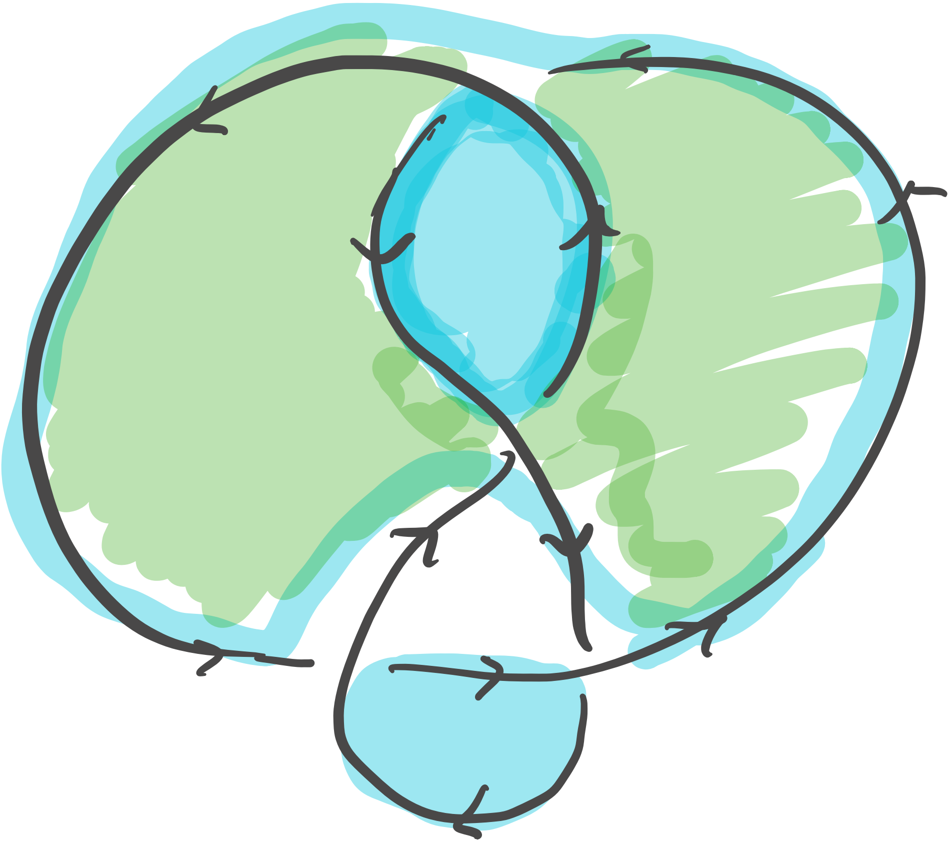
1 boundary component

F = punctured klein bottle.

$K = \partial F$

F is not orientable!





$T^2 - \mathring{B}^2$.

Thm (Hirsch) Every closed orientable 3-mfld embeds smoothly into S^5.

Cor (Rohlin) Every orientable 3-mfld is the boundary of orientable 4-mfld.

(closed nonor. 3-mflds also embed in S^5. and they are also always ∂s ! ($\mathbb{RP}^2$ is not a ∂ at all)

Proof of Rohlin

By Hirsch, M^3 closed orientable embeds in S^5.

By Alexander Duality,

$$H_1(S^5 - M^3) \cong H^{5-1-1}(M;\mathbb{Z}) \cong \mathbb{Z}$$

So $H_1(S^5 - \text{nbhd}(M^3)) \cong \mathbb{Z}$.

So build a map

$f: S^5 - \text{nbhd of } M^3 \longrightarrow S^1$

and pull back a regular value to get an orientable 4-mfld $f^{-1}(*)$ whose ∂ is $\cong M^3$. $\square$.

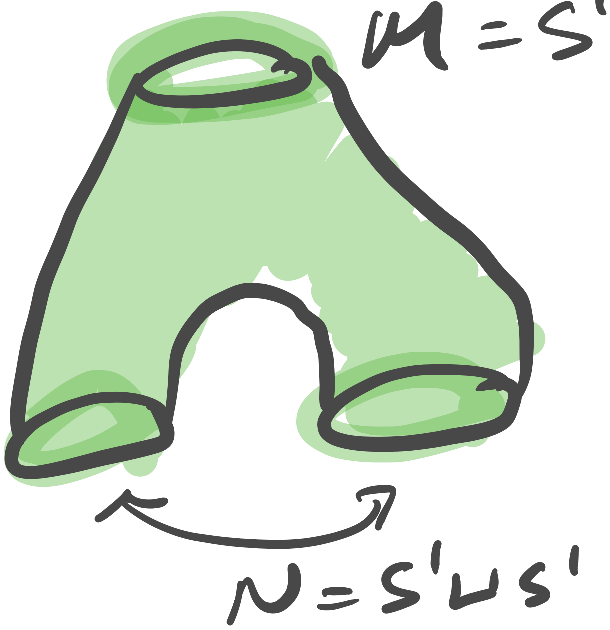
This a theorem in the study of "cobordism."

Basic cobordism stuff.

Look at set of all closed unoriented n-mflds with following equivN relation:

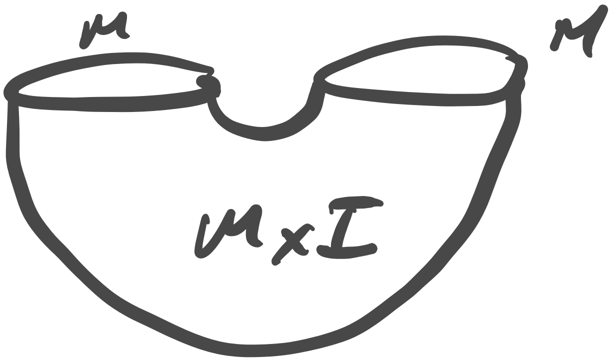
$$M \sim N \text{ if } M \sqcup N \cong \partial W$$

where W is a connected compact $n+1$ mfld with ∂.



Notice:

$M \sqcup M \sim \emptyset$



So $M \sqcup M \sim S^n$

$\sim \emptyset$.

$S^n \sim \emptyset$

$\emptyset = 0$ then

The set of equivalence classes is a abelian group. $M = M^{-1}$

operation is $\sqcup / \sim$

N_n = unoriented n-dim'l cobordism gp.

2-torsion. $2M = 0$.

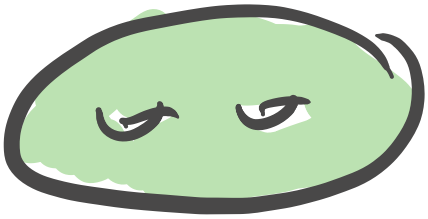
$$N_0 = \mathbb{Z}/2\mathbb{Z} \atop {\shortparallel \atop \langle * \rangle}$$

$N_1 = 0$

N_2

$$\boxed{\begin{matrix}\mathbb{RP}^2 \\ \nparallel \\ 0\end{matrix}}$$

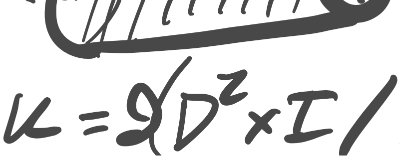
Klein bottle



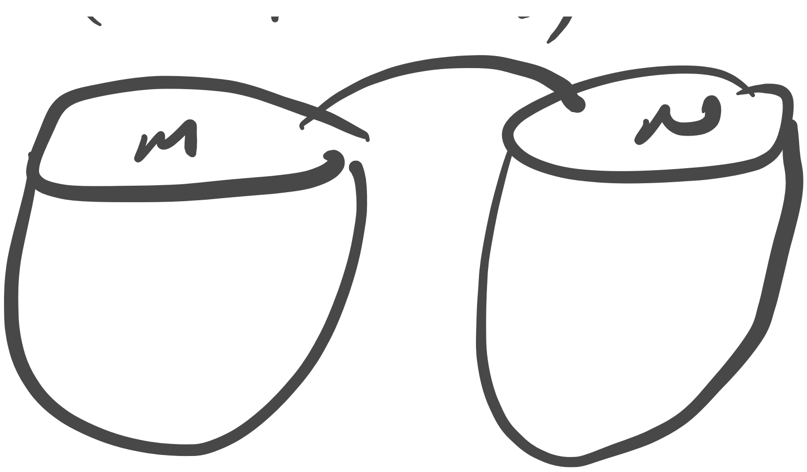
$S_g = \partial H_g$

$H_g = \underbrace{\text{nbhd}(\text{graph} \subset S^3)}_{\text{handlebody.}}$

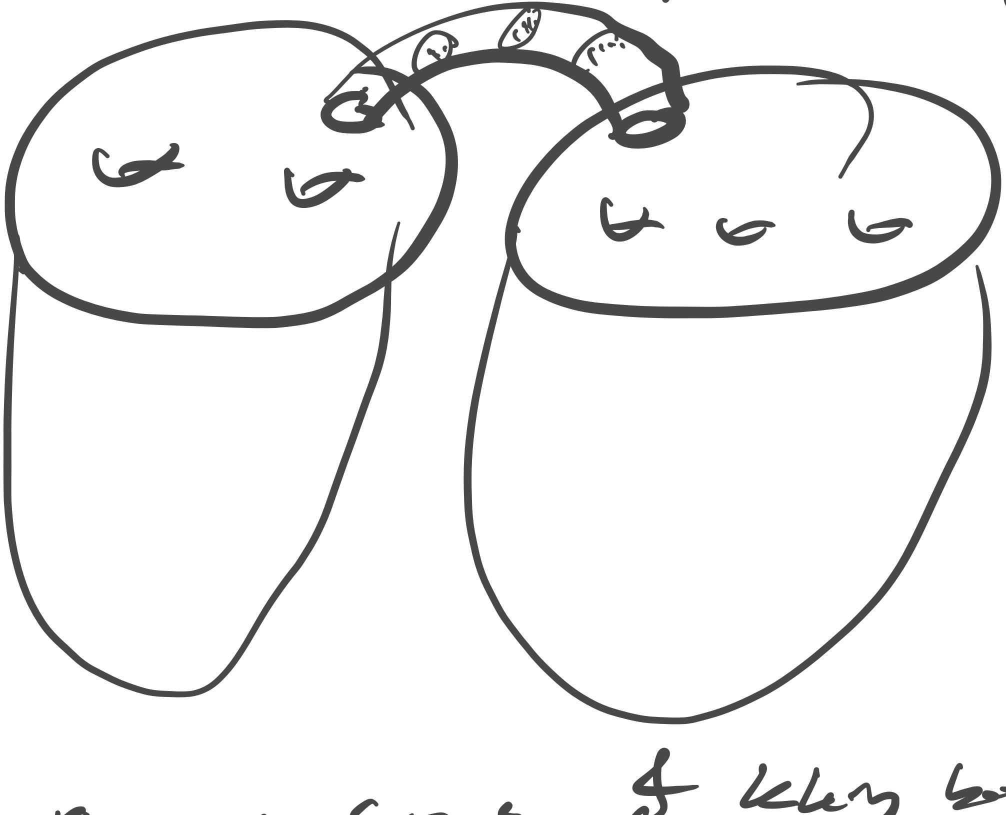
$S^1 \times I/_{x\sim -x \text{ on ends}} = \partial$ twisted solid torus.



$K = \partial(D^2 \times I/_{x\sim -x \text{ on ends}})$



$M\#N$ = Delete small open balls from both and glue spheres together.



Connect sums of Klein bottles bound and others don't

$\mathbb{RP}^2$ not a ∂.

Thm $\chi(\partial M^n)$ is even.

Double M to get DM.

Mayer-Vietoris: $\mathbb{Q}$ coefficients.

$$\begin{tikzcd}
H_{k+1}(DM) \arrow[r, "\partial_{k+1}"] & H_k(\partial M) \arrow[r] \arrow[d, equal] & H_k(M)\oplus H_k(M) \arrow[r] \arrow[d, equal] & H_k(DM) \arrow[r, "\partial_k"] \arrow[d, equal] & H_{k-1}(\partial M) \\
 & \mathbb{Q}^{b_k(\partial M)} & \mathbb{Q}^{b_k(M)}\oplus\mathbb{Q}^{b_k(M)} & \mathbb{Q}^{b_k(DM)} &
\end{tikzcd}$$

$$\chi(M) = \sum (-1)^k b_k(M)$$

$$\chi(DM) = \sum (-1)^k b_k(DM)$$

$$b_k(DM) = 2b_k(M) - \left(b_k(\partial M) - \dim_{\mathbb{Q}} \operatorname{Im}(\partial_{k+1})\right) + \dim_{\mathbb{Q}} \operatorname{Im} \partial_k$$

Alternating sum $\Rightarrow$ $\chi(DM) = 2\chi(M) - \chi(\partial M)$

If $\dim M$ odd, then $\chi(\partial M) = 0$.

by poincaré duality.

So $\chi(\partial M)$ even.

If $\dim M$ even, then $\dim \partial M$ odd

$\Rightarrow$ $\chi(\partial M) = 0$. $\square$.

all cobordant to $\mathbb{RP}^2$ or $\emptyset$.

$$\underline{\mathbb{RP}^2} \# \underbrace{\mathbb{RP}^2 \# \mathbb{RP}^2}_{\underline{K}} \# \underbrace{\dots}_{\_\_} \# \underbrace{\mathbb{RP}^2 \# \mathbb{RP}^2}_{\_}$$

$N_2 = \mathbb{Z}/2\mathbb{Z} = \langle \mathbb{RP}^2 \rangle$.

$N_3 = 0$ (Rochlin–Wall–Lickorish)

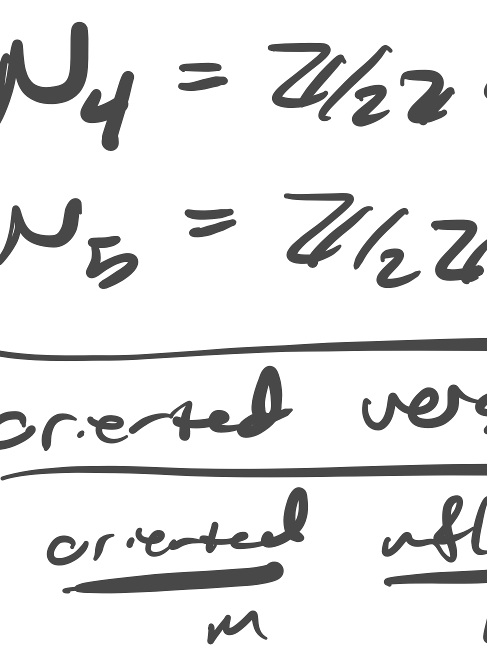
$N_4 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

$N_5 = \mathbb{Z}/2\mathbb{Z}$.

<u>oriented version</u>

<u>oriented</u> <u>mfld</u>.

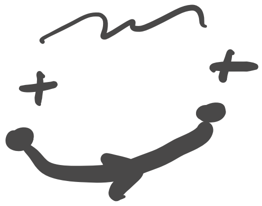
M $\quad M^{-1} = -M$



Now it's not nec.
2-torsion.

∂M might not be null cobordant.

This gp Ω_n.



c. bord.

$\Omega_0 = \mathbb{Z}$

$\Omega_1 = 0$

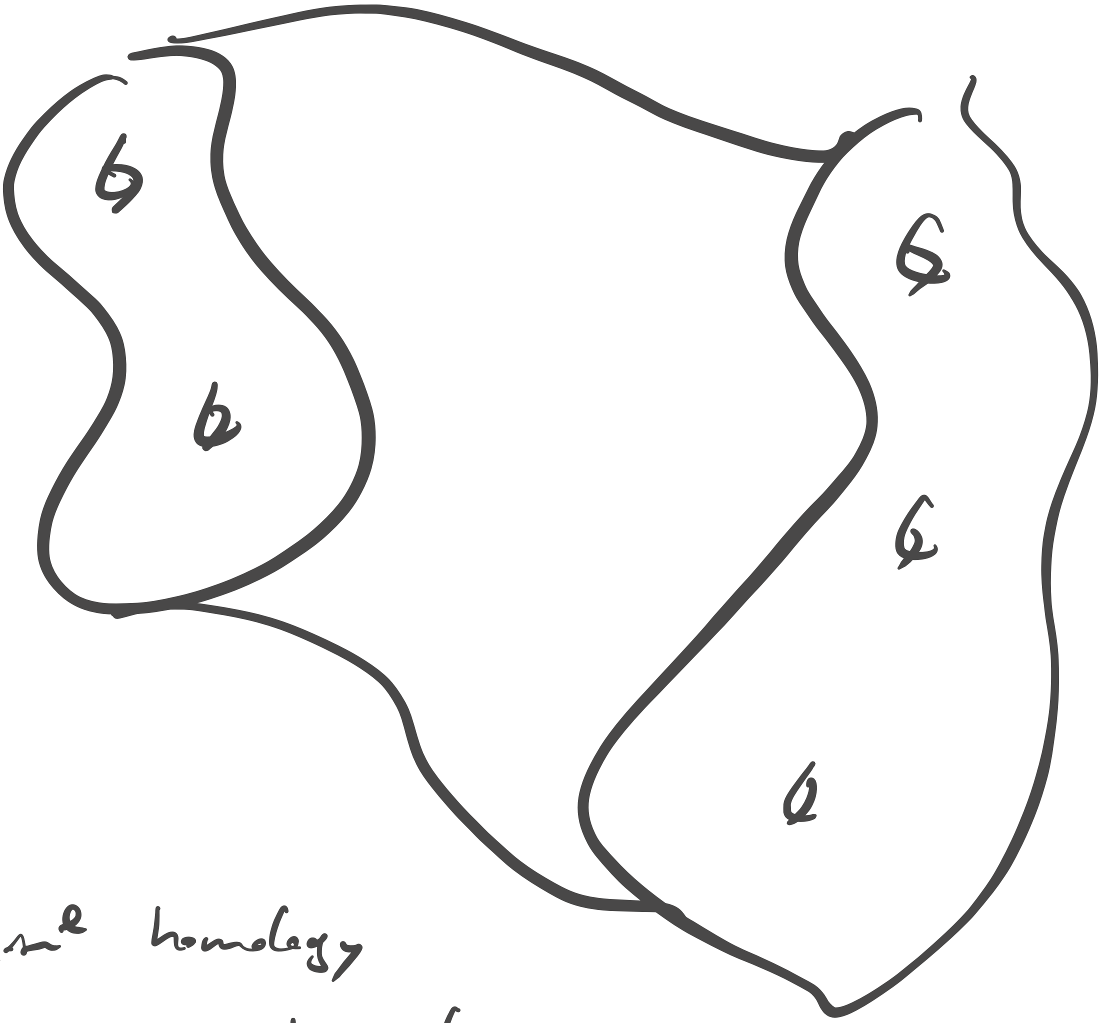
$\Omega_2 = 0$

$\Omega_3 = 0$

$\Omega_4 = \mathbb{Z} = \langle \mathbb{C}P^2 \rangle$

$\Omega_5 = \mathbb{Z}/2\mathbb{Z}$

Originally cobordism came up as attempt to construct homology for mflds using only mflds.



2-dim'l homology
classes homologous
iff ∃ a map of a
cobordism between them
into your mfld X
whose homology you're
studying.

A few things about the PD isomorphism in top. case.

Cap Product.

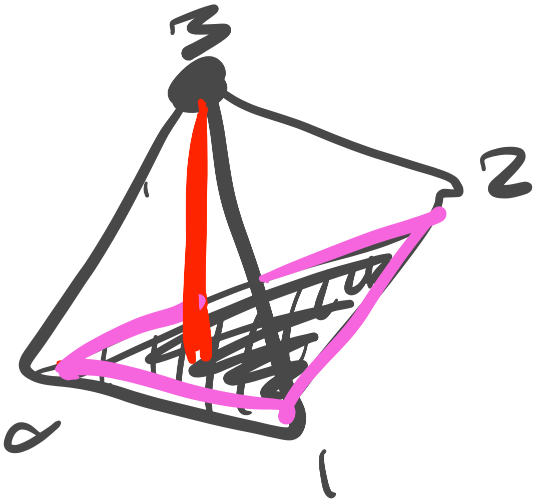
X any Rg, R

$\exists$ R-linear product

$$\frown : C_k(X;R) \times C^l(X;R) \to C_{k-l}(X;R)$$

$$(\underset{\substack{\text{sx} \\ \sigma|_{[v_0,\dots,v_k]}}}{\sigma}, \varphi)$$

$$\longmapsto \varphi(\sigma|_{[v_0,\dots,v_l]})\,\sigma|_{[v_l,\dots,v_k]}.$$



φ 2-chain

Check $\partial(\sigma \frown \varphi)$

$$= (-1)^{\ell}(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

so $\partial(\underbrace{\text{cycle} \frown \text{cocycle}}_{\Rightarrow \text{Cycle}}) = 0$

cap prod. of a cycle C

and coboundary $\varphi = \delta\Phi$.

then $(-1)^{\ell+1} C \frown \varphi = \partial(C \frown \Phi)$

⇒ cap prod of cycle and cobound. is boundary

also if φ cocycle, $C = \partial D$

then $(-1)^{\ell} \partial D \frown \varphi = \partial(D \frown \varphi)$

So cap of bound. and cocycle is ∂.

$\Rightarrow$

$$\frown : H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-1}(X;R)$$

R-linear in each variable

Also rel. versions

$$H_k(X,A) \times H^\ell(X) \xrightarrow{\frown} H_{k-\ell}(X,A)$$

$$\underbrace{H_k(X,A) \times H^\ell(X,A) \xrightarrow{\frown} H_{k-1}(X)}$$

$$C_k(X) \times C^\ell(X) \xrightarrow{\frown} C_{k-\ell}(X)$$

restricts to 0 on

$C_k(A) \times C^\ell(X,A)$ so

get $C_k(X,A) \times C^\ell(X,A) \to C_{k-\ell}(X)$

Naturality: $f: X \to Y$

$$\begin{tikzcd}
H_k(X) \arrow[d, "f_*"] & \times & H^l(X) \arrow[r, "\cap"] & H_{k-l}(X) \\
H_k(X) & \times & H^l(Y) \arrow[u, "f^*"'] \arrow[r] & H_{k-l}(Y)
\end{tikzcd}$$

$$f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*\varphi).$$

Poincaré Duality.

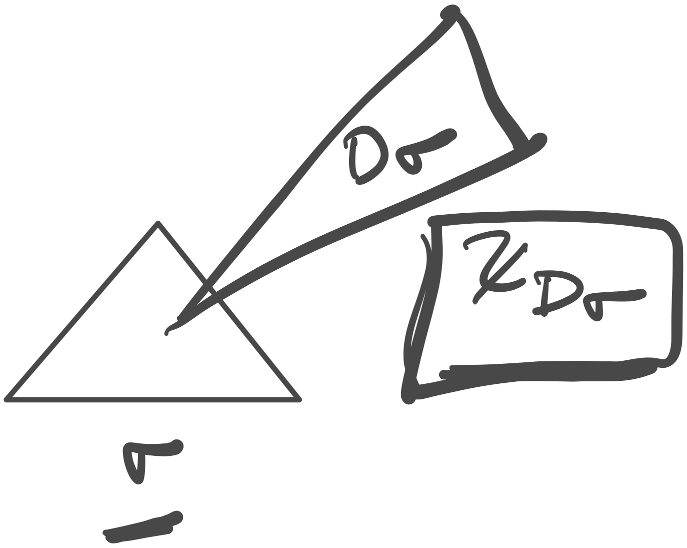
M R-orientable n-mfld w/ fundamental class $[M] \in H_n(M;R)$, then

$D: H^k(M;R) \to H_{n-k}(M;R)$

given by $D(\alpha) = [M] \cap \alpha$

is an isomorphism $\forall k$.

$$\varphi \xmapsto{D} \underbrace{\varphi([0,\dots,k])[v_k,\dots,v_n]}_{\text{where } [0,\dots,n] \text{ chosen as x chosen by } [M].}.$$



One idea from the proof is cohomology with compact supports.

$\exists$ lots of noncompact mflds.

Duality for noncompact mflds.

$$H^k_c(M;R) \cong H_{n-k}(M;R)$$

for R-oriented mflds

Cohomology but only allow cochains supported on finitely many cells.

Next time briefly talk about this.

$$H^1_c(\mathbb{R};\mathbb{Z}) = \mathbb{Z}.$$

Poincaré Duality

M R-orientable with fund. class

$[M] \in H_n(M;R)$ then

$D: H^k(M;R) \longrightarrow H_{n-k}(M;R)$

$D(\alpha) = [M] \cap \alpha$.

$[v_0, \dots, v_n] = \alpha([v_0, \dots, v_k])[v_k, \dots, v_n]$.

One tool in general proofs is cohomology w/ compact supports.

Let X space, G abelian Group, or ring, then $\Delta^i(X;G)$ = singular cochains.

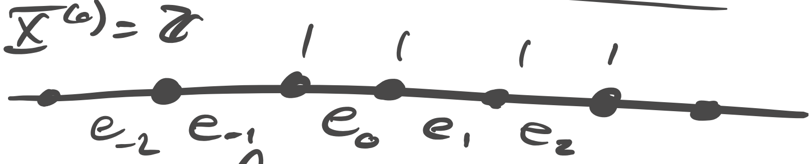
$\Delta^i_c(X;G)$ "compactly supported cochains" all cochains that vanish on all but finitely many singular simplices.

[Alt. definition: could take all cochains φ for which $\exists$ compact $K \subset X$ s.t. φ vanishes on σ if σ misses K.]

$\Delta_c^i(X;G)$ also form a chain cx.

homology of this chain cx is called cohomology of X w/ compact supports, written $H_c^*(X;G)$.

Ex. $X = \mathbb{R}$.



A simplicial 0-cochain φ is a cocycle only if it has same value on all vertices.

So if φ lies in Δ_c^0 and it's a cocycle, then it is $\equiv 0$.

