GEOMETRIC PARTITIONS OF

DEFINABLE SETS AND THE CAUCHY-CROFTON FORMULA

By

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Abstract

Let $\mathbb{F}$ be an ordered field extension of $\mathbb{R}$. I prove that each $\mathbb{Q}$-bounded definable set in an o-minimal expansion of $\mathbb{F}$ has a partition into pieces that satisfy the Whitney arc property, and that any bounded definable set can be decomposed into cells built out of functions with bounded derivatives.

I show that length is bounded in definable families of curves and I formulate and prove a Cauchy-Crofton formula for $\mathbb{Q}$-bounded definable curves in an o-minimal expansion of $\mathbb{F}$.

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Contents

Chapter 1

Introduction

A semialgebraic subset of $\mathbb{R}^n$ is a finite union of sets given by polynomial equations and inequalities. For example, the intervals $(a, b) \subset \mathbb{R}$ and the open disk $\{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ are semialgebraic sets. Semialgebraic sets behave well under nice mappings: the projection of a semialgebraic set is semialgebraic (this is the Tarski-Seindenberg theorem, see [10]). These sets also possess good stratifications properties. A broader class of sets is the class of all globally subanalytic subsets of $\mathbb{R}^n$.

The collection of all semialgebraic subsets of $\mathbb{R}^n$, for $n \geq 0$, is an example of an *o-minimal* structure over the real field (a consequence of Tarski's quantifier elimination for real closed fields) as is the collection of all globally subanalytic subsets of $\mathbb{R}^n$ (a consequence of Gabrielov's theorem of the complement [2]). The subject of o-minimality started in the early 1980's, when Van den Dries [9] realized that many of the properties of semialgebraic sets and subanalytic sets could be deduced from a small collection of axioms. These axioms were subsequently studied in detail by Knight, Pillay and Steinhorn [5].

An **o-minimal structure** $\mathcal{S}$ over an ordered field $\mathbb{F}$ is a collection $\mathcal{S}^n$ of subsets of $\mathbb{F}^n$, for each $n \geq 0$, satisfying the following conditions:

(i) $\mathcal{S}^n$ is a Boolean algebra of subsets of $\mathbb{F}^n$.

(ii) $A \times \mathbb{F}$, $\mathbb{F} \times A \in \mathcal{S}^{n+1}$ whenever $A \in \mathcal{S}^n$.

(iii) If $A \in \mathcal{S}^{n+1}$, then $\pi_n(A) \in \mathcal{S}^n$, where $\pi_n : \mathbb{F}^{n+1} \longrightarrow \mathbb{F}^n$ is the projection onto the first n coordinates.

(iv) The diagonals $\{(x_1, \ldots, x_n) : x_1 = x_n\}$ belong to $\mathcal{S}^n$ for every n.

(v) $\{(x, y) : x < y\} \in \mathcal{S}^2$.

(vi) $\{(x, y, z) : z = x + y\}$, $\{(x, y, z) : z = xy\} \in \mathcal{S}^3$.

(vii) The sets in $\mathcal{S}^1$ are precisely the finite unions of intervals (a, b), $a, b \in \mathbb{F} \cup \{-\infty, +\infty\}$ and points of $\mathbb{F}$.

The sets belonging to an o-minimal structure are called the **definable** sets. A function is definable if its graph is a definable set.

There are many examples of o-minimal structures over the reals other than the semialgebraic and subanalytic sets. For instance, there are o-minimal structures over the real field in which the exponential function and all Pfaffian functions are definable (Wilkie [13]); or more generally, where non-spiralling leaves of definable hyperplane fields are definable (Speissegger [8]).

I will concentrate on discussing o-minimal structures over ordered field extensions of the real field. Proper ordered field extensions of $\mathbb{R}$ necessarily contain positive elements that are smaller than every positive real number, as well as elements that are bigger than any real number. Ordered field extensions of the real field have played an important role in real algebraic geometry since Artin's solution of Hilbert's seventeenth problem.

In the first chapter of this thesis, I study techniques for partitioning definable sets into definable pieces satisfying certain geometric properties; this chapter generalizes results of Kurdyka's [6]. A typical example of a theorem of this section states that any definable set can be partitioned into pieces that are "almost flat". The main result of this chapter is that definable sets can be partitioned into pieces whose metric structure is equivalent to that induced by the ambient space.

The second chapter concerns a generalization of the Cauchy-Crofton formula from integral geometry to arbitrary o-minimal structures over a field extension of $\mathbb{R}$. I use this generalization to show that the lengths of the curves in a bounded definable family of curves are uniformly bounded.

The classical Cauchy-Crofton formula expresses the length of a compact embedded curve $\gamma \subset \mathbb{R}^n$ as an average over the affine Grassmannian of all affine hyperplanes in $\mathbb{R}^n$ of the number of points of intersection of each affine hyperplane with γ (see Howard [4]).

Given an o-minimal structure over an ordered field $\mathbb{F}$, Berarducci and Otero [1] define a measure on a certain Boolean algebra of subsets of $\mathbb{F}^n$, $n \geq 0$, and show that the definable subsets of $\mathbb{F}^n$ that are bounded by a box with rational corners are measurable in this sense. Using this measure, I define the length of a definable curve whose image is contained in a box with rational corners, that is, a curve with $\mathbb{Q}$-bounded image. The main result of chapter two is the following:

Theorem 1.1. *There is a constant $C \in \mathbb{R}_{>0}$ such that for every definable injective curve $\gamma : [0,1]_{\mathbb{F}} \longrightarrow \mathbb{F}^n$ with $\gamma([0,1])$ $\mathbb{Q}$-bounded,*

$$\text{length}(\gamma) = C \int_{AGr_{n-1}(\mathbb{F}^n)} \#(\gamma \cap L)\, dL.$$

(Here, $\#(\gamma \cap L)$ is the number of points of intersection of γ and L, and $AGr_{n-1}(\mathbb{F}^n)$ is the affine Grassmannian of hyperplanes in $\mathbb{F}^n$). I define the integral on the right by using the Berarducci-Otero measure and I show that it coincides with an analogous integral over the real affine Grassmannian for a suitable real curve $\overline{\gamma}$ obtained from γ.

The study of this generalization of the Cauchy-Crofton formula was motivated by an attempt to generalize a result found in [6]: subanalytic sets can be stratified in such a way that each stratum has the "Whitney arc property". A set $A \subset \mathbb{R}^n$ satisfies the **Whitney arc property (WAP)**, if there is a number $K > 0$ such that any two points x, y of A can be joined by a curve γ in A satisfying $\text{length}(\gamma) \leq K|x-y|$. This property was introduced by Whitney in [12], where the author shows that if f is a function of class C^m defined on a region R that has the Whitney arc property, and if all the m-th order partials of f can be defined on the boundary B of R so that they are continuous in $R \cup B$, then f can be extended to a C^m function on all of $\mathbb{R}^n$.

Chapter one generalizes Kurdyka's result to sets that are definable in an o-minimal structure over a field extension of the real field. Moreover, I show that there is a definable family of curves witnessing the WAP. More precisely:

Theorem 1.2. *Let $A \subset \mathbb{F}^n$ be a definably connected definable set, and asumme that A is contained in a box with rational corners. Then there is a $K \in \mathbb{Q}_{>0}$, which depends only on n, and definable pairwise disjoint sets A_i, for $i = 1, \ldots, s$, such that*

$$A = \bigcup_{i=1,\ldots,s} A_i$$

and for each i, there is a definable family of curves

$$\lambda^i \subset A_i^2 \times ([0,1] \times A_i)$$

with the property that for every pair of points $x, y \in A_i$, $\lambda^i_{x,y}$ is a curve in A_i joining x and y such that $\text{length}(\lambda^i_{x,y}) \leq K|x-y|$. *In particular, A_i has the WAP.*

Chapter 2

Preliminaries

2.1 Definable sets

This section contains some of the basic definitions and theorems about definable sets. We refer the reader to [10] for details.

From now on, we fix an o-minimal structure $\mathcal{S}$ over an ordered field $\mathbb{F}$. Continuity and differentiability of functions are defined by the standard limits. Many of the standard theorems of differential calculus, like the intermediate value theorem and the mean value theorem, hold for definable functions. Definable continuous images of closed and bounded sets are closed and bounded; definable, continuous functions on a closed and bounded set achieve maximum and minimum values.

A definable set A is **definably connected** if it is not the union of two disjoint definable open subsets of A. The only definably connected subsets of $\mathbb{F}$ are intervals. Images of definably connected sets under definable and continuous maps are definably connected. A definable set A is **definably path connected** if any pair of points of A can be connected by a definable path in A. Definably connected sets are definably path connected.

A point in the closure of a subset of $\mathbb{R}^n$ is the limit of a sequence of points in the set. The analogous fact for definable sets in $\mathbb{F}^n$ is the following:

Fact 2.1 (Curve Selection)**.** *Let C be a definable set, and let $c \in \overline{C} \setminus C$. Then there is a definable, continuous, injective map $\gamma : (0,1) \longrightarrow C$ such that $\lim_{t \longrightarrow 0} \gamma(t) = a$.*

A cell in $\mathbb{F}^n$ is a special kind of definable set:

Definition 2.2. *Let $(i_1, \ldots, i_m)$ be a sequence of zeros and ones and $k \in \mathbb{N}$. An $(i_1, \ldots, i_m)$ C^k- **cell** is a subset of $\mathbb{F}^m$ defined inductively as follows:*

(i) *A (0) C^k-cell is a point $\{r\} \subset \mathbb{F}$, a (1) C^k-cell is an interval $(a, b) \subset \mathbb{F}$, where $a, b \in \mathbb{F} \cup \{-\infty, +\infty\}$.*

(ii) *An $(i_1, \ldots, i_m, 0)$ C^k-cell is the graph $\Gamma(f)$ of a definable C^k function $f : X \longrightarrow \mathbb{F}$, where X is a $(i_1, \ldots, i_m)$ C^k-cell; an $(i_1, \ldots, i_m, 1)$ C^k-cell is a set*

$$(f, g)_X := \{(x, r) \in X \times \mathbb{F} : f(x) < r < g(x)\},$$

where X is an $(i_1, \ldots, i_m)$ C^k-cell and $f, g : X \longrightarrow \mathbb{F}$ are definable C^k functions on X such that for all $x \in X$, $f(x) < g(x)$; we also allow $f = -\infty$ or $g = +\infty$.

Observe that the definition of cells depends on the ordering of the coordinates of $\mathbb{F}^n$. It is easy to check that cells are definably connected.

Definition 2.3. *A C^m **cell decomposition** of $\mathbb{F}^n$ is a special partition of $\mathbb{F}^n$ into C^m-cells. The definition is given by induction on n:*

(i) *A C^m cell decomposition of $\mathbb{F}$ is a collection*

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, \infty), \{a_1\}, \ldots, \{a_k\}\}$$

where $a_1 < \cdots < a_k$ are points of $\mathbb{F}$.

(ii) *A C^m cell decomposition of $\mathbb{F}^{n+1}$ is a partition $\mathcal{A}$ of $\mathbb{F}^n$ into C^m-cells such that the set $\{\pi_n(A) : A \in \mathcal{A}\}$ is a C^m cell decomposition of $\mathbb{F}^n$, where $\pi_n : \mathbb{F}^{n+1} \longrightarrow \mathbb{F}^n$ is the projection onto the first n coordinates.*

A fundamental fact about definable sets is that they can be partitioned into C^m-cells. More precisely:

Theorem 2.4 (Cell Decomposition theorem [10], [5])**.** *The following holds:*

(i) *Given definable sets $A_1, \dots, A_k \subset \mathbb{F}^n$, there is a C^m cell decomposition of $\mathbb{F}^n$ partitioning each A_i.*

(ii) *For every definable function $f : A \longrightarrow \mathbb{F}$, with $A \subset \mathbb{F}^n$, there is a C^m cell decomposition $\mathcal{D}$ of $\mathbb{F}^n$ partitioning A such that the restriction $f|B : B \longrightarrow \mathbb{F}$, for each $B \in \mathcal{D}$ with $B \subset A$, is a C^m function.*

The **dimension** of a definable set $A \subset \mathbb{F}^n$ is defined by

$$\dim(A) := \max\{i_1 + \dots, +i_n : A \text{ contains an } (i_1, \dots, i_n)\text{-cell}\}.$$

For a definable set $S \subset \mathbb{F}^{m+n}$ and a point $a \in \mathbb{F}^m$, the **fiber** of S over a is the set

$$S_a := \{x \in \mathbb{F}^n : (a, x) \in S\}.$$

We consider S as describing the family of sets $(S_a)_{a \in \mathbb{F}^m}$, also called a **definable family** in $\mathbb{F}^n$ with parameters in $\mathbb{F}^m$. In this situation, the cell decomposition theorem shows the existence of a bound on the number of definably connected components of S_a:

Fact 2.5. *Let $S \subset \mathbb{F}^{m+n}$ be a definable set. Then there is an $N \in \mathbb{N}$ such that each fiber S_a has at most N definably connected components.*

We say that $A \subset \mathbb{F}^d$ is $\mathbb{Q}$-**bounded** if $A \subset [-q, q]^d$ for some $q \in \mathbb{Q}_{>0}$.

Definition 2.6. *Let $A \subset \mathbb{F}^n$, $B \subset \mathbb{F}^m$ be definable sets. Let $\lambda \subset A \times ([0,1] \times B) \subset \mathbb{F}^n \times \mathbb{F}^{1+m}$ be a definable set such that for every $x \in A$, the fiber λ_x is the graph of a function $\lambda_x : [0,1] \longrightarrow B$. We view λ as describing the family of curves $\{\lambda_x\}_{x\in A}$. Such a family is a* ***definable family of curves*** *(in B, parametrized by A). When there is no risk of confusion, we denote the image of λ_x also by λ_x. λ is $\mathbb{Q}$-**bounded** if*

$$\bigcup_{x\in A} \lambda_x([0,1])$$

is a $\mathbb{Q}$-bounded subset of $\mathbb{F}^m$.

2.2 The Berarducci-Otero integral

In [1], a theory of measure and integration in o-minimal structures over a field is developed. This section contains a description of the corresponding integral.

Definition 2.7. *$B \subset \mathbb{F}^d$ is a* ***polyrectangle*** *of dimension d if B is a finite union of rectangles $[q_1, r_1) \times \cdots \times [q_d, r_d)$ with rational coordinates q_i, r_i. The set $\mathcal{PR}^{(d)}(\mathbb{F})$ is the set of polyrectangles of dimension d of $\mathbb{F}$. The* ***volume*** *of a rectangle $[q_1, r_1) \times \cdots \times [q_d, r_d)$ is*

$$\mu([q_1, r_1) \times \cdots \times [q_d, r_d)) = \pi_{i=1}^{d}(r_i - q_i).$$

If a polyrectangle P is the disjoint union of rectangles R_i, $i = 1, \ldots, m$, then $\mu(P) := \sum_{i=1}^{m} \mu(R_i)$.

