Computable Properties of Decomposable and Completely Decomposable Groups

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Kyle Riggs

COMPUTABLE PROPERTIES OF DECOMPOSABLE AND COMPLETELY DECOMPOSABLE GROUPS

We consider the class of torsion-free abelian groups and show that the class of decomposable torsion-free abelian groups is $\mathbf{\Sigma^1_1}$-complete. Thus, this property cannot be characterized by a first-order formula in the language of arithmetic.

We also consider the class of completely decomposable groups in which each element is divisible by only finitely many distinct primes. We attempt to classify the isomorphism classes of groups which meet this description and can be constructed algorithmically. Conventional definitions used in computability theory fail to capture these structures, so we discuss new approaches to this problem.

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CHAPTER 1

Introduction

Studying computable structures can yield results telling us the difficulty of classifying some of their most fundamental properties. For example, Downey and Montalbán [**4**] studied the isomorphism problem for torsion-free abelian groups. They found that the set of isomorphic pairs of countable torsion-free abelian groups is $\mathbf{\Sigma^1_1}$ -complete. I have proven similar results for the class of countable decomposable torsion-free abelian groups [**7**] (namely, that this class of groups is also $\mathbf{\Sigma^1_1}$-complete). This means that the problem of classifying these groups is as hard as it can possibly be. We will never be able to completely characterize them the way we have the decomposable torsion groups.

Another question that arises in effective algebra is: what groups can be presented computably? This question is best answered by looking at a specific class of groups and identifying which groups in that class have computable copies. Khisamiev [**6**] looked at countable reduced torsion groups, which are uniquely determined by their Ulm sequences. He was able to characterize the Ulm sequences of length $< \omega^2$ which can occur in a computably presented group. Ash, Knight, and Oates [2], working slightly later, independently duplicated his results. My studies focus on completely decomposable groups, specifically those in which each element is only divisible by finitely many distinct primes. This work builds on a result by Downey, Kach, Goncharov, Knight, Kudinov, Melnikov, and Turetsky [**2**] and attempts to expand their result to a larger class of groups.

1. Summary

These topics will be discussed as follows. In Chapter 2 we will cover some basic notions of computability theory and group theory.

In Chapter 3 we will discuss the decomposability problem for torsion-free abelian groups and show that it is $\mathbf{\Sigma^1_1}$-complete.

In Chapter 4 we will consider groups of the form

$$H_S = \bigoplus_{D \in S} Q_D$$

where S is a collection of finite sets of primes and Q_D is the subgroup of Q generated by

$$\langle \frac{1}{p^k} : p \in D, k > 0 \rangle$$

We will discuss necessary criteria for S in order for there to be a computable presentation of H_S.

In Chapter 5 we will define limitwise monotonic functions, as well as some invariants on this concept. We will see that these allow us to give a sufficient and necessary condition for S such that there is a computable presentation of H_S.

CHAPTER 2

Background

A group is computable if its domain can be enumerated effectively and the binary operation of the group is computable (via the enumeration). Such an enumeration is called a *computable presentation.* In other words, a computable group has a computable word problem. In this chapter we will define several properties of groups and discuss complexity hierarchies, a core concept of computability theory.

1. Algebra Background

DEFINITION 2.1. Let $(G, +)$ be an abelian group with identity 0.

1) G is *torsion* if every element has finite order.

2) G is *torsion-free* if every element has infinite order.

3) G is *mixed* if it contains elements of both finite and infinite order.

We will exclusively discuss torsion-free abelian groups. The term *basis* will refer to a maximal linearly independent subset of a group as a Z-module. The *rank* of a group is the cardinality of any of its bases. Baer [**1**] showed that a torsion-free abelian group has rank 1 if and only if it is isomorphic to a subgroup of Q.

DEFINITION 2.2. Let G be an abelian group

1) Let A and B be subgroups of G. We say that G is the *direct sum* of A and B (denoted $G = A \oplus B$) if, for every element $g \in G$ there are unique elements $a \in A$ and $b \in B$ such that $g = a + b$. We call the subgroups A, B *direct summands.* If both A and B are nontrivial, we say that G is *decomposable.*

2) Let $\{G_i\}_{i \in I}$ be a (possibly infinite) set of subgroups of G. We say that G is the direct sum of the G_i's if, for every element $g \in G$ there are unique elements $\{g_i\}_{i \in I}$

with $g_i \in G_i$ such that $g_i = 0$ for all but finitely many i, and $g = \sum_i g_i$. If G_i is a subgroup of Q for every i, we say that G is *completely decomposable*.

To study these groups it will be necessary to define the characteristic and type of an element.