So $H_c^0(\mathbb{R}) = 0$.

Define: $S:\Delta^1_c(\mathbb{R}) \longrightarrow G$

that sends φ to the sum of its values on all 1-sxs.

only makes sense on $\Delta^1_c(\mathbb{R})$ but not $\Delta^1(\mathbb{R})$.

If $\varphi \in \Delta^1_c(\mathbb{R})$ is a coboundary,

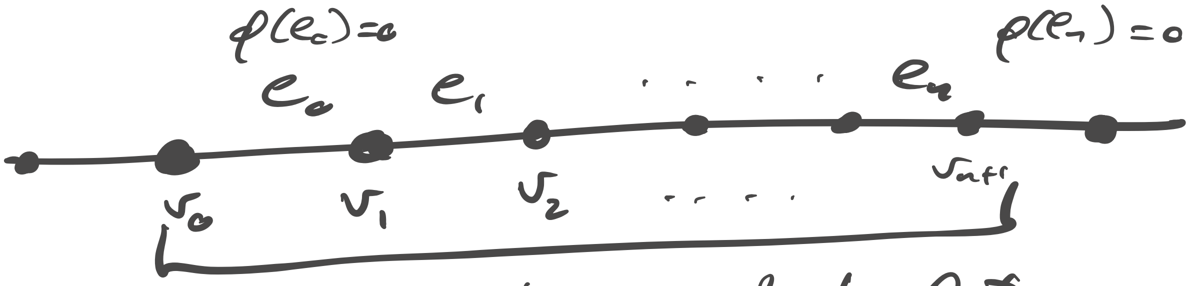
i.e. $\varphi = \delta\Phi$ for $\Phi \in \Delta^0_c(\mathbb{R})$

$= \Phi(\partial\,\cdot)$

Then $S(\varphi) = S(\Phi(\partial\,.))$

$$= \sum_{\sigma \in supp(\varphi)} \Phi(\partial\sigma)$$

$$= 0$$



all 1-simplices on which $\varphi \neq 0$ are in this interval.

Φ s.t. $\varphi = \delta \Phi$.

$$
\begin{aligned}
S(\varphi) = {} & \Phi(v_1) - \Phi(v_0) \\
& + \Phi(v_2) - \Phi(v_1) \\
& \cdots \Phi(v_{n+1}) - \Phi(v_n). \\
= {} & \underbrace{\Phi(v_{n+1}) - \Phi(v_0)}_{\text{but values at endpoints are zero.}}.
\end{aligned}
$$

So if coboundary $S(\varphi) = 0$.

So get map

$$S : H^1_c(\mathbb{R}) \to G.$$

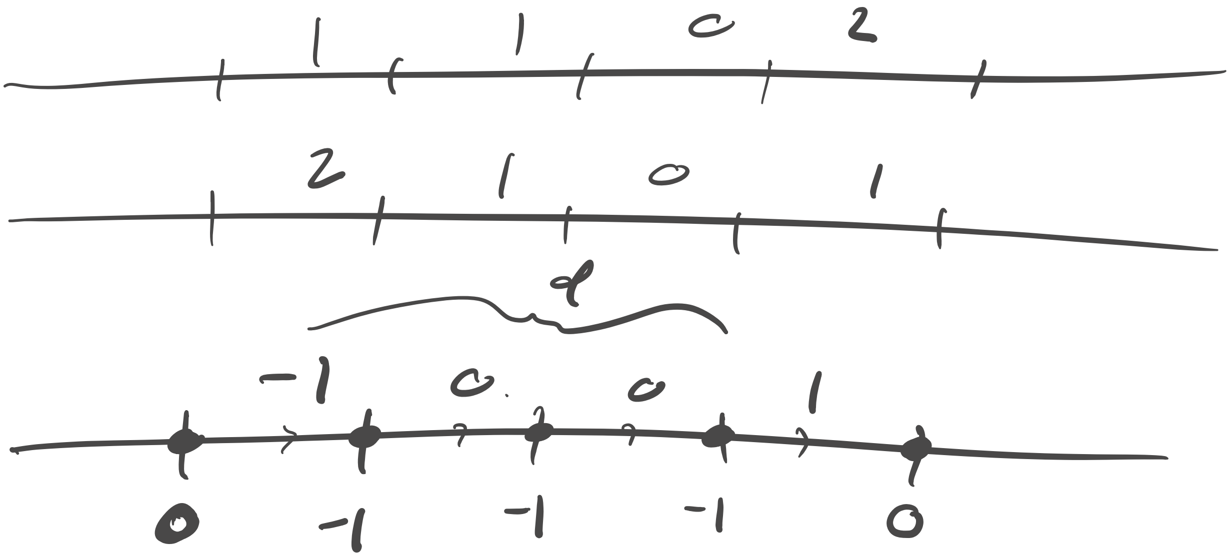
Everything in $H^1_c(\mathbb{R})$ is a cocycle. (no $\Delta^2_c(\mathbb{R})$).

And so $S : H^1_c(\mathbb{R}) \to G$ is surjective: $\forall g \in G$, cocycle:

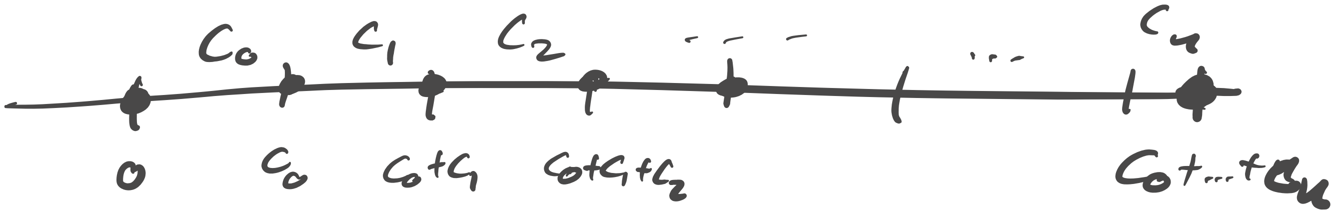
<u>Claim</u>: S injective.

i.e. if $S(\phi) = S(\psi)$ then $\phi \sim \psi$.

$\Leftrightarrow$ if $S(\phi) = 0$ then ϕ coboundary.



Φ



$$c_0+\dots+c_n = S(\phi) = 0.$$

$\phi(e_i) = c_i$.

if $S(\phi)=0$, then assigning partial sums to vertices gives a Φ s.t.

$$\delta\Phi = \phi.$$

So $H^1_c(\mathbb{R};G) = G$.

Duality for noncompact mflds without ∂:

$$H^k_c(M;R) \cong \tilde{H}_{n-k}(M;R)$$

when M is R orientable.

alt. $$H_c^*(X;G) = \varinjlim_{\substack{K \text{ cpct} \\ K \subset X}} H^*(X, X-K; G)$$

$$\downarrow$$

inverse limit.

Connection w/ cap product.

have cap product and we have cup product.

$\boxed{\begin{array}{l} \alpha \in C_{k+\ell} \\ \varphi \in C^k \\ \psi \in C^\ell \end{array}}$

$$\underset{\text{homology}}{\alpha} \cap \underset{\text{cohomology}}{\varphi} \quad \underline{\text{homology class.}}$$

$$\psi(\alpha \cap \varphi) = (\varphi \cup \psi)(\alpha).$$

($\underset{C^k \to C^{k+\ell}}{\varphi \cup \cdot}$ is dual to $\cdot \cap \varphi$) $\mathrm{Hom} \to \mathrm{Hom}$.

PD has implications for R-oriented mfld M

$$\underline{(*)}\ H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R$$

$$(\varphi, \psi) \longmapsto \varphi \cup \psi[M].$$

<u>Bilinear form</u>.

A bilinear form $A \times B \to R$

is <u>nonsingular</u> if the induced maps

$$A \longrightarrow \mathrm{Hom}(B, R)$$
$$B \longrightarrow \mathrm{Hom}(A, R)$$

given by fixing something in each factor are both isomorphisms.

<u>Thm</u> cup product pairing (*) is nonsingular for closed R-orientable mflds when R is a field or $R = \mathbb{Z}$ and you mod out by torsion.

Pf

$$\begin{tikzcd} H^{n-k}(M;R) \arrow[r, "h", "\cong \text{ by hyp.}"'] & \mathrm{Hom}(H_{n-k}(M;R),R) \arrow[d, "D^*", "\cong"'] \\ & \mathrm{Hom}(H^k(M;R),R) \end{tikzcd}$$

D^* = hom dual of duality map.

$D(\Phi)$
$= [M]\cap\Phi$

$$(\dagger)\quad D^*h(\psi) = (\varphi \mapsto \psi([M]\cap\varphi)$$
$$= (\varphi\cup\psi)([M])).$$

h iso by hypothesis.

D^* iso by Poincaré duality

but by $(\dagger)$, this implies that cup product pairing is non singular.

□

$$\begin{tikzcd} H_{n-k} \arrow[r] & R \\ H^k \arrow[u, "D"] \arrow[r] & R \end{tikzcd}$$

Cor. M closed conn. n-mfld

$\forall \alpha \in H^k(M;\mathbb{Z})$ of infinite order, not a multiple (a "primitive" element)

$\exists \beta \in H^{n-k}(M;\mathbb{Z})$ s.t.

$\alpha \smile \beta$ generates $H^n(M;\mathbb{Z})$

And over a field, this is true for *any* nonzero α.

Pf. α generates a summand of H^k so

$\exists \varphi : H^n(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$ s.t. $\varphi(\alpha) = 1$.

This is realized by cup prod. w/ some β and evaluating on $[M]$, by nonsingularity of the cup product pairing.

So $\alpha \smile \beta$ generates $H^n(M;\mathbb{Z})$ $\square$

Revisit $H^*(\mathbb{C}P^n;\mathbb{Z})$

$\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ iso on H^i $i \leq 2n-2$.

Assume by induction that $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$ generated by α^i for $i<n$.

By corollary, $\exists m$ s.t.

$\alpha \smile \underbrace{m\alpha^{n-1}}_{\beta} = m\alpha^n$ generates $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$.

That's only true if $m = \pm 1$

(since 1 not $= mx$).

So $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$.

$\square$

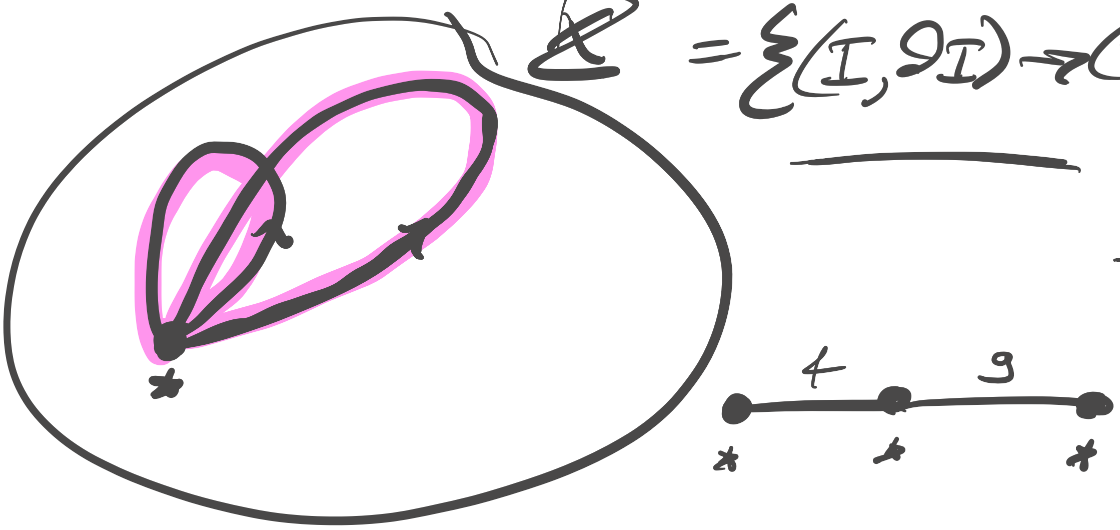
Homotopy Theory.

Homotopy gps.

X space.

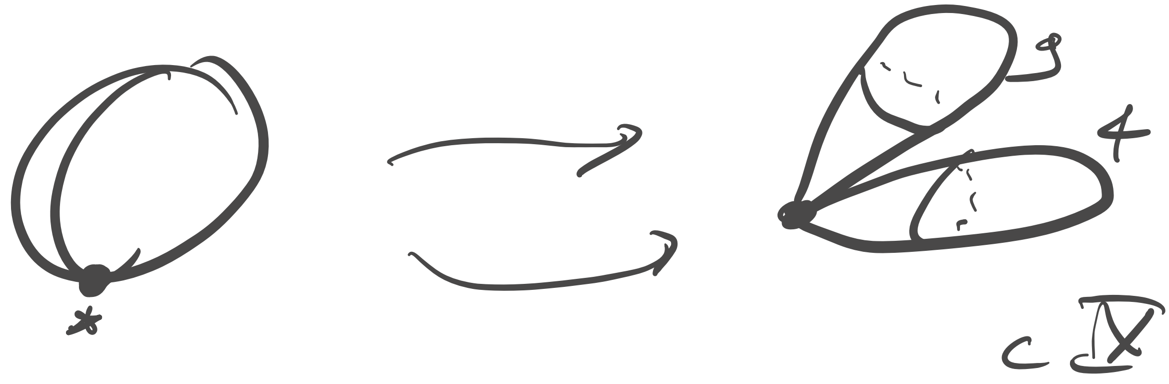
$$\pi_1(X, *) = \{(S^1, *) \to (X, *)\}/\text{htpy}.$$

$$= \{(I, \partial I) \to (X, *)\}/\text{htpy}.$$

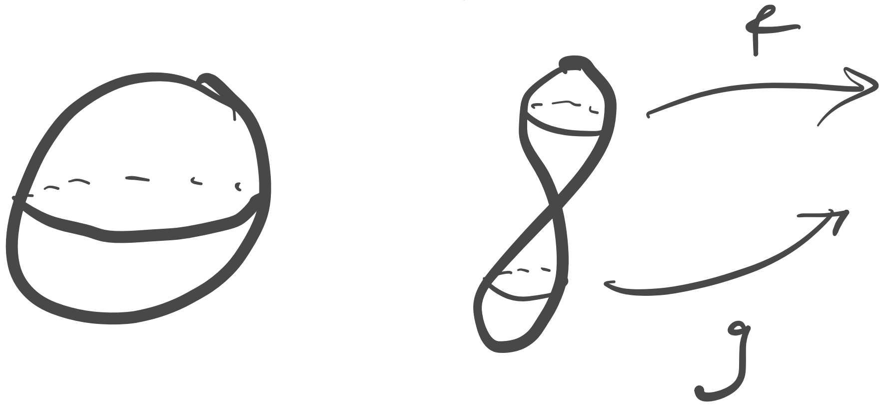


This is a group with concatenation of paths as our operation.

$$\{(S^n,*) \to (X,*)\}/\text{htpy of pairs}$$



want "multiply" the two spheres.

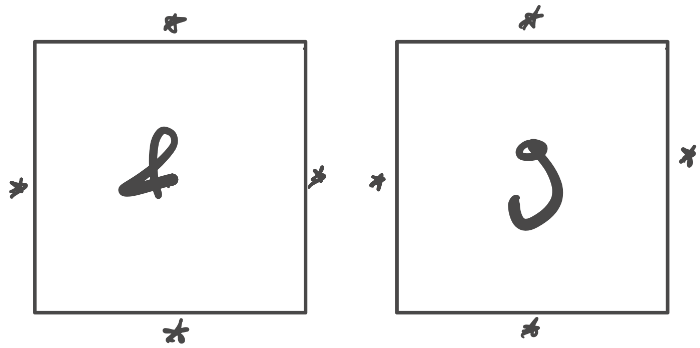


Define: $\pi_n(X,*)$

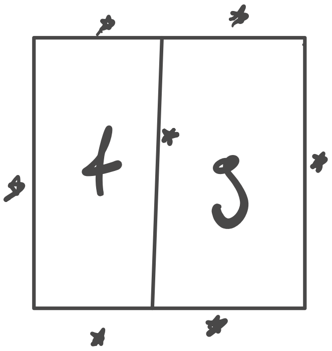
$$= \{(S^n,*) \to (X,*)\} \text{ htpy of pairs}$$

$$= \{(I^n,\partial I^n) \to (X,*)\}/\text{htpy}.$$

If $n \geq 1$ this is a group where the operation is concatenation in first coordinate. Assume $n \geq 2$.



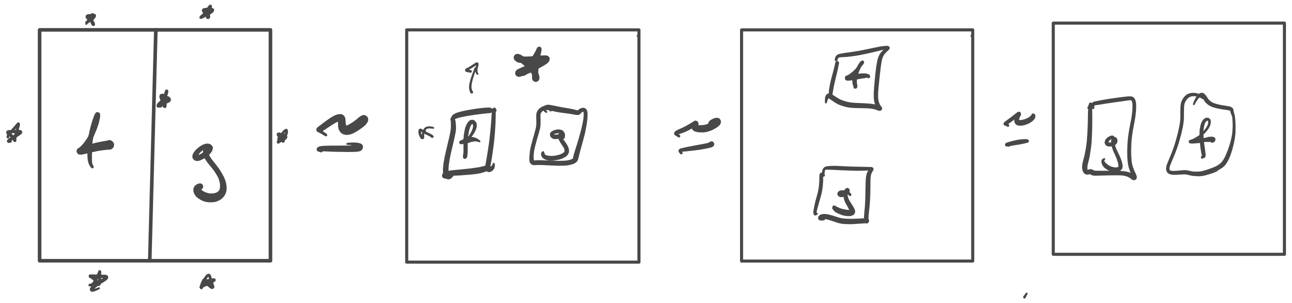
$f+g$:



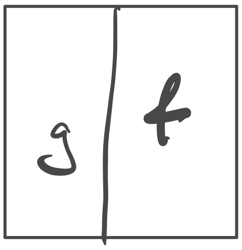
$$f+g(s_1,\dots,s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & s_1 \in [0,\tfrac{1}{2}] \\ f(2s_1-1, s_2, \dots, s_n) & s_1 \in [\tfrac{1}{2},1] \end{cases}$$

Why + sign?

$n \geq 2$, $\pi_n(X,*)$ is abelian.



$\simeq$



.

These gps are really mysterious.

First interesting examples comes from spheres, whose homotopy gps are still, and probably forever, unknown:

$\xrightarrow{i}$ $\pi_i(S^n)$

$n \downarrow$	1	2	3	4	5	6 ⋯		
1	$\mathbb{Z}$	0	0	0	0	0		
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	0	0	$\mathbb{Z}$					
4	0	0	0	$\mathbb{Z}$				
5	0	0	0	0	⋱			

Homotopy Groups.

X space. $X \neq \emptyset$.

$(X, *)$. $* \in X$.

Homotopy groups

$$\pi_n(X,*) = \{(S^n,*) \to (X,*)\}/\underset{\text{of pair}}{\text{htpy}}$$

$$= \{(I^n, \partial I^n) \to (X,*)\}/\text{htpy}$$

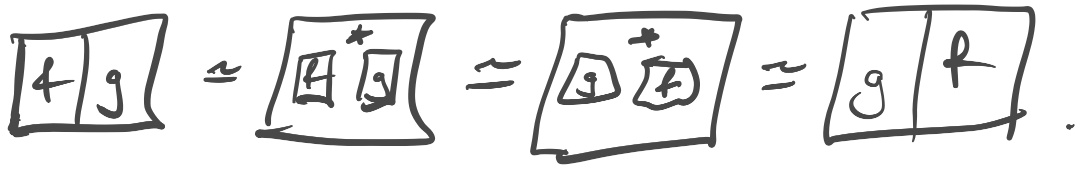
$n \geq 1$, this is a group.

$n = 0$, this is the set of path components of X.

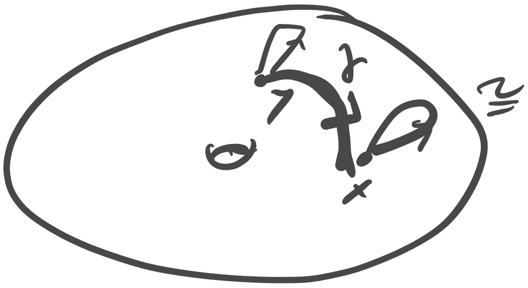
(If X is a topological gp, then $\pi_0(X,*)$ is also a group.)

$$\pi_0(X,*) \overset{}{=} X/\text{cmpt containing identity}.$$

If $n \geq 2$, then $\pi_n(X,*)$ is abelian:

$f+g =$ 

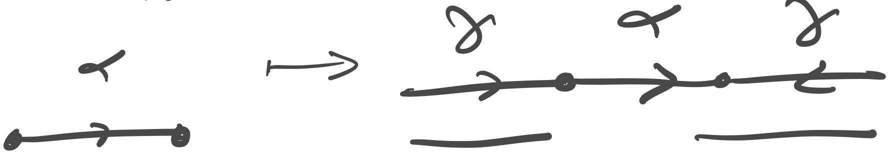
π_n is a functor from spaces to grps.

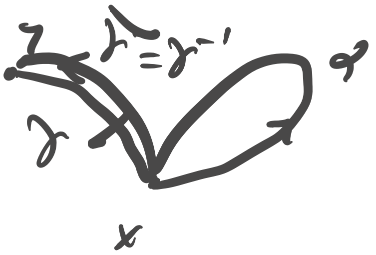


$\pi_1(X, x)$
$\cong \pi_1(X, y)$ if X path connected or if x, y in same path component.

γ path joining y to x

$\beta_\gamma : \pi_1(X, y) \to \pi_1(X, y)$

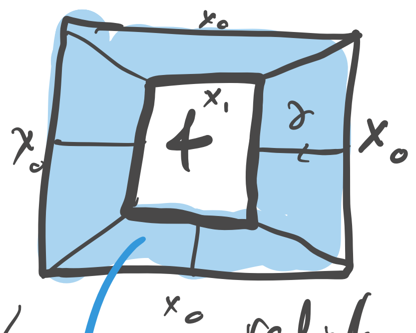


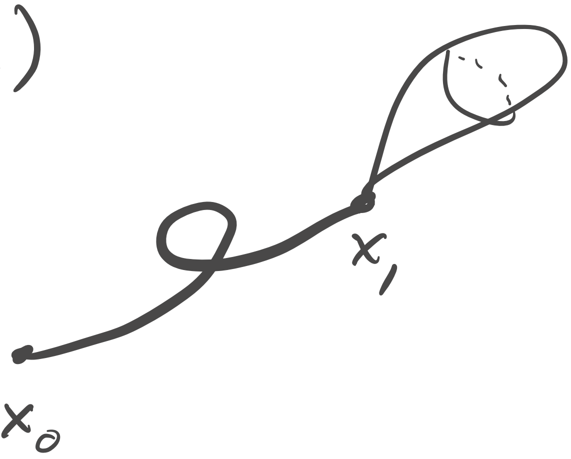


If X path connected, have basepoint change isomorphisms:

$\gamma : I \to X$ $\quad x_0 = \gamma(0)$, $x_1 = \gamma(1)$

$[f] \in \pi_n(X, x_1)$





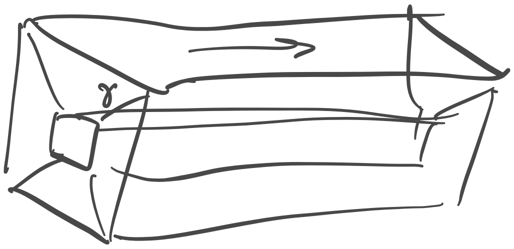
radially insert γ.

Shell is just $\cong \partial I^n \times I$

$\gamma f : (I^n, \partial I^n) \to (X, x_0)$

$[\gamma f] \in \pi_n(X, x_0)$.