Definition 2.8. *For a $\mathbb{Q}$-bounded set $A \subset \mathbb{F}^d$ we define:*

The ***outer measure*** *of A: $\mu^*(A) := \inf\{\mu(P) : P \supset A, P \in \mathcal{PR}^{(d)}(\mathbb{F})\}$. The* ***inner***

measure *of A:* $\mu_*(A) := \sup\{\mu(P) : P \subset A, P \in \mathcal{PR}^{(d)}(\mathbb{F})\}$*. Here the infimum and supremum are taken in* $\mathbb{R}$*.*

A $\mathbb{Q}$-bounded set A is **measurable** if $\mu_*(A) = \mu^*(A)$, and in this case the **measure** of A is defined as $\mu(A) := \mu_*(A) = \mu^*(A)$. One of the main results in [1] is the following

Theorem 2.9. *Let A be a $\mathbb{Q}$-bounded definable subset of $\mathbb{F}^d$. Then A is measurable. Moreover, if $\dim(A) < d$ then $\mu(A) = 0$.*

Let $A \subset \mathbb{F}^d$ be $\mathbb{Q}$-bounded. For $f : \mathbb{F}^d \longrightarrow \mathbb{F}_{\geq 0}$, we define

$$\int_A f := \mu([0, f)_A),$$

provided that $[0, f)_A := \{(x, y) : x \in A, 0 \leq y < f(x)\}$ is measurable. For general $f : \mathbb{F}^d \longrightarrow \mathbb{F}$, we put

$$\int_A f := \int_A f^+ - \int_A f^-,$$

provided both terms on the right exist, where f^+ and f^- are, respectively, the positive and negative part of f.

This integral can be used to define the **length** of a definable C^1 curve $\gamma : (a, b) \longrightarrow \mathbb{F}^n$, with (a, b) and $\text{Im}(\gamma')$ $\mathbb{Q}$-bounded, by

$$\text{length}(\gamma) := \int_a^b |\gamma'(x)|.$$

The Berarducci-Otero integral is additive and real valued but in general, it is not true that $\int_A cf = c \int_A f$ for $c \in \mathbb{F}$. This formula holds for rational c.

2.3 The Cauchy-Crofton formula

A reference for standard facts about Lie groups is Warner [11]. For facts about group invariant integration see Helgason [3], and for a proof of the Cauchy-Crofton formula see Howard [4].

We denote the Lie algebra of a Lie group G by $\mathfrak{g}$. A **representation** of G is a Lie group homomorphism from G into $GL(V)$, for a finite dimensional real vector space V. G acts on itself on the left by conjugation and the identity element $e \in G$ is a fixed point of this action. For $g \in G$, let $\mu_g : G \longrightarrow G$ be conjugation by g, that is $\mu_g(x) := gxg^{-1}$. The map $\mathrm{Ad}_G : G \longrightarrow GL(\mathfrak{g})$ given by

$$\mathrm{Ad}_G(g)(v) := d\mu_g(v), \quad g \in G, v \in \mathfrak{g}$$

is a representation of G. It is called the **adjoint representation** of G.

Let $C_c(G)$ be the collection of compactly supported continuous real valued functions on G. A **measure** on a Lie group G is an $\mathbb{R}$-linear mapping $C_c(G) \longrightarrow \mathbb{R}$ such that: for each compact $K \subset G$, there is a constant M_K such that for every continuous f with compact support contained in K,

$$\int_G f \leq M_K \sup_{x \in G} |f(x)|.$$

Let $l_g : G \longrightarrow G$ and $r_g : G \longrightarrow G$ denote, respectively, left and right multiplication by g, that is $l_g(x) = gx$ and $r_g(x) = xg$ for all $x \in G$. A measure $\int_G$ on G is **bi-invariant** if for all $g \in G$ and all continuous, compactly supported $f : G \longrightarrow \mathbb{R}$,

$$\int_G f \circ l_g = \int_G f \quad \text{and} \quad \int_G f \circ r_g = \int_G f.$$

The following proposition tells us when bi-invariant measures exist.

Fact 2.10. *A Lie group G has a bi-invariant measure if and only if $|\det(\mathrm{Ad}_G(g))| = 1$ for all $g \in G$.*

See [3] Chapter 1, pg. 88 for the proof.

For example, semisimple Lie groups and compact Lie groups have bi-invariant measures.

Let G and H be Lie groups. An **action by automorphisms** of G on H on the left is a C^∞ map $\tau : G \times H \longrightarrow H$ such that:

(i) $\tau(g, h_1h_2) = \tau(g, h_1)\tau(g, h_2)$, for all $g \in G$ and $h_1, h_2 \in H$.

(ii) $\tau(g_1g_2, h) = \tau(g_1, \tau(g_2, h))$, for all $g_1, g_2 \in G$ and $h \in H$.

(iii) $\tau(e, h) = h$, for all $h \in H$, where $e \in G$ is the identity element.

$\tau(g, h)$ is also commonly denoted as $g \cdot h$.

Let G and H be lie groups with an action by automorphisms $g \cdot h$ of G on H on the left. The **semidirect product** $H \rtimes G$ of H and G is the Lie group whose underlying manifold is the product manifold $H \times G$ and with product operation

$$(h_0, g_0)(h_1, g_1) = (h_0(g_0 \cdot h_1), g_0g_1).$$

We identify G and H with subgroups of K by $h = (h, e_G)$ and $g = (e_H, g)$ where $g \in G$, $h \in H$ and e_G, e_H are the identity elements of G and H respectively. Conjugation of h by g in $K := H \rtimes G$ corresponds to the action of G on H, in other words $ghg^{-1} = g \cdot h$; in particular, μ_g maps H into H.

Take $g \in G$ and $v \in \mathfrak{h}$. Since μ_g maps H into H and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g}$,

$$\mathrm{Ad}_K(g)(v) = d\mu_g(v) = d(\mu_g\big|H)(v) = dl_g(v),$$

where $l_g : H \longrightarrow H$ is the map $h \longrightarrow g \cdot h$.

Let $G = \mathbb{R}^n \rtimes O_n(\mathbb{R})$, and let v be in the Lie algebra of $\mathbb{R}^n$, that is $v \in T_0\mathbb{R}^n = \mathbb{R}^n$. Then:

(i) $\mathrm{Ad}_G(g)(v) = gv$, for $g \in O_n(\mathbb{R})$.

(ii) $\mathrm{Ad}_G(g)(v) = v$, for $g \in \mathbb{R}^n$.

The first part follows since $\mathrm{Ad}_G(g)(v) = dl_g(v)$ for $g \in O_n(\mathbb{R})$, and $l_g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the linear map $x \longrightarrow gx$. For (ii), notice that for $g \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, $\mu_g(h) = h$. Let $e_1, \ldots, e_n$ be a basis for $T_0\mathbb{R}^n$, and let $v_1, \ldots, v_d$ a basis for the lie algebra of $O_n(\mathbb{R})$. By (i) and (ii), for any $g = (b, A) \in G$, the matrix representation of $\mathrm{Ad}_G(g)$ with respect to this basis is of the form:

$$\begin{pmatrix} B & * \\ 0 & C \end{pmatrix},$$

where the colums of B are the vectors

$$\mathrm{Ad}_G(g)(e_i) = \mathrm{Ad}_G(b)(\mathrm{Ad}_G(A)(e_i)) = \mathrm{Ad}_G(b)(Ae_i) = Ae_i,$$

and therefore $\det(B) = \det(A) = \pm 1$.For $g \in \mathbb{R}^n$, and $X \in O_n(\mathbb{R})$,

$$\mu_g(X) = (g - Xg, X),$$

and therefore C is the identity matrix. For $g \in O_n(\mathbb{R})$, conjugation by g maps $O_n(\mathbb{R})$ into itself, and therefore $* = 0$ and C is the matrix representation of $\mathrm{Ad}_{O_n(\mathbb{R})}(g)$ in the basis $v_1, \ldots, v_d$. Since $O_n(\mathbb{R})$ is compact it has a bi-invariant measure and therefore $\det(\mathrm{Ad}_{O_n(\mathbb{R})}(g) = \pm 1$ for all $g \in O_n(\mathbb{R})$. This shows that for any $g \in G = \mathbb{R}^n \rtimes O_n(\mathbb{R})$,

$$\det(\mathrm{Ad}_G(g)) = \det(B)\det(C) = \pm 1, \tag{2.1}$$

so $\mathbb{R}^n \rtimes O_n(\mathbb{R})$ has a bi-invariant measure.

Let G be a Lie group, and H a closed subgroup. A measure in G/H is **G-invariant** if for all $g \in G$,

$$\int_{G/H} f = \int_{G/H} f \circ L_g,$$

where $L_g : G/H \longrightarrow G/H$ is left multiplication by g, that is $L_g(xH) = gxH$. A G-invariant measure in G/H is unique up to a constant factor. By the change of variables formula, the existence of a top form ω in G that is G-invariant up to sign implies the existence of a G-invariant measure.

The existence of a G-invariant measure in a homogeneous space G/H is related to the determinants of the adjoint representations of G and H. More precisely,

Fact 2.11. *G/H has a G-invariant measure if and only if for all $h \in H$,*

$$|\det(\mathrm{Ad}_G(h))| = |\det(\mathrm{Ad}_H(h))|.$$

See [3], Chapter 1, Theorem 1.9 for the proof.

The reason the determinants of the adjoint representations play a role in the existance of a G-invariant measure is that:

Fact 2.12. *For $h \in H$,*

$$\det((dL_h)_H) = \frac{\det(\mathrm{Ad}_G(h))}{\det \mathrm{Ad}_H(h))}.$$

See [3] Chapter 1, lemma 1.7 for the proof.

Lets consider again $G = \mathbb{R}^n \rtimes O_n(\mathbb{R})$, the group of isometries of $\mathbb{R}^n$, and let H be the stabilizer of $\langle e_1, \ldots, e_{n-1}\rangle$. H is isomorphic to

$$(\mathbb{R}^{n-1} \rtimes O_{n-1}(\mathbb{R})) \times O_1\mathbb{R}.$$

Therefore by (2.1) for all $g \in G, h \in H$, $\det(\mathrm{Ad}_G(g)) = \pm 1$ and $\det(\mathrm{Ad}_H(h)) = \pm 1$. This implies that G/H has a G-invariant measure, and for all $h \in H$,

$$\det((dL_h)_H) = \pm 1. \tag{2.2}$$

The Cauchy-Crofton formula expresses the length of a compact, embedded curve in $\mathbb{R}^n$ as the average number of points of intersection of the curve with a hyperplane in $\mathbb{R}^n$. Observe that the set of all affine hyperplanes in $\mathbb{R}^n$ can be identified with G/H.

Theorem 2.13. *Let $G = \mathbb{R}^n \rtimes O_n(\mathbb{R})$ and let $H = (\mathbb{R}^{n-1} \rtimes O_{n-1}(\mathbb{R})) \times O_1\mathbb{R}$ be the stabilizer of $\langle e_1, \ldots, e_{n-1}\rangle$ in G. Fix a G-invariant measure dL on G/H. Then, there is a constant $C \in \mathbb{R}$ such that for any compact embedded 1-dimensional submanifold γ of $\mathbb{R}^n$,*

$$\text{length}(\gamma) = C \int_{G/H} \#(\gamma \cap L) dL,$$

where for $L \in G/H$, $\#(\gamma \cap L)$ is the number of points of intersection of γ with the plane $g\langle e_1, \ldots, e_{n-1}\rangle$ for any $g \in G$ with $L = gH$.

For the proof see [4] 3.18.

2.4 The Affine Grassmannian.

The **affine Grassmannian**, $AGr_{n-1}(\mathbb{F}^n)$, is the set of all affine hyperplanes in $\mathbb{F}^n$. We will see that it can be regarded as an algebraic subset of the vector space V of affine linear transformations of $\mathbb{F}^n$.

An **affine orthogonal projection** is an affine linear map $p : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ with $p^2 = p$ and $\ker(p - p(0)) \perp \text{Im}(p - p(0))$. In coordinates, the set of affine linear maps is $\mathbb{F}^n \times M_{n\times n}(\mathbb{F})$ and the set of affine orthogonal projections of rank $n-1$ is:

$$\{(b, A) \in \mathbb{F} \times M_{n\times n}(\mathbb{F}) : Ab = 0,\ A^2 = A,\ rk(A) = n-1,\ \text{and } A^t = A\}. \qquad (2.3)$$

To see this, let $A \in M_{n\times n}(\mathbb{F})$. If $A^t = A$ then $\ker A \perp \operatorname{Im} A$: take $v \in \ker A$ and write

$$A = (A_1 \ldots A_n) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

where $A_1, \ldots, A_n$ are column vectors, and $a_1 \ldots, a_n$ are row vectors. Now if $v \in \ker A$, then $a_i \cdot v = 0$ for all $i = 1, \ldots, n$. But $a_i = A_i$, so $A_i \cdot v = 0$ for $i = 1, \ldots, n$, that is $v \perp \operatorname{Im} A$. If $\ker A \perp \operatorname{Im} A$ and $A^2 = A$ then $A^t = A$: Since $\ker A \perp \operatorname{Im} A$, $\mathbb{F}^n = \ker A \oplus \operatorname{Im} A$. Let $v_1, \cdots, v_{n-1}$ be an orthonormal basis of $\operatorname{Im} A$, and V_n an orthonormal basis of $\ker A$. Since $A^2 = A$, for $i = 1, \ldots, n-1$ we have $Av_i = v_i$. Thus the matrix of A in the basis $v_1, \ldots, v_n$ is the symmetric matrix

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where I is the $(n-1) \times (n-1)$ identity matrix. Therefore A is symmetric.

To every hyperplane l there corresponds the affine orthogonal projection p_l onto l, and the map $l \longrightarrow p_l$ is a bijection from $AGr_{n-1}(\mathbb{F}^n)$ onto the set of affine orthogonal projections of rank $n-1$. Thus equation (2.3) shows that we can regard the affine Grassmannian as an algebraic variety, and hence as a definable set (in any o-minimal structure over a field $\mathbb{F}$).

Similarly, the **Grassmannian** $Gr_k(\mathbb{F}^n)$ consisting of the k linear subspaces of $\mathbb{F}^n$ can be regarded as an algebraic subvariety of $M_{n\times n}(\mathbb{F})$:

$$Gr_k(\mathbb{F}^n) = \{A \in M_{n\times n}(\mathbb{F}) : A^t = A, A^2 = A \text{ and } rkA = k\}.$$

We consider $M_{n\times n}(\mathbb{F})$ a normed linear space with the operator norm $|A| := \sup_{|v|=1} |Av|$. Then $Gr_k(\mathbb{F}^n)$ has finite diameter as a subset of $M_{n\times n}(\mathbb{F})$ and for any $\epsilon \in \mathbb{Q}_{>0}$ we can cover $Gr_k(\mathbb{F}^n)$ by a finite number of balls of radius ϵ.

Definition 2.14. *$M \subset \mathbb{F}^n$ is a* ***definable embedded submanifold*** *of dimension d if there are a finite number of definable open sets $U_1, \ldots, U_i \subset \mathbb{F}^n$ such that*

(i) *$M \subset \bigcup_{j=1}^{i} U_j$.*

(ii) *For each $j = 1, \ldots, i$ there is a definable open set $V_i \subset \mathbb{F}^n$ and a definable C^1 map with C^1 inverse $h_j : U_j \longrightarrow V_j$ such that*

$$h_j(U_j \cap M) = V_j \cap (\mathbb{F}^k \times \{0\}).$$

We will use over and over the fact that C^1 cells are definable embedded submanifolds.