DEFINITION 2.3. Given an abelian group G, an element $x \in G$, and a prime p, we say that p divides x in G and write "$p|x$" if there is an element $y \in G$ such that $py = x$. The *height* of p at x is given by

$$h_p(x) = \sup\{k : p^k|x\}$$

We call

$$\chi_G(x) = (h_2(x), h_3(x), h_5(x), ...)$$

the *characteristic* of x in G.

DEFINITION 2.4. We define an equivalence relation on characteristics by saying that $\chi_G(x) \sim \chi_G(y)$ if

- for all p, $h_p(x) = \infty \Leftrightarrow h_p(y) = \infty$, and
- $h_p(x) = h_p(y)$ for all but finitely many p

We call the equivalence classes *types*. In other words, x and y have the same type iff there exist integers m and n such that $\chi_G(mx) = \chi_G(ny)$.

The isomorphism class of a rank 1 torsion-free abelian group is determined by the type of any nonzero element. Thus, the isomorphism class of a completely decomposable group is determined by the types of its summands.

DEFINITION 2.5. We can put a partial order on types by declaring for two types α, β that $\alpha \preceq \beta$ if, given any element a of type α and any element b of type β,

- for all p, $h_p(a) = \infty \Rightarrow h_p(b) = \infty$ and
- $h_p(a) \leq h_p(b)$ for all but finitely many p

A nonzero element has *strictly maximal type* if no nonzero element linearly independent from it has a greater or equal type.

DEFINITION 2.6. In an abelian group G, a subgroup H is called *pure* if for every $x \in H$ and $m \in \omega$, if m divides x in G, m also divides x in H. If S is a set of elements in G, the *pure subgroup generated by* S is the smallest pure subgroup of G containing S.

For example, Z is a pure subgroup of $Z \oplus Z$. In fact, in a decomposable group any summand is a pure subgroup. However, Z is not a pure subgroup of Q (the only nontrivial pure subgroup of Q is Q itself).

As a final note, there will be many equations where we write a group element g as a linear combination of other group elements $\{x_l\}$. Rather than write an equation

$$mg = \sum_l m_l x_l$$

with integer coefficients, it will often be more convenient to write an equivalent equation

$$g = \sum_l q_l x_l$$

with rational coefficients. This can be confusing because for each x_l there may not be an element $h_l = q_l x_l$ in the group we are considering. When it is important it will be made clear whether each h_l exists as well.

2. Computable Functions and Groups

A computable function is a partial function from the set of natural numbers ω to itself that can be determined by a Turing machine. In any language with countably many symbols, the"programming" that goes into constructing a Turing machine is just a finite string of symbols. This means that there are only countably many programs, so we can assign each one a natural number in a computable way (that is based on the programming language). We call this number the *index* of the function.

We let ϕ_e denote the computable function with index e, and W_e denote its domain. If $W_e = \omega$, we say that ϕ_e is *total*. For $x \in \omega$, if $x \in W_e$, we say that ϕ_e *halts* on x and write $\phi_e(x)\downarrow$.

The process of taking two natural numbers e and x and determining if ϕ_e halts on x is not computable. However, if we allow the computation of $\phi_e(x)$ to work for a finite amount of time (usually counted in "steps"), we can computably say whether or not ϕ_e halts on x after s many steps.

A group is computable if there is a computable program that enumerates the elements of the groups and determines the group's binary relation. Like computable functions, we can assign natural numbers to computable groups as well. A computable function f from the set of computable functions to the set of computable groups takes the index of a computable function ϕ_e and gives the index of a computable group $G_{f(e)}$. The description of f often involves stages where we run the function ϕ_e on finitely many inputs for a finite amount of time. If ϕ_e halts on one or more inputs, this will often have an effect on how we enumerate the group. At later stages, we may increase the number of inputs on which we are running ϕ_e, as well as increase the amount of time we allow the function to run. If there is a finite description of how this is done and we only consider computable relations, then the function f is computable.

3. Complexity Hierarchies

The arithmetical hierarchy was developed to describe the complexity of properties based on their formulas. A computable set (or relation) $S \subset \omega$ is said to be Σ^0_0 (or Π^0_0). A set S_1 is Σ^0_{n+1} if it can be characterized by a formula of the form

$$x \in S_1 \Leftrightarrow (\exists y \in \omega)\, R_1(x, y)$$

where R_1 is a Π^0_n relation. Likewise, a set S_2 is Π^0_{n+1} if it can be characterized by a formula of the form

$$x \in S_2 \Leftrightarrow (\forall y \in \omega)\, R_2(x, y)$$

where R_2 is a Σ^0_n relation.