<u>Note</u>: htpy of γ $\rightsquigarrow$ htpy of γf rel ∂I



Facts: 1) $\gamma(f+g) \simeq \gamma f + \gamma g$

2) $(\gamma \eta) f \simeq \gamma(\eta f)$

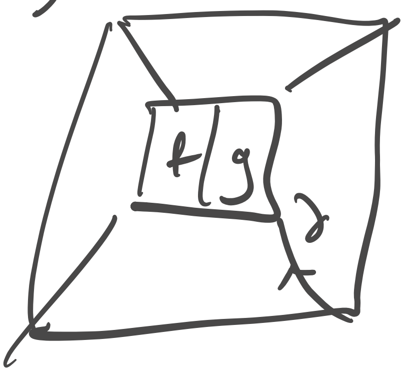
3 $1 \cdot f \simeq f$

$\downarrow$

const. paths

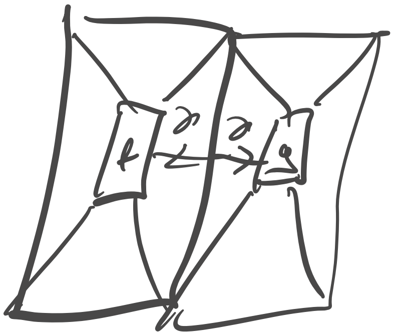
2, 3 clear. ✓

To see 1)



$\gamma(f+g)$

$\gamma f + \gamma g$



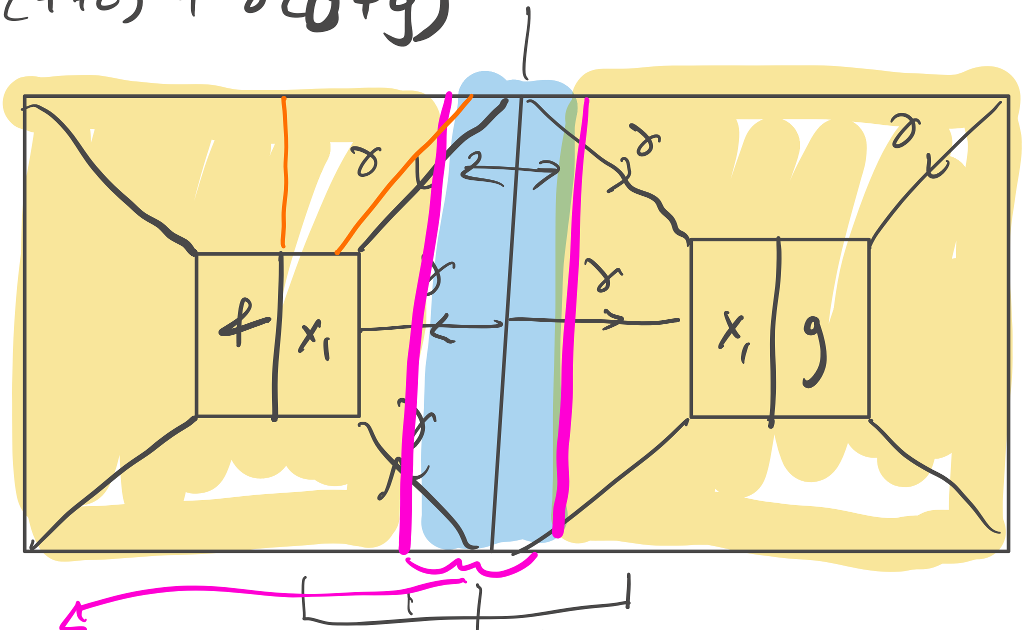
To formally see $\gamma(f+g) \simeq \gamma f + \gamma g$

First: htpe f to $f+0 = \boxed{f \mid x_1} \simeq f$

g to $0+g = \boxed{x_1 \mid g} \simeq g$

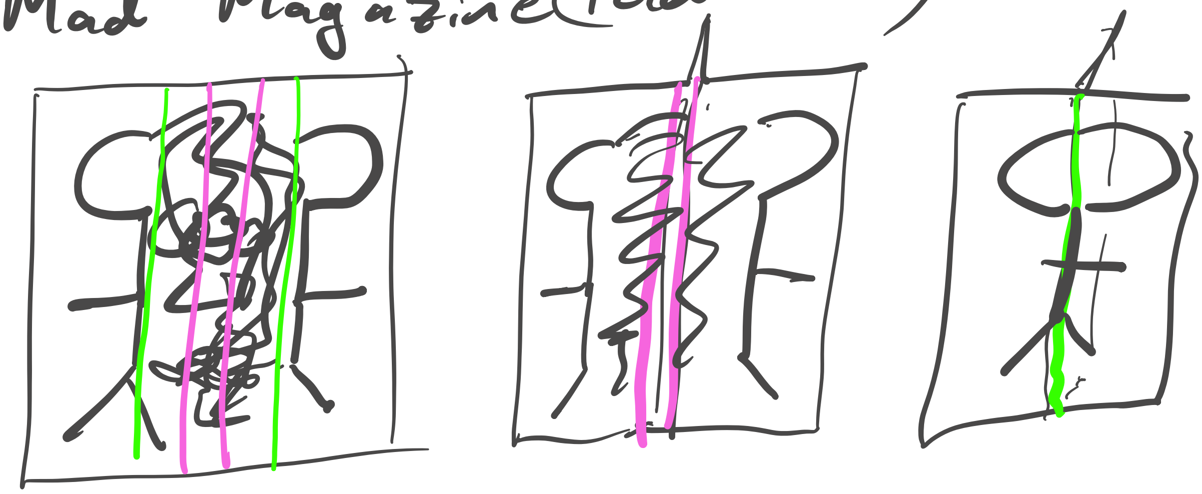
$\gamma(f+0) + \gamma(0+g)$

$=$



$$h_t(s_1,\dots,s_n) = \begin{cases} \gamma(f+0)((2-t)s_1,\dots,s_n), & s_1\in[0,\tfrac{1}{2}] \\ \gamma(0+g)((2-t)s_1+t-1, s_2,\dots,s_n), & s_1\in[\tfrac{1}{2},1]. \end{cases}$$

Mad Magazine (Fold overs?)



Get map

$$\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$$
$$[f] \longmapsto [\gamma f].$$

homomorphism by 1.

2, 3 $\Rightarrow$ β_γ is isomorphism

w/ $\beta_{\bar{\gamma}} = \beta_{\gamma^{-1}}$ is its inverse

---

If γ is a loop at $*$,

Then since

$$\beta_{\gamma\eta} = \beta_\gamma \beta_\eta$$

$[\gamma] \mapsto \beta_\gamma$ is a homo.

$$\pi_1(X, *) \to \mathrm{Aut}(\pi_n(X, *)).$$

each γ give automorphism

$$\beta_\gamma : \pi_n(X, *) \to \pi_n(X, *).$$

If $n=1$, this is the map from

$$\pi_1(X,*) \to \text{Inn}(\pi_1(X,*)).$$

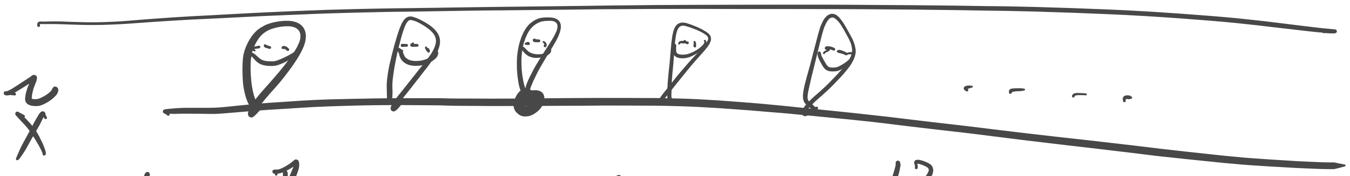
So we have π_1 acting on π_n, so for $n \geq 2$, $\pi_n(X,*)$ is a "$\pi_1(X,*)$-module."

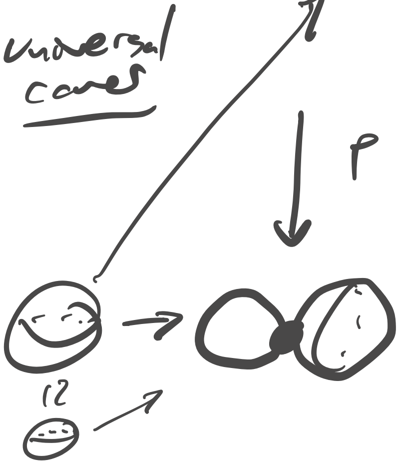
If G is a group, a G-module is a $\mathbb{Z}G$-module, where $\mathbb{Z}G$ is the *integral group ring* of G:

$$\mathbb{Z}G := \left\{ \sum_{i=1}^{N} c_i g_i \,\middle|\, c_i \in \mathbb{Z},\ g_i \in G \right\}$$

Any time G acts on abelian grp A, A is a $\mathbb{Z}G$-module by extending the action linearly.

So for $n \geq 2$, $\pi_n(X,*)$ is $\pi_1(X,*)$-module.





$p_* : \pi_n(\tilde{X}) \twoheadrightarrow \pi_n(X)$ surj. cuz lift spheres.

injective too since can lift htpies.

$\overset{\wr\|}{\bigvee_{\alpha\in\mathbb{Z}} S^2_\alpha}$

$X = S^1 \vee S^2$

$\pi_1(S^1 \vee S^2, *) = \mathbb{Z}$ by VK's theorem

what is $\pi_2(S^1 \vee S^2, *) = ?$

See in a minute, $\pi_2(X,*)$

$\cong \pi_2(\tilde{X}, \tilde{*})$

The π_1 action on π_2 is just action of deck group of $\tilde{X}$ on $\pi_n(\tilde{X})$.

So as a $\underset{\overset{\|}{\mathbb{Z}G}}{\mathbb{Z}\mathbb{Z}}$ module, $\pi_2(X)$ is cyclic.

$G = \mathbb{Z}$

$$\overset{\mathbb{Z}[t,t^{-1}]}{\pi_2(S^1 \vee S^2) = \boxed{\mathbb{Z}\pi_1(X).}}$$

→ Finitely generated as a $\pi_1 X$ module.

This example shows that finite complexes can have infinitely generated $\pi_n(X)$.

Might conjecture that
$\pi_n(X,*)$ is a f.g. $\pi_1(X,*)$-module.
You'd be wrong!

Thm (Stallings)
∃ finite complexes s.t. $\pi_2(X)$ is not f.g. as a $\pi_1(X)$-module.

This issue is related to the cohomology of $\pi_1(X)$, defined by dualizing a free res. of $\mathbb{Z}\pi_1$ and taking homology

$$\cdots \to F_1 \to F_0 \to \mathbb{Z}\pi_1(X) \to 0$$

$$\begin{tikzcd} & {} \arrow[d] & & \\ \cdots & F_0^* \arrow[l] & \mathbb{Z}\pi_1(\tilde{X})^* \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

take homology.

Group rings are fascinating!

$\mathbb{Z}(\mathbb{Z}/p\mathbb{Z})$ has zero divisors.

$\underline{\boxed{t^p}-1}$ factors ⇝ zero divisors.

$t^p-1=0$ in $\mathbb{Z}(\mathbb{Z}/p\mathbb{Z})$, $\mathbb{Z}/p\mathbb{Z}=\langle t\rangle$

factor this poly get zero divisor.

$(t-1)(t+1)=0$.

Kaplansky's Conjecture: G finitely present

If G is torsion free, then

$\mathbb{Z}G$ has no zero divisors.

Thm (Hertweck 2001)

$\exists$ finite gps G, G' s.t.

$\mathbb{Z}G \cong \mathbb{Z}G'$ but $G \not\cong G'$.

$|G| = 2^{25} \cdot 97^2$.

Want explicit description of the action of $\mathbb{Z}\pi_1(X,*)$ on $\pi_n(X,*)$:

$$\left(\sum_{i=1}^{N} c_i \gamma_i\right) f = \sum_{i=1}^{N} c_i (\gamma_i f)$$

If $\pi_1(X,*)$ action is trivial,

we say X is "n-simple".

if it's n-simple for all n,

say that X is simple

or say X is "abelian."

(if action by inner autos is trivial, then $\pi_1 X$ is abelian.)

in particular, if $\pi_1 X = 1$, then X is abelian.

π_n is a functor.

$$\varphi : X \to Y \rightsquigarrow \varphi_* : \pi_n(X, *) \to \pi_n(Y, \varphi(*))$$

and $(\varphi\psi)_* = \varphi_* \psi_*$ and

$$1_* = 1.$$

if φ_t is htpy,

then $(\varphi_0)_* = (\varphi_1)_*$.

So, in particular, pointed homotopy equivalences induce isos on all the htpy gps.

Same is true if ignore basepoints: i.e. if $X \xrightarrow{\simeq} Y$ a homotopy equiv. then $\pi_n(X) \cong \pi_n(Y)$.

Prop: If $p:(\tilde{X},\tilde{*}) \to (X,*)$ is a covering map, then p_* is an isomorphism on π_n $\forall n \geq 2$.

Pf. We can lift any map $f:S^n \to X$ to a map $\tilde{f}:S^n \to \tilde{X}$, i.e.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

since $\pi_1 S^n = 1$

So p_* is surjective.

Can also lift homotopies,
cuz $S^n \times I$ is simply connected

so p_* is injective. $\square$

For homology, interesting homology can appear in covers.

E.g. $\exists$ homology spheres w/ covers that have infinite H_1.

E.g. $K =$ Klein bottle.

$$\begin{tikzcd}
T^2 \arrow[d] & H_2 = \mathbb{Z} \\
K & H_2 = 0.
\end{tikzcd}$$

Corollary of covering space isomorphism

If $\tilde{X}$ is a contractible cover of X (hence $\tilde{X}$ is universal cover) then $\pi_n(X, x) = 0$ $\forall n \geq 2$.

(See later that there is a converse for this.)

Ex. $\pi_n(\underset{\substack{\downarrow \\ n\text{-torus}}}{T^k}) = 0$ for $n \geq 2$.

Defn X is aspherical if
$$\pi_n(X) = 0 \quad \forall\, n \geq 2.$$

Borel Conjecture. If M is an aspherical n-mfld, then M is determined by $\pi_1(M)$. i.e. Does $\pi_1 M \cong \pi_1 N$ for

M, N closed aspherical n-mflds $\Rightarrow$ that $M \cong N$?

Prop. $X = \prod_{\alpha \in \Delta} X_\alpha$, where X_α is path connected, then

$$\pi_n\Big(\prod_{\alpha \in \Delta} X_\alpha\Big) \cong \prod_{\alpha \in \Delta} \big(\pi_n(X_\alpha)\big).$$

Pf. A map $Y \to \prod_{\alpha \in \Delta} X_\alpha$ is the same thing as a collection of maps $Y \to X_\alpha$.

So for $Y = S^n, S^n \times I$, this proves them. $\square$

Next time:

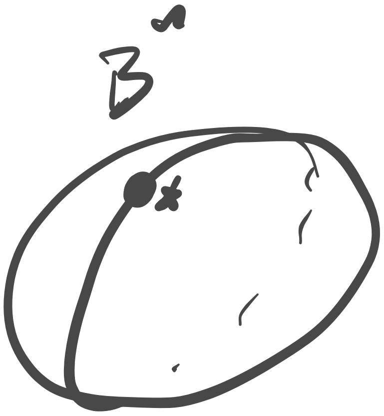
relative groups.

$\pi_n(X, A, *)$

$* \in A \subset X.$

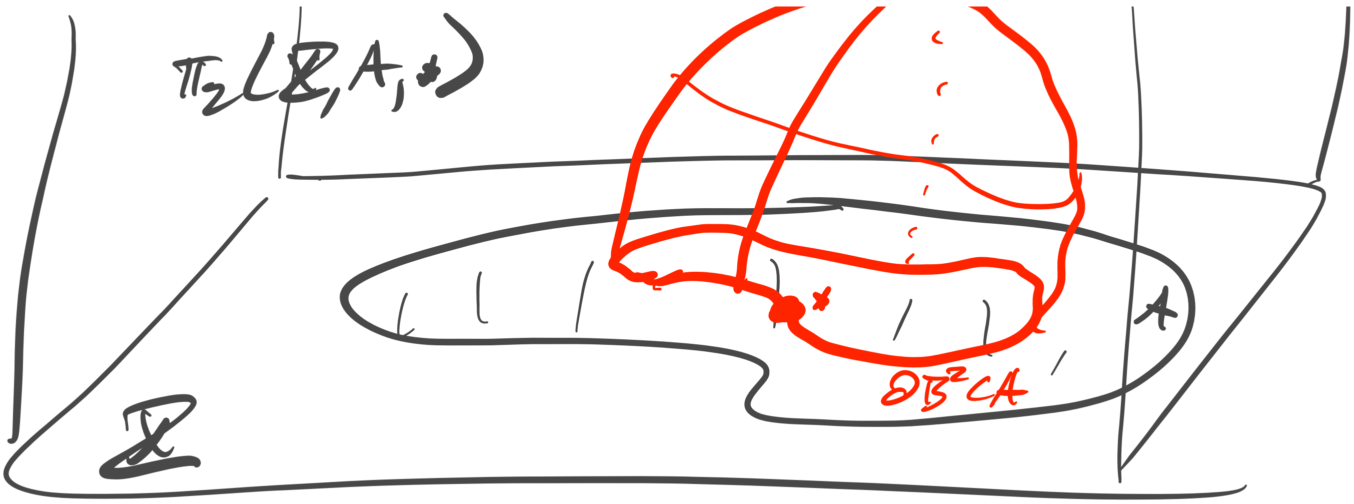
Can think of

$$\pi_n(X,*) \text{ as } \{(S^n,*) \to (X,*)\}/\simeq$$
$$= \{(I^n, \partial I^n) \to (X,*)\}/\simeq$$
$$= \{(B^n, \partial B^n) \to (X,*)\}/\simeq$$



$$\pi_n(X, A, *) = \{(B^n, \partial B^n, *) \to (X, A, *)\}/\simeq \text{ of triples}$$

e.g.


$\pi_2(\mathfrak{X}, A, *)$
$\mathfrak{X}$
A
$*$
$\partial B^2 \subset A$

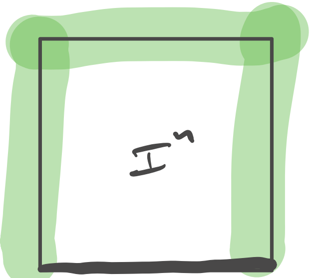
Homotopy Groups

$$\pi_n(X,*) = \{(I^n,\partial I^n) \to (X,*)\}/_{\substack{\text{htpy}\\ \text{of}\\ \text{pairs.}}}$$

Relative groups

$X \neq \emptyset$

$* \in A \subset X$



$J^{n-1} = \partial I^n - I^{n-1}$

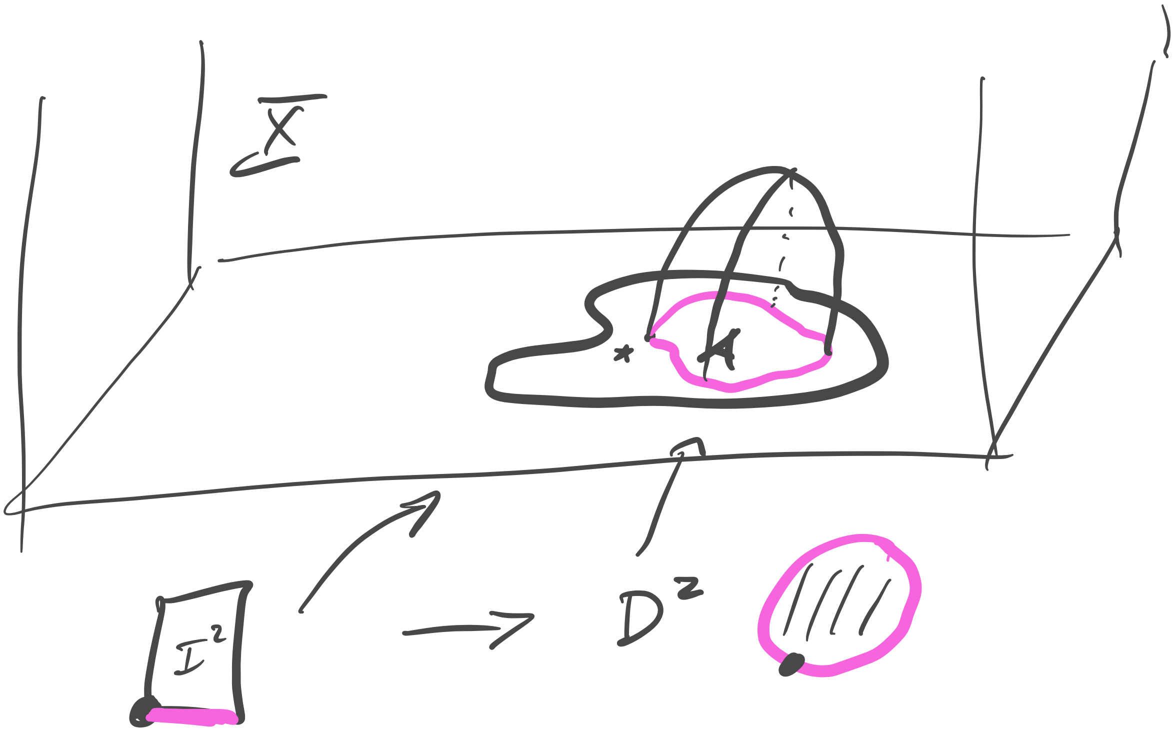
$I^{n-1} = \{(s_1,\dots,s_n) \mid s_n = 0\}$

For $n \geq 1$, let

$$\pi_n(X,A,*) = \{(I^n,\partial I^n, J^{n-1}) \to (X,A,*)\}/_{\substack{\text{htpy}\\ \text{of triples.}}}$$

$\pi_0(X,A,*)$ undefined.

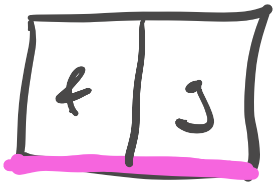
Note $\pi_n(X,*,*) = \pi_n(X,*)$.



Product just like before, except when $n=1$, since s_n plays a special role and so if $s_n = s_1$ then we can't do the concatenation.

If $n \geq 2$, $\pi_n(X, A, *)$ is group.

If $n \geq 3$, $\pi_n(X, A, *)$ is abelian



If $n=1$, $I^1 = \{0,1\}$, $I^0 = \{0\}$, $J^0 = \{1\}$

$\pi_1(X, A, *)$

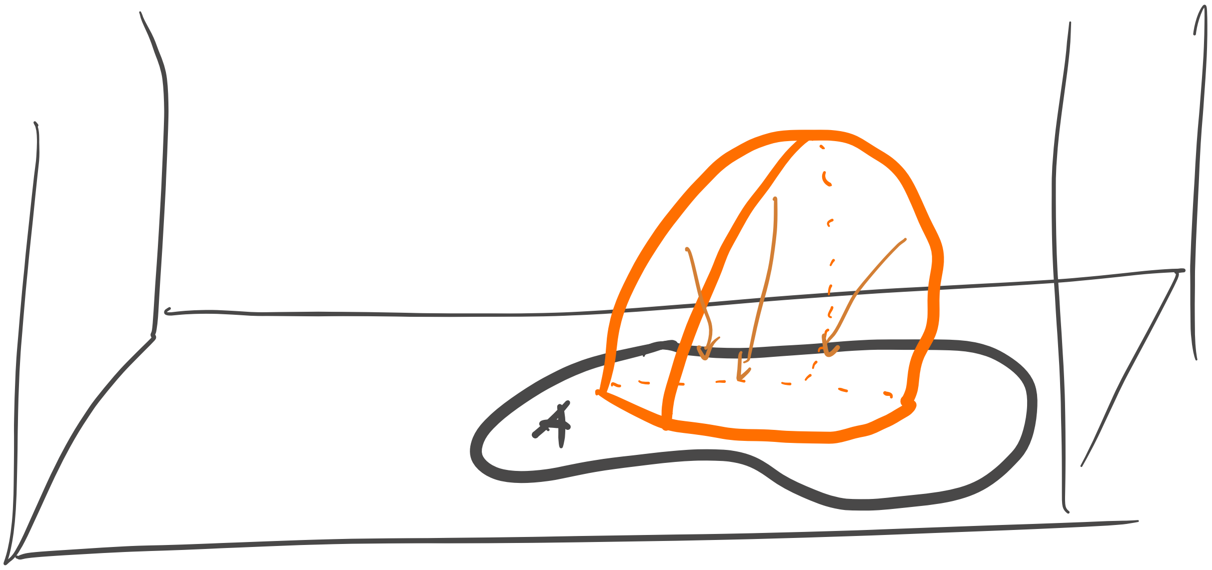
= htpy classes of paths going arbitrary points in A to $*$.

$I^n / J^{n-1} \cong D^n$ so can think of $\pi_n(X, A, *)$ as htpy classes of maps

$(D^n, \partial D^n, *) \longrightarrow (X, A, *)$.

With this perspective, can think of addition as

What does $0 \in \pi_n(X, A, *)$ mean?



If X can htpc disk into A, keeping its ∂ in A $\Longleftrightarrow$ it's trivial in $\pi_n(X, A, *)$.

Compression Criterion:

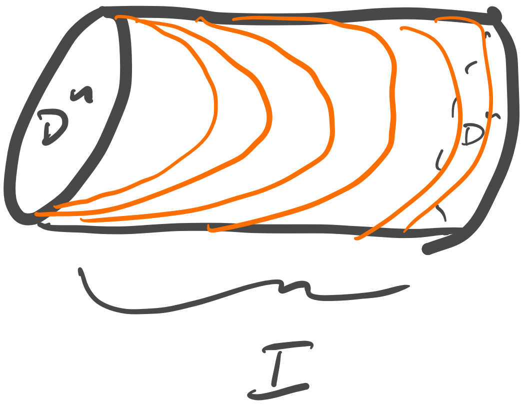
A map $f : (D^n, S^{n-1}, *) \to (X, A, *)$ is 0 in $\pi_n(X, A, *)$, iff it's htpc $\underline{\text{rel } S^{n-1}}$. to a map g w/ image in A.

means fixing values on ∂

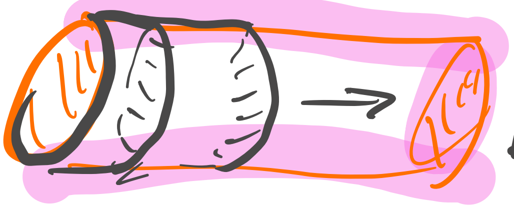
Pf. If there's such a htpy, can then follow w/ a htpy gotten by precomposing with def. retr. $D^n \searrow *$ and so $[f]=0 \in \pi_n(X, A, *)$.

If $[f]=0$ in $\pi_n(X,A,*)$ then f is htpc through maps of triples by a htpy $F: D^n \times I \to X$ takes D^n into A.

Then $\exists$ a htpy F' that keeps values fixed on ∂D^n:



1-parameter family of disks in $D^n \times I$ beginning w/ $D^n \times \{0\}$ and ending with $\underbrace{D^n \times \{1\} \cup S^{n-1} \times I}_{\cong D^n}$.



$D_t = S^{n-1} \times [0,t] \cup D^n \times \{t\}$

$$F' : D^n \times I \to X$$

$$F'(d,t) = F|_{D_t}.$$

F' is a htpy rel ∂ of f to a map that lands in A.

$\pi_n(X,A,*)$ also functorial.

$$\varphi : (X,A,*) \to (Y,B,*)$$

$$\rightsquigarrow \varphi_* : \pi_n(X,A,*) \to \pi_n(Y,B,*)$$

$$(\varphi\psi)_* = \varphi_* \psi_* , \quad \mathbb{1}_* = \mathbb{1},$$

$\varphi_* = \psi_*$ if $\varphi \simeq \psi$ as maps of triples.