Definition 2.15. *Let M be a definable embedded submanifold of $\mathbb{F}^n$ of dimension k. The* ***Gauss map*** *for M is the map $G : M \longrightarrow Gr_k(\mathbb{F}^n)$ given by*

$$G(p) = T_pM.$$

It is well known that for a definable embedded submanifold M of $\mathbb{F}^n$ the Gauss map is continuous and definable.

Chapter 3

Geometric Partitions of Definable Sets

3.1 Motivations

Thruoghout this chapter we fix an o-minimal expansion $\mathcal{F}$ of a field $(\mathbb{F}, <, 0, +, -, 1, \cdot)$, where $\mathbb{F}$ is a field extension of the real field.

Consider the situation of the unit circle in $\mathbb{R}^2$ with the south pole removed. There are points on this circle that are very close together in the plane, but far apart on the circle. However, if we also remove the north pole, we are left with two semicircles, and a semicircle satisfies the property that any pair of points x, y on the semicircle can be connected by a path in the semicircle of length at most $\sqrt{6}|x - y|$, where $|x - y|$ is the Euclidean distance. In other words, the circle can be partitioned into pieces that satisfy the WAP. That any bounded definable set can be partitioned into definable pieces that satisfy the WAP will be proved in the second section of this chapter. In the first section of this chapter, we find a partition of definable sets into pieces that are, after a change of coordinates, C^1-cells with bounded differentials. In the second section, in addition to a partittion into pieces that have the WAP, we give a partition into pieces that are Lipschitz cells (after a change of coordinates). In the last section, I discuss uniform locall connectedness for a definable set and prove that Lipschitz cells are uniformly locally connected .

3.2 K-C^1-cell partition of definable sets

In this section we define K-C^1-cell and establish the existence of a partition of any definable, bounded subset of $\mathbb{F}^n$ into pieces that are M_n-C^1-cells after a coordinate change in $O_n(\mathbb{F})$. M_n is a bound on the norm of the differentials of the functions defining the cell, that depends only on n. The proof of this result is by induction, the heart of which is Lemma 3.7, needed to deal with the top dimensional case. I adapt Kurdyka's lemma in [6] to fit this more general situation.

The image under a definable and continuous map of a definable, closed and bounded set is closed and bounded. Therefore we can define the norm of linear maps:

Definition 3.1. *Let $L : V \longrightarrow W$ be a linear map between normed $\mathbb{F}$–vector spaces. Define*

$$|L| := \sup_{|v|=1} L(v).$$

A C^1-cell is a K-C^1**-cell**, where $K \in \mathbb{F}$, if the C^1 functions that define the cell satisfy $|df| \leq K$. More precisely:

Definition 3.2. *Let $(i_1, \ldots, i_m)$ be a sequence of zeros and ones, and $K \in \mathbb{F}_{>0}$. An $(i_1, \ldots, i_m)$-K-C^1-**cell** is a subset of $\mathbb{F}^m$ defined inductively as follows:*

(i) *A (0)-K-C^1-cell is a point $\{r\} \subset \mathbb{F}$, a (1)-K-C^1-cell is an interval $(a, b) \subset \mathbb{F}$, where $a,\ b \in \mathbb{F}$.*

(ii) *An* $(i_1, \ldots, i_m, 0)$*-*K*-*C^1*-cell is the graph* $\Gamma(f)$ *of a definable* C^1*-function* $f : X \longrightarrow \mathbb{F}$ *with* $|df| \leq K$*, where* X *is an* $(i_1, \ldots, i_m)$*-*K*-* C^1*-cell; an* $(i_1, \ldots, i_m, 1)$*-*K*-*C^1*-cell is a set*

$$(f, g)_X := \{(x, r) \in X \times \mathbb{F} : f(x) < r < g(x)\},$$

where X *is an* $(i_1, \ldots, i_m)$*-*K*-*C^1*-cell and* $f, g : X \longrightarrow \mathbb{F}$ *are definable* C^1*-functions on* X *with* $|df|, |dg| \leq K$ *such that for all* $x \in X$*,* $f(x) < g(x)$*.*

We next define the distance between subspaces of $\mathbb{F}^n$. For $X, Y \in Gr_k(\mathbb{F}^n) \subset \mathrm{End}_{\mathbb{F}}(\mathbb{F}^n)$, let S and T be the orthogonal projections onto X and Y respectively. The **distance function** in the Grassmannian is given by the inclusion above:

$$\delta(X, Y) := |S - T|.$$

For a line P in $\mathbb{F}^n$ and $X \in Gr_k(\mathbb{F}^n)$, define

$$\delta(P, X) := |v - \pi_X(v)|,$$

where π_X is the orthogonal projection onto X, and v is a generator of P of norm 1. Notice that $\delta(P, X) = 0$ if and only if $P \subset X$, $0 \leq \delta(P, X) \leq 1$ and $\delta(P, X) = 1$ if and only if $P \perp X$.

Remark 3.3. *If* $\delta(P, X) > \epsilon$ *and* $w \in X$ *is a unit vector with* $\pi_P(w) \neq 0$*, where* π_P *is the orthogonal projection onto* P*, then*

$$|\pi_P(w) - w| \geq |\pi_P(w) - \pi_X(\pi_P(w))| > |\pi_P(w)|\epsilon.$$

Therefore, if $|\pi_P(w)| \leq 1/2$*, then* $|\pi_P(w) - w| \geq 1/2$*, and otherwise* $|\pi_P(w) - w| \geq \frac{1}{2}\epsilon$*; in either case, we have* $|\pi_P(w) - w| \geq \frac{1}{2}\epsilon$*.*

Lemma 3.4. *Let $N \in \mathbb{N}_{>0}$. Then there exists an $\epsilon_n \in \mathbb{Q}_{>0}$ such that for every $X_1, \ldots, X_{2n} \in Gr_{n-1}(\mathbb{F}^n)$, there is a line P in $\mathbb{F}^n$ such that whenever $Y_1, \ldots, Y_{2n} \in Gr_{n-1}(\mathbb{F}^n)$ and*

$$\delta(X_i, Y_i) < \epsilon_n, \quad i = 1, \ldots, 2n,$$

then

$$\delta(P, Y_i) > \epsilon_n, \quad i = 1, \ldots, 2n.$$

Proof. Choose $\epsilon_n \in \mathbb{Q}_{>0}$ such that the sets

$$S_i := \{v \in S^{n-1} : |v - \pi_{X_i}(v)| \leq 2\epsilon_n\},$$

where π_{X_i} is the orthogonal projection onto X_i, do not cover all of S^{n-1}, that is,

$$\bigcup_{i=1}^{2n} S_i \neq S^{n-1}.$$

Such an ϵ_n exists for $\mathbb{F} = \mathbb{R}$: we can take ϵ_n small enough such that the union of the S_i has very small volume compared to the volume of the sphere. The same ϵ_n will necessarily work for any field $\mathbb{F}$ containing $\mathbb{R}$.

Now, we choose

$$v \in S^{n-1} - \bigcup_{i=1}^{2n} S_i$$

and let $P := \langle v \rangle$. Then

$$\delta(P, Y_i) = |v - \pi_{Y_i} v| \geq |v - \pi_{X_i} v| - |\pi_{X_i} v - \pi_{Y_i} v| > \epsilon_n,$$

as required. □

Definition 3.5. *Let $\epsilon > 0$. A definable embedded submanifold M of $\mathbb{F}^n$ is ϵ-**flat** if for each $x, y \in M$ we have $\delta(T_x M, T_y M) < \epsilon$.*

Lemma 3.6. *Let $\epsilon \in \mathbb{Q}_{>0}$, , and let $S \subset \mathbb{F}^m$ be definable. Then S can be partitioned into a finite number of ϵ-flat C^1-cells.*

Proof. By induction on $n := \dim S$; the claim is clear for sets S with $\dim S = 0$. Assume the claim holds for definable sets S with $\dim S \leq n-1$. By the inductive hypothesis and C^1-cell decomposition, it is enough to consider the case where S is an n dimensional C^1-cell. Cover $Gr_n(\mathbb{F}^m)$ by a finite number of balls B_i of radius $\frac{\epsilon}{2}$ (here we use that ϵ is rational), and consider the Gauss map $G : S \longrightarrow Gr_n(\mathbb{F}^m)$. Take a cell decomposition of $\mathbb{F}^m$ partitioning each $G^{-1}(B_i)$. Then the n-dimensional cells contained in S are ϵ-flat, and the cells of dimension less than n can be partitioned into ϵ-flat C^1-cells by induction. □

Lemma 3.7. *Let $\epsilon \in \mathbb{Q}_{>0}$, and let $A \subset \mathbb{F}^n$ be an open and bounded definable set. Then there are open, pairwise disjoint cells $A_1, \ldots, A_p \subset A$ such that*

(i) $\dim(A - \cup A_i) < n$.

(ii) *For each i, there are definable, pairwise disjoint sets $B_1, \ldots, B_k$ (with k depending on i) such that*

  (a) $k \leq 2n$;

  (b) *each B_j is a definable subset of ∂A_i and an ϵ-flat, $(n-1)$-dimensional, C^1-submanifold of $\mathbb{F}^n$;*

  (c) $\dim(\partial A_i - \cup_{j=1}^k B_j) < n - 1$.

Proof. By induction on n. The lemma is clear for $n = 1$. Assume that $n > 1$ and the lemma holds for smaller values of n.

Take a cell decomposition of $\overline{A}$ into C^1-cells. Let C be an open cell in this decomposition; it suffices to prove the lemma for C. Note that $C = (f,g)_X$, where X is an open cell in $\mathbb{F}^{n-1}$ and f, g are definable C^1-functions on X. Take finite covers of $\Gamma(f)$ and $\Gamma(g)$ by open sets U_i and V_j, respectively, such that each $U_i \cap \Gamma(f)$ and each $V_j \cap \Gamma(g)$ is $\frac{\epsilon}{2}$-flat (to do this, take a finite cover of the Grassmannian by $\frac{\epsilon}{2}$-balls and pull it back via the Gauss maps for $\Gamma(f)$ and $\Gamma(g)$). The collection of all sets $\pi(U_i) \cap \pi(V_j)$ is an open cover $\mathcal{O}$ of X, where $\pi : \mathbb{F}^n \longrightarrow \mathbb{F}^{n-1}$ is the projection onto the first $n-1$ coordinates. By the cell decomposition theorem, there is a C^1-cell decomposition of X partitioning each set in $\mathcal{O}$. Let S be an open cell in this decomposition, and let $C_0 := (f,g)_S$. It suffices to prove the lemma for C_0. By the inductive hypothesis, we can find $A'_1, \dots, A'_p \subset S$ and $B'_1, \dots, B'_k \subset \partial A'_i$ satisfying the conditions (i) and (ii) above. Define

$$A_i := (f,g)_{A'_i}, \qquad i = 1, \dots, p.$$

Then $\dim(C_0 - \cup_{i=1}^p A_i) < n$. For $j = 1, \dots, k$, the set $(B'_j \times \mathbb{F}) \cap \partial A_i$ is definable. Take a C^1-cell decomposition of this set, and let B_j be the union of the $(n-1)$-dimensional cells in this decomposition (note that B_j may be empty). Then B_j is an ϵ-flat C^1-submanifold of $\mathbb{F}^n$ and

$$\dim(((B'_j \times \mathbb{F}) \cap \partial A_i) - B_j) < n - 1.$$

Define $B_{k+1} := \Gamma(f\big|A'_i)$ and $B_{k+2} := \Gamma(g\big|A'_i)$; by construction these are ϵ-flat. It is routine to see that $\partial A_i \subset B_{k+1} \cup B_{k+2} \cup (\partial A'_i \times \mathbb{F})$. Thus

$$\begin{aligned}\partial A_i - \cup_{j=1}^{k+2} B_j &\subset ((\partial A'_i \times \mathbb{F}) \cap \partial A_i) - \cup_{j=1}^k B_j \\ &= (E \cup \cup_{j=1}^k (B'_j \times \mathbb{F}) \cap \partial A_i) - \cup_{j=1}^k B_j \\ &\subset \cup_{j=1}^k ((B'_j \times \mathbb{F}) \cap \partial A_i - B_j) \cup E,\end{aligned}$$

where E is a definable set with $\dim(E) < n-1$. Therefore $\dim(\partial A_i - \cup_{j=1}^{k+2} B_j) < n-1$. Since $k \leq 2(n-1)$, we get $k+2 \leq 2n$ and the lemma is proved. □

Theorem 3.8. *Let $A \subset \mathbb{F}^n$ be definable and bounded. Then there are definable, pairwise disjoint sets A_i, $i = 1, \ldots, s$, such that $A = \cup_i A_i$ and for each A_i, there is a change of coordinates $\sigma_i \in O_n(\mathbb{F})$ such that $\sigma_i(A_i)$ is an M_n-C^1-cell, where $M_n \in \mathbb{Q}_{>0}$ is a constant depending only on n.*

Proof. By induction on n; for $n = 1$ the theorem is clear. We assume that $n > 1$ and that the theorem holds for smaller values of n. We also proceed by induction on $d := \dim(A)$. It's clear for $d = 0$; so we assume that $d > 0$ and the theorem holds for definable bounded subsets B of $\mathbb{F}^n$ with $\dim(B) < d$.

Case I: $\dim(A) = n$. In this case A is an open, bounded, definable subset of $\mathbb{F}^n$, so by using the inductive hypothesis and Lemma 3.7, we can reduce to the case where there are pairwise disjoint, definable $B_1, \ldots, B_k \subset \partial A$ such that $k \leq 2n$, $\dim(\partial A - \cup_{j=1}^{k} B_j) < n-1$ and each B_j is an ϵ_n-flat submanifold, where ϵ_n is as in Lemma 3.4. By Lemma 3.4, there is a hyperplane L such that for each B_j and all $x \in B_j$, we have $\delta(L^\perp, T_x B_j) > \epsilon_n$. Take a cell decomposition $\mathcal{B}$ of $\mathbb{F}^n$, with respect to orthonormal coordinates in the L, $L^\perp$ axis, partitioning each B_j. Let

$$\mathcal{S} := \{C \in \mathcal{B} : \dim(C) = n-1, C \subset \cup_{j=1}^{k} B_j$$

and note that $\dim(\partial A \setminus \cup_{C \in \mathcal{S}} C) < n-1$. Furthermore,

$$BAD := \{x \in A : \pi_L^{-1}(\pi_L(x)) \cap \partial A \not\subset \cup_{C \in \mathcal{S}} C\}$$

has dimension smaller than n. To see this, let $\mathcal{D}$ be a cell decomposition of $\partial A \setminus \cup_{C \in \mathcal{S}} C$. If $x \in A$ and $\pi_L^{-1}(\pi_L(x)) \cap \partial A \not\subset \cup_{C \in \mathcal{S}} C$, then there exists $y \in \pi_L^{-1}(\pi_L(x)) \cap \partial A$ such that $y \notin \cup_{C\in\mathcal{S}} C$. Thus, there is a $D \in \mathcal{D}$ with $y \in D$, and therefore $x \in \pi_L^{-1}(\pi_L(D))$. This shows that $BAD \subset \cup_{D\in\mathcal{D}}\pi_L^{-1}(\pi_L(D))$; but $\dim(D) < n-1$ for each $D \in \mathcal{D}$ so BAD is a set of dimension strictly smaller than n.