In other words, n represents how many times the formula alternates quantifiers over ω (or some other infinite computable set), and a Σ^0_n formula starts with an existential quantifier, while a Π^0_n formula starts with a universal quantifier.

For example, given a computable group G, an element $g \in G$, and a fixed prime p, there is a Π^0_2 formula that says whether p infinitely divides g

$$p \,|^{\infty}\, g \Leftrightarrow (\forall k \in \omega)\ (\exists h \in G)\ p^k h = g$$

To see another example, let $[G]^{<\omega}$ denote the set of all finite sets in G. The following Π^0_2 formula describes the property of being a basis of G. For $\bar{x} \in [G]^{<\omega}$,

$$\begin{aligned} BASIS(\bar{x}) \Leftrightarrow [(\forall y \in G)\ (\exists \bar{q} \in Q^{<\omega})\ (|\bar{q}| = |\bar{x}| \wedge y = \sum_i q_i x_i) \\ \wedge (\forall \bar{q} \in Q^{<\omega})\ (|\bar{q}| = |\bar{x}| \wedge \sum_i q_i x_i = 0) \Rightarrow \bar{q} = \bar{0}] \end{aligned}$$

A set is Δ^0_n if it is both Σ^0_n and Π^0_n. A Δ^0_0 set is clearly computable, but so is any Δ^0_1 set S. This is because membership in S can be defined by a Σ^0_1 formula or a Π^0_1 formula:

$$x \in S \Leftrightarrow (\exists y \in \omega) R_1(x, y) \Leftrightarrow (\forall z \in \omega) R_2(x, z)$$

where both R_1 and R_2 are computable relations. This means for any $x \in \omega$, there is either a y such that $R_1(x, y)$ holds or a z such that $R_2(x, z)$ does not hold. Because we know we will find such a y or such a z, we can search through the natural numbers until we do, and this will tell us whether or not $x \in S$.

We should note that any formula that is a finite boolean combination of Σ^0_n and Π^0_n formulas is a Δ^0_{n+1} formula. To demonstrate why this is true, we observe how we can rewrite a conjunction of a Σ^0_n formula and a Π^0_n formula.

$$(\exists y)R_1(x,y)\wedge(\forall z)R_2(x,z) \Leftrightarrow (\exists y)(\forall z)[R_1(x,y)\wedge R_2(x,z)] \Leftrightarrow (\forall z)(\exists y)[R_1(x,y)\wedge R_2(x,z)]$$

Any set that is characterized by a Δ^0_n formula for some n is said to be *arithmetical*. If we allow quantifiers over functions from ω to ω (or between any two computable sets), then our formula will be *analytic*. We say that a formula is Σ^1_1 if it is of the form

$$(\exists f \in \omega^\omega)\ R(f)$$

where is R is any arithmetical formula.

For any complexity class Γ, we say that a set $A \in \Gamma$ is Γ-complete if any other set $B \in \Gamma$ can be "coded" into A. That is to say, there is a computable function $f : \omega \to \omega$ such that $x \in B$ iff $f(x) \in A$.

An example of a Σ^0_1-complete set is the index set of computable functions which halt on their own index:

$$K = \{e : \phi_e(e)\downarrow\} = \{e : e \in W_e\}$$

An example of a Σ^1_1-complete set is the index set of computable trees in $\omega^{<\omega}$ with an infinite path.

The Borel hierarchy is a complexity class for Polish spaces (like ω^ω) which defines $\boldsymbol{\Sigma^0_1}$ sets to be open sets and $\boldsymbol{\Pi^0_1}$ sets to be closed sets. In this hierarchy, the set of trees in $\omega^{<\omega}$ with an infinite path is $\boldsymbol{\Sigma^1_1}$-complete.

4. Turing Degrees

Definition 2.7. Given a set $S \subset \omega$, we say that a Turing machine has oracle S if it has a priori knowledge as to whether any number n is an element of S. Given two sets $X, Y \subset \omega$, we say that X is *Turing reducible* to Y ($X \leq_T Y$) if there is a Turing machine with oracle Y that can compute membership in X.

We say that X is *Turing equivalent* to Y ($X \equiv_T Y$) if $X \leq_T Y$ and $Y \leq_T X$. We call the equivalence classes of this relation the *Turing degrees*.

We denote the Turing degree of computable sets by $\mathbf{0}$, and the Turing degree of the halting problem $\mathbf{0'}$. Given any Turing degree $\mathbf{d}$and any set $S \in \mathbf{d}$, we use $\mathbf{d}'$ to denote the Turing degree of the halting problem for a Turing machine with oracle S. This degree is called the *jump* of $\mathbf{d}$, and the jump of $\mathbf{d}'$ itself is called the second jump of $\mathbf{d}$. Most of the proofs in Chapter 5 involve constructions using an oracle in $\mathbf{0}''$.