Thm There is a long exact sequence

$$\to \pi_n(A,*) \xrightarrow{i_*} \pi_n(X,*) \xrightarrow{j_*} \pi_n(X,A,*) \xrightarrow{\partial} \pi_{n-1}(A,*)$$

$$\to \dots$$

i, j inclusions $\quad i:(A,*)\to(X,*)$

$j:(X,*,*)\to(X,A,*)$.

∂ just restricts

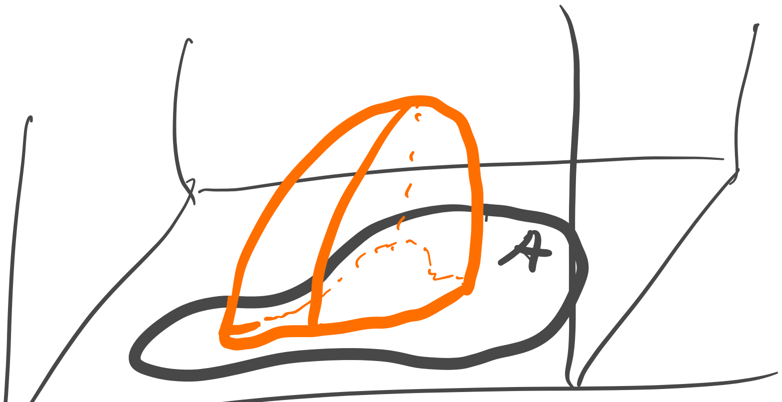
$(I^n,\partial I^n, J^{n-1})$

$\to (X,A,*)$

to I^{n-1}.

(or $(D^n,S^{n-1},*)\to(X,A,*)$

to S^{n-1}.)



map of D^n $\quad f:(D^n,\underline{\partial D^n})\to(X,\underline{A})$

$$\underbrace{\partial D^n \longrightarrow A}_{\partial f}$$

Proof of exactness.

$$\xrightarrow{\partial} \pi_n(A,*) \xrightarrow{i_*} \pi_n(X,*) \xrightarrow{j_*} \pi_n(X,A,*) \xrightarrow{\partial} \pi_{n-1}(A,*) \to$$

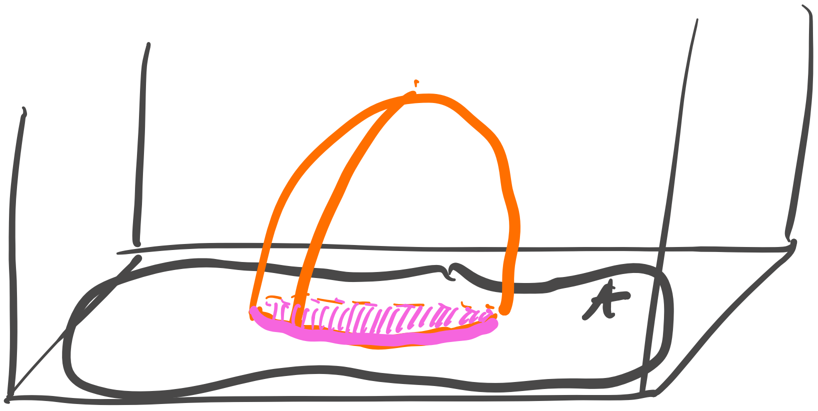
something here dying in $\pi_n(X)$ means that the map of the sphere $f: S^n \to A \subset X$ can be extended to a map $F: D^{n+1} \to X$. But $F(\partial D^{n+1}) \subset A$, and so F represents an element of $\pi_{n+1}(X,A,*)$ and $\partial F = f$. So f is in the image of ∂.

$\ker i_* \subset \operatorname{Im} \partial$.

something in here dying in $\pi_n(X,A,*)$ means that sphere is htpc rel basepoint to a map into A. That means that the map is in $i_* \pi_n(A,*)$. So $\ker j_* \subset \operatorname{Im} i_*$.

Something dying under ∂ means that $S^{n-1} = \partial D^n \to A$ is null htpc in A. So I can extend the original $(D^n, \partial D^n) \to (X, A)$

to a map $S^n \to X$, which is repr. an element of $\pi_n(X,*)$. So $\ker \partial \subset \operatorname{Im} j_*$.



$$\xrightarrow{\partial} \pi_n(A,*) \xrightarrow{i_*} \pi_n(X,*) \xrightarrow{j_*} \pi_n(X,A,*) \xrightarrow{\partial} \pi_{n-1}(A,*) \to$$

Something in image of ∂ extends over a ball mapping into X, and so is null-htpc in X. So in $\ker i_*$.

Something in image of i_* is htpc into A, and so it's in the kernel of j_*.

Something is in image of j_*, then the ∂ map takes it to zero.

$\operatorname{Im} \partial \subset \ker i_*$

$\operatorname{Im} i_* \subset \ker j_*$

$\operatorname{Im} j_* \subset \ker \partial$.

$\square$

Ex. $\underset{\text{cone}}{CX} \simeq *$

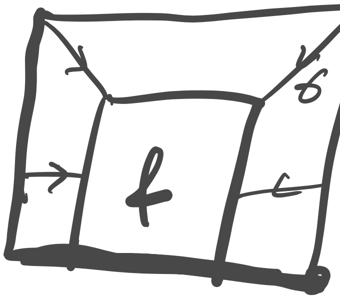
LES of (CX, X)

$\rightsquigarrow \quad \pi_n(CX, X, *) \cong \pi_{n-1}(X, *)$

$\forall n \geq 1$.

Let $n=2$, realize any group G as relative $\pi_2(CX, X)$, since every G is $\pi_1(X, *)$ for some X.

Again $\pi_1(X, *)$ acts on these things:



π_1 acts on entire LES of pair.
So LES is a LES of π_1-modules.

Def^n: $(X,*)$ is **n-connected**

if $\pi_i(X,*)=0$ $\forall i\le n$.

0-conn. = path connected

1-conn = simply connected.

TFAE: ① Every $S^i\to X$ null htpc

② Every $S^i\to X$ extends to D^{i+1}

③ $\pi_i(X,x)=0$ $\forall x\in X$.

TFAE: (for $i>0$)

① Every $(D^i,\partial D^i)\to(X,A)$ htpc **rel ∂** to $D^i\to A$.

② Every $(D^i,\partial D^i)\to(X,A)$ htpc through maps of pairs to $D^i\to A$

③ Every $(D^i,\partial D^i)\to(X,*)$

htpc through maps of pairs to constant map $D^i \to *$.

④ $\pi_i(X, A, *) = 0 \quad \forall * \in A$.

If $i=0$, (1)-(3) are equivalent to each cmpt of X containing an elt of A since D^0 is a pt and $\partial D^0 = \emptyset$.

Df
(X, A) is n-conn. if 1-4 hold for $0 < i \leq n$ and (1)-(3) for $i = 0$.

Whitehead's Theorem

If $f: X \to Y$ is a map between connected CW complexes that induces isomorphisms

$f_*: \pi_n(X,*) \to \pi_n(Y,*)$ $\forall n$,

then f is a htpy equivalence.

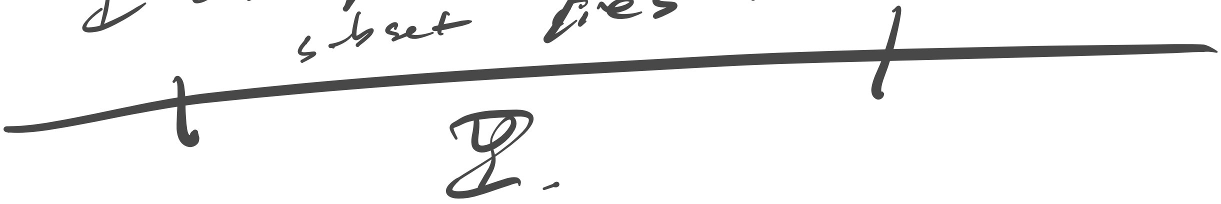
If f is inclusion of a subcomplex then X is a deformation retract of Y.

(If X, Y aren't CW cxs, theorem can fail, but say that X, Y are weak equivalent if $\exists$ map like f, i.e. inducing $\cong$ on all π_n.)

Ex. Let X be a point and let Y be the long line.

$\pi_n(Y,*) = 0 \ \forall n.$

Y has property that every cpct subset lives in an interval.



$i: X \to Y$ will induce isomorphisms on all π_n.

But Y is not contractible.

Thm Basically everything is weak-equivalent to a cell cx.

So as far as algebraic topology is concerned, cell cxs are everything.

Warning. There has to be a map that induces the isos on π_n.

---

i.e. two non-h.e. spaces can have isomorphic π_n $\forall n$.

Ex. $X = \mathbb{RP}^2$. $Y = S^2 \times \mathbb{RP}^\infty$.

$\pi_1(X) = \mathbb{Z}/2\mathbb{Z} = \pi_1(Y)$

and the universal covers $\tilde{X} = S^2$,

$$\tilde{Y} = S^2 \times S^\infty.$$

$S^\infty \simeq *$, so $\tilde{X} \simeq \tilde{Y}$.

$$\text{so} \quad \begin{array}{c} \pi_n(\tilde{X}) \cong \pi_n(X) \quad \forall n \geq 2 \\ \text{\rotatebox{90}{$\cong$}} \\ \pi_n(\tilde{Y}) \cong \pi_n(Y) \end{array}$$

X, Y have all same htpy gps.

But $\underbrace{\mathbb{RP}^2}_{\text{this don't}} \not\simeq \underbrace{S^2 \times \mathbb{RP}^\infty}_{\text{has homology in infinitely many dimensions}}$

So there is no map

$$\mathbb{RP}^2 \to S^2 \times \mathbb{RP}^\infty$$

realizing the isos.

Ex. (later) S^2 and $\underbrace{S^3 \times \mathbb{CP}^\infty}$

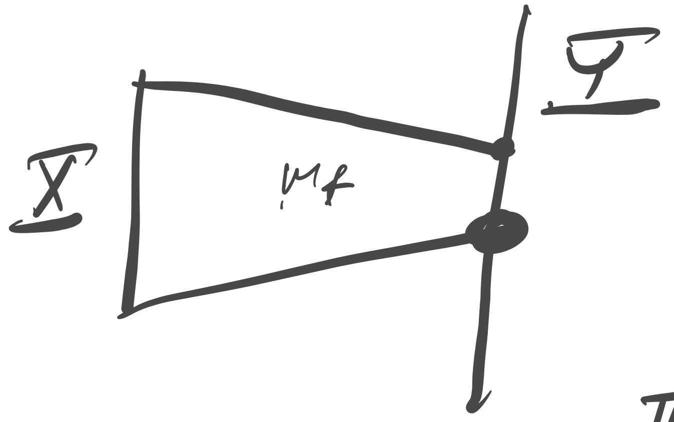
However! If $X^{\neq\emptyset} \simeq$ cell cx and all the htpy gps vanish:

i.e. $\pi_n(X) = 0 \quad \forall n \geq 0.$

then $X \simeq *$. i.e X contractible.

Pf. $i : * \hookrightarrow X$ induces isos on all π_ns and so Whitehead's theorem says $* \simeq X$.

i_* is h.e.



$f : X \to Y$.

$\pi_n(M_f, X) = e$.

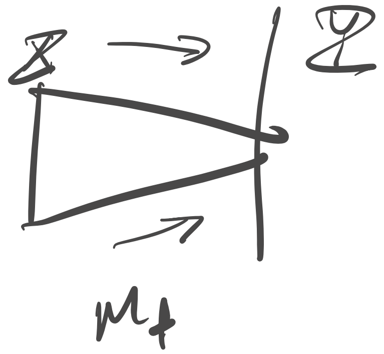
M_f

If f induces isos on $\pi_n \; \forall n$, def. retract M_f onto X.

Whitehead's Thm

If $f: X \to Y$ between connected CW cxs induces isomorphisms $f_*: \pi_n(X) \to \pi_n(Y)$ $\forall n$, then f htpy equiv.

If f is inclusion of subcx then X def. retract of Y.



hyp. $\Rightarrow$

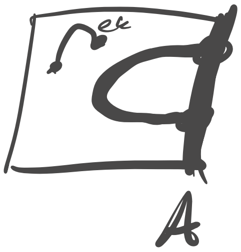
$M_f \searrow X$.

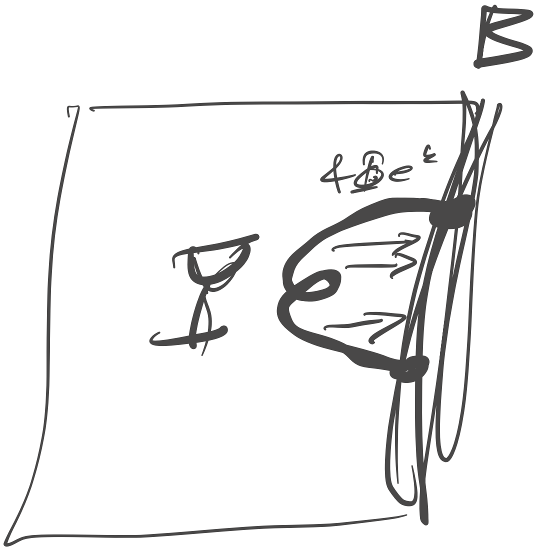
Compression lemma

(X, A) be CW pair. (Y, B) any pair w/ $B \neq \emptyset$.

Suppose $\forall n$ s.t. $X - A$ has cells of dim^n n, assume that $\pi_n(Y, B, y_0) = 0$ $\forall y_0 \in B$. ($\pi_0(Y, B, y_0) = 0$ means (Y, B) 0-connected)

Then every map $f : (X, A) \to (Y, B)$ is homp rel A to a map $f : X \to B$.





Pf. Assume inductively that we've already htped so that $X^{k-1}\to B$.

Φ char. map of cell e^k in $X-A$.

$$e^k \xrightarrow{\Phi} X-A \xrightarrow{f} Y$$

in $k-1$ shel, which goes to B by hypothesis

Then $f\Phi : (D^k, \partial D^k) \to (Y, B)$ can be htped rel ∂D^k to take D^k into B since $\pi_k(Y,B,y_0)=0$.

(By previous "compression lemma.")

This induces a htpy of f on $X^{k-1}\cup e^k$ rel X^{k-1}.

Do this for all $e^k\subset X-A$ simultaneously while doing constant htpy on A to get htpy of $f|_{X^k\cup A}$ to a map $X^k \cup A \longrightarrow B$.

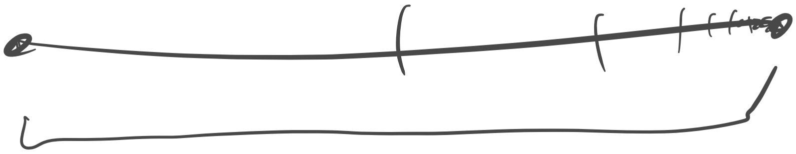
By the htpy extension property for subcxs (Prop. 0.16).

This gives a htpy of f on all of X so that $X^k \cup A \longrightarrow B$.

If X has finite dimension, or even if cells of $X - A$ are of bdd dimension. Do this finitely many times.

If not, just do the htpies at kth stage on interval

$$\left[1-\frac{1}{2^k}, 1-\frac{1}{2^{k+1}}\right].$$



$\square$

Pf of Whitehead's Thm

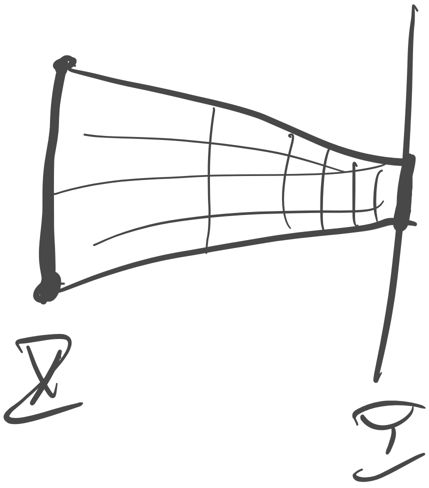
If f is the inclusion of a subcomplex, consider (Y, X). If f induces isos on all htpy groups, then by looking at LES of pair (Y, X), we have $\pi_n(Y, X) = 0$ $\forall n$

$$\begin{tikzcd}
{} \arrow[r, "\partial = 0"] & \pi_n(X) \arrow[r, "i_*", "\cong"'] & \pi_n(Y) \arrow[r, "j_*", "0"'] & \pi_n(Y, X) \arrow[r, "0"] \arrow[d, equal] & \pi_{n-1}(X) \\
& & & 0 &
\end{tikzcd}$$

By lemma, $Y \searrow X$, since we can homotope the identity to $\mathbb{1}$ on X.

For proof of first part, use mapping cylinders:

Let $M_f = X \times I \sqcup Y /_{(x,1) \sim f(x)}$.



We have inclusions $X \hookrightarrow M_f$
$Y \hookrightarrow M_f$.

And $M_f \searrow Y$, and so $M_f \simeq Y$

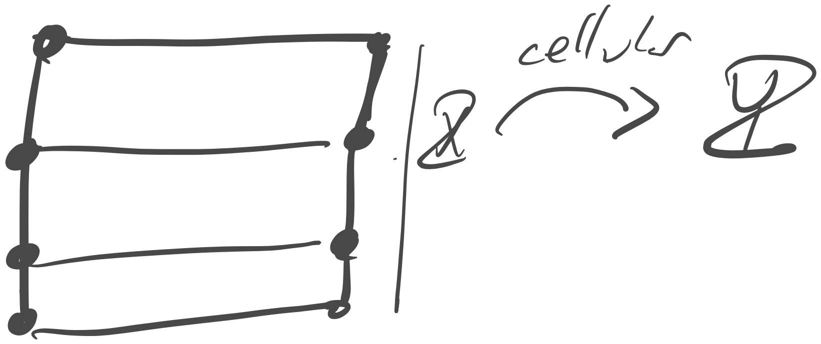
Want $M_f \searrow X$.

f induces isomorphisms on all π_n iff $\pi_n(M_f, X) = 0$ $\forall n$.

If f is cellular, i.e. $f(X^k) \subset Y^k$

then M_f is a cell complex

and X is a subcomplex.



$\Rightarrow$ M_f cell cx.

In that case, previous argument

$\Rightarrow$ $M_f \searrow X$.

Thm (Cellular approx)
Every map between cell cxs
is htpc to a cellular map.

Then finish proof by saying

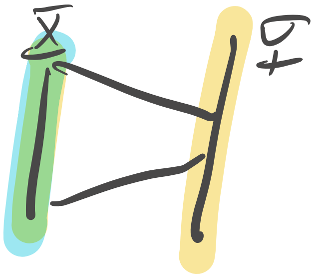
$f \sim g$, g cellular.

$M_f \simeq M_g$ (chapter 0.).

and so $M_g \searrow X$. ✓

Can Avoid cellular approximation if you want as follows:

$(X \cup Y, X) \hookrightarrow (M_f, X)$.

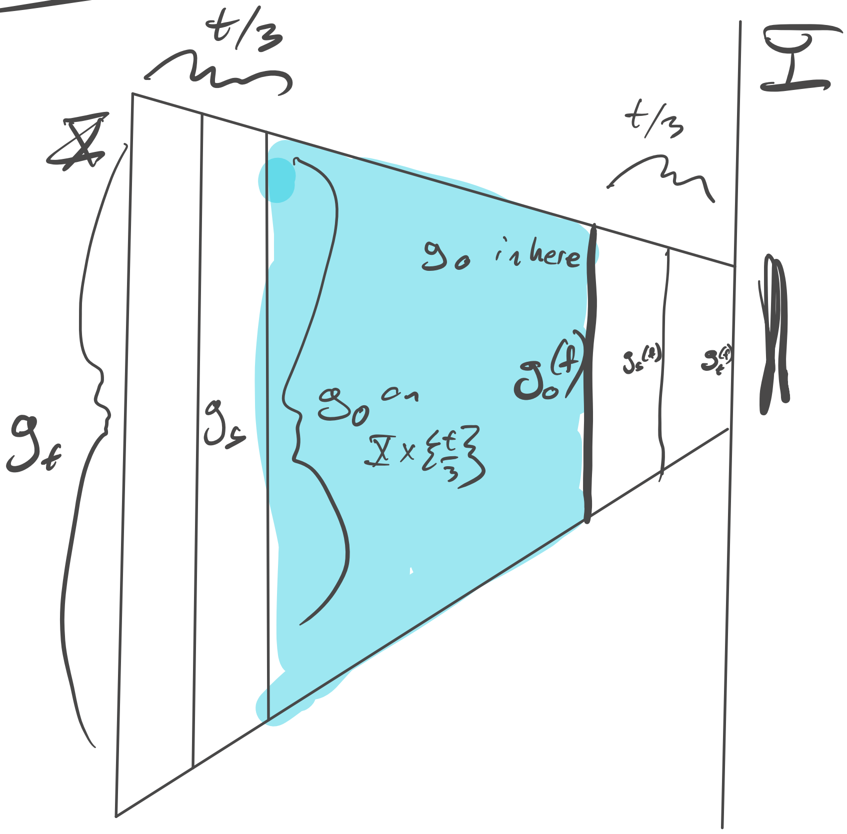


By lemma, htpe this map rel X to a map into X.

$$\boxed{(Y, B) = (M_f, X) \qquad \pi_n(M_f, X) = 0}$$

Pair $(M_f, X \cup Y)$ has htpy extension property: If $g_0 : M_f \to Z$ is a map to a space Z and $g_t\big|_{X \cup Y}$ is a htpy, then can extend htpy to a htpy on all of M_f.

time t:



At time t, what does g_t do on M_f?

So we can extend our htpy above to a htpy of $\mathbb{1}$ to $g : M_f \to M_f$

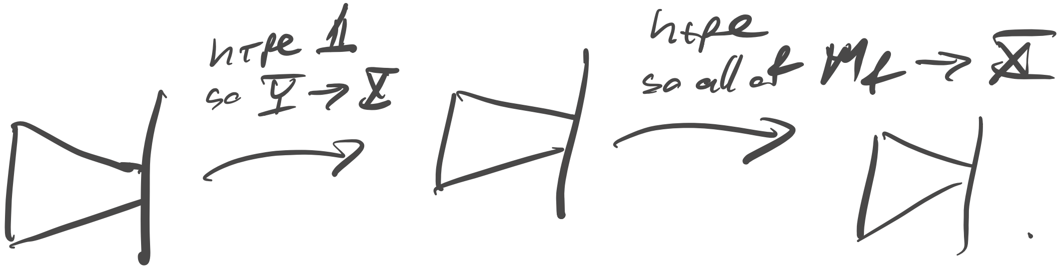
takes $X \cup Y$ into X.

Apply lemma again to the comp.

$$(X \times I \sqcup Y, X \times \partial I \cup Y) \longrightarrow (M_f, X \cup Y) \xrightarrow{g} (M_f, X)$$

to get def retract of M_f to X.

<u>recap:</u>



$\square$

Need the map in Whitehead's theorem.

Useful. Extension Lemma.

CW pair (X,A) and $f : A \to Y$ w/ Y path conn., then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ $\forall n$ s.t. $X - A$ has cells of dimension n.

Pf. If already defined on $(n-1)$ skel. Then can extend over n cell iff ∂ is $\simeq *$ $\square$.

Cellular Approximation.

Recall proof that $\pi_1(S^n) = 1$ if $n > 1$.

$S^n = B_1^n \cup_{\partial B_1^n = \partial B_2^n} B_2^n$ $\quad B_i^n$ an n-ball.

Pick basepoint in $\mathring{B}_1^n$.

Consider map $f:(S^1,*) \to (S^n,*)$.

Pick point $p \in \mathring{B}_2^n$.

If f misses p, then $f \simeq *$. cuz $S^n - p \cong \mathbb{R}^n$.

So we htpe f to g s.t. g misses p.

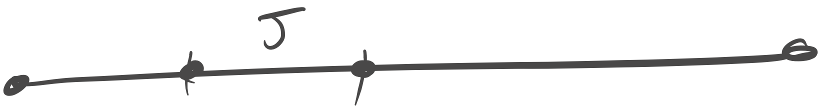
$f^{-1}(\mathring{B}_2^n)$ open. = union of open intervals in S^1.

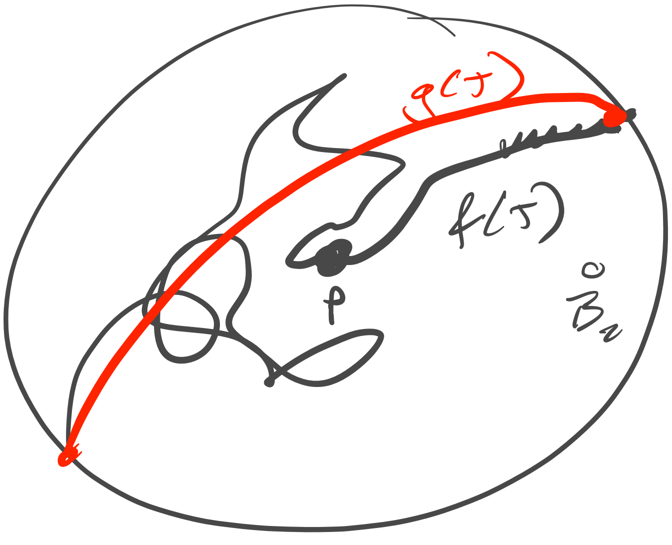
and $f^{-1}(p)$ is compact.

So $f^{-1}(p)$ is completely contained in a finite union of open intervals in $f^{-1}(\mathring{B}_2^n)$.