Let $U_1, \ldots, U_l$ be the elements of $\{\pi_L(C) : C \in \mathcal{S}\}$. Then the set

$$x \in A : x \notin \pi_L^{-1}(\cup_{i=1}^l U_i)\}$$

is containes in BAD, and therefore has dimension smaller than n: suppose $x \in A \setminus \pi_L^{-1}(\cup_{i=1}^l U_i)$, since A is bounded and open $\pi_L^{-1}(\pi_L(x)) \cap \partial A \neq$, take $y \in \pi_L^{-1}(\pi_L(x)) \cap \partial A$ since $\pi_L(x) \not\cup_{C\in\mathcal{S}} \pi_L(C)$ we have $y \notin \cup_{C\i\mathcal{S}} C$, thus $x \in BAD$.

By using the inductive hypothesis, we only need to find the required partition for each of the set $A \cap \pi_L^{-1}(U_i)$, $i = 1, \ldots, l$. Fix $i \in \{1, \ldots, l\}$ and let $U := U_i$. Take $C \in \mathcal{S}$ with $\pi_L(C) = U$. Then $C = \Gamma(\phi)$ for a definable C^1-map $\phi : U \longrightarrow L^\perp$ and for all $x \in C$,

$$T_x C = \{(v, d\phi(v)) : v \in T_{\pi_L(x)} U\}.$$

Let $v \in T_{\pi_L(x)} U$ be a unit vector; since $\delta(L^\perp, T_x C) > \epsilon_n$ and $|(v, d\phi(v))| = \sqrt{1 + |d\phi(v)|^2}$, it follows from Remark 3.3 that

$$\frac{1}{2}\epsilon_n \leq \frac{1}{\sqrt{1+|d\phi(v)|^2}} |\pi_{L^\perp}((v, d\phi(v))) - (v, d\phi(v))| = \frac{1}{\sqrt{1+|d\phi(v)|^2}} |v|.$$

Therefore,

$$|d\phi(v)| \leq \sqrt{\frac{1}{4\epsilon_n^2} - 1}.$$

Let

$$M_n := \max\left\{M_{n-1}, \sqrt{\frac{1}{4\epsilon_n^2} - 1}\right\}.$$

We have proved that for each $C_j \in \mathcal{S}$ with $\pi_L(C_j) = U$ there is a definable C^1-map $\phi_j : U \longrightarrow \mathbb{F}$, such that $C_j = \Gamma(\phi_j)$ and $|d\phi_j| < M_n$.

By the inductive hypothesis, there is a partition $\mathcal{P}$ of U such that each piece $P \in \mathcal{P}$ is an M_{n-1}-C^1-cell after a change of coordinates of L.

$$A = \coprod_{\substack{P \in \mathcal{P} \\ (\phi_r, \phi_s)_S \subset A}} (\phi_r, \phi_s)_P, \tag{3.1}$$

and $(\phi_r, \phi_s)_P$ is an M_n-C^1-cell after a coordinate change.

Case II: $\dim(A) < n$. In this case, after partitioning A into cells which are ϵ_n-flat, we may assume that A is an ϵ_n-flat submanifold, where ϵ_n is as in Lemma 3.4. As in case I, there is a hyperplane L such that A is the graph of a function $f : U \longrightarrow \mathbb{F}$, $U \subset L$ and $|df| < M_n$. By the inductive hypothesis, we can partition U into M_{n-1}-C^1-cells. The graphs of f over the cells in this partition give the required partition of A. □

3.3 Lipschitz partition of definable sets

A K-tame cell is a K-C^1-cell which is also a K-Lipschitz cell (see below for definitions). An element $x \in \mathbb{F}$ is **finite** if there is an $N \in \mathbb{N}$ with $|x| < N$. We show that any pair of points x, y in a K-tame cell can be connected by a definable path γ with speed bounded by $|x - y|$ times a constant that depends on K, and is finite if K is (Lemma 3.10). Using this fact about tame cells, we can prove that K-C^1-cells are L-tame, where L is a constant depending on K, that is finite if K is; therefore any definable bounded set can be partitioned into pieces that are Lipschitz cells after a change of coordinates. We conclude by proving that $\mathbb{Q}$-bounded definable sets can be decomposed into pieces that have the WAP.

Definition 3.9. *A function $f : A \longrightarrow \mathbb{F}^n$, where $A \subset \mathbb{F}^m$, is **Lipschitz** if there is a $c \in \mathbb{F}_{>0}$ such that for every $x, y \in A$, $|f(x) - f(y)| \leq c|x - y|$. Such a constant c is a*

Lipschitz constant *for f, and f is called c-**Lipschitz**. Let $(i_1, \dots, i_m)$ be a sequence of zeros and ones and let $L \in \mathbb{F}$. An $(i_1, \dots, i_m)$-L-**Lipschitz cell** is a definable subset of $\mathbb{F}^m$ obtained by induction on m as follows:*

(i) *A (0)-L-Lipschitz cell is a point $\{r\} \subset \mathbb{F}$, a (1)-L-Lipschitz cell is an interval $(a, b) \subset \mathbb{F}$, where $a,\ b \in \mathbb{F}$.*

(ii) *An $(i_1, \dots, i_m, 0)$-L-Lipschitz cell is the graph of a definable L-Lipschitz function $f : X \longrightarrow \mathbb{F}$, where X is an $(i_1, \dots, i_m)$-L-Lipschitz cell; an $(i_1, \dots, i_m, 1)$-L-Lipschitz cell is a set*

$$(f, g)_X := \{(x, r) \in X \times \mathbb{F} : f(x) < r < g(x)\},$$

where X is an $(i_1, \dots, i_m)$-L-Lipschitz cell and $f, g : X \longrightarrow \mathbb{F}$ are definable L-Lipschitz functions on X such that for all $x \in X$, $f(x) < g(x)$.

*An L-C^1-cell is L-**tame** if the Lipschitz functions f that define the cell are L-Lipschitz.*

Suppose we have a definable C^1-function $f : S \longrightarrow \mathbb{F}$. Given a bound on $|df|$, we might expect to be able to prove that f is Lipschitz. In general this is false, but if S has the WAP, then f is Lipschitz. This observation is the key to the proof that K-C^1-cells are tame.

Lemma 3.10. *Fix $L \in \mathbb{F}_{>0}$ and $n \in \mathbb{N}_{>0}$. Then, there is a constant $K(n, L) \in \mathbb{F}_{>0}$ depending only on n and L, that is finite if L is, such that for every L-tame cell $C \subset \mathbb{F}^n$ there is a definable family of curves $\gamma \subset C^2 \times ([0, 1] \times C)$ such that: For all $x, y \in C$, $\gamma_{x,y} : [0, 1] \longrightarrow C$ is a C^1-curve with*

(i) *$\gamma_{xy}(0) = x, \gamma_{xy}(1) = y$;*

(ii) $|\gamma'_{xy}(t)| \leq K(n,L)|x-y|$, *for all* $t \in [0,1]$.

Proof. By induction on n. For $n = 1$ the lemma is clear. Take $n \geq 1$, and assume that the lemma holds for n. Let $C \subset \mathbb{F}^{n+1}$ be an L-tame cell. Then $C = \Gamma(f)$ or $C = (g,h)_X$ for some L-tame cell $X \subset \mathbb{F}^{n-1}$ and definable, C^1, L-Lipschitz functions f, g, h with $g < h$, and $|df|, |dg|, |dh| \leq L$. By induction, there are a constant $k := K(n-1, L)$ and a definable family of C^1-curves β in X with the required properties. Let $\pi_n : \mathbb{F}^{n+1} \longrightarrow \mathbb{F}^n$ be the projection onto the first n coordinates.

If $C = \Gamma(f)$, we lift β to C via f: fix $x, y \in C$ and let $\gamma_{x,y}(t) := (\alpha(t), f(\alpha(t)))$, where for all $t \in (0,1)$ $\alpha(t) := \beta_{\pi_n(x),\pi_n(y)}(t)$. Then we have

$$\begin{aligned} |\gamma'_{xy}(t)| &\leq |\alpha'(t)| + |df(\alpha'(t))| \\ &\leq (1+L)|\alpha'(t)| \leq (1+L)k|\pi_n(x) - \pi_n(y)| \leq (1+L)k|x-y|. \end{aligned}$$

If $C = (g,h)_X$, we lift β as follows: Fix $x, y \in C$ and let $\alpha := \beta_{\pi_n(x),\pi_n(y)}$. Let $\pi : \mathbb{F}^{n+1} \longrightarrow \mathbb{F}$ be the projection onto the last coordinate and take $u, v \in (0,1)$ with

$$\begin{aligned} \pi(x) &= uh(\alpha(0)) + (1-u)g(\alpha(0)) \\ \pi(y) &= vh(\alpha(1)) + (1-v)g(\alpha(1)). \end{aligned}$$

Let $l(t) := tv + (1-t)u$, for $t \in [0,1]$. We define

$$\gamma_{x,y}(t) := (\alpha(t), l(t)h(\alpha(t)) + (1-l(t))g(\alpha(t))),$$

and note that

$$\begin{aligned} |\gamma'_{xy}(t)| &\leq |\alpha'(t)| + |l'(t)h(\alpha(t)) + l(t)dh(\alpha'(t)) - l'(t)g(\alpha(t)) + (1-l(t))dg(\alpha'(t))| \\ &\leq k|x-y| + |(v-u)(h(\alpha(t)) - g(\alpha(t)))| + 2Lk, \end{aligned}$$

since $l(t), 1 - l(t)$ are between 0 and 1 and $|dh(\alpha'(t))|, |dg(\alpha'(t))| \leq L|\alpha'(t)|$. Let $f := h - g$; we want to bound $|(v-u)f(\alpha(t))|$. Let $a := \alpha(0)$, $b := \alpha(1)$; then

$$\begin{aligned}\pi(x) - \pi(y) &= vf(b) - uf(a) + g(b) - g(a) \\ &= (v-u)f(\alpha(t)) + v(f(b) - f(\alpha(t))) - u(f(a) - f(\alpha(t))) \\ &+ g(b) - g(a).\end{aligned}$$

So

$$\begin{aligned}(v-u)f(\alpha(t)) &= \pi x - \pi y - v(f(b) - f(\alpha(t))) + u(f(a) - f(\alpha(t))) \\ &+ g(a) - g(b).\end{aligned}$$

But

$$\begin{aligned}|f(b) - f(\alpha(t))| &\leq 2L|b - \alpha(t)| = 2L|1-t|\left|\frac{\alpha(1) - \alpha(t)}{1-t}\right| \\ &\leq 2L|\alpha'(t_0)|\end{aligned}$$

for some t_0 between t and 1. Similarly, there is a t_1 between t and 1 such that

$$|f(a) - f(\alpha(t))| \leq 2L|\alpha'(t_1)|.$$

Since $u, v \in [0,1]$, we get

$$\begin{aligned}|(v-u)f(\alpha(t))| &\leq |\pi y - \pi x| + 4Lk|x-y| + L|a-b| \\ &\leq |x-y| + 4Lk|x-y| + L|x-y|;\end{aligned}$$

thus $|\gamma'_{xy}(t)| \leq K(n,L)|x-y|$ for some constant $K(n,L)$ depending only on n and L which is finite if L is. The collection of the curves γ_{xy} for $x, y \in C$ constitutes the required family of curves. $\square$

Theorem 3.11. *Let $L > 0$, and let $C \subset \mathbb{F}^n$ be a L-C^1-cell. Then C is a $k(n, L)$-tame cell, where $k(n, L)$ depends only on n, L, and is finite if L is.*

Proof. By induction on n; the theorem is clear for $n = 1$. Assume that $n > 1$ and that the theorem holds for $n - 1$. Then $C = \Gamma(f)$ or $C = (g, h)_X$, where $X \subset \mathbb{F}^{n-1}$ is a $k(n-1, L)$-tame cell and f, g, h are C^1-functions on X such that $|df|, |dg|, |dh| \leq L$. We need to show that f, g, h are Lipschitz.

Since X is a k-tame cell, $k := k(n-1, L)$, it follows from Lemma 3.10 that there is a constant $K(n-1, k)$ such that whenever $x, y \in X$, there is a definable, C^1-curve γ joining x and y with $|\gamma'(t)| \leq K(n-1, k)|x - y|$ for all $t \in [0, 1]$. Let $g := f \circ \gamma$; then there is a $t_0 \in (0, 1)$ such that

$$\begin{aligned} |f(x) - f(y)| &= |g(1) - g(0)| = |g'(t_0)| \\ &= |df(\gamma'(t_0))| \leq L|\gamma'(t_0)| \leq LK(n-1, k)|x - y|. \end{aligned}$$

showing that f is $LK(n-1, k)$-Lipschitz. Similarly, g, h are $LK(n-1, k)$-Lipschitz, so $k(n, L) := LK(n-1, k)$ is the desired constant. □

Definition 3.12. *A definable set $A \subset \mathbb{F}^n$ has the **Whitney Arc Property** (WAP) if there is a constant $K \in \mathbb{F}_{>0}$ such that for every $x, y \in A$ there is a definable curve $\gamma : [0, 1] \longrightarrow A$ with $\gamma(0) = x$, $\gamma(1) = y$ and* $\text{length}(\gamma) < K|x - y|$.

Remark 3.13. *Theorem 3.11 shows that the assumption of Lipschitzness in the definition of a tame cell is redundant, but for it we used the fact that tame cells have the WAP (Lemma 3.10).*

Corollary 3.14. *Let $A \subset \mathbb{F}^n$ be a definable set. Then there is a partition $A = \cup_{i=1}^{s}$ such that for each i the set A_i is definable and there is a change of coordinates $\sigma_i \in O_n(\mathbb{F})$ with $\sigma_i(A_i)$ an L-Lipschitz cell, where $L \in \mathbb{Q}_{>0}$ is a constant depending only on n.*

Proof. By Theorem 3.11, this follows from Theorem 3.8. □

Recall that $A \subset \mathbb{F}^n$ is $\mathbb{Q}$-bounded if there is a $q \in \mathbb{Q}_{>0}$ such that $A \subset [-q, q]^n$.

Theorem 3.15. *Let $A \subset \mathbb{F}^n$ be a $\mathbb{Q}$-bounded, definably connected definable set. Then there is a $K \in \mathbb{Q}_{>0}$, which depends only on n, and definable pairwise disjoint sets A_i, $i = 1, \ldots, s$ such that*

$$A = \bigcup_{i=1}^{s} A_i$$

and for each i there is a definable family of curves

$$\lambda^i \subset A_i^2 \times ([0, 1] \times A_i)$$

with the property that for any pair of points $x, y \in A_i$, $\lambda^i_{x,y}$ is a curve in A_i joining x and y with $\text{length}(\lambda^i_{x,y}) \leq K|x - y|$. *In particular, A_i has the WAP.*

Proof. Let $A = \cup A_i$ be as in Theorem 3.8 so that for each A_i there is a $\sigma \in O_n(\mathbb{F})$ with $\sigma(A_i)$ a M_n-C^1-cell. By Lemma 3.10 and Theorem 3.11 there is a constant K, depending only on n and finite since M_n is, and a definable family of curves γ in $\sigma(A_i)$ with $|\gamma'_{xy}| \leq K|x - y|$ for all $x, y \in \sigma(A_i)$, therefore

$$\text{length}(\gamma_{xy}) = \int_0^1 |\gamma'_{xy}| \leq K|x - y|.$$

Since $\sigma \in O_n(\mathbb{F})$, A_i has the required family of curves. □

3.4 Lipschitz cells are uniformly locally connected

Next we will introduce *uniformly locally connected sets*. We prove that Lipschitz cells are uniformly locally connected. In the next chapter, we will use the Cauchy-Crofton formula to prove that any ULC set has the WAP. But the ULC property is of independent topological interest. There is a rough analogy between the ULC property for definable sets and the normality property of algebraic varieties, which is related to Zariski's main theorem and obstructions to extending rational maps (Mumford [7]).