CHAPTER 3

The Decomposability Problem for Torsion-Free Abelian Groups

Often the best way to study an abelian group is by writing it as a direct sum of its indecomposable subgroups, so determining whether a group is decomposable is a problem at the heart of abelian group theory.

It is known that the only indecomposable torsion groups are the cocyclic groups (cyclic groups of the form $Z(p^k)$, and their direct limit $Z(p^\infty)$), and that every mixed group is decomposable. Torsion-free groups of rank 1 are indecomposable, but beyond this no classification has been found. It had been conjectured by some (Kudinov, Melnikov) that this is because the class of torsion-free decomposable groups is non-arithmetical. We will show that this is indeed the case for groups of infinite rank. This means that there is no characterization for decomposable torsion-free abelian groups simpler than the one given in Definition 2.2. To understand why this is true, we must first consider a nontrivial example of an indecomposable group.

1. An Example of an Indecomposable Group

The following example can be found in Fuchs [**5**]. Let G_0 be the free abelian group generated by two elements, x_1 and x_2. For every $k > 0$, we add elements of the form

$$\frac{x_1}{3^k} \text{ and } \frac{x_2}{5^k}$$

to G_0. We also add the element $\frac{x_1+x_2}{2}$ to the group. We denote by G the group generated by all these elements.

Note that $\{x_1, x_2\}$ is still a basis for this group, and that

$$\chi_G(x_1) = (0, \infty, 0, 0, 0, ...) \text{ and } \chi_G(x_2) = (0, 0, \infty, 0, 0, ...)$$

Furthermore, any element of the form $q_1x_1 + q_2x_2$ with both coefficients nonzero has type $(0, 0, 0, ...)$ because it would only be divisible by finitely many primes, and not infinitely divisible by any prime. Thus, x_1 and x_2 both have strictly maximal type.

PROPOSITION 3.1. *In a decomposable group G $(= A \oplus B)$, if $x \in G$ decomposes as $x = a + b$ and an integer m divides x, then m divides a and b as well.*

PROOF. Let $y \in G$ be such that $my = x$, and suppose y decomposes $y = a_1 + b_1$. Then we see that

$$ma_1 + mb_1 = my = x = a + b$$

$ma_1 \in A$ and $mb_1 \in B$, so $ma_1 = a$ and $mb_1 = b$. □

COROLLARY 3.2. *In a decomposable group $G = A \oplus B$, every element of strictly maximal type must be contained in a direct summand.*

PROOF. Suppose that x is an element of strictly maximal type that is not in A or B. Then we can write $x = a + b$ with $a \in A$ and $b \in B$, and both a and b nonzero. Because x has strictly maximal type, there are a prime p and an integer k such that p^k divides x, but neither a nor b. However, this contradicts proposition 3.1. □

We claim that G is indecomposable. We assume $x_1 \in A$ and $x_2 \in B$. Now consider the decomposition of the element

$$\frac{x_1 + x_2}{2} = a + b$$

with $a \in A$ and $b \in B$. It is clear that $2a = x_1$ and $2b = x_2$, but there are no elements in G which satisfy either of these equations. Thus, the group is indecomposable.

The proofs contained in this chapter will mimic this technique of creating elements of strictly maximal type and then introducing elements which force them to be contained in the same direct summand. We call these elements *links*.

DEFINITION 3.3. Let x and y be two elements of strictly maximal type in a torsion-free abelian group G. If there is a prime p which divides the sum $x + y$ but neither x nor y, then the element $\frac{x+y}{p}$ is a link connecting x and y. We say that x and y are connected by a *chain* of links if there are elements $x_1, x_2, ..., x_n$ such that the sequence $\{x_0 = x, x_1, x_2, ..., x_n, x_{n+1} = y\}$ has the property that for $0 \leq i \leq n$, there is a link connecting x_i and x_{i+1}.

The following proposition gives us a simple way to construct indecomposable groups.

PROPOSITION 3.4. *If a torsion-free group has a basis of elements of strictly maximal type, with each pair of them having a link or a chain of links connecting them, then it is indecomposable.*

PROOF. By Corollary 3.2, every element of strictly maximal type must be contained in a direct summand, and any two of these elements with a link between them must be in the same direct summand. Transitively, this is also true of any two elements with a chain of links connecting them. Thus, the entire basis is contained in a single direct summand