Change f on those intervals:

htpe f on each interval J to get a map that misses p on





these intervals, and so resulting map $g: S^1 \to S^n$ misses p and so $f \simeq g \simeq *$ so $[f] = 1 \in \pi_1$

□

Same kind of idea is how you prove cellular approximation.

hits only finitely many cells $e^k \subset Y$.

for each $e^k \subset Y$ $k > n$, want to htpe f on e^n so it misses some point in e^k.

X

Y

Then htpe e^n into ∂e^k.

Do this until $X^n \to Y^n$.

Thm (Cellular Approximation)
Every $f: X \to Y$ of cell cxs X, Y is homotopic to a cellular map.
If f is already cellular on subcx $A \subset X$, can take htpy to be stationary on A.

Cor $\pi_n(S^k) = 0$ if $n < k$.

Pf Use std cell structure on S^ℓ with 1 0-cell & 1 n-cell.
Then any cell. map $S^n \to S^k$ is constant when $n < k$. ☐

Proof of cell. approx.

Suppose $f: X \to Y$ is cellular on X^{n-1}.

Let e^n n-cell in X.

$\bar{e}^n$ is compact

So $f(\bar{e}^n)$ is cpct.

So $f(\bar{e}^n)$ meets only finitely many cells of Y.

So $f(e^n)$ " " " "

" " ".

Let e^k be a cell of largest dim that it hits.

Can assume that $k > n$ or else map is already cellular on e^n.

Want ~~to~~

Deform $f|_{X^{n-1} \cup e^n}$, keeping fixed on X^{n-1}, so that $f|_{e^n}$ misses $p \in e^k$. (we'll see next time that we can do this).

Then deform $f|_{X^{n-1} \cup e^n}$ rel X^{n-1} to miss e^k entirely.

After finitely many of these, we get $f|_{X^{n-1} \cup e^n}$ misses all cells of dim $k > n$.

So we do this for all n-cells simultaneously to get htpy of $f|_{X^n}$ rel $X^{n-1} \cup A^n$ to a cell map. Then use h.e.p. to get htpy of f so that it's cellular on $X^n \cup A^n$. letting $n \to \infty$ and doing these htpies faster and faster, i.e. on $[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}]$ get desired htpy. $\square$

Need

Lemma $f : I^n \to Z_1$ where

$$Z_1 = W \cup e^k$$

Then $f \simeq$ rel $f^{-1}(W)$ to f_1 so that $\exists\, \Delta^k \subset e^k$ w/ $f^{-1}(\Delta^k)$ a union of finitely many convex polyhedra on each of which, f_1 is restriction of a linear surjection $\mathbb{R}^n \to \mathbb{R}^k$.

Cellular Approximation.

Recall that a map $f: X \to Y$ between cell cxs is **cellular** if $f(X^k) \subset Y^k$ $\forall k \geq 0$.

Thm Every $f: X \to Y$ between cell cxs is htpc to a cellular map. If f is already cellular on subcx $A \subset X$, htpy can be taken stationary on A.

Lemma $f: I^n \to Z_1$ where

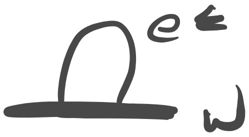
$Z_1 = W \cup e^k$

Then $f \simeq \text{rel } f^{-1}(W)$ to f_1 so that $\exists\, \Delta^k \subset e^k$ w/ $f_1^{-1}(\Delta^k)$ a union of finitely many convex polyhedra on each of which, f_1 is restriction of a linear surjection $\mathbb{R}^n \to \mathbb{R}^k$.

So, if $k > n$, then there are no linear surjections $\mathbb{R}^n \to \mathbb{R}^k$ and so f_1 will miss Δ^k completely.

So lemma completes proof of cellular approximation by picking copy of I^n in e^n.

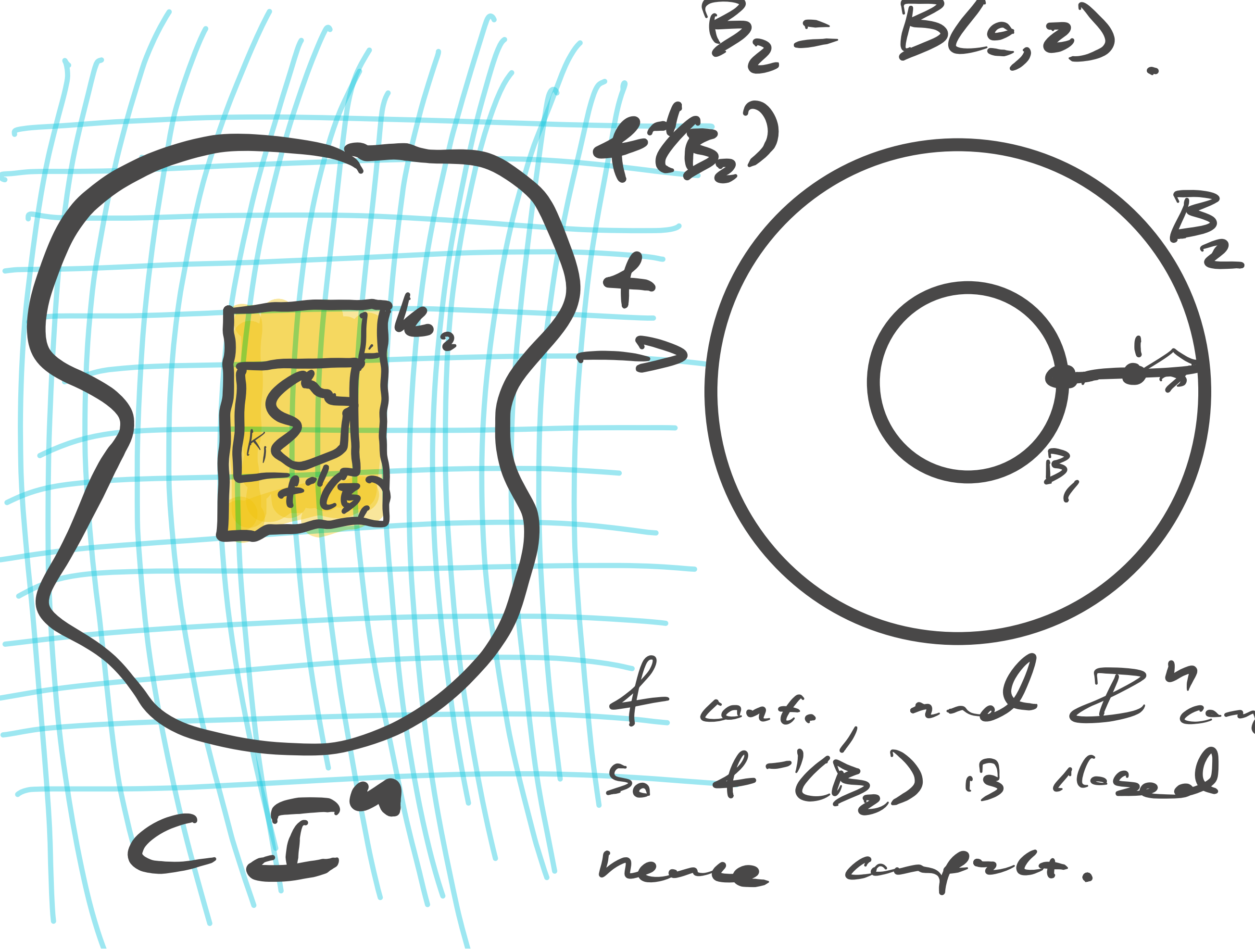
Proof of lemma: $I^n \longrightarrow$



Identify e^k w/ $\mathbb{R}^k$.

in e^k let $B_1 = \overline{B(\underline{0},1)}$

$B_2 = \overline{B(\underline{0},2)}$.



f cont., and I^n compact, so $f^{-1}(B_2)$ is closed hence compact.

So f is uniformly continuous on $f^{-1}(B_2)$ by the Heine-Cantor Theorem. So $\exists\, \varepsilon > 0$ s.t.

$$\underset{\text{on } I^n}{\|x-y\|_{\text{Eucl.}}} < \varepsilon \quad \text{then} \quad \underset{\text{on } \mathbb{R}^k}{\|f(x)-f(y)\|_{\text{Eucl.}}} < \tfrac{1}{2}$$

$\forall\, x, y \in f^{-1}(B_2)$.

Subdivide I^n into cubes of diam $< \varepsilon$.

K_1 = union of cubes hitting $f^{-1}(B_1)$

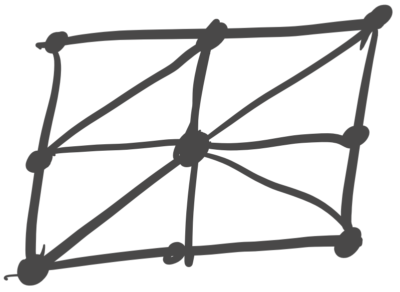
K_2 = " " " hitting K_1.

By defs:

$$f^{-1}(B_1) \subset K_1 \subset \underbrace{K_2 \subset f^{-1}(B_2)}_{\substack{\text{by our choice of} \\ \text{diameter of cubes.}}}$$

since every cube in K_2 hits K_1, and so distance between image of such a cube and $Z - B_2$ is >0.

Make our "cubulation" of I^n into a simplicial structure by letting midpoints of all obvious k-cubes of I^n into vertices and taking cx hulls:



let $g : K_2 \longrightarrow e^k = \mathbb{R}^k$ be the map that's equal to f on all the vertices in our simplicial structure and linear on sxs.

Pick a cont. map $\varphi : K_2 \to [0,1]$

s.t. $\varphi(\partial K_2) = 0$ and $\varphi(K_1)=1$

(do this by Urysohn's lemma.)

Define htpy $f_t : K_2 \to \mathbb{C}^k$.

$$\underline{(1-t\varphi)f} + (t\varphi)g.$$

<u>Note:</u> $f_0 = f$

$$f_1|_{K_1} = g|_{K_1}$$

and f_t is stationary on ∂K_2

because on K_2 we have $\varphi \equiv 0$.

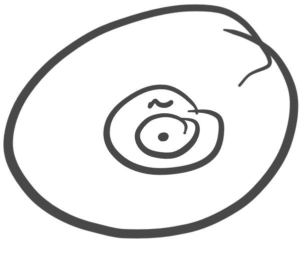
So we extend f_t to htpy

$f_t : I^n \to Z_1$ by demanding

that it's stationary on $I^n - K_2$.

Now: We claim $\exists$ nbhd N of 0 in $B_1 \subset \mathbb{R}^k$ s.t.

$$f_1^{-1}(N) \subset K_1$$



That happens

$$\underbrace{\Leftrightarrow f_1 \text{ takes complement of } K_1 \text{ to the complement of } N.}$$

__To see this:__

Note that on $I^n - K_2$ that's satisfied, cuz $f_1 = f$ there and f sends $I^n - K_2$ to $\mathbb{R}^k - B_1$.

For elements of $K_2 - K_1$, argue as follows:

pick simplex σ in K_2.

$f(\sigma)$ maps into a ball B_δ of radius

$\frac{1}{2}$. B_σ is convex! ($e^k = \mathbb{R}^k$)

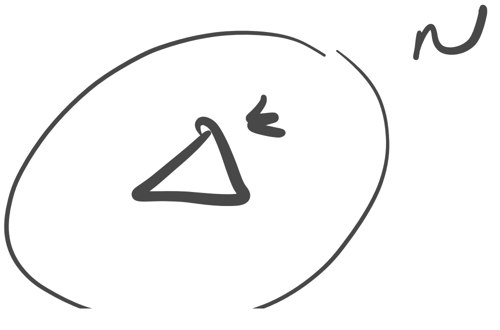
So g also maps σ into B_σ cuz it's linear on σ!

So therefore f_1 takes σ into B_σ.

Now, if σ is not in K_1, then B_σ meets the exterior of B_1 and so it is disjoint from a nbhd of $\underline{0}$ in B_1.

There are only finitely many σs like that. So we get a nbhd N as desired.

Finally: for $\Delta^k \subset N$



$$\underbrace{f_1^{-1}(\Delta^k)} \subset K_1$$

union of intersections with simplices σ of K_1, and each of those intersections is a cx polyhedron $L_\sigma^{-1}(\Delta^k)$ where $L_\sigma : \mathbb{R}^n \to \mathbb{R}^k$ is the linear map $g|_\sigma$.

Now, pick Δ^k disjoint from all of the nonsurjective L_σs. We can do that, since the image of a nonsurjective L_σ lies in a hyperplane, and so we only need to miss finitely many hyperplanes.

Then result is a linear surjection. $\square$

This technique of "straightening" a map to $\mathbb{R}^k$ by looking at image of a mesh of points and extending linearly comes up a lot in geometry.

Cor. Cellular approximation for pairs:

$$f: (X, \underset{\text{sub cx}}{A}) \longrightarrow (Y, \underset{\text{sub cx}}{B}) \text{ map.}$$

First deform $A \to B$ to be cellular.

Extend deformation to $(X, A) \to (Y, B)$ using htpy extension property.

Then htpe the maps to be cellular via homotopy that's stationary on A. $\square$

Cor. A CW-pair (X,A) is n-connected if all the cells in $X - A$ have dim $> n$.

In particular, (X, X^n) is n-connected and so $X^n \hookrightarrow X$ induces isomorphisms on π_i when $i < n$ and a surjection on π_n.

Pf. apply cell. approximation to maps $(D^i, \partial D^i) \to (X,A)$.

$\Rightarrow$ first statement.

Rest of statement follows from LES of (X,A).

Next in text: CW approximation of spaces. X, Y are spaces. (not nec. cell.)

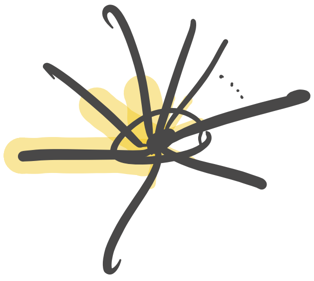
If $f: X \to Y$ induces isomorphisms on all homotopy gps, say that f is a *weak equivalence*.

CW approximation M for X would be a CW cx M and a weak equivalence

$$M \to X.$$

---

often in topology and gp theory you have infinite cell complexes and they aren't metrizable.

$\mathbb{X} =$  $= \bigvee_{\alpha \in \mathbb{N}} I_\alpha$.

This is not metrizable.

for Example $U = \bigcup [0, \frac{1}{\alpha})$ is open in $\mathbb{X}$.

There is <u>no</u> metric that induces this topology, cuz every metric nbhd of $\underline{0}$ will completely contain all but finitely many of the $[0, \frac{1}{\alpha})$.

$\exists$ obvious metric on $\mathbb{X}$: make each interval I_α unit length and take path metric.

$\mathbb{X}'$

X weak equiv to X!

often a cell cx might be easier to understand via a weak equivalent non-cellular space.

Next up. Excision for π_n?

Excision doesn't hold for π_n.

In certain cases, there is a version excision.

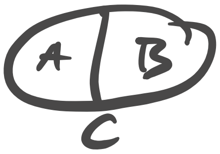
Excision

Hurewicz

Fiber Bdles

Cohomology.

Excision? $X =$



$$\tilde{H}_k(X,B) \cong \tilde{H}_k(A,C).$$

For htpy gps things aren't as nice:

Ex: $$X = S^2 = \underset{A}{D^2} \underset{\underset{C}{\partial D^2}}{\cup} \underset{B}{D^2}$$

$$\pi_3(S^2, \underset{\underset{B}{\shortparallel}}{D^2}) \cong \pi_3(S^2,*)$$

$\pi_3(S^2,*) \overset{?}{\cong} \mathbb{Z}$ } by Hopf fibration (see Ex3 later)

By LES:

$$\underset{\underset{0}{\shortparallel}}{\pi_3(D^2)} \to \pi_3(S^2,*) \xrightarrow{\cong} \pi_3(S^2,D^2) \xrightarrow{\partial} \underset{\underset{0}{\shortparallel}}{\pi_2(D^2)}$$

$$\pi_3(A,C) = \pi_3(A,\partial A) = \underline{\pi_3(D^2,\partial D^2).}$$

$$\underset{\underset{0}{\shortparallel}}{\pi_3(D^2)} \to \underset{0 \neq \mathbb{Z}.}{\underset{\shortparallel}{\pi_3(D^2,\partial D^2)}} \to \pi_2(\underset{\underset{S^1}{\overset{?}{\cong}}}{\partial D^2}) = 0$$

But! Excision does work in a small range:

Thm Let X CW cx. $X = A \cup B$, A, B subcxs and $A \cap B = C \neq \emptyset$ also subcx.

If (A, C) is n-connected and (B, C) is m-connected, where $m, n \geq 0$, then

$$\underbrace{c_*}_{\text{inclusion}_*} : \pi_i(A, C) \to \pi_i(X, B)$$

is isomorphism for $i < m+n$ and surjection for $i = m+n$.

Skip the proof. Technical. Uses our lemma from cellular approximation.

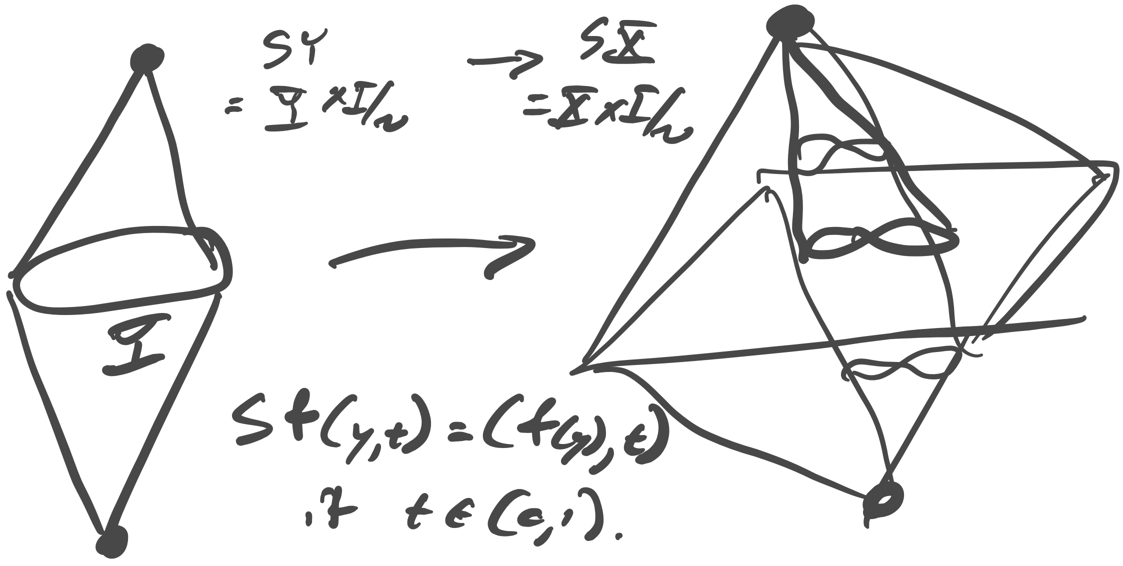
There's a suspension map

$$S: \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$
$$f \longrightarrow Sf$$

More generally, given Σ and a map $f: \Upsilon \to \Sigma$ there is a "suspension" of the map:

$$Sf: S\Upsilon \to S\Sigma.$$

$$\begin{tikzcd} S\Upsilon \arrow[r] & S\Sigma \end{tikzcd}$$
$= \Upsilon \times I/\sim \qquad = \Sigma \times I/\sim$



$$Sf(y,t) = (f(y), t)$$
if $t \in (0,1)$.

Sends suspension vertices to " " .

Cor. Freudenthal Suspension Thm.

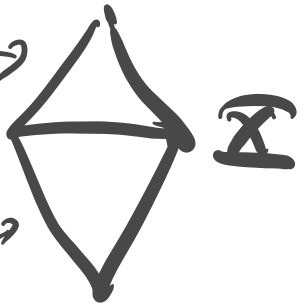
Suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$

$$f \longmapsto Sf$$

is an isomorphism for $i<2n-1$ and a surjection for $i=2n-1$.

Actually true for $\pi_i(X) \to \pi_{i+1}(SX)$ when X is $(n-1)$-connected CW cx.

Pf. $$SX = C_+X \cup_X C_-X$$



$S: \pi_i(X) \to \pi_{i+1}(SX)$ and this map is the "same" as the map

$$\pi_i(X) \underset{\cong}{\xleftarrow{\partial}} \pi_{i+1}(C_+X, X):$$

since

$$\begin{tikzcd} \pi_{i+1}(C_+X) \arrow[r] \arrow[d, equal] & \pi_{i+1}(C_+X, X) \arrow[r, "\partial", "\cong"'] & \pi_i(X) \arrow[r] & \pi_i(C_+X) \arrow[d, equal] \\ 0 & & & 0 \end{tikzcd}$$

since $C_+X \simeq *$

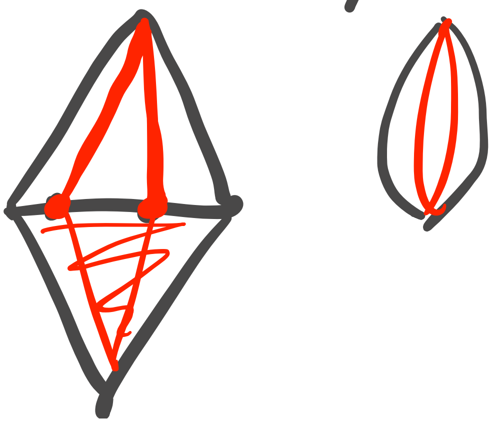
and $\pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX)$

since

$$\begin{tikzcd} \pi_{i+1}(C_-X) \arrow[r] \arrow[d, equal] & \pi_{i+1}(SX) \arrow[r] & \pi_{i+1}(SX, C_-X) \arrow[r] & \pi_i(C_-X) \arrow[d, equal] \\ 0 & & & 0 \end{tikzcd}$$

Therefore:

$$\begin{tikzcd} \pi_i(X) & \pi_{i+1}(C_+X, X) \arrow[l, "\partial"', "\cong"] \arrow[r, "i_*"] & \pi_{i+1}(SX, C_-X) \arrow[d, "\wr\wr"] \\ f \arrow[rr, "\text{suspension map}"'] & & \pi_{i+1}(SX) \end{tikzcd}$$



By LES of (C_+X, X), this pair is n-connected if X is $(n-1)$-connected. So, by theorem, middle map is isomorphism for $i+1 < 2n$ and surjection for $i+1 = 2n$. $\square$

<u>Cor.</u> $\pi_n(S^n) \cong \mathbb{Z}$ generated by $\mathbb{1}$ for all $n \geq 1$. In particular, $\deg : \pi_n(S^n) \to \mathbb{Z}$ is iso.

<u>Pf.</u> We have a suspension sequence

$$\pi_1(S^1) \xrightarrow{S} \pi_2(S^2) \xrightarrow{S} \pi_3(S^3) \to \cdots$$

By cor. first map is surjective
(for $n=1$, $n = 2n-1$,
for $n \geq 2$, $n < 2n-1$.)

and the rest are isomorphisms.

$\pi_1(S^1) = \mathbb{Z}$ generated by $\mathbb{1}$,
and so $\pi_n(S^n)$, for $n \geq 2$, is
a cyclic gp that's independent of
n, generated by $\mathbb{1}$.
But $\forall n$, there exist maps
of arbitrary degree, and nonzero
degree maps are not nullhtpc.
So that cyclic gp is infinite.

The degree map itself $\pi_n(S^n) \to \mathbb{Z}$
is iso since $z \to z^k$ on S^1
has degree k and so do its
suspensions by Prop 2.33.

$\square$

Ex. $\pi_n(\vee_\alpha S^n_\alpha)$ $n \geq 2$ is free abelian w/ basis htpy classes of $S^n_\alpha \xhookrightarrow{i_\alpha} \vee_\alpha S^n_\alpha$.

If fntely many summands,

$$\vee_\alpha S^n = \left(\prod_\alpha S^n_\alpha\right)^{(n)} \leftarrow \text{n-skeleton}$$

where we take usual cell structure $\cdot \cup B^n$ on S^n and prod. cell sts. on prod.

The dimensions of cells of $\prod_\alpha S^n_\alpha$ are all multiples of n.

So $(\prod_\alpha S^n_\alpha, \vee_\alpha S^n_\alpha)$ is $(2n-1)$-connected, by cell. approx.

So,

$$i_*: \pi_n(\vee S^n_\alpha) \to \pi_n(\prod_\alpha S^n_\alpha)$$

is an isomorphism:

$$\begin{tikzcd}
\pi_{n+1}(\prod S^n_\alpha, \vee S^n_\alpha) \arrow[r] \arrow[d, phantom, "\cong"] & \pi_n(\vee_\alpha S^n_\alpha) \arrow[r, "i_*", "\cong"'] & \pi_n(\prod_\alpha S^n_\alpha) \arrow[d] \\
0 & & \pi_n(\prod S^n_\alpha, \vee S^n_\alpha) \arrow[d, phantom, "\cong"] \\
 & & 0
\end{tikzcd}$$

Now, since $\pi_n(\prod_\alpha S^n_\alpha) = \bigoplus_\alpha \pi_n(S^n_\alpha)$

we're **done.**

If there are infinitely many S^n_αs,

$\exists$ homom. $\Phi: \bigoplus \pi_n(S^n_\alpha) \to \pi_n(\vee S^n_\alpha)$

induced by inclusions.