Definition 3.16. *Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in (0, \infty)^n$. The α-**box centered at** a is the open box*

$$B(a, \alpha) := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in (a_i - \frac{\alpha_i}{2}, a_i + \frac{\alpha_i}{2}) \right\}.$$

Definition 3.17. *A definable set $U \subset \mathbb{F}^n$ is **uniformly locally connected** if there is an $\alpha = (\alpha_1, \dots, \alpha_n) \in (0,1)^n$ such that for every $u \in \overline{U}$ and every $\delta \in (0,1)$, the set $U \cap B(u, \delta\alpha)$ is definably connected.*

Lemma 3.18. *Let $C \subset \mathbb{F}^n$ be a definable set, $k > 0$, and $f : C \longrightarrow \mathbb{F}$ a k-Lipschitz, definable function on C. Then f has a definable k-Lipschitz extension $\overline{f} : \overline{C} \longrightarrow \mathbb{F}$.*

Proof. Let $c \in \partial C$. By curve selection (Chapter 2, fact 2.1), there is a definable, continuous curve $\gamma : (0,1) \longrightarrow C$, such that $\lim_{t \longrightarrow 0} \gamma(t) = a$. Now $\lim_{t \longrightarrow 0} f(\gamma(t))$ exists in $\mathbb{F} \cup \{\pm\infty\}$, and since f is Lipschitz $\lim_{t \longrightarrow 0} f(\gamma(t))$ must actually be in $\mathbb{F}$. Suppose $\alpha : (0,1) \longrightarrow C$ is another definable, continuous curve with $\lim_{t \longrightarrow 0} \alpha(t) = c$. If $l_1 := \lim_{t \longrightarrow 0} f(\gamma(t))$ does not equal $l_2 := \lim_{t \longrightarrow 0} f(\alpha(t))$ let $\epsilon := |l_1 - l_2|/2$. Let $\delta > 0$ be such that $|f(\gamma(t) - l_1| < \epsilon/2$ and $|f(\alpha(t)) - l_2| < \epsilon/2$ for all $t \in (0, \delta)$. Then, for $t \in (0, \delta)$,

$$\begin{aligned} 2\epsilon &= |l_1 - l_2| \\ &\leq |f(\gamma(t)) - l_1| + |f(\alpha(t) - l_2| + |f(\alpha(t)) - f(\gamma(t))| \\ &< \epsilon + |f(\alpha(t)) - f(\gamma(t))| \end{aligned}$$

That is $|f(\alpha(t)) - f(\gamma(t))| > \epsilon$, but this is impossible since $\gamma(t), \alpha(t)$ converge to c as t goes to 0 and f is Lipschitz. We define

$$\overline{f}(c) := \lim_{t \longrightarrow 0} f(\gamma(t)).$$

To see that $\overline{f}$ is Lipschitz, let $c, d \in \overline{C}$, let α, γ be continuous, definable curves in C which converge to c, d respectively as $t \longrightarrow 0$, then

$$\begin{aligned} |\overline{f}(c) - \overline{f}(d)| &= \lim_{t \longrightarrow 0} |f(\alpha(t)) - f(\gamma(t))| \\ &\leq \lim_{t \longrightarrow 0} k|\alpha(t) - \gamma(t)| \\ &= k|c - d|, \end{aligned}$$

as required. □

In what follows, $\pi_n : \mathbb{F}^{n+1} \longrightarrow \mathbb{F}^n$ will denote the projection from $\mathbb{F}^{n+1}$ onto the first n coordinates and $\pi : \mathbb{F}^{n+1} \longrightarrow \mathbb{F}$ the projection onto the last coordinate.

Lemma 3.19. *Let $X \subset \mathbb{F}^n$ be a definable set, $f, g, h : X \longrightarrow \mathbb{F}$ definable, continuous functions with continuous extensions $\overline{f}, \overline{g}, \overline{h} : \overline{X} \longrightarrow \mathbb{F}$. Assume that $g(x) < h(x)$ for all $x \in X$.*

(i) *If $C = \Gamma(f)$, then $\partial C = \Gamma(\overline{f}\big|_{\partial X})$.*

(ii) *If $C = (g, h)_X$, then*

$$\begin{aligned} \partial C =& \{x \in \mathbb{F}^{n+1} : \pi_n(x) \in \partial X \text{ and } \overline{g}(\pi_n(x)) \leq \pi(x) \leq \overline{h}(\pi_n(x))\} \\ &\cup \Gamma(h) \cup \Gamma(g). \end{aligned}$$

Proof. For (i), let $(x, \overline{f}(x)) \in \Gamma(\overline{f}\big|_{\partial X})$. By curve selection (Chapter 2, fact 2.1), there is a definable, continuous curve $\gamma : (0,1) \longrightarrow X$ with $\gamma(t) \longrightarrow x$ as $t \longrightarrow 0$. Thus

$f(\gamma(t)) \longrightarrow \overline{f}(x)$ as $t \longrightarrow 0$, that is $(x, \overline{f}(x)) \in \overline{C}$. It is clear that $(x, \overline{f}(x)) \notin C$. Suppose now that $c \in \partial C$, and let $\gamma : (0,1) \longrightarrow C$ be a definable, continuous curve in C converging to c as $t \longrightarrow 0$. Let

$$x := \lim_{t \longrightarrow 0} \pi_n(\gamma(t)),$$

then

$$(x, \overline{f}(x)) = (\lim_{t \longrightarrow 0} \pi_n(\gamma(t)), \lim_{t \longrightarrow 0} f(\pi_n(\gamma(t)))) = (\pi_n(c), \pi(c)) = c,$$

and $x \notin X$ since $c \notin C$.

Now we prove (ii). $\Gamma(h), \Gamma(g)$ are contained in ∂C. Thus (i) implies that $\Gamma(\overline{g}\big|_{\partial X}) = \partial(\Gamma(g)) \subset \overline{\Gamma(g)} \subset \overline{C}$, moreover, $\Gamma(\overline{g}\big|_{\partial X}) \subset \partial C$, and similarly for h. Let $x \in \mathbb{F}^{n+1}$ with $\pi_n(x) \in \partial X$, and $\overline{g}(\pi_n(x)) \leq \pi(x) \leq \overline{h}(\pi_n(x))$. We may assume the inequalities are strict. Let

$$t_0 := \frac{\pi(x) - \overline{g}(\pi_n(x))}{\overline{h}(\pi_n(x)) - \overline{g}(\pi_n(x))},$$

and let $\gamma : (0,1) \longrightarrow X$ be a definable, continuous curve in X converging to $\pi_n(x)$ as $t \longrightarrow 0$, then $t_0 \in (0,1)$, and $t \longrightarrow (\gamma(t), t_0 h(\gamma(t)) + (1 - t_0) g(\gamma(t)))$ is a curve in C converging to $(\pi_n(x), t_0 \overline{h}(\pi_n(x)) + (1 - t_0)\overline{g}(\pi_n(x))) = x$ as $t \longrightarrow 0$. For the other inclusion, let $c \in \partial C$, then $\pi_n(c) \in \overline{X}$. Moreover if $\gamma : (0,1) \longrightarrow C$ is a definable, continuous curve in C converging to c as $t \longrightarrow 0$, then $\overline{g}(\pi_n(\gamma(t))) < \pi(\gamma(t)) < \overline{h}(\pi_n(\gamma(t)))$. Taking the limit as $t \longrightarrow 0$ we get $\overline{g}(\pi_n(c)) \leq \pi(c) \leq \overline{h}(\pi_n(c))$. Assume that $\pi_n(c) \in X$, since $c \notin C$, we have $\pi(c) \notin (g(\pi_n(c)), h(\pi_n(c)))$, thus $c \in \Gamma(h) \cup \Gamma(g)$. If $\pi_n(c) \in \partial X$, then c belongs to the right hand side of the equation in (ii). $\square$

Theorem 3.20. *Every Lipschitz cell $C \subset \mathbb{F}^n$ is uniformly locally connected.*

Proof. For $n = 1$ the theorem is obvious. Assume now that $n > 1$ and the theorem holds for $i = 1, \ldots, n$. Let $C \subset \mathbb{F}^{n+1}$ be a Lipschitz cell. Then, there is a Lipschitz cell

$X = \pi_n(C) \subset \mathbb{F}^n$ such that either $C = \Gamma(f)$ where f is a definable, c-Lipschitz function on X, or $C = (g,h)_X$, where $g(x) < h(x)$ for all $x \in X$, g is a definable, c_1-Lipschitz function on X and h is a definable, c_2-Lipschitz function on X; in this case, we let $c = \max\{c_1, c_2\}$. By Lemma 3.18, f, g, h extend to c-Lipschitz functions $\overline{f}, \overline{g}$ and $\overline{h}$ on $\overline{X}$, respectively.

By the inductive hypothesis, X is uniformly locally connected, so there is a tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of positive reals, such that for all $x \in \overline{X}$ and $\delta \in (0,1)$, the set $X \cap B(x, \delta\alpha)$ is definably connected. Let $d > c\max\{\alpha_1, \ldots, \alpha_n\}$, $\delta \in (0,1)$, $a \in \overline{C}$ and B be the box $B(a, \delta(\alpha, d))$. To finish the proof of the theorem it is enough to prove that $B \cap C$ is definably path connected.

Claim 3.21. *If $C = \Gamma(f)$, and $x \in C \cap \partial B$, then $\pi(x) \in (\pi(a) - \delta d/2, \pi(a) + \delta d/2)$.*

Proof. By Lemma 3.19, $\overline{C} = \Gamma(\overline{f})$, thus since $a \in \overline{C}$, $\pi(a) = \overline{f}(\pi_n(x))$. Also, since $x \in \partial B$, $\pi_n(x) \in \overline{B(\pi_n(a), \alpha\delta)}$, so that $|\pi_n(x) - \pi_n(a)| \leq \frac{\delta}{2}\max\{\alpha_j\}$. If $\pi(x) \notin (\pi(a) - \frac{\delta d}{2}, \pi(a) + \frac{\delta d}{2})$ then $|f(\pi_n(x)) - \overline{f}(\pi_n(a))| = |\pi(x) - \pi(a)| \geq \frac{\delta d}{2}$. Therefore,

$$\begin{aligned} \frac{|f(\pi_n(x)) - \overline{f}(\pi_n(a))|}{|\pi_n(x) - \pi_n(a)|} &\geq \frac{\frac{\delta d}{2}}{|\pi_n(x) - \pi_n(a)|} \\ &\geq \frac{d}{\max\{\alpha_1, \ldots, \alpha_n\}} > c, \end{aligned}$$

contradicting that $\overline{f}$ is c-Lipschitz. □

Since the image of a closed, bounded, definable set under a continuous, definable map is closed and bounded ([10], Chapter 6, 1.10), $\pi_n(\overline{C}) = \overline{\pi_n(C)}$. Thus $\pi_n(a) \in \overline{X}$. Also, $\pi_n(B)$ is the box $B(\pi_n(a), \delta\alpha) \subset \mathbb{F}^n$. Furthermore, for x_1, $x_2 \in B \cap C$, $\pi_n(x_1)$, $\pi_n(x_2) \in X \cap \pi_n(B)$. By induction X is uniformly locally connected, so there is a definable path $\gamma : [0,1] \longrightarrow X \cap \pi_n(B)$ joining $\pi_n(x_1)$, and $\pi_n(x_2)$. We will construct a definable path in $C \cap B$ between x_1, and x_2, showing that $C \cap B$ is definably path connected.

Case I: $C = \Gamma(f)$. $f \circ \gamma$ is a definable path in C joining x_1, and x_2. If $f \circ \gamma([0,1]) \not\subseteq B$, then there is an $x \in X \cap \pi_n(B)$ such that $f(x) \notin (\pi(a) - \frac{\delta d}{2}, \pi(a) + \frac{\delta d}{2})$. Without loss of generality, assume $f(x) \geq \pi(a) + \frac{\delta d}{2}$. Since $f \circ \gamma$ is continuous and definable, the value $\pi(a) + \frac{\delta d}{2}$ should be achieved, but then there will be an $x \in X \cap \pi_n(B)$ with $(x, f(x)) \in \partial B \cap C$, and $\pi((x, f(x))) = f(x) \notin (\pi(a) - \frac{\delta d}{2}, \pi(a) + \frac{\delta d}{2})$, contradicting the previous claim. Thus, $f \circ \gamma([0,1]) \subset B$, and $f \circ \gamma$ is a definable path in $C \cap B$ joining x_1, and x_2 as wanted.

Case II: $C = (g, h)_X$. Recall that B is the $\delta(\alpha, d)$-box $B(a, \delta(\alpha, d))$. Let $l, u : X \cap \pi_n(B) \longrightarrow \mathbb{F}$ be the functions

$$l(x) := \max\{g(x), \pi(a) - \frac{\delta d}{2}\}, \qquad u(x) := \min\{h(x), \pi(a) + \frac{\delta d}{2}\}.$$

Claim 3.22. *For all $x \in X \cap \pi_n(B)$, $u(x) > l(x)$.*

Proof. Suppose there is an $x \in X \cap \pi_n(B)$ with $u(x) = l(x)$. Since $g(x) \neq h(x)$, we must have that either $u(x) = h(x)$, and $l(x) = \pi(a) - \frac{\delta d}{2}$, or $l(x) = g(x)$, and $u(x) = \pi(a) + \frac{\delta d}{2}$. Assume the first possibility. Consider the box $B(a_1, \delta(\alpha, d))$ around $a_1 = (\pi_n(a), \overline{h}(\pi_n(a))) \in \overline{\Gamma(h)}$, and note that this box projects onto $\pi_n(B)$, and that by lemma 3.19 (ii), $\pi(a) \leq \overline{h}(\pi_n(a))$. By assumption the function $h\big|(\pi_n(B) \cap X)$ takes the value $l(x) = \pi(a) - \delta d/2$, as well as values arbitrarily close to $\overline{h}(\pi_n(a))$; but $\pi(a) - \delta d/2 \leq \overline{h}(\pi_n(a)) - \delta d/2 < \overline{h}(\pi_n(a))$ and h is a continuous, definable function on the definably connected set $X \cap \pi_n(B)$ so the value $\overline{h}(\pi_n(a)) - \delta d/2$ must be achieved, say at a point $y \in X \cap \pi_n(B)$. We can now apply the previous claim to $x_1 = (y, \overline{h}(\pi_n(a)) - \delta d/2) \in \Gamma(h) \cap \partial B_1$, where B_1 is the box $B(a_1, \delta(\alpha, d))$, to obtain that $\overline{h}(\pi_n(a)) - \delta d/2 \in$

$(\overline{h}(\pi_n(a)) - \delta d/2, \overline{h}(\pi_n(a)) + \delta d/2)$, a contradiction. The second possibility is handled in a similar manner. Thus for all $x \in X \cap \pi_n(B)$, $u(x) \neq l(x)$, but u, l are continuous and definable, therefore the sets

$$\{x \in X \cap \pi_n(B) : \; u(x) > l(x)\}, \qquad \{x \in X \cap \pi_n(B) : \; u(x) < l(x)\},$$

are open and definable. $X \cap \pi_n(B)$ is their union. Since B is a box with center at a point of $\overline{C}$, we can find c with $c \in B \cap C$. Then $l(c) < \pi(c) < u(c)$, but $X \cap \pi_n(B)$ is definably connected, so

$$X \cap \pi_n(B) = \{x \in X \cap \pi_n(B) : \; u(x) > l(x)\}$$

as wanted. $\square$

Let $x \in X \cap \pi_n(B)$, and let $r := l(x) + 1/2(u(x) - l(x))$. Then

$$\begin{aligned} g(x),\ \pi(a) - \frac{\delta d}{2} &\leq l(x) \\ &< r < u(x) \leq \pi(a) + \frac{\delta d}{2},\ h(x), \end{aligned}$$

so that $(x, r) \in C \cap B$, and $\pi_n((x, r)) = x$. Thus we have $X \cap \pi_n(B) = \pi_n(C \cap B)$. From this we get that

$$\begin{aligned} &C \cap B = \\ &\{x \in \mathbb{F}^{n+1} : \pi_n(x) \in \pi_n(B \cap C), \text{ and } l(\pi_n(x)) < \pi(x) < u(\pi_n(x))\} = \\ &\{x \in \mathbb{F}^{n+1} : \pi_n(x) \in X \cap \pi_n(B), \text{ and } l(\pi_n(x)) < \pi(x) < u(\pi_n(x))\}. \end{aligned}$$

Let $F(x, t) := (x, tu(x) + (1 - t)l(x))$. F is continuous and by Lemma 3.19, $F : (X \cap \pi_n(B)) \times [0, 1] \longrightarrow \overline{C \cap B}$. Let t be such that $F(\pi_n(x_1), t) = x_1$, for instance,

$$t := \frac{\pi(x_1) - l(\pi_n(x_1))}{u(\pi_n(x_1)) - l(\pi_n(x_1))}.$$

Join x_1 to $F(\pi_n(x_2), t)$ by the path $F_t \circ \gamma$, where $F_t : X \cap \pi_n(B) \longrightarrow C \cap B$ is given by $F_t(x) := F(x, t)$. Then join $F(\pi_n(x_2), t)$ to x_2 by a vertical segment. This gives a definable path in $C \cap B$ joining x_2 and x_1 as wanted. $\square$

Chapter 4

The Cauchy-Crofton formula

4.1 Motivations

In this chapter we define the length of a definable $\mathbb{Q}$-bounded curve. The curves in the family of Lemma 3.10 have bounded derivatives. In general, the curves in an arbitrary bounded family of curves may not have bounded derivatives, but we will use the Cauchy-Crofton formula to show that they have bounded length. We also state a generalization of the Cauchy-Crofton formula by using the Berarducci-Otero measure and prove it by reducing to the real case.

4.2 The length of a definable $\mathbb{Q}$-bounded curve.

We reparametrize any definable $\mathbb{Q}$-bounded curve by a piecewise C^1 map. This particular reparametrization satisfies other properties that will allow us to reduce the generalized Cauchy-Crofton formula to the real case.

Recall that $A \subset \mathbb{F}^n$ is $\mathbb{Q}$-bounded if it is contained in a box with rational coordinates. An element x of $\mathbb{F}^n$ is **finite** if it is bounded in magnitude by some natural number, **infinite** otherwise and **infinitesimal** if $|x| < r$ for every $r \in \mathbb{R}_{>0}$ in this case we write $x \approx 0$. For a finite $x \in \mathbb{F}$ the **monad** of x, $\mu(x)$ consists of all $y \in \mathbb{F}$ with $y - x \approx 0$, we write $y \approx x$ for $y \in \mu(x)$ and we say that y is **infinitesimally close** to x. For finite

$x \in \mathbb{F}$ the **standard part** of x is $st(x) := \sup\{r \in \mathbb{R} : r < x\}$. If $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ is finite, $st(x) := (st(x_1), \ldots, st(x_n))$.

Lemma 4.1. *Let $f : (a, b) \longrightarrow (c, d)$ be a definable and twice differentiable function, where $(a, b),\ (c, d) \subset \mathbb{F}$ are $\mathbb{Q}$-bounded. Then f' is finite outside a finite union of monads.*

Proof. For $r \in \mathbb{F}_{>0}$, let

$$A_r := \{x \in (a, b) : |f'(x)| > 1/r\}.$$

This is a definable family of sets. By the mean value theorem, any interval contained in A_r, $r \approx 0$, is of infinitesimal length. Thus A_r, $r \approx 0$ is contained in a finite union of monads, let n_r be the minimum number of monads containing A_r. By Chapter 2, Fact 2.5, there is an $N \in \mathbb{N}$ such that A_r, $r \in \mathbb{F}_{>0}$, is a union of at most N disjoint intervals and points; let $s \approx 0$ be such that

$$n_s = \max_{r \approx 0} n_r,$$

and let A be the finite union of the n_s monads containing A_s. For $r < s$, $A_r \subset A_s$ so $A_r \subset A$. For $r > s$, $r \approx 0$, $A_s \subset A_r$ so n_s is at most n_r; since n_s is maximal, $n_r = n_s$ and therefore A_r must be contained in A. f' is finite away from A. □

For $(a, b) \subset \mathbb{F}$, $(a, b)_{\mathbb{R}}$ consists of the reals numbers between a and b, notice this may be an open, closed or half closed interval. For a function $f : (a, b) \longrightarrow \mathbb{F}^n$, $(a, b) \subset \mathbb{F}$, with $\mathbb{Q}$-bounded image we define $\overline{f} : (a, b)_{\mathbb{R}} \longrightarrow \mathbb{R}$ by $\overline{f}(x) = st(f(x))$. Similarly, for a function $f : A \longrightarrow \mathbb{F}^n$ of m variables which maps finite elements into finite elements, we define $\overline{f} : A_{\mathbb{R}} \longrightarrow \mathbb{R}^n$ by $\overline{f}(x) = st(f(x))$ where $A_{\mathbb{R}} = A \cap \mathbb{R}^m$.

Lemma 4.2. *Let $f : (a, b) \longrightarrow (c, d)$ be a definable and differentiable function, where $(a, b),\ (c, d) \subset \mathbb{F}$ are $\mathbb{Q}$-bounded. Suppose that for $x \not\approx a, b$ both f' and f'' are finite. Then*

$\overline{f}$ *is differentiable on the interior of* $(a,b)_{\mathbb{R}}$ *and for* $x \in (a,b)$ *with* $st(x) \in \mathrm{Int}((a,b)_{\mathbb{R}})$, $st(f'(x)) = \overline{f}'(st(x))$.

Proof. We first consider the case where $0 \in (a,b)$ but $0 \not\approx a, b$, $f(0) = 0$, and $f'(0) = 0$. Let $\epsilon \in \mathbb{R}_{>0}$. If $\delta \approx 0$ and $\delta > 0$ then

$$|\frac{f(h)}{h}| < \epsilon$$

whenever $|h| < \delta$. Otherwise there would be a $\delta > 0$, $\delta \approx 0$ and $h \in (a,b)$, $|h| < \delta$ with

$$|\frac{f(h)}{h}| > \epsilon,$$

thus by the mean value theorem there is an x between 0 and h such that

$$|f'(x)| = |\frac{f(h)}{h}| > \epsilon,$$

and a z between 0 and x with

$$|f''(z)| = |\frac{f'(x)}{x}|$$

but this last fraction is infinite. This shows that the set

$$\{\delta \in \mathbb{F}_{>0} : \text{ for all } h \in (a,b), |h| < \delta \implies |\frac{f(h)}{h}| < \epsilon/2\}$$

contains all positive infinitesimals. This set is also definable, so by the cell decomposition theorem it is a finite union of intervals and points and therefore it must contain a positive real δ. This shows that $\overline{f}$ is differentiable at 0 and $\overline{f}'(0) = 0$.

For x_0 in (a,b) with $st(x_0) \in \mathrm{Int}((a,b)_{\mathbb{R}})$, consider the function

$$g(x) := f(x_0 - x) - f(x_0) - f'(x_0)x.$$

Since $g(0) = 0$, $g'(0) = 0$ and 0 is not infinitesimally close to the endpoints of $\mathrm{Dom}(g)$, $\overline{g}$ is differentiable at 0 and $\overline{g}'(0) = 0$. It follows that $\overline{f}$ is differentiable at $st(x_0)$ with derivative $st(f'(x_0))$. ∎

Theorem 4.3. *Let $\gamma : (a, b) \longrightarrow \mathbb{F}^n$ be a definable curve with $\mathbb{Q}$-bounded image. Then, there are $a_0 = a < \cdots < a_k = b$ such that each restriction $\gamma\big|(a_i, a_{i+1})$ is either constant or has a reparametrization σ with σ' finite, $\sigma''(x)$ finite for $x \not\approx a_i, a_{i+1}$, and with $\overline{\sigma}$ an embedded C^1-curve in $\mathbb{R}^n$.*

Proof. By the C^1-cell decomposition theorem γ is piecewise C^1, so without loss of generality we can assume that γ is C^1. Also $\gamma' = 0$ in a finite union of intervals and points, and γ is constant on those intervals where $\gamma' = 0$; thus we may assume that $\gamma' \neq 0$. Similarly we can assume that γ is injective.

$Gr_1(\mathbb{F}^n)$ is the disjoint union of the definable sets

$$A_i := \{l \in Gr_1(\mathbb{F}^n) : l = \langle v \rangle,\ |v_i| \geq |v_j| \text{ for } j \geq i, \text{ and } |v_i| > |v_j| \text{ for } j < i\}.$$

Let $\phi : (a, b) \longrightarrow Gr_1(\mathbb{F}^n)$ be the Gauss map of γ. The sets $\phi^{-1}(A_i)$ are definable, and therefore are a union of intervals and points. Suppose that I is one of these intervals and let $J := \gamma_i(I)$. Since $\gamma_i' \neq 0$ on I, J contains an interval; and since J is a finite union of intervals and points the intermediate value theorem shows that J is a single interval. Moreover, J is $\mathbb{Q}$-bounded because γ is. We define $\sigma_I : J \longrightarrow \mathbb{F}^n$ as $\sigma_I := \gamma \circ \gamma_i^{-1}$. σ_I is a C^1 function since $\gamma_i|I$ is invertible with C^1 inverse. Moreover, σ_I' is finite: for $x \in J$, $\sigma_I'(x)$ generates the line $\langle \gamma'(\gamma_i^{-1}(x)) \rangle \in A_i$ thus $(\sigma_I)_i'(x) \geq (\sigma_I)_j'(x)$, but $(\sigma_I)_i'(x) = 1$.

By Lemma 4.1, there are points $b_0, \ldots, b_k$ such that $J = (b_0, b_k)$, σ_I'' and σ_I''' exist on (b_i, b_{i+1}) and are finite except possibly on the monads of b_i and b_{i+1}. Lemma 4.2 shows that for the restriction σ of σ_I to one of this subintervals $\overline{\sigma}$, $\overline{\sigma'}$ are differentiable and $\overline{\sigma}' = st(\sigma')$, $\overline{\sigma'}' = st(\sigma'')$. Since $st(\sigma') = \overline{\sigma'}$ it follows that $\overline{\sigma}$ is twice differentiable. Finally, $(\overline{\sigma})_i(t) = t$, therefore for $(c, d) \subset \mathrm{Dom}(\overline{\sigma})$ we have

$$\overline{\sigma}((c, d)) = \{x \in \mathbb{R}^n : c < x_i < d\} \cap \mathrm{Im}(\overline{\sigma})$$

showing that $\overline{\sigma}$ is an embedding. $\square$

4.3 The integral on $AGr_{n-1}(\mathbb{F}^n)$

We will define a top form on a big open subset of $AGr_{n-1}(\mathbb{F}^n)$. This will be done by choosing a top form at a given point and translating it by using a section of the action map of the group of motions on $AGr_{n-1}(\mathbb{F}^n)$. The integral on $AGr_{n-1}(\mathbb{F}^n)$ is then defined by integration of this top form, for $\mathbb{F} = \mathbb{R}$ the integral is the one invariant under the group of motions, that is, it is the one for which the Cauchy-Crofton's formula (Chapter 2, Theorem 2.13) holds.

Recall from Chapter 2 that to every hyperplane l in $\mathbb{F}^n$ there corresponds the affine orthogonal projection p_l onto l, and the map $l \longrightarrow p_l$ is a bijection from $AGr_{n-1}(\mathbb{F}^n)$ onto the set of affine orthogonal projections of rank $n-1$. In coordinates, the set of affine linear maps is $\mathbb{F}^n \times M_{n\times n}(\mathbb{F})$ and the set of affine orthogonal projections of rank $n-1$ is:

$$AGr_{n-1}(\mathbb{F}^n) = \{(b, A) \in \mathbb{F} \times M_{n\times n}(\mathbb{F}) : Ab = 0,\ A^2 = A,\ rk(A) = n-1,\ \text{and } A^t = A\}.$$

In this way we regard the affine Grassmannian as an algebraic variety, and hence as a definable set.

The group of motions of $\mathbb{F}^n$, $G = \mathbb{F}^n \rtimes O_n(\mathbb{F})$, acts transitively on $AGr_{n-1}(\mathbb{F}^n)$ by

$$g \cdot l := \{gx : x \in l\}$$

for l a hyperplane in $\mathbb{F}^n$. Define $g \cdot p_l := p_{g\cdot l}$. Then we can check that $g \cdot p_l$ is the composition of functions gp_lg^{-1}: clearly $(gp_lg^{-1})^2 = gp_lg^{-1}$. Also if we write $p_l = (a, A)$ and $g = (b, B)$, where $a, b \in \mathbb{F}^n$, $B \in O_n(\mathbb{F})$, and A is a rank $n-1$ matrix with

$Aa = 0$, $A^2 = A$ and $A^t = A$. Then $gp_lg^{-1} = (c, BAB^{-1})$ for some $c \in \mathbb{F}^n$, thus $(BAB^{-1})^t = BAB^{-1}$ so

$$\ker(gp_lg^{-1} - gp_lg^{-1}(0)) \perp \mathrm{Im}(gp_lg^{-1} - gp_lg^{-1}(0)).$$

Therefore gp_lg^{-1} is an orthogonal projection. Since $\mathrm{Im}(gp_lg^{-1}) = g \cdot l$, we get

$$gp_lg^{-1} = p_{g \cdot l}. \tag{4.1}$$

We define $\tau : U \longrightarrow G$, where

$$U := \{(b, (A_1 \dots A_n)) \in AGr_{n-1}(\mathbb{F}^n) : b \neq 0 \text{ and } rk(A_1 \dots A_{n-1}) = n - 1\}$$

by

$$\tau(b, (A_1 \dots A_n)) = (b, (GS(A_1, \dots, A_{n-1}), \frac{b}{|b|})).$$

$GS(A_1, \dots, A_{n-1})$ is the n by $n-1$ matrix whose column vectors are those obtained by applying the Gram-Schmidt process to the vectors $A_1, \dots, A_{n-1}$ and

$$(GS(A_1, \dots, A_{n-1}), \frac{b}{|b|})$$

is the $n \times n$ matrix with $\frac{b}{|b|}$ as the last column. τ is an algebraic map. Note that U is an open subset of $AGr_{n-1}(\mathbb{F}^n)$ with lower dimensional complement.