Φ is surjective since any $f: S^n \to \vee S^n_\alpha$

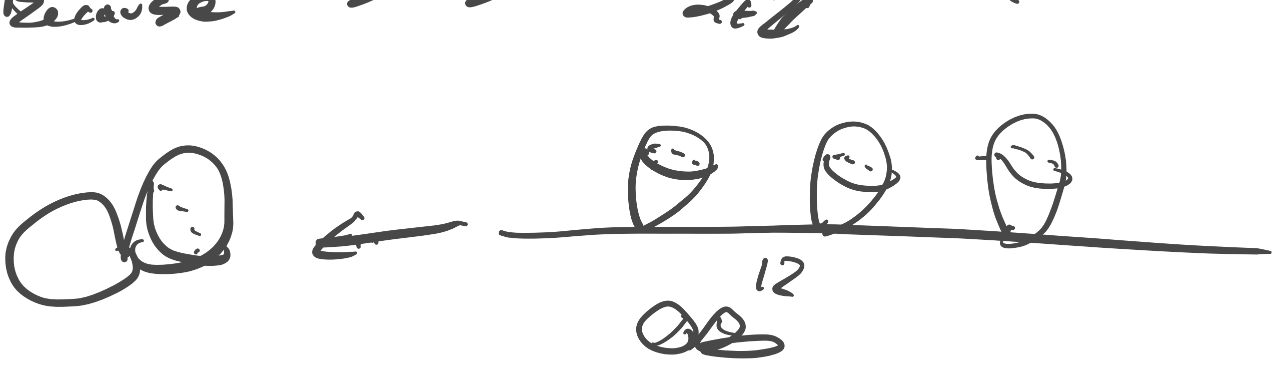
has compact image and so is in
the image of Φ by finite case.
A null htpy is also compact and
so Φ injective by the case. $\square$.

This is going to be very important.
(that we can realize all free
abelian groups as π_n.)

Cor. $\pi_n(S^1 \vee S^n) =$ free abelian gp w/ $\mathbb{Z}$s worth of $\mathbb{Z}$s.
$n \geq 2$

$$\begin{array}{c} \pi_n(S^1 \vee S^n) \\ \parallel \\ \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \end{array}$$

Because $\widetilde{S^1 \vee S^n} \simeq \bigvee_{\alpha\in\mathbb{Z}} S^n_\alpha$.



Actually a free π_1-module.

$$\pi_1(S^1 \vee S^n) \cong \mathbb{Z} = \langle t \rangle$$

$$\underbrace{\pi_n(S^1 \vee S^n)}_{\text{cyclic } \pi_1\text{-module.}} = \mathbb{Z}[t, t^{-1}].$$

Cor. $\pi_k(X)$ isn't nec. finitely generated even when X cpct cell cx.

Again: Warning. $\pi_k(X)$ of cpct. cell cx X is not nec. finitely generated even as a π_1-module.

Ex: $\pi_3(S^1 \vee S^2)$ is not f.g. as a π_1-module.

Eilenberg-MacLane spaces.

An Eilenberg-MacLane space is a space X w/ $\pi_n(X) \cong G$ and $\pi_k(X) = 0$ $\forall k \neq n$, denoted

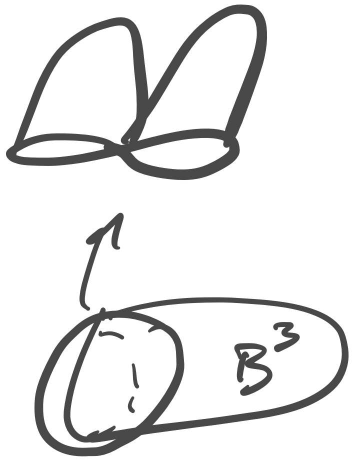
$K(G, n)$.

(It follows from Whitehead's theorem that cellular $K(G,n)$s are uniquely determined up to htpy eq. by G & n.)

Thm If G any gp, $\exists$ a $K(G,1)$.

(Add 2 cells to a wedge of circles and Van Kampen's theorem allows you to add them so that π_1(resulting space) $\cong G$.

Then add higher dimensional cells to kill all the higher htpy gps.



Add 3-cells w/ attaching maps the generators of $\pi_2(X^2)$. This kills π_2 but doesn't change $\pi_1(X) = \pi_1(X^2)$.
Add 4-cells to kill π_3

.....

$\lrcorner$

For any abelian G, and $n \geq 2$ there exists a $K(G,n)$.

Prop 4.28. Suppose CW pair (X,A) is r-conn. and A is s-conn. w/ $r,s \geq 0$, then

$q_* : \pi_i(X,A) \to \pi_i(X/A)$ ind. by quotient $q : X \to X/A$ is isomorphism for $i \leq r+s$ and surjective for $i = r+s+1$.

Pf. $X \cup CA$. $CA \simeq *$.

So $X \cup CA \to (X \cup CA)/CA \cong X/A$

is a htpy equivalence.

(by prop. 0.17).

So $\exists$ comm. diag:

$$\begin{tikzcd}
\pi_i(X,A) \arrow[r] & \pi_i(X \cup CA, CA) \arrow[r] & \pi_i(X \cup CA/CA) = \pi_i(X/A) \\
& \pi_i(X \cup CA) \arrow[u, "\cong \text{LES}"'] \arrow[ur, "\cong"'] \arrow[u, phantom, shift left=4ex, "\circ"] &
\end{tikzcd}$$

(CA, A) is $(s+1)$-connected when

A is s-conn. by LES of pair.

Now we apply excision to first

map in diagram. $\square$

Ex. Suppose $n \geq 2$ and

$$X = \bigvee_\alpha S^n_\alpha \cup \bigcup_\beta e^{n+1}_\beta$$ by attaching maps $\varphi_\beta : S^n \to \bigvee_\alpha S^n_\alpha$.

By cell approx. $\pi_i(X) = 0$ if $i<n$.

Claim that $\pi_n(X) \cong \pi_n(\bigvee_\alpha S^n_\alpha) / \langle [\varphi_\beta] \rangle$.

(note that any subgp. of $\pi_n(\bigvee_\alpha S^n_\alpha)$ is realized as such a $\langle [\varphi_\beta] \rangle$.)

Pf of claim: Look LES:

$$\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\bigvee_\alpha S^n_\alpha) \to \pi_n(X) \to 0$$

Now! $X / \bigvee_\alpha S^n_\alpha$ is a wedge of $(n+1)$-spheres. So by preceding proposition $\pi_{n+1}(\underline{X}, \vee_\alpha S^n_\alpha)$ is free w/ basis the characteristic maps of e^{n+1}_β. But ∂ of these are precisely the attaching maps. $\square$.

Now, we can construct a space w/ $\pi_n(\underline{X}) \cong G$ and $\pi_k(\underline{X}) = 0$ if $k < n$.

To kill $\pi_k(\underline{X})$ when $k > n$, add higher dim$^{\underline{l}}$ cells:

Suppose $\pi_n(\underline{X}) \cong G$.

If $[f] \neq 0 \in \pi_{n+1}(\underline{X})$, attach a cell e^{n+2}_f so that

$\partial e_f^{n+2} = f$.

Do this for all such f.

By cell approx. this doesn't change π_n. (π_n determined by $n+1$ skeleton, by cell. approx.).

So this way we can construct a $K(G,n)$ for any abelian G and $n \geq 2$. $\square$

Let G_n be any sequence of gps s.t. G_n is abelian when $n \geq 2$. Then

$$\pi_n\left(\prod_{n=1}^{\infty} K(G_n,n)\right) = G_n.$$

$\mathbb{Z}/7\mathbb{Z}$ $\quad \mathbb{Q}$ $\quad \mathbb{R}$ $\quad \ldots$

for $n=1$,

Naturally occurring $K(G,1)$s are really common.

$S^1 = K(G,1)$

S_g orientable surface of genus $g \geq 1$ is a $K(\pi_1(S_g),1)$.

Hyperbolic mflds.

<u>**Thm**</u> If X is a cellular $K(G,1)$ and G has torsion, then X is infinite dimensional.

(e.g. try and build a $K(\mathbb{Z}/2\mathbb{Z},1)$ start w/ $\mathbb{RP}^2$ and add cells to kill higher dim'l π_n. you <u>have to</u> add cells in infinitely many dimensions.)

Cor. $\pi_1(S_g)$ has no torsion.

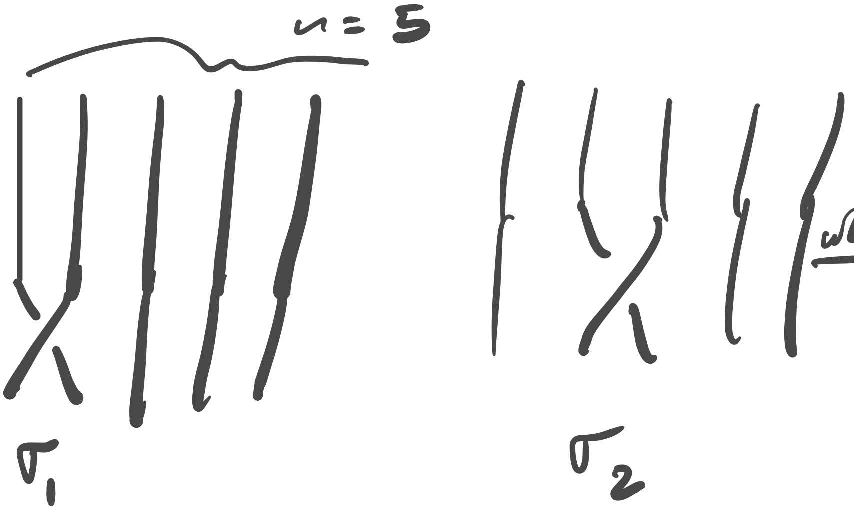
Braid gp. $B_n = \langle \sigma_0, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i$

if $|i-j| \geq 2$

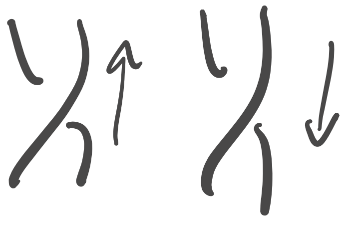
and

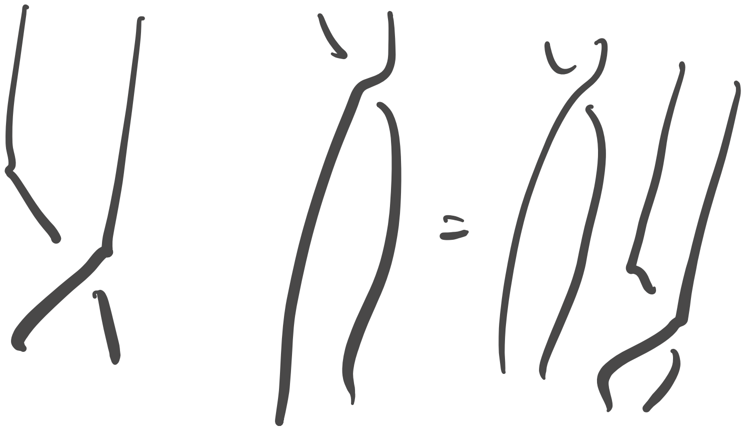
$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \rangle$

is torsion free.

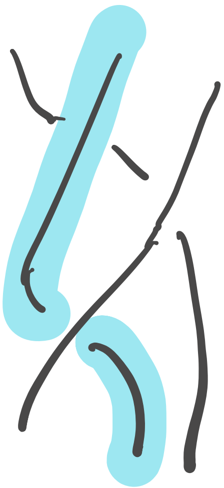


Prove that $B_n = \pi_1(X)$ where X is aspherical.

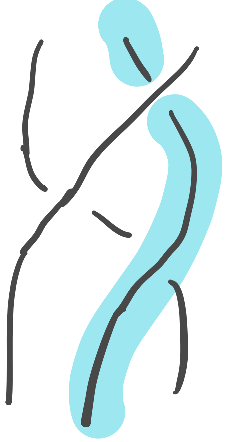


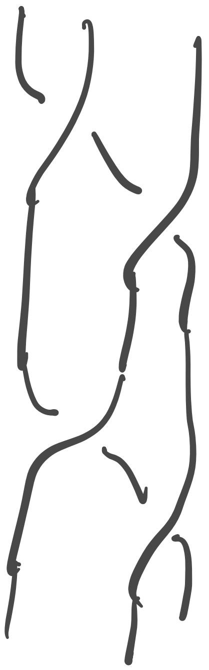


$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$



=





$K(G,n)$s.

Naturally occuring $K(G,n)$s.

$n=1$. aspherical mflds common.

$n \geq 2$.

Nice examples of $K(\mathbb{Z},n)$s.

Symmetric product of a space

$$SP_n(X) = X^n / S_n$$

where S_n is symmetric gp

$S_n \circlearrowright X^n$ in obvious way.

So $SP_n(X) = \{ n$ not nec distinct unordered points on $X \}$.

$$X^n \hookrightarrow X^{n+1}$$

$$\rightsquigarrow SP_n(X) \hookrightarrow SP_{n+1}(X)$$

let $SP(X) = \bigcup_{n \in \mathbb{Z}} SP_n$

"infinite symmetric product on X."

<u>Amazing Thm</u>:

$\pi_i(SP(X)) \cong H_i(X;\mathbb{Z})$ $\forall i>0$.

$\forall$ conn. CW cx.

Cor. if X sphere, then

$SP(S^n) = K(\mathbb{Z},n)$.

Notice $\pi_n(X,*) \to H_n(X,*)$

a map $S^n \to X$ is

a singular cycle.

Hurewicz Theorem.

Thm X (n-1)-conn. $n \geq 2$.

Then $\tilde{H}_i(X) = 0$ for $i < n$,

and $\pi_n(X) \cong H_n(X) \cong \tilde{H}_n(X)$.

If (X, A) is (n-1)-conn. $n \geq 2$,

with A 1-connected and

$A \neq \emptyset$, then $H_i(X, A) = 0$

for $i < 0$ and $\pi_n(X, A)$

$\cong H_n(X, A)$.

N.B.

Beyond that there's no relationship:

e.g. S^n complicated π_k but

boring H_k

$\mathbb{C}P^\infty$ has interesting H_k

but boring π_k.

Observation. Use Hurewicz Thm to compute π_2 sometimes.

If X space with a universal cover $\tilde{X}$, then $\pi_2(X) \cong \pi_2(\tilde{X})$ and if $\tilde{X}$ is nice enough that we can compute $H_2(\tilde{X})$ then we'll know $\pi_2(\tilde{X}) \cong \pi_2(X)$.

Pf. Just do the case where X cell. cx. and (X, A) cell. pair. (for gen. case you can use cell. approximations to X and (X, A).).

Relative case reduces to absolute case since $\pi_i(X, A) \cong \pi_i(X/A)$ ism by 4.28. and

$H_i(X, A) \cong \tilde{H}_i(X/A) \quad \forall i.$

<u>Need a cor.</u> 4.16: If (X, A) is n-connected cw pair, then $\exists$ a cw pair $(Z, A) \simeq (X, A)$ rel A s.t. all cells of $(Z-A)$ have dim $> n$. Sketch of pf. Start with A and add cells to surject $\pi_{n+1}(X)$, then π_{n+2} etc. ... $\square$

So we assume X has $X^{(n-1)} = *$.

So $\tilde{H}_i(X) = 0 \quad i < n.$

Now to show that $\pi_n(X) \cong H_n(X)$, throw away cells of dim $> n+1$ since those are irrelevant by cell. approx. and cell homology.

$\left(\pi_n(X) \cong \pi_n(X^{(n+1)}) \right.$ and

$\left. H_n(X) \cong H_n(X^{(n+1)}). \right)$

So $X = \left(\bigvee_\alpha S^n_\alpha\right) \bigcup_\beta e^{n+1}_\beta$ w/

attaching maps preserve basepoints.

Then $\pi_n(X)$

$$= \operatorname{coker}\left(\partial : \pi_{n+1}(X, X^{(n)}) \to \pi_n(X^{(n)}) \right)$$

$$\bigoplus_\beta \mathbb{Z} \longrightarrow \bigoplus_\alpha \mathbb{Z}.$$

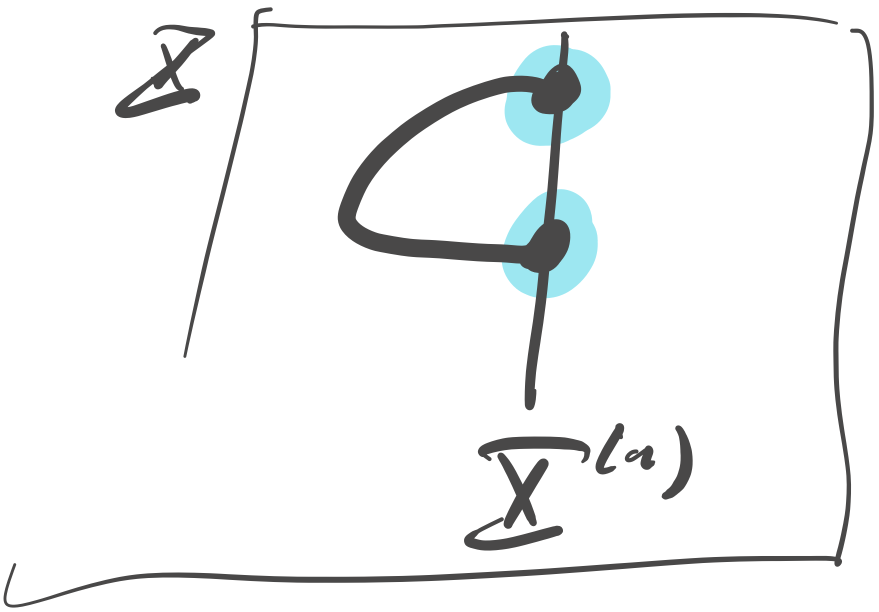
By:

LES of $(X, X^{(n)})$

$$\begin{tikzcd} {} \arrow[r] & \pi_{n+1}(X, X^{(n)}) \arrow[r, "\partial"] & \pi_n(X^{(n)}) \arrow[r] & \pi_n(X) \arrow[d] \\ & & & \pi_n(X, X^{(n)}) \arrow[d, phantom, "\parallel"] \\ & & & 0 \end{tikzcd}$$

But $\partial : \pi_{n+1}(X, X^{(n)}) \to \pi_n(X^{(n)})$

$$\pi_n(X^{(n)}) \cong \bigoplus_\alpha \mathbb{Z} = \pi_n(\vee S^n_\alpha)$$



This ∂ map is exactly the cellular ∂ operator

$$d : H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})$$

since for e_β^{n+1}, coefficients of de_β^{n+1} are the degrees of $q_\alpha \varphi_\beta$ where φ_β is attaching map for e_β^{n+1} and q_α is the map that crushes away all the spheres in $X^{(n)}$ other than S^n_α, and also the fact that $\pi_n(S^n) \cong \mathbb{Z}$ given by degree.

So since there are no $(n-1)$-cells,

$$H_n(X) \cong \operatorname{coker} d_n \cong \operatorname{coker} \partial . \quad \square$$

Cor. $f : X \to Y$ between 1-conn. cell cxs is a htpy equivalence if $f_* : H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.

Pf. Can assume f is $i : X \hookrightarrow Y$. (use a mapping cylinder).

In that case

$\pi_1(\Upsilon, \Xi) = 0$.

Relative Hurewicz Thm tells us that 1st nonzero $\pi_n(\Upsilon, \Xi)$ is the 1st nonzero $H_n(\Upsilon, \Xi)$.

But all of the latter gps are 0 by hypothesis. So all the $\pi_n(\Upsilon, \Xi)$ vanish.

So $i_* : \pi_n(\Xi) \to \pi_n(\Upsilon)$ is isomorphism $\forall\, n$.

So by Whitehead's theorem,

$i : \Xi \to \Upsilon$ htpy equiv.

$\square$

Bundles. If we have a "SES of spaces":

$$A \hookrightarrow X \to X/A.$$

$\rightsquigarrow$ LES of Homology grps.

But don't get one for π_n b/c excision fails.

Nice class of "SESs of spaces" where you <u>do</u> get a LES.

Def. $p : E \to B$ has the <u>htpy lifting property</u> w.r.t. X if, given a htpy

$$g_t : X \to B$$

and $\tilde{g}_0 : X \to E$ lift of g_0

i.e. $p\,\tilde{g}_0 = g_0$.

$$\begin{tikzcd} & E \arrow[d, "p"] & & & E \arrow[d] \\ X \arrow[ur, "\tilde{g}_0"] \arrow[r, "g_0"'] \arrow[r, phantom, bend left=40, "\circ"] & B & \arrow[r, squiggly, "HLP"] & X \arrow[ur, "\tilde{g}_t"] \arrow[r, "g_t"'] & B \end{tikzcd}$$

Then $\exists$ a htpy $\tilde{g}_t : X \to E$ lifting g_t.

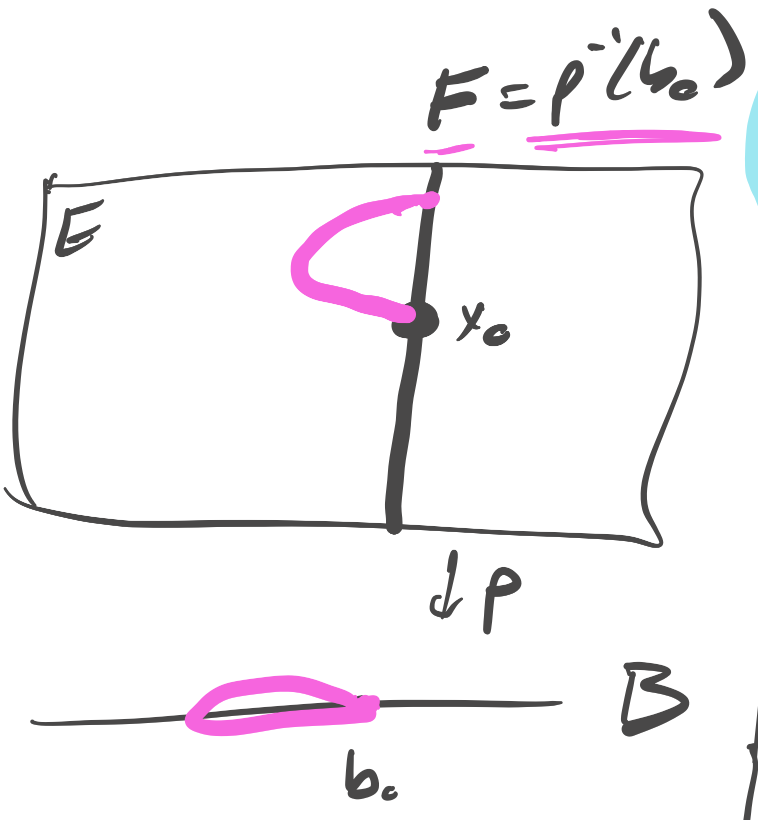
A **fibration** is a map $p: E \to B$ having h.l.p. $\forall$ spaces X.

Ex. $B \times F \to B$ usual proj.

$$\tilde{g}_t(x) = (g_t(x), h(x))$$

if $\tilde{g}_0(x) = (g_0(x), h(x))$

Thm Suppose $p: E \to B$ has h.l.p. w.r.t. to disks D^k $\forall k \geq 0$. Pick $b_0 \in B$, $x_0 \in F = p^{-1}(b_0)$.

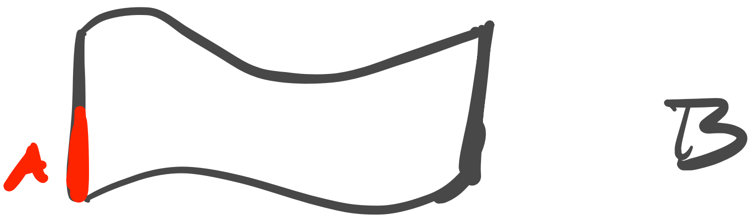


Then $p_*: \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is isomorphism $\forall n \geq 1$. And if B is 0-connected there is a LES

$$\pi_n(F,x_0) \to \pi_n(E,x_0) \xrightarrow{p_*} \pi_n(B,b_0) \to \pi_{n-1}(F,x_0) \to \dots \to \pi_0(E,x_0) \to 0$$

Def. (Rel. version)

$p: E \to B$ has h.l.p. for (X, A) if $f_t : X \to B$ is a htpy and we're given a lift $\tilde{g}_0$ of f_0 and a lift $\tilde{g}_t : A \to E$ of $f_t|_A : A \to B$, then we can extend $\tilde{g}_t$ to a lift $\tilde{f}_t$ of f_t.

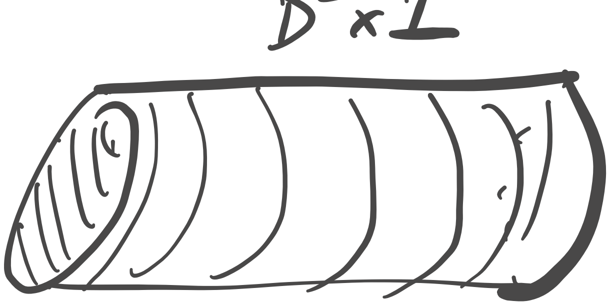


Htpy lifting prop. for D^k is equivalent to h.l.p. for $(D^k, \partial D^k)$.

since

$$(D^k \times I,\ D^k \times \{0\})$$

$$\cong (D^k \times I,\ D^k \times \{0\} \cup \partial D^k \times I)$$



By inducting over skeleton, can show that h.l.p. for CW pairs is equivalent to h.l.p. for disks.

Def. A space with h.l.p for disks D^k $\forall k \geq 0$ is <u>Serre fibration</u>.

Pf of Theorem.

① p_* surjective.

Let $[f:(I^n,\partial I^n)\to(B,b_0)]\in\pi_n(B,b_0$

Constant map to our basepoint x_0 is a lift of f to E on $J^{n-1}\subset I^n$. so rel. 4.1.1 for $(I^{n-1},\partial I^{n-1})$ extends $\tilde{f}:I^n\to E$ w/ $\tilde{f}(\partial I^n)\subset F$ since $f(\partial I^n)=b_0$.

So $\tilde{f}\in\pi_n(E,F,x_0)$ w/ $p_*([\tilde{f}])$
$=[f]$

since $p\tilde{f}=f$.