Let $l_0 := \langle e_1, \dots, e_{n-1} \rangle$, and denote p_{l_0} by p_0. If $\pi : G \longrightarrow AGr_{n-1}(\mathbb{F}^n)$ is the map

$$\pi(g) = g \cdot p_0,$$

then $\pi \circ \tau = id|_U$, in other words τ is a section for the action of G in $AGr_{n-1}(\mathbb{F}^n)$. For, by equation (4.1), if l is such that $p = p_l$, then $\tau(p) \cdot p_0 = p$ if and only if $\tau(p) \cdot l_0 = l$ and this last equation is clear. Let H be the stabilizer of l_0. Since G is acting transitively on $AGr_{n-1}(\mathbb{F}^n)$ we can view $AGr_{n-1}(\mathbb{F}^n)$ as the quotient G/H, in particular $AGr_{n-1}(\mathbb{F}^n)$

is a non-singular subvariety of $\mathbb{F}^{n(n+1)}$. The map π above is precisely the projection $\pi : G \longrightarrow G/H$.

Now we define a top form in U. Let $\omega_1, \ldots, \omega_n$ be a basis of the dual of the tangent space to $AGr_{n-1}(\mathbb{R}^n)$ at p_0. Viewing $AGr_{n-1}(\mathbb{R}^n)$ as a subset of $AGr_{n-1}(\mathbb{F}^n)$, $\omega_1, \ldots, \omega_n$ are a basis of the cotangent space of $AGr_{n-1}(\mathbb{F}^n)$ at p_0 which take real values on tangent vectors to $AGr_{n-1}(\mathbb{R}^n)$. For tangent vectors $X_1, \ldots, X_n$ to $AGr_{n-1}(\mathbb{F}^n)$ at $p \in U$, we define

$$\omega_p(X_1, \ldots, X_n) := \det(\omega_i(dL_{(\tau(p))^{-1}} X_j))$$

where, for $g \in G$, $L_g : AGr_{n-1}(\mathbb{F}^n) \longrightarrow AGr_{n-1}(\mathbb{F}^n)$ is the map $p \longrightarrow g \cdot p$. Notice that L_g extends to a map in all of V. The map $L : G \times V \longrightarrow V$ given by $L(g, p) = g \cdot p$ is an algebraic map, therefore dL is algebraic. For $g \in G$, $p \in V$ and X tangent to V at p, $(dL_g)_p(X) = dL_{(g,p)}(0, X)$ and therefore $(dL_g)_p(X)$ is an algebraic function of g, p, X. Since τ is algebraic, $dL_{(\tau(p))^{-1})}(X)$ is an algebraic function of p, X and therefore ω is algebraic.

Now we define an algebraic chart $\alpha : U \longrightarrow \{v \in \mathbb{F}^n : v \cdot e_n \neq 0\}$ by

$$\alpha(b, A) = b.$$

Let $h : \{v \in \mathbb{F}^n : v \cdot e_n \neq 0\} \longrightarrow \mathbb{F}$ be the function such that

$$\delta\alpha^{-1}(\omega) = h dr_1 \wedge \cdots \wedge dr_n,$$

where $\delta\alpha^{-1}(\omega)$ is the pull back of ω, and $r_1, \ldots r_n$ are the standard coordinate functions on $\mathbb{F}^n$. h is an algebraic function. For a definable function f on $AGr_{n-1}(\mathbb{F}^n)$ we define

$$\int_{AGr_{n-1}(\mathbb{F}^n)} f := \int_{\mathbb{F}^n} (f \circ \alpha^{-1})|h|, \tag{4.2}$$

whenever the integral on the right exists in the Berarducci-Otero sense. We will show that if f has $\mathbb{Q}$-bounded support and image, then the integral on (4.2) exists.

Lemma 4.4. *For a $\mathbb{Q}$-bounded set $S \subset \mathbb{F}^n$, $h(S)$ is $\mathbb{Q}$-bounded.*

Proof. For $b \in \{v \in \mathbb{F}^n : v \cdot e_n \neq 0\}$, $\alpha^{-1}(b) = (b, A)$, where A is the matrix of the orthogonal projection onto $b^\perp$. If $b = (x_1, \ldots, x_n)$ we can directly check that the matrix $A = (a_{ij})_{ij}$ is given by

$$a_{ii} = \frac{x_1^2 + \cdots + \widehat{x_i^2} + \cdots + x_n^2}{\sqrt{x_1^2 + \cdots + x_n^2}},$$

where $\widehat{x_i^2}$ means that the term x_i^2 is omitted from the sum, and

$$a_{ij} = -\frac{x_i x_j}{\sqrt{x_1^2 + \cdots + x_n^2}}.$$

Therefore the partial derivatives

$$\frac{\partial a_{ij}}{\partial x_k}, \qquad i, j, k = 1, \ldots, n$$

are bounded in magnitude by a natural number, and thus there is a $C \in \mathbb{N}$ such that for all $b \in \mathrm{Dom}(h)$,

$$|(d\alpha^{-1})_b| \leq C.$$

Now, conjugation by $(b, A) \in G$ is an affine linear map on $\mathbb{F}^n \times M_{n\times n}(\mathbb{F})$. For, if $(x, Y) \in \mathbb{F}^n \times M_{n\times n}(\mathbb{F})$ then the composition $(b, A)(x, Y)(b, A)^{-1}$ is given by

$$\begin{aligned}(b + Ax, Ay)(-A^{-1}b, A^{-1}) &= (b + Ax - AYA^{-1}b, AYA^{-1}) \\ &= (b, 0) + (Ax - AYA^{-1}b, AYA^{-1}).\end{aligned}$$

Therefore the differential of conjugation by $(b, A) \in G$ is the linear map

$$(x, Y) \longrightarrow (Ax - AYA^{-1}b, AYA^{-1}),$$

which has operator norm bounded by $2+|b|$ since $|A|=1$,

$$|Ax + AYA^{-1}b| \leq |x| + |Y||b| \quad \text{and}$$
$$|AYA^{-1}| \leq |Y|.$$

Now,

$$\begin{aligned} h(b) &= \omega_{\alpha^{-1}(b)}(d\alpha^{-1}(\frac{\partial}{\partial r_1}), \ldots, d\alpha^{-1}(\frac{\partial}{\partial r_n})) \\ &= \det(\omega_i(dL_{(\tau(\alpha^{-1}(b)))^{-1}}(d\alpha^{-1}(\frac{\partial}{\partial r_j})))), \end{aligned}$$

an therefore it follows from the bounds on the differential of conjugation, and of $d\alpha^{-1}$ that $|h|$ is bounded by a natural number near $|b|=0$.

Since $\omega_1, \ldots, \omega_n$ are cotangent vectors to $AGr_{n-1}(\mathbb{R}^n)$, h takes real values on real entries, and therefore the image of a $\mathbb{Q}$-bounded set is also $\mathbb{Q}$-bounded. □

Proposition 4.5. *Let f be a definable function on $AGr_{n-1}(\mathbb{F}^n) \subset \mathbb{F}^n \times M_{n\times n}(\mathbb{F})$ with $\mathbb{Q}$-bounded support and image. Then*

$$\int_{AGr_{n-1}(\mathbb{F}^n)} f$$

exits.

Proof. $f \circ \alpha^{-1}$ vanishes outside a $\mathbb{Q}$-bounded set, and by Lemma 4.4 h maps the support of f to a $\mathbb{Q}$-bounded set. Therefore $(f \circ \alpha^{-1})|h|$ is a $\mathbb{Q}$-bounded function vanishing outside a $\mathbb{Q}$-bounded set , it follows by Chapter 2, Theorem 2.9 that $(f \circ \alpha^{-1})|h|$ is integrable. □

Now we show that for $\mathbb{F} = \mathbb{R}$, the integral (4.2) coincides with the G-invariant integral on $AGr_{n-1}(\mathbb{R}^n)$.

The real affine Grassmannian $AGr_{n-1}(\mathbb{R}^n)$ sits inside $AGr_{n-1}(\mathbb{F}^n)$, it consists of all the pairs $(b, A) \in AGr_{n-1}(\mathbb{F}^n)$ with real entries. If $\omega_{\mathbb{R}}$, $\alpha_{\mathbb{R}}$ and $h_{\mathbb{R}}$ are the form, function and chart constructed above for the real field, then $h_{\mathbb{R}} = h|AGr_{n-1}(\mathbb{R}^n) = \overline{h}$ and $\alpha_{\mathbb{R}} = \alpha|AGr_{n-1}(\mathbb{R}^n) = \overline{\alpha}$. We show that $\omega_{\mathbb{R}}$ is G-invariant up to sign, and therefore the integral in (4.2) is the G-invariant measure on $AGr_{n-1}(\mathbb{R}^n)$.

Lemma 4.6. *For every $g \in G$, the pullback of $\omega_{\mathbb{R}}$ by L_g is $\omega_{\mathbb{R}}$ up to sign. That is*

$$\delta L_g(\omega_{\mathbb{R}}) = \pm\omega_{\mathbb{R}}.$$

Proof.

$$\begin{aligned}(\delta L_g(\omega_{\mathbb{R}}))_p(X_1, \ldots, X_n) &= (\omega_{\mathbb{R}})_{g\cdot p}(dL_g(X_1), \ldots, dL_g(X_n)) \\ &= \det(\omega_i(dL_{(\tau(g\cdot p))^{-1}} dL_g(X_j))).\end{aligned}$$

Also, $\tau(p) \cdot p_0 = p$, thus

$$\tau(g \cdot p) \cdot p_0 = g \cdot p = g\tau(p) \cdot p_0,$$

and therefore there is an $h \in H$ with

$$(\tau(g \cdot p))^{-1} g\tau(p) = h.$$

Thus

$$dL_{\tau(g\cdot p)^{-1}} dL_g(X_j) = dL_{h\tau(p)^{-1}g^{-1}} dL_g(X_j) = dL_h dL_{\tau(p)^{-1}}(X_j),$$

so

$$\begin{aligned}\det(\omega_i(dL_{\tau(g\cdot p)^{-1}} dL_g(X_j))) &= \det(dL_h) \det(\omega_i(dL_{\tau(p)^{-1}}(X_j))) \\ &= \pm(\omega_{\mathbb{R}})_p(X_1, \ldots, X_n)\end{aligned}$$

since by Chapter 2, equation (2.2), $\det(dL_h) = \pm 1$ for all $h \in H$. $\square$

4.4 The generalized Cauchy-Crofton formula

For a $\mathbb{Q}$-bounded definable curve γ in $\mathbb{F}^n$, the length of γ is the average number of points of intersection of γ with an affine hyperplane. The proof is a reduction to the standard Cauchy-Crofton formula for the length of a curve in $\mathbb{R}^n$. The main point is that the number of points of intersection of γ with a hyperplane L defined over the reals is the same as the number of points of intersection of the standard part of γ, $\overline{\gamma}$, with the real points of L, as long as L is not tangent to the curve $\overline{\gamma}$.

Lemma 4.7. *Let $f : B \longrightarrow [0, q]$ be a definable function, where $B \subset \mathbb{F}^n$ is a $\mathbb{Q}$-bounded box and $q \in \mathbb{Q}$, then $\overline{f}$ is Riemann integrable, and*

$$\int_B f = \int_{B_{\mathbb{R}}} \overline{f}$$

Proof. Let P be a polyrectangle. If $P \supset [0, f)$, then $P \supset [0, \overline{f})$. Thus, $\mu^*([0, f)) \geq \mu^*([0, \overline{f}))$. Also, if $P \subset [0, f)$ then $P \subset [0, \overline{f})$. Thus $\mu_*([0, f)) \leq \mu_*([0, \overline{f}))$. Since $[0, f)$ is μ-measurable we get

$$\mu^*([0, \overline{f})) \leq \mu([0, f)) \leq \mu_*([0, \overline{f})).$$

But $\mu^*([0, \overline{f})) \geq \mu_*([0, \overline{f}))$, thus $[0, \overline{f})$ is μ-measurable, so $\overline{f}$ is Riemann integrable. Moreover,

$$\int_{B_{\mathbb{R}}} \overline{f} = \mu([0, f)) = \int_B f.$$

□

Let $\gamma : (0, 1) \longrightarrow \mathbb{F}^n$ be a definable curve with $\mathbb{Q}$-bounded image γ' finite and $\gamma''(x)$ finite for $x \not\approx 0, 1$. Note that it follows from Lemma 4.7 that

$$\text{length}(\gamma) = \text{length}(\overline{\gamma}).$$

We will assume that $\overline{\gamma}$ is an embedded C^1 curve in $\mathbb{R}^n$.

Lemma 4.8. *Let $f : AGr_{n-1}(\mathbb{F}^n) \longrightarrow \mathbb{F}$ be the function $f(L) = \#(\gamma \cap L)$. Then*

$$\int_{AGr_{n-1}(\mathbb{F}^n)} f = \int_{AGr_{n-1}(\mathbb{R}^n)} f|AGr_{n-1}(\mathbb{R}^n).$$

Proof.

$$\begin{aligned}\int_{AGr_{n-1}(\mathbb{F}^n)} f &= \int_{\mathbb{F}^n} (f \circ \alpha^{-1})|h| \\ &= \int_{\mathbb{R}^n} \overline{(f \circ \alpha^{-1})|h|} \\ &= \int_{\mathbb{R}^n} (f \circ \alpha_{\mathbb{R}}^{-1})|h_{\mathbb{R}}| \\ &= \int_{AGr_{n-1}(\mathbb{R}^n)} f|AGr_{n-1}(\mathbb{R}^n),\end{aligned}$$

where the second equality is given by Lemma 4.7, noting that $f(L)$ is finite whenever L intersects γ transversely, and that the set

$$\{b \in \mathbb{F}^n : f(\alpha^{-1}(b)) \neq 0\}$$

is contained in a $\mathbb{Q}$-bounded box because γ is $\mathbb{Q}$-bounded. □

We would like to compare the integral in Lemma 4.8 above with

$$\int_{AGr_{n-1}(\mathbb{R}^n)} \#(\overline{\gamma} \cap L)\, dL.$$

For this, let $g : AGr_{n-1}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ be the function $L \longrightarrow \#(\overline{\gamma} \cap L)$.

Suppose that $L \in AGr_{n-1}(\mathbb{R}^n) \subset AGr_{n-1}(\mathbb{F}^n)$ is a hyperplane which intersects $\overline{\gamma}$ transversely, and denote by $L_{\mathbb{R}}$ the set of $\mathbb{R}$ points of L. Let $p \in \overline{\gamma} \cap L$, then there are $t_0 < t_1$ such that $\overline{\gamma}|[t_0, t_1] \cap L = \{p\}$ and $\overline{\gamma}(t_0)$, $\overline{\gamma}(t_1)$ lie on opposite sides of L. Then $\gamma(t_0)$, $\gamma(t_1)$ must lie on opposite sides of L, so there is a $t \in (t_0, t_1)$ such that $\gamma(t) \in L$.

Since $t \approx st(t)$ and γ' is finite, $\gamma(st(t)) \approx \gamma(t)$. Thus $st(\gamma(st(t))) = st(\gamma(t)) \in st(L) = L_{\mathbb{R}}$, i.e. $\overline{\gamma}(st(t)) \in L$. But L intersects $\overline{\gamma}$ only at p when the parameter runs in $[t_0, t_1]$ and $st(t) \in [t_0, t_1]$ so $\overline{\gamma}(st(t)) = p$, in particular $\gamma(t) \approx p$. It follows that $g(L) \leq f(L)$ whenever L is transverse to $\overline{\gamma}$.

On the other hand, if $f(L) > g(L)$ then there are two infinitesimally close points of γ in L, that is, there are $\gamma(t_0),\ \gamma(t_1) \in L$ with $\gamma(t_0) \approx \gamma(t_1)$ and say $t_0 < t_1$. Assume $t_0 \not\approx 0, 1$. By Lemma 4.2, for all $s, t \approx t_0$, and $i = 1, \ldots, n$, $\gamma_i'(s) \approx \gamma_i'(t)$. By the mean value theorem, there are $u_1, \ldots, u_n \in (t_0, t_1)$ such that $\gamma_i'(u_i)(t_1 - t_0) = \gamma_i(t_1) - \gamma_i(t_0)$, therefore for all $t \approx t_0$,

$$\gamma'(t) \approx \frac{1}{t_1 - t_0}(\gamma(t_1) - \gamma(t_0)).$$

This means that $st(\gamma'(t))$ is paralel to $st(\gamma(t_1) - \gamma(t_0))$, in other words, if l is the secant line through $\gamma(t_0),\ \gamma(t_1)$, we must have $st(l)$ tangent to $\overline{\gamma}$ at $st(t_0)$. In particular, we have that L is tangent to $\overline{\gamma}$ at some point of $\overline{\gamma}$. Thus $f(L) \leq g(L)$ whenever L is transverse to $\overline{\gamma}$ and not infinitesimally close to $\gamma(0), \gamma(1)$.

We have shown that f and g agree almost everywhere, thus

$$\int_{AGr_{n-1}(\mathbb{R}^n)} \#(\gamma \cap L)\, dL = \int_{AGr_{n-1}(\mathbb{R}^n)} \#(\overline{\gamma} \cap L)\, dL.$$

But $\overline{\gamma}$ is an embedded curve, so by the Cauchy-Crofton formula

$$\text{length}(\overline{\gamma}) = C \int_{AGr_{n-1}(\mathbb{R}^n)} \#(\overline{\gamma} \cap L)\, dL,$$

and since $\text{length}(\overline{\gamma}) = \text{length}(\gamma)$, we have

Proposition 4.9. *There is a constant $C \in \mathbb{R}_{>0}$ such that for every definable curve $\gamma : (0,1) \longrightarrow \mathbb{F}^n$ with $\mathbb{Q}$-bounded image, γ' finite, $\gamma''(x)$ finite for $x \not\approx 0, 1$. and $\overline{\gamma}$ an* *embedded C^1 curve in $\mathbb{R}^n$,*

$$\text{length}(\gamma) = C \int_{AGr_{n-1}(\mathbb{F}^n)} \#(\gamma \cap L)\, dL.$$

Let $\gamma : (0,1) \longrightarrow \mathbb{F}^n$ be a definable injective curve with $\mathbb{Q}$-bounded image. Suppose that $0 = a_0 < \cdots < a_k = 1$ is a partition of $(0,1)$ such that:

Each restriction $\gamma\big|(a_i, a_{i+1})$, $i = 0, \ldots, k-1$, has a reparametrization α_i with α_i' finite, $\alpha_i''(x)$ finite for $x \not\approx 0, 1$, and $\overline{\alpha}_i$ an embedded C^1 curve in $\mathbb{R}^n$.

Then

$$\sum_{i=0}^{k-1} \int_{AGr_{n-1}(\mathbb{F}^n)} \#(\alpha_i \cap L) dL = \int_{AGr_{n-1}(\mathbb{F}^n)} \#(\gamma \cap L) dL.$$

Therefore we can define

$$\text{length}(\gamma) := \sum_{i=0}^{k-1} \text{length}(\alpha_i),$$

and this is independent of the partition and reparametrization. Moreover we have the generalized Cauchy-Crofton formula:

Theorem 4.10. *There is a constant $C \in \mathbb{R}_{>0}$ such that for any definable injective curve $\gamma : (0,1) \longrightarrow \mathbb{F}^n$ with $\mathbb{Q}$-bounded image,*

$$\text{length}(\gamma) = C \int_{AGr_{n-1}(\mathbb{F}^n)} \#(\gamma \cap L) dL.$$

4.5 Length in definable families of curves

We now prove that there is a bound for the lengths of the curves in a $\mathbb{Q}$bounded definable family. It should be noted that this result does not depend on the full strength of the generalized Cauchy-Crofton formula (Theorem 4.10). We conclude by using this result to prove that uniformly locally connected sets have the WAP.

Proposition 4.11. *If $A \subset \mathbb{F}^m$ is definable and definably connected, then there is a definable family of curves $\lambda \subset A^2 \times ([0,1] \times A)$ such that for every $a, a' \in A$, $\lambda_{(a,a')}(0) = a$, $\lambda_{(a,a')}(1) = a'$, and $\lambda_{(a,a')}$ is piecewise C^1.*

Proof. We use induction on m. The case $m = 1$ is trivial. For $m > 1$, assume first that A is a cell. By induction, we may assume that A is an open cell in $\mathbb{F}^m$, for, if A itself is not open, then A is the graph of a function $g : U \longrightarrow \mathbb{F}$, $U \subset \mathbb{F}^{m-1}$, and we may lift the paths in U to paths in A by using g. Let C be the projection of A into $\mathbb{F}^{m-1}$ so that $A = (f, g)_C$ for some definable functions f, g on C. By induction there is a definable family of curves Λ in C with the required property. Assume that f, g take values in $\mathbb{F}$ (the other cases are handled similarly). Let (y, r), $(z, s) \in A$ with $y, z \in C$. We first connect (y, r) to $(y, (f(y) + g(y))/2)$ by a vertical path in A. The path $\Lambda_{(y,z)}$ in C connecting y and z lifts to the path

$$t \longrightarrow (\Lambda_{(y,z)}(t), (f(\Lambda_{(y,z)}(t)) + g(\Lambda_{(y,z)}(t)))/2)$$

connecting $(y, (f(y)+g(y))/2)$ to $(z, (f(z)+g(z))/2)$. The last point can be connected to (z, s) by a vertical path in A. Concatenating these three paths, we get a path $\lambda_{((y,r),(z,s))}$ in A connecting (y, r) and (z, s). The collection of these paths constitutes the required definable family.

In the general case, since A is definably connected, we can write it as the union of cells $C_1, \ldots, C_k$, where for $i < k$ either C_i intersects the closure of C_{i+1}, or C_{i+1} intersects the closure of C_i ([10] Chapter 3, (2.19)). By definable choice ([10] Chapter 6, (1.2)), we can definably pick an element $e(C_i, C_{i+1})$ in $C_i \cap \overline{C_{i+1}}$ (if $C_i \cap \overline{C_{i+1}} \neq \emptyset$) and a definable curve $\gamma_i : (0, \epsilon] \longrightarrow C_{i+1}$ such that $\lim_{t \longrightarrow 0} \gamma_i(t) = e(C_i, C_{i+1})$. Combining this with the fact that the result was already proved for cells we get the desired family λ. □

Theorem 4.12. *Let* $\lambda \subset A \times ([0,1] \times B) \subset \mathbb{F}^n \times \mathbb{F}^{m+1}$ *be a definable and* $\mathbb{Q}$*-bounded family of curves. Then there is a* $K \in \mathbb{R}_{>0}$ *such that for any* $x \in A$, $\text{length}(\lambda_x) \leq K$.

Proof. Let $\lambda \subset A \times ([0,1] \times B) \subset \mathbb{F}^n \times \mathbb{F}^{m+1}$ be a definable family of curves. By Chapter 2, Fact 2.5, there is a natural number N such that for any affine $(m-1)$-plane $L \subset \mathbb{F}^m$ and $x \in A$, if $L \cap \lambda_x$ is finite then it contains at most N points.

Let

$$\Lambda := \bigcup_{x \in A} \lambda_x([0,1]).$$

Take $x \in A$, then by the Cauchy-Crofton formula (Theorem4.10), there is a $C \in \mathbb{R}_{>0}$ such that

$$\text{length}(\lambda_x) = C \int_{AGr_{m-1}(\mathbb{F}^m)} \#(\lambda_x \cap L) dL \leq C \int_{L \cap \lambda_x \neq \emptyset} N dL \leq NC \int_{L \cap \Lambda \neq \emptyset} dL.$$

The last integral is finite since λ is $\mathbb{Q}$-bounded, thus

$$K := NC \int_{L \cap \overline{\Lambda} \neq \emptyset} dL$$

is the required constant. □

Remark 4.13. *Note that since* $\text{length}(\lambda_x) = \text{length}(\overline{\lambda_x})$*, the ordinary Cauchy-Crofton formula (Chapter 2, Theorem 2.13) together with a bound for* $\#(\overline{\lambda_x} \cap L)$ *is all that is needed for the proof of Theorem 4.12. That such a bound exists was part of the proof of the generalized Cauchy-Crofton formula.*

Corollary 4.14. *If* $A \subset \mathbb{F}^n$ *is definable,* $\mathbb{Q}$*-bounded, and definably connected, then there is a definable family of curves* $\lambda \subset A^2 \times ([0,1] \times A)$ *and* $K > 0$ *such that for any pair of points* $x, y \in A$, $\lambda_{(x,y)}$ *is a piecewise* C^1 *curve in* A *joining* x *and* y *with* $\text{length}(\lambda_{(x,y)}) \leq K$.

Recall that for $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0, \infty)^n$. The α- box centered at a is the open box

$$B(a, \alpha) := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \in (a_i - \frac{\alpha_i}{2}, a_i + \frac{\alpha_i}{2}) \right\}.$$

Recall as well that a definable set $U \subset \mathbb{F}^n$ is uniformly locally connected if there is an $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$ such that for every $u \in \overline{U}$ and every $\delta \in (0,1)$, the set $U \cap B(u, \delta\alpha)$ is definably connected.

For convenience we use the max norm in $\mathbb{F}^n$, that is, for $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$, $|x| := \max\{|x_i| : i = 1, \ldots, n\}$.

Proposition 4.15. *If $A \subset \mathbb{F}^n$ is a definable, $\mathbb{Q}$-bounded, definably connected, and uniformly locally connected set, then there is a $K > 0$ and a definable family of curves $\gamma \subset A^2 \times [0,1] \times A$ such that for $x, y \in A$, $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$, and* $\text{length}(\gamma_{x,y}) \leq K|x - y|$.

Proof. For $\lambda > 0$ and $a \in \mathbb{F}^n$ let $f_{a,\lambda} : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ be the dilation about a, that is $f_{a,\lambda}(x) = \lambda(x - a) + a$. Since A is uniformly locally connected, there is a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$ such that for every $a \in \overline{A}$ and $\delta \in (0,1)$, the set $B(a, \delta\alpha) \cap A$ is definably connected. For $a \in \overline{A}$ and $\delta \in (0,1)$ define

$$B_{\delta,a} := f_{a,1/\delta}(B(a, \delta\alpha) \cap A).$$

This set is definably connected since $B(a, \delta\alpha) \cap A$ is. By Proposition 4.11, there is a definable family of curves $\lambda^{\delta,a} \subset B_{\delta,a}^2 \times ([0,1] \times B_{\delta,a})$ such that for $b, b' \in B_{\delta,a}$, $\lambda_{b,b'}^{\delta,a}$ is piecewise C^1, $\lambda_{b,b'}^{\delta,a}(0) = b$ and $\lambda_{b,b'}^{\delta,a}(1) = b'$. Consider

$$\lambda := \{(\delta, a, x, y, \epsilon, z) \in ((0,1) \times A \times (\mathbb{F}^n)^2) \times ([0,1] \times \mathbb{F}^n) : (x, y, \epsilon, z) \in \lambda^{\delta,a}\}.$$

This is a definable and $\mathbb{Q}$-bounded family of curves. Thus by Theorem 4.12, there is a $K_1 > 0$ such that for every $\delta \in (0,1)$, $a \in \overline{A}$, and $b, b' \in B_{\delta,a}$, length $\left(\lambda^{\delta,a}_{b,b'}\right) \leq K_1$.

Similarly, by Corollary 4.14 there is a definable family of curves $\Lambda \subset A^2 \times ([0,1] \times A)$ and $K_2 > 0$ such that for each $x, y \in A$, $\Lambda_{x,y} : [0,1] \longrightarrow A$ is a piecewise C^1 curve in A joining x and y, and length$(\Lambda_{x,y}) \leq K_2$.

Now let x, y be distinct points in A, and assume that $|x-y| < \min\{\alpha_j/3\}$. Let

$$\delta := \frac{3|x-y|}{\min\{\alpha_j\}}, \quad y' := f_{x,1/\delta}(y) = \frac{1}{\delta}(y-x) + x.$$

Then $\delta \in (0,1)$ and for any j,

$$|x_j - y_j| < \frac{3}{2}\frac{\alpha j}{\min\{\alpha_j\}}|x-y| = \frac{\delta\alpha_j}{2}.$$

Thus, $y \in B(x, \delta\alpha)$, that is, $y' \in B_{\delta,x}$. Consider the curve $\lambda^{\delta,x}_{x,y'}$ in $B_{\delta,x}$, joining x and y', and let $\gamma_{x,y} : [0,1] \longrightarrow \mathbb{F}^n$ be defined by

$$\gamma_{x,y}(t) := f_{x,\delta}(\lambda^{\delta,x}_{x,y}(t)) = \delta(\lambda^{\delta,x}_{x,y'}(t) - x) + x.$$

Then $\gamma_{x,y}(t)$ is a curve in A joining x and y, and moreover,

$$\text{length}(\gamma_{x,y}) = \delta \,\text{length}\left(\lambda^{\delta,x}_{x,y'}\right) \leq \delta K_1 = \frac{3K_1}{\min\{\alpha_j\}}|x-y|.$$

Now assume that $|x-y| \geq \frac{1}{3}\min\{\alpha_j\}$, and let $\gamma_{x,y} := \Lambda_{x,y}$. Then γ is a curve in A joining x and y and

$$\text{length}(\gamma_{x,y}) \leq \frac{K_2}{|x-y|}|x-y| \leq \frac{3K_2}{\min\{\alpha_j\}}|x-y|.$$

The collection of curves $\gamma_{x,y}$ constitutes the required definable family.

$$K := \frac{3\max\{K_1, K_2\}}{\min\{\alpha_j\}},$$

is the required constant. □

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