Injectivity is similar:

if $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ s.t. $p_*[\tilde{f}_0] = p_*[\tilde{f}_1]$, let

$G = (I^n \times I, \partial I^n \times I) \to (B, b_0)$

be htpy between $p\tilde{f}_0$ and $p\tilde{f}_1$.

We have a partial lift $\tilde{G}$ given by $\tilde{f}_0$ on $I^n \times \{0\}$,

$\tilde{f}_1$ on $I^n \times \{1\}$,

constant map to x_0 on $J^{n-1} \times I$.

So h.l.p. extends this to a lift

$\tilde{G} : I^n \times I \to E$.

to give htpy $\tilde{f}_t$ between $\tilde{f}_0$ and $\tilde{f}_1$.

So p_* inj.

To Get LES:

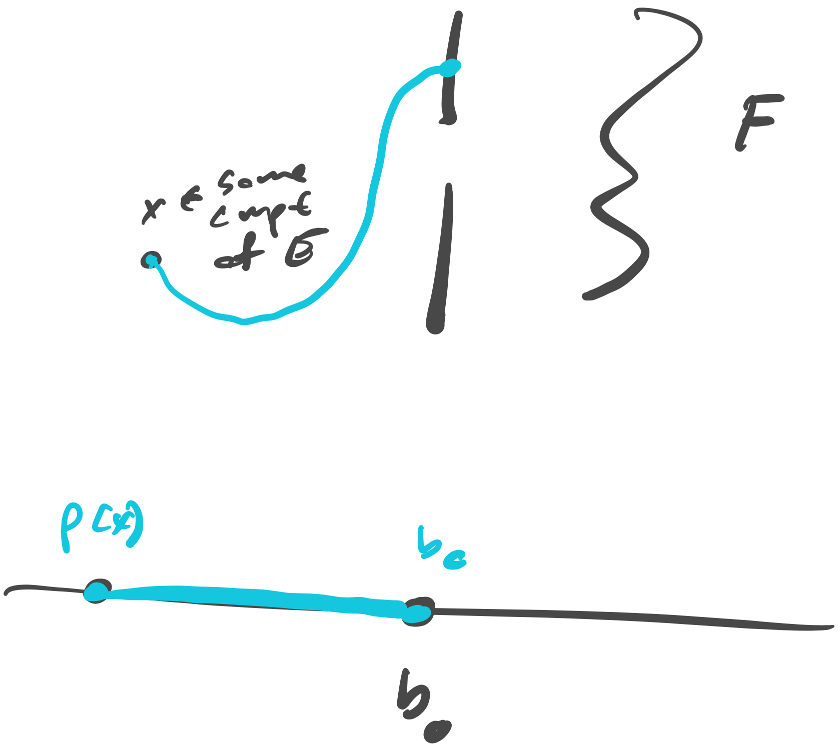
plug $\pi_n(B, b_0)$ in for $\pi_n(E, F, x_0)$ in htpy LES for pair (E, F).

So $\pi_n(E, x_0) \to \pi_n(E, F, x_0)$ in exact sequence is just the composition

$$\pi_n(E, x_0) \to \pi_n(E, F, x_0) \xrightarrow{p_*} \pi_n(B, b_0)$$

which is just $p_* : \pi_n(E, x_0) \to \pi_n(B, b_0)$.

Surjectivity of $\pi_0(F, x_0) \to \pi_0(E, x_0)$ follows since B is path connected:



So given $x \in E$, path btw $p(x)$ and b_0 in B. Lift to path in E beginning at x ending on F. So every cmpt of E contains a pt of F.

$\square$

K(G,n)s.

Naturally occurring K(G,n)s.

n=1. aspherical mflds common.

$n \geq 2$.

Nice examples of $K(\mathbb{Z},n)$s.

Symmetric product of a space

$$SP_n(X) = X^n / S_n$$

where S_n is symmetric gp

$S_n \circlearrowright X^n$ in obvious way.

So $SP_n(X) = \{ n$ not nec distinct unordered points on $X \}$.

$$X^n \hookrightarrow X^{n+1}$$

$$\rightsquigarrow SP_n(X) \hookrightarrow SP_{n+1}(X)$$

let $SP(X) = \bigcup_{n \in \mathbb{Z}} SP_n$

"infinite symmetric product on X."

Amazing Thm:

$$\pi_i(SP(X)) \cong H_i(X;\mathbb{Z}) \quad \forall i>0.$$

$\forall$ conn. CW cx.

Cor. If X sphere, then

$$SP(S^n) = K(\mathbb{Z},n).$$

Notice $\pi_n(X,*) \to H_n(X,*)$

a map $S^n \to X$ is a singular cycle.

Hurewicz Theorem.

Thm X $(n-1)$-conn. $n \geq 2$.

Then $\tilde{H}_i(X) = 0$ for $i < n$,

and $\pi_n(X) \cong H_n(X) \cong \tilde{H}_n(X)$.

If (X, A) is $(n-1)$-conn. $n \geq 2$,
with A 1-connected and
$A \neq \emptyset$, then $H_i(X, A) = 0$
for $i < 0$ and $\pi_n(X, A)$
$\cong H_n(X, A)$.

N.B.
Beyond that there's no relationship:
e.g. S^n complicated π_k but
boring H_k

$\mathbb{C}P^\infty$ has interesting H_k
but boring π_k.

Observation. Use Hurewicz Thm to compute π_2 sometimes.

If X space with a universal cover $\tilde{X}$, then $\pi_2(X) \cong \pi_2(\tilde{X})$ and if $\tilde{X}$ is nice enough that we can compute $H_2(\tilde{X})$ then we'll know $\pi_2(\tilde{X}) \cong \pi_2(X)$.

Pf. Just do the case where X cell. cx. and (X, A) cell. pair. (for gen. case you can use cell. approximations to X and (X, A).).

Relative case reduces to absolute case since $\pi_i(X, A) \cong \pi_i(X/A)$ isn by 4.28. and

$H_i(X, A) \cong \tilde{H}_i(X/A) \quad \forall i.$

Need a cor. 4.16: If (X, A) is n-connected cw pair, then $\exists$ a cw pair $(Z, A) \simeq (X, A)$ rel A s.t. all cells of $(Z - A)$ have dim $> n$. Sketch of pf. Start with A and add cells to surject $\pi_{n+1}(X)$, then π_{n+2} etc. ... $\square$

So we assume X has $X^{(n-1)} = *$.

So $\tilde{H}_i(X) = 0 \quad i < n.$

Now to show that $\pi_n(X) \cong H_n(X)$, throw away cells of dim$^n > n+1$ since those are irrelevant by cell. approx. and cell homology.

($\pi_n(X) \overset{\sim}{=} \pi_n(X^{(n+1)})$ and

$H_n(X) \overset{\sim}{=} H_n(X^{(n+1)})$.)

So $X = \left(\bigvee_\alpha S^n_\alpha\right) \bigcup_\beta e^{n+1}_\beta$ w/

attaching maps preserve basepoints.

Then $\pi_n(X)$

$$= \operatorname{coker}\left(\partial : \pi_{n+1}(X, X^{(n)}) \to \pi_n(X^{(n)})\right)$$

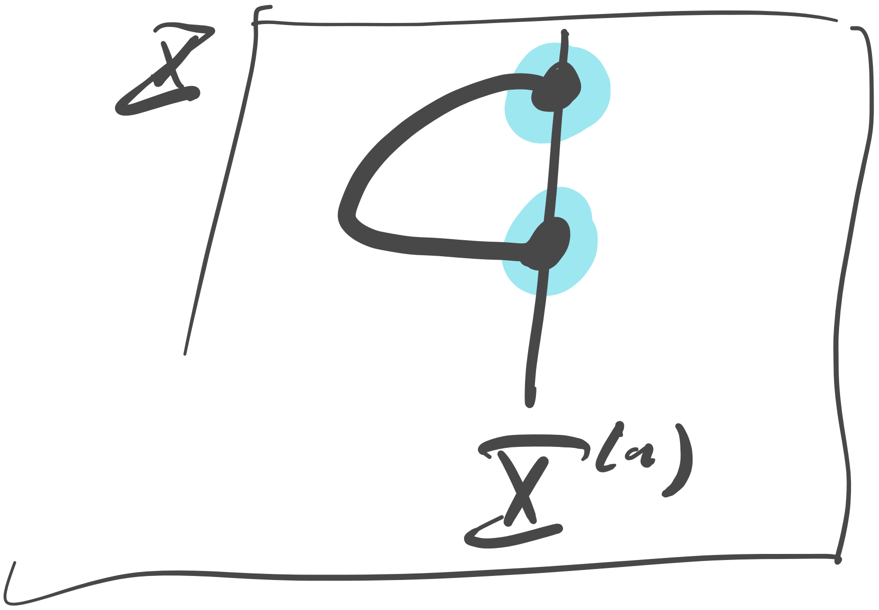
$$\bigoplus_\beta \mathbb{Z} \longrightarrow \bigoplus_\alpha \mathbb{Z}.$$

By:
LES of $(X, X^{(n)})$

$$\begin{tikzcd}
{} \arrow[r] & \pi_{n+1}(X, X^{(n)}) \arrow[r, "\partial"] & \pi_n(X^{(n)}) \arrow[r] & \pi_n(X) \arrow[d] \\
 & & & \pi_n(X, X^{(n)}) \arrow[d, phantom, "\shortparallel"] \\
 & & & 0
\end{tikzcd}$$

But $\partial : \pi_{n+1}(X, X^{(n)}) \to \pi_n(X^{(n)})$

$$\pi_n(X^{(n)}) \cong \bigoplus_\alpha \mathbb{Z} = \pi_n(\vee S^n_\alpha)$$



This ∂ map is exactly the cellular ∂ operator

$$d : H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})$$

since for e_β^{n+1}, coefficients of de_β^{n+1} are the degrees of $q_\alpha \varphi_\beta$ where φ_β is attaching map for e_β^{n+1} and q_α is the map that crushes away all the spheres in $X^{(n)}$ other that S^n_α, and also the fact that $\pi_n(S^n) \cong \mathbb{Z}$ given by degree.

So since there are no $(n-1)$-cells,

$$H_n(X) \cong \operatorname{coker} d_n \cong \operatorname{coker} \partial \qquad . \ \square$$

Cor. $f : X \to Y$ between 1-conn. cell cxs is a htpy equivalence if $f_* : H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.

Pf. Can assume f is $i : X \hookrightarrow Y$. (use a mapping cylinder).

In that case

$\pi_1(\Upsilon, X) = 0.$

Relative Hurewicz Thm tells us that 1st nonzero $\pi_n(\Upsilon, X)$ is the 1st nonzero $H_n(\Upsilon, X)$.

But all of the latter gps are 0 by hypothesis. So all the $\pi_n(\Upsilon, X)$ vanish.

So $i_* : \pi_n(X) \to \pi_n(\Upsilon)$ is isomorphism $\forall$ n.

So by whitehead's theorem,

$i : X \to \Upsilon$ htpy equiv. $\square$

Bundles. If we have a "SES of spaces":

$$A \hookrightarrow X \to X/A.$$

$\rightsquigarrow$ LES of Homology gps.

But don't get one for π_n cuz excision fails.

Nice class of "SESs of spaces" where you **do** get a LES.

Def. $p : E \to B$ has the **htpy lifting property** w.r.t. X if, given a htpy

$$g_t : X \to B$$

and $\tilde{g}_0 : X \to E$ lift of g_0

i.e. $p\,\tilde{g}_0 = g_0$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{g}_0"] \arrow[r, "g_0"'] \arrow[r, phantom, shift left=3, "\circ"] & B \end{tikzcd}
\quad \overset{\text{HLP}}{\rightsquigarrow} \quad
\begin{tikzcd} & E \arrow[d] \\ X \arrow[ur, "\tilde{g}_t"] \arrow[r, "g_t"'] & B \end{tikzcd}$$

then $\exists$ a htpy $\tilde{g}_t : X \to E$.

lifting g_t.

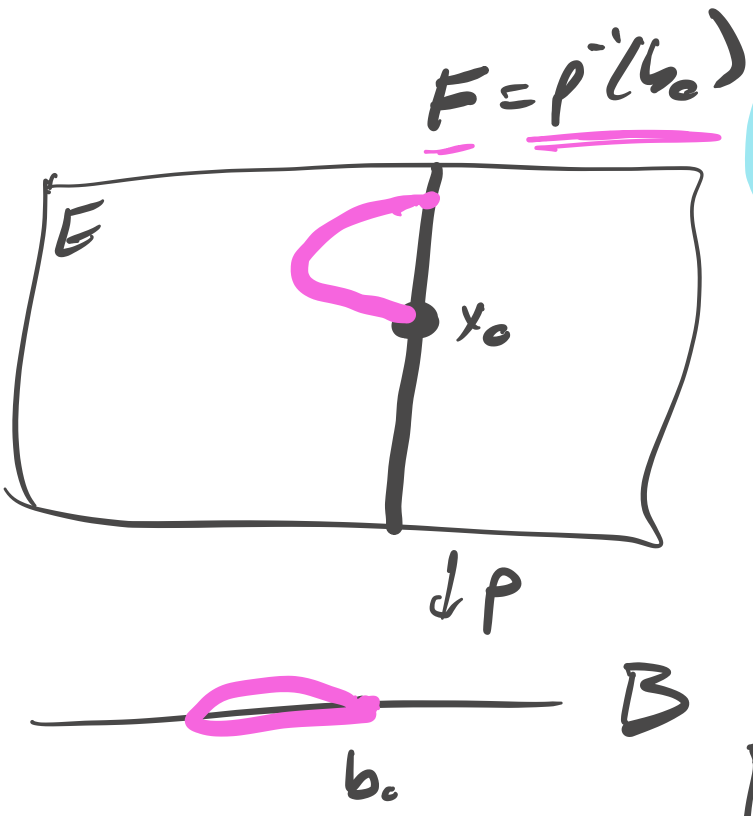
A **fibration** is a map $p : E \to B$ having h.l.p. $\forall$ spaces X.

Ex. $B \times F \to B$ usual proj.

$$\tilde{g}_t(x) = (g_t(x), h(x))$$

if $\tilde{g}_0(x) = (g_0(x), h(x))$

Thm Suppose $p : E \to B$ has h.l.p. w.r.t. to disks D^k $\forall k \geq 0$. Pick $b_0 \in B$, $x_0 \in F = p^{-1}(b_0)$.

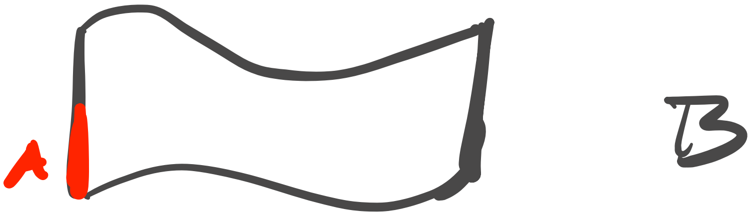


Then $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is isomorphism $\forall n \geq 1$. And if B is 0-connected there is a LES

$$\pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \ldots \to \pi_0(E, x_0) \to 0$$

Def. (Rel. version)

$p: E \to B$ has h.l.p. for (X, A) if $f_t : X \to B$ is a htpy and we're given a lift $\tilde{g}_0$ of f_0 and a lift $\tilde{g}_t : A \to E$ of $f_t|_A : A \to B$, then we can extend $\tilde{g}_t$ to a lift $\tilde{f}_t$ of f_t.



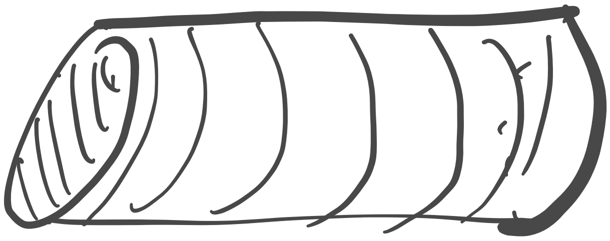
Htpy lifting prop. for D^k is equivalent to h.l.p. for $(D^k, \partial D^k)$.

since

$$(D^k \times I, D^k \times \{0\})$$

$$\cong (D^k \times I, D^k \times \{0\} \cup \partial D^k \times I)$$

$D^k \times I$



By inducting over skeleton, can show that h.l.p. for CW pairs is equivalent to h.l.p. for disks.

Def. A space with h.l.p for disks D^k $\forall k \geq 0$ is Serre fibration.

Pf of Theorem.

① p_* surjective.

Let $[f:(I^n,\partial I^n)\to(B,b_0)]\in\pi_n(B,b_0)$

Constant map to our basepoint x_0 is a lift of f to E on $J^{n-1}\subset I^n$. So rel. 4.1.1 for $(I^{n-1},\partial I^{n-1})$ extends $\tilde{f}:I^n\to E$ w/ $\tilde{f}(\partial I^n)\subset F$ since $f(\partial I^n)=b_0$.

So $\tilde{f}\in\pi_n(E,F,x_0)$ w/ $p_*([\tilde{f}]) = [f]$

since $p\tilde{f}=f$.

Injectivity is similar:

if $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$

s.t. $p_*[\tilde{f}_0] = p_*[\tilde{f}_1]$, let

$G = (I^n \times I, \partial I^n \times I) \to (B, b_0)$

be htpy between $p\tilde{f}_0$ and

$p\tilde{f}_1$.

We have a partial lift $\tilde{G}$ given

by $\tilde{f}_0$ on $I^n \times \{0\}$,

$\tilde{f}_1$ on $I^n \times \{1\}$,

constant map to x_0 on

$J^{n-1} \times I$.

So h.l.p. extends this to a lift

$\tilde{G} : I^n \times I \to E$.

to give htpy $\tilde{f}_t$ between $\tilde{f}_0$ and $\tilde{f}_1$.

So p_* inj.

To Get LES:

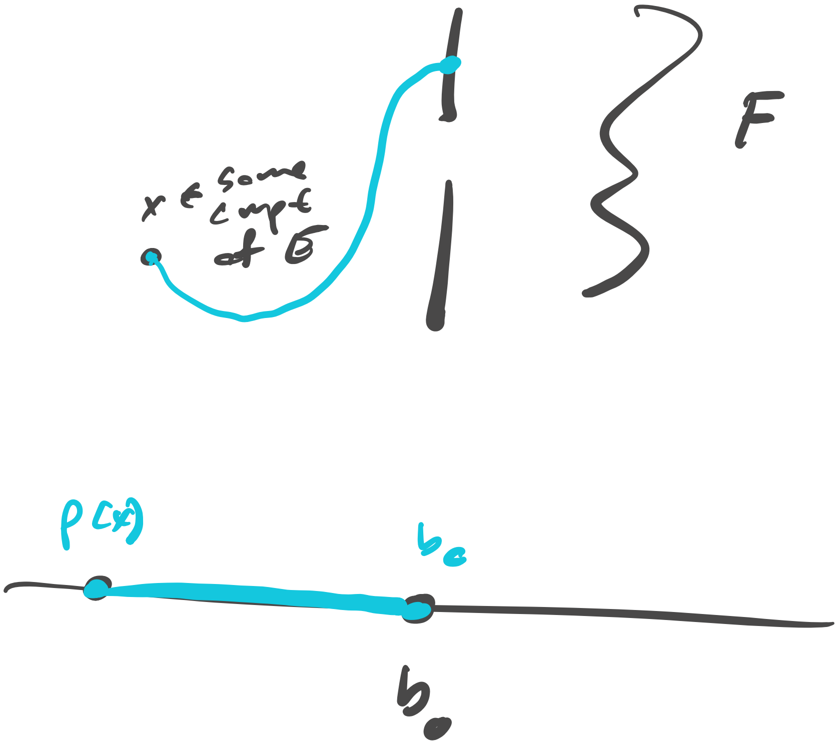
plug $\pi_n(B,b_0)$ in for $\pi_n(E,F,x_0)$ in htpy LES for pair (E,F).

So $\pi_n(E,x_0)\to\pi_n(E,F,x_0)$ in exact sequence is just the composition

$$\pi_n(E,x_0)\to\pi_n(E,F,x_0)\xrightarrow{p_*}\pi_n(B,b_0)$$

which is just $p_*:\pi_n(E,x_0)\to\pi_n(B,b_0)$.

Surjectivity of $\pi_0(F,x_0)\to\pi_0(E,x_0)$ follows since B is path connected:



So given $x \in E$, path btw $p(x)$ and b_0 in B. Lift to path in E beginning at x ending on F. So every cmpt of E contains a pt of F. $\square$

Serre fibrations.

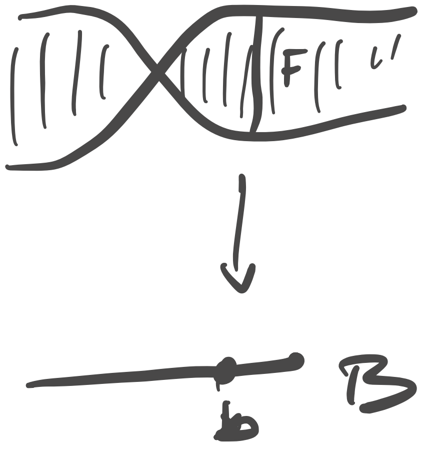
fibration $p: E \to B$ has h.l.p. w.r.t. all X.

Serre fibration $p: E \to B$ has h.l.p. w.r.t. all D^k.

Given serre fibration, there exists a LES

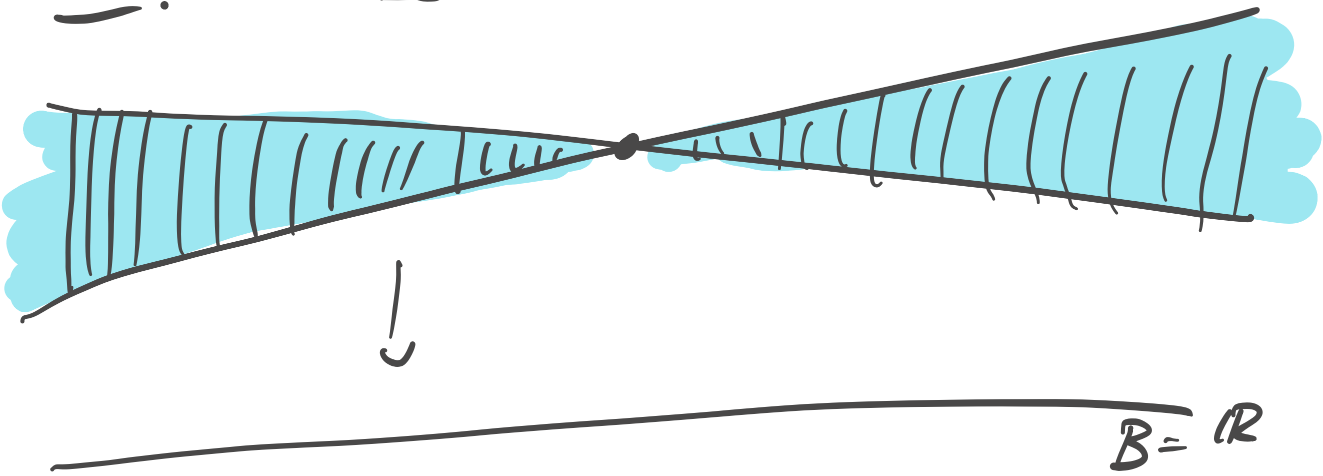
$$\pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \quad \pi_0(E) \to 0$$

where F is fiber of p over basepoint of B.



Ex. $F \times B \to B$.

Ex: Some fibration.



A **fiber bundle** is a space E, a fiber space F, and a projection $p: \underset{\text{total space}}{E} \to \underset{\text{Base}}{B}$ s.t. each point in B has a nbhd $\mathcal{U}$ with a homeomorphism

$$h: p^{-1}(\mathcal{U}) \to \mathcal{U} \times F$$ "local trivialization"

s.t.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[ddr, "p"'] & & U \times F \arrow[ddl, "\text{proj. onto first factor.}"] \\ & \circ & \\ & U & \end{tikzcd}$$

Often write $F \to E \to B$

and say " let $F \to E \to B$ be a fiber bundle."

"F bundle over B"

Bundle is "trivial" if it's a product.

Ex. $\mathbb{R} \times S^1$

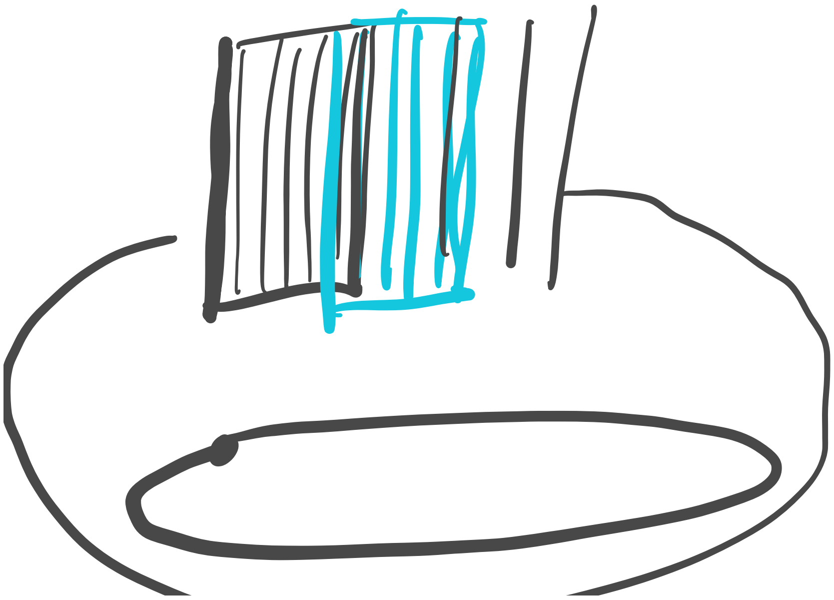
Ex. Möbius band

$\mathbb{R}$-bdle over S^1

$p^{-1}(U) \to F$

$\cong$

$U \times F$

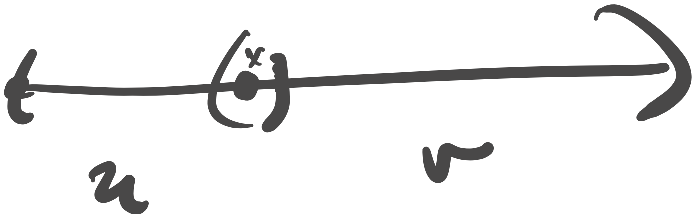


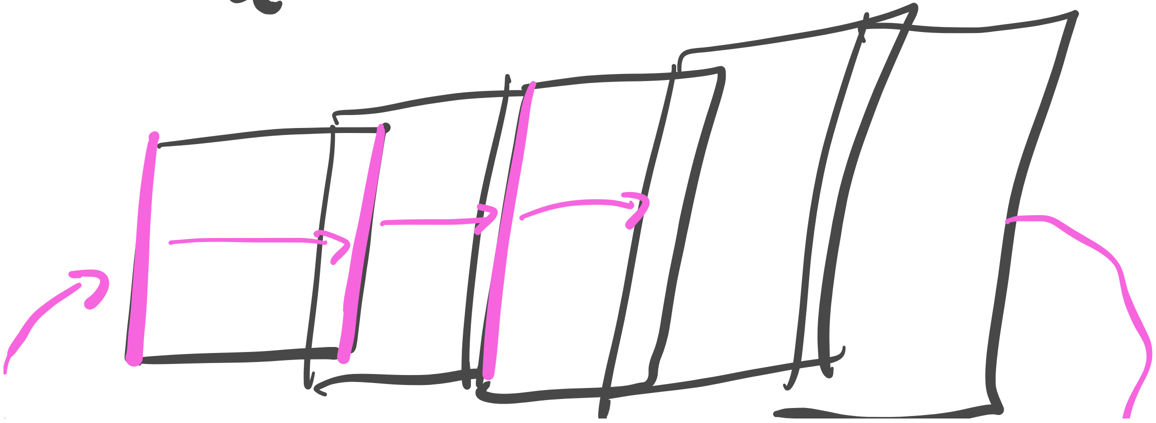
$p^{-1}(U) \xrightarrow{h_U} U\times F \qquad V\times F \xleftarrow{h_V} p^{-1}(V)$

$$\begin{tikzcd}
p^{-1}(U\cap V) \arrow[r, "h_U"] \arrow[dr, "p"'] \arrow[dr, phantom, shift left=4, "\circ"] & U\cap V\times F \arrow[d, "\mathrm{proj}"] & p^{-1}(V) \arrow[l, "h_V"'] \arrow[dl, "p"] \arrow[dl, phantom, shift right=4, "\circ"] \\
 & U\cap V &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{on } p^{-1}(x) \arrow[r, "h_U"] \arrow[d, phantom, "\cong" rotate=90] & \{x\}\times F \arrow[r, "h_V^{-1}"] & p^{-1}(x). \\
F & &
\end{tikzcd}$$

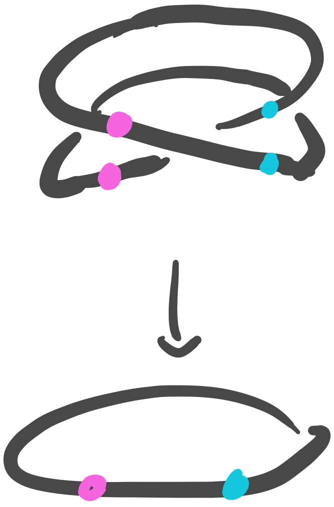
No requirement that $h_V^{-1} h_U = \mathbb{1}$.





Thm $\pi_1(B)$ acts on fibers by homeomorphisms modulo htpy.

Möbius band

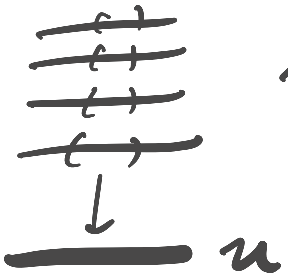


restrict to ∂ we get a fiber bdle

$$S^0 \to S^1 \to S^1$$

2-fold covering space.

EX. Fiber bdle w/ discrete fibers are covering spaces.

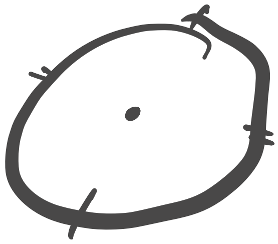


$$\mathcal{U} \times F = \bigsqcup_F \mathcal{U}.$$

Conversely if $p: E \to B$ is a covering space s.t. fibers all have same cardinality, then $p: E \to B$ fiber bundle. e.g. if $p: E \to B$ is a covering space and B conn.

Ex. Proj space:

$$\begin{tikzcd}
S^0 \arrow[r] & S^n \arrow[r] & \mathbb{R}P^n = \mathbb{R}^{n+1} - \{0\} / \mathbb{R}^* \\
& \mathbb{R}^{n+1} - \{0\} \arrow[u, phantom, "\cap" rotate=180] \arrow[ur] &
\end{tikzcd}$$



Same thing with $\mathbb{C}P^n$

$$\begin{tikzcd}
S^1 \arrow[r] & S^{2n+1} \arrow[r, "p"] \arrow[d, phantom, "\shortparallel"] & \mathbb{C}P^n = S^{2n+1} / z \sim \lambda z \\
& \|z\| = 1 \text{ in } \mathbb{C}^{n+1} &
\end{tikzcd}$$

where $\lambda \in S^1 \subset \mathbb{C}$.

$p: S^{2n+1} \to \mathbb{C}P^n$

takes $(z_0, \dots, z_n)$ to $[z_0 : \dots : z_n] \in \mathbb{C}P^n$.

let $U_i \subset \mathbb{C}P^n$ be $U_i = \{[z_0 : \dots : z_n] \mid z_i \neq 0\}$

want $p^{-1}(U_i) \xrightarrow{h_i} U_i \times S^1$

$$h_i(z_0, \dots, z_n) = \left([z_0 : \dots : z_n], z_i/|z_i|\right)$$

takes fibers to fibers.

it's homeo as

$$([z_0 : \dots : z_n], \lambda) \longmapsto \lambda |z_i| z_i^{-1} (z_0, \dots, z_n)$$

is the inverse.

Same thing works for $n = \infty$, so

$$S^1 \to S^\infty \to \mathbb{C}P^\infty.$$

Cor. $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$.

LES of fibration

$$\begin{tikzcd}
{} \arrow[r] & \pi_n(S^1) \arrow[r] & \pi_n(S^\infty) \arrow[r] \arrow[d, equal] & \pi_n(\mathbb{CP}^\infty) \arrow[r] & \pi_{n-1}(S^1) \arrow[d, equal] \\
 & & 0 & & 0 \\
 & & S^\infty \simeq * & & \text{if } n-1 \geq 1
\end{tikzcd}$$

So if $n-1>1$ i.e. $n>2$,

then $\pi_n(\mathbb{CP}^\infty)=0$

Also if $n=2$, then $\pi_2(\mathbb{CP}^\infty)$

$$\cong \pi_1(S^1) \cong \mathbb{Z}.$$

$$\pi_1(\mathbb{CP}^\infty) = 0 = \pi_0(\mathbb{CP}^\infty).$$

Beautiful example:

EX. $n=1$

$$S^1 \to S^3 \to \mathbb{C}P^1$$

So we have f.b.

$$S^1 \to S^3 \to S^2.$$

Hopf fibration.

$$p : S^3 \to S^2$$

$$(z_0, z_1) \longmapsto z_0/z_1 \in \mathbb{C} \cup \{\infty\} \cong S^2$$

Polar coord.

$$p(r_0 e^{i\theta_0}, r_1 e^{i\theta_1})$$

$$= r_0/r_1 \cdot e^{i(\theta_0 - \theta_1)}$$

$$\Big| \; r_0^2 + r_1^2 = 1$$

for fixed $\rho = r_0/r_1 \in (0,\infty)$

θ_1 & θ_2 form a torus

T_ρ in S^3.

as ρ varies the tori fill up

S^3. becoming degenerate at T_0

& T_∞, which are circles.

Each T_ρ union of fibers where

the difference $\theta_0 - \theta_1$ is constant.

Similar bdles

Ex. $\underbrace{S^3}_{\text{unit quaternions}} \to S^{4n+3} \to \mathbb{HP}^n$

n=1, get $S^3 \to S^7 \to S^4 = \mathbb{HP}^1$

Ex. Octonions give you

$$S^7 \to S^{15} \to S^8$$

These are the only spheres that are sphere bundles over spheres.

$$\left.\begin{array}{l} S^1 \to S^3 \to S^2 \\ S^3 \to S^7 \to S^4 \\ S^7 \to S^{15} \to S^8 \end{array}\right\}$$

$$S^0 \to S^1 \xrightarrow{n} S^1$$

~~Hopf~~ fibration useful:

LES of fibration ⇝

$$\pi_2(S^2) \cong \pi_1(S^1)$$

$$\pi_n(S^3) \cong \pi_n(S^2) \quad \forall n \geq 3.$$

if $n=3$, we have $\pi_3(S^2) = \mathbb{Z}$ generated by Hopf map $p : S^3 \to S^2$.

Low-dimensional topology.

Let S_g be a surface of genus g.

Surface bundles are really important in l.d. topology:

$$S_g \to E \to X.$$

3-dimensions: take $X = S^1$.

$$S_g \to M^3 \to S^1$$

Thm (Agol-Wise)

Most 3-mflds have a finite cover that is homeomorphic to a surface bdle over the circle.

Famous open Q:

Does there exist a sfce bdle over a surface

$$\begin{tikzcd} S_g \arrow[r] & E \arrow[r] \arrow[d] & S_h \\ & \text{4-mfld.} & \end{tikzcd}$$

s.t. $\pi_1(E) \not\supset \mathbb{Z}\oplus\mathbb{Z}$.

Common to consider "all bdles" at once.

Ex. Say you like line bdles,

i.e. $\mathbb{C} \to E \to X$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & E \arrow[r, "p"] & \mathbb{C}P^n \arrow[d, phantom, "\shortparallel"] \\
 & & \text{lines in } \boxed{\mathbb{C}^{n+1}}
\end{tikzcd}$$

Bdle of lines over $\mathbb{C}P^n$

where $p^{-1}(z) =$ line that corresponds to z.

Canonical bdle over $\mathbb{C}P^n$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & E \arrow[r] & \mathbb{C}P^\infty \\
\mathbb{C} \arrow[r] & E' \arrow[r] & X \arrow[u]
\end{tikzcd}$$

Notion of a pull back bdle.

$$\begin{tikzcd} F \arrow[r] & E' \arrow[r] & X \arrow[d] \\ F \arrow[r] & E \arrow[r] & B \end{tikzcd}$$

} pull back fibration.

Interested in bdles of given F, often there is a "classifying space" for all of them, meaning there is a universal fiber bdle

$$F \to U \to M$$

s.t. every fibration $F \to E \to B$ is pullback of a map

$$B \to M.$$

For example: Riemann's moduli space of curves $\mathcal{M}$ is "kinda" the classifying space for bdles of the form

$$\underbrace{S_g}_{\text{surface of genus } g.} \to E \to B$$

Next time: Sketch out ideas behind proving the following:

Thm $\forall$ abelian gp G, there is a natural bijection

$$T: \underbrace{\langle X, K(G,n)\rangle}_{\text{htpy classes of pointed maps}} \to H^n(X,G)$$

$\forall$ CW X and $n \geq 0$

where T is of the form

$$T([f]) = f^*(\alpha)$$ where

$\alpha \in H^n(K(G,n);G)$ is a

certain "fundamental class."

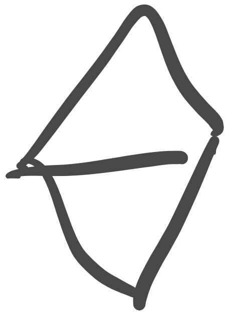
$$H^n(K(G,n);G)$$

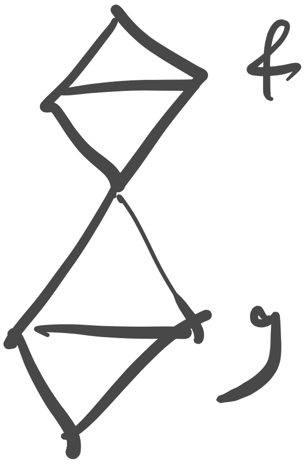
$$= \mathrm{Hom}(\underbrace{H_n(K(G,n))}, G)$$

$$\begin{tikzcd}
H_n(K(G,n)) & \pi_n(K(G,n)) \arrow[l, "\text{Hurewicz}"', "\cong"] \arrow[d, phantom, sloped, "\cong"] \\
 & G
\end{tikzcd}$$

$$\alpha = (\text{Hurewicz})^{-1}.$$

$$\langle \underline{S\overset{l}{X}}, \textcircled{K} \rangle$$



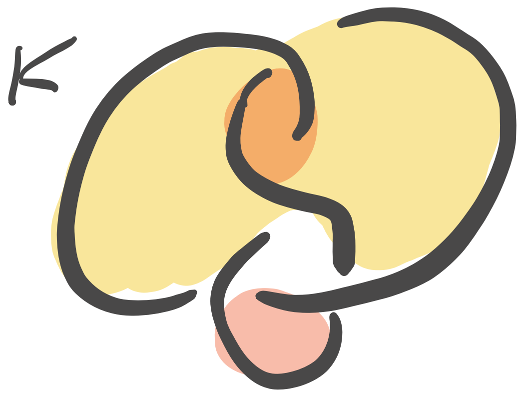


$$\boxed{\langle \underline{SX}, \underline{K} \rangle} = \boxed{\langle \underline{X}, \underbrace{\Omega \textcircled{K}}_{\text{space of all loops in } K.} \rangle}$$

$$\Omega K(G,n) = K(G,n-1).$$

$$Gr(k,\mathbb{R}^n) \hookrightarrow Gr(k,\mathbb{R}^{n+1})$$

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r] & E \arrow[r] & Gr(k) \\
 & & X \arrow[u]
\end{tikzcd}$$



$S^3 - K$ is fibered

$T^2 - * \to S^3 - K \to S^1$

Homotopy Theoretic Construction of Cohomology

<u>Thm</u> $\forall$ abelian G, there is a natural bijection

$$T : \langle X, K(G,n) \rangle \longrightarrow H^n(X;G)$$

ptd htpy classes of maps $(X,*) \to (K(G,n),*)$

whenever X cell cx and $n>0$.

where T is of the form $T([f])$

$= f^*\alpha$ where $\alpha \in H^n(K(G,n);G)$ is the "fundamental class."

$H^n(K(G,n);G) \cong \mathrm{Hom}(H_n(K(G,n)),G)$

α is the inverse of the Hurewicz isomorphism

$G = \pi_n(K(G,n),*) \to H_n(K(G,n))$.

More concretely, assume that $(K(G,n))^{n-1} = *$ then α is the cocycle that assigns to each n cell "the characteristic map $e^n \to K(G,n)$, considered as an element of $\pi_n(K(G,n),*)$ $\cong G$.

Note: If X connected, can forget basepoint and you have

$$[X, K(G,n)] \longleftrightarrow H^n(X;G).$$

unpointed maps up to htpy

Saw direct proof in dim 1. where $G=\mathbb{Z}$.

$$H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(H_1(X),\mathbb{Z})$$

$$\cong \mathrm{Hom}(\pi_1(X),\mathbb{Z})$$

$$= \langle X, S^1 \rangle.$$

by directly constructing $X \to S^1$.

Can prove this directly.

More natural approach:

1) $h^n(X) = \langle X, K(G,n) \rangle$

This is a contravariant functor

Show that $h^n(X)$ is a reduced cohomology theory on pointed cell complexes.

$$\begin{tikzcd} f\colon X \arrow[r] \arrow[dr, "\circ"'] & Y \arrow[d] \\ & K(G,n) \end{tikzcd}$$

so f induces a map

$\langle Y, K(G,n) \rangle$

$\xrightarrow{f_*} \langle X, K(G,n) \rangle$.

2) If a red. cohomology theory h^* on cell cxs has coeffs $h^n(S^0) = 0 \ \forall n \neq 0$,

then there are natural isomorphisms

$$h^n(X) \cong \tilde{H}^n(X; h^0(S^0))$$

$\forall$ CW cxs X and all n.

To do i), need to know that

$\langle X, K(G,n)\rangle$ is a gp,

and abelian, too.

Let $K = K(G,n)$.

If X S^n, then

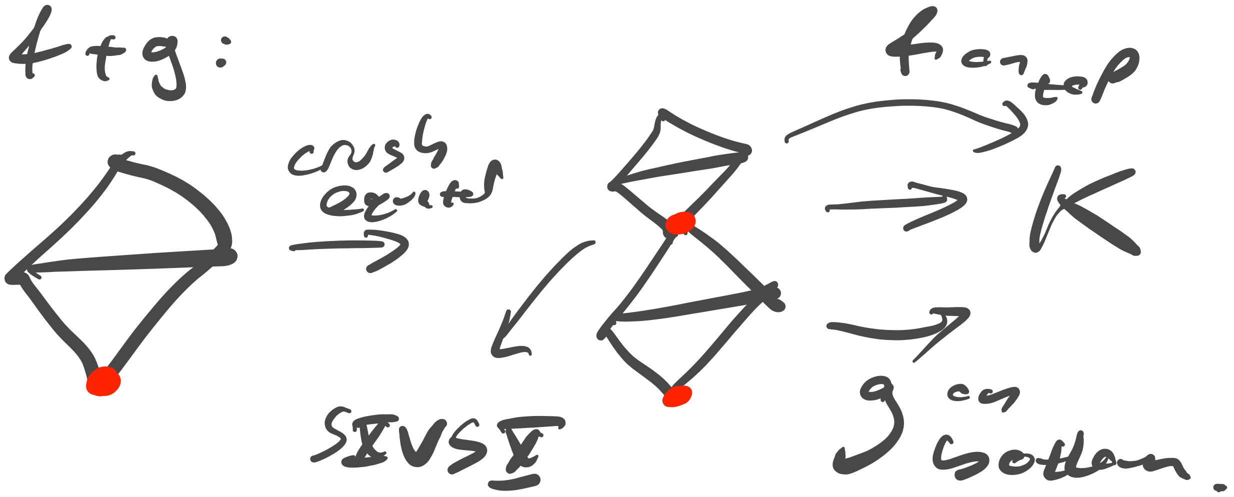
$\langle S^n, K\rangle = \pi_n(K)$.

That's a gp when $n > 0$.

If $\mathfrak{X}$ anything

$\langle S\mathfrak{X}, K\rangle$ then there's an binary operation on this set.

f+g:



$f, g \rightsquigarrow$ map $f+g : S\mathfrak{X} \to K$.

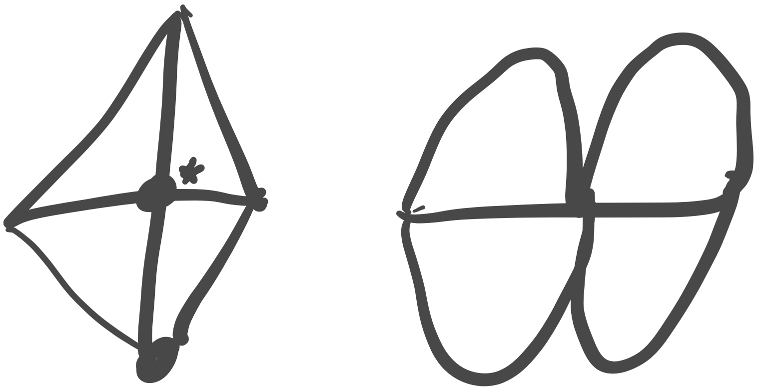
$f:(S\mathfrak{X},*)\to(K,*)$ $g:(S\mathfrak{X},*)\to(K,*)$.

Not really a well defined operation on $\langle S\overline{X}, K\rangle$.

<u>Trick</u>: Pick basepoint $* \in \overline{X}$.

$\Sigma \overline{X} = S\overline{X} / * \times I$ "reduced suspension"

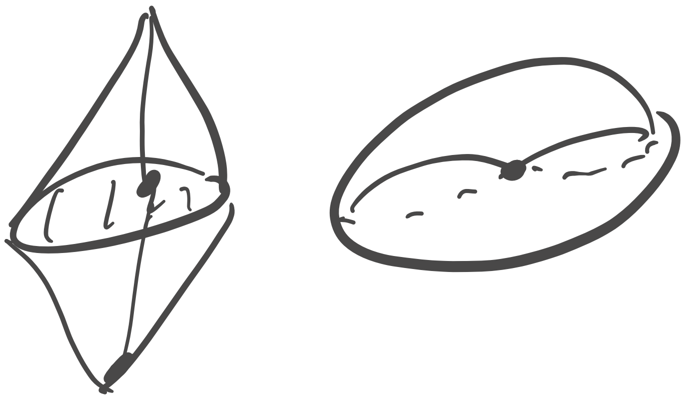
has nat natural basepoint $* \times I$.



$\overline{X}$ cell. and

$*$ 0-cell

then $\Sigma \overline{X} \simeq S\overline{X}$.



Briali.

So $\langle S\overline{X}, K\rangle \cong \langle \Sigma\overline{X}, K\rangle$.

Now $\langle \Sigma X, K \rangle$ is naturally a group. For inverses invert interval.

$\langle X, K \rangle$.

$\langle \Sigma X, K \rangle$

We want X on LHS. Might be ok w/ changing K.

ΩK be the loop space of K.

$$\Omega K = \{ \underset{\text{continuous}}{(S^1, *) \to (K, *)} \}$$

with topology induced by

inclusion $\Omega K \subset K^I$

$= \{ I \to K \}$

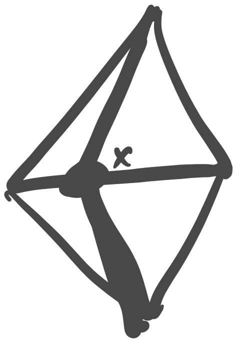
w/ compact open topology.

A ptd map

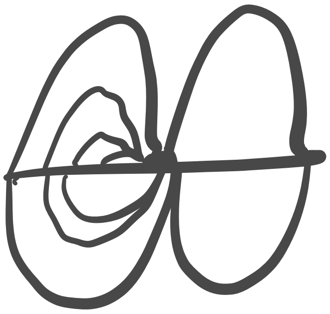
$\Sigma X \to K$ is the

same as a ptd map

$X \to \Omega K$.



SX



so $\langle \Sigma X, K\rangle \leftrightarrow \langle X, \Omega K\rangle$

If $X = S^n$, then

$$\pi_{n+1}(K) \cong \langle \Sigma S^n, K\rangle$$
$$\cong \langle S^n, \Omega K\rangle$$
$$\cong \pi_n(\Omega K)$$

so $\Omega K(G,n) = K(G,n-1)$.

$X \longmapsto \Omega X$ is a functor.

$f : X \to Y$

$\rightsquigarrow \Omega f : \Omega X \to \Omega Y$ by composing loops.

$f \simeq g \rightsquigarrow \Omega f \simeq \Omega g$.

So $X \simeq Y \Rightarrow \Omega X \simeq \Omega Y$.

Thm (Milner) If X has finitely many cells in each dimension, then $\Omega X \simeq$ to a cell cx w/ same property.

$\langle \Sigma X, K \rangle$ has bin. operation

$\Downarrow$

$\shortparallel$

$\langle X, \Omega K \rangle$ $\leftarrow$ binary operation here.

More directly:

$\circ : \Omega K \times \Omega K \to \Omega K$

given by concatenation of loops.

So $\langle X, \Omega K \rangle$ is a gp w/

$$(f+g)(x) = f(x)\cdot g(x).$$

This might not be abelian.

So: take K to be a $K(G, n+2)$

Consider

$$\langle X, \Omega^2 K\rangle = \langle X, \Omega(\Omega K)\rangle$$

$\forall n \geq 1$
n-fold loopspace $\Omega^n K$:

Fact: $K^{I\times Z} \cong (K^I)^Z$ for

l.c. cpct. Hausdorffs.

So $(\Omega^2 K) \subset (K^I)^I \cong K^{I^2}$

$$\Omega^2 K = \{\text{maps } I^2 \to K \text{ send } \partial I^2 \to *\}$$

Then:

$\langle X, \Omega^2 K\rangle$ is abelian

by same argument for $\pi_2(X)$.

more generally

$\langle X, \Omega^n K \rangle$ abelian $n \geq 2$.

$$\Omega\Omega K(G,n) = \Omega K(G,n-1)$$
$$= K(G,n-2).$$

Sequence of spaces

$K_n \left(= K(G,n)\right)$

w/ property that there are

htpy eqs $K_n \to \Omega K_{n+1}$

Funny property:

If we forget initial piece

of the list

$\Omega^2 K_3, \Omega K_3, K_3, K_4, \ldots$

and moreover can extend the list to negative values of n.

$K(G,-100)$

A seq. of spaces like this is called an Ω-spectrum.

Thm If $\{K_n\}$ is any Ω-spectrum then $\underline{X} \longmapsto h^n(\underline{X}) = \langle \underline{X}, K_n \rangle$ for $n \in \mathbb{Z}$, is a reduced cohomology theory on ptd spaces.

The converse is true

Thm (Brown Representability)

Every reduced cohomology is of the form $\langle X, K_n \rangle$ for some Ω-spectrum.

Thm h^* is unred. cohomology theory on CW pairs and $h^n(*) = 0$ when $n \neq 0$ then $h^n(X, A) \cong H^n(X, A; h^0(pt))$ $\forall$ pairs (X, A) $\forall n$.

Similar theorem for homology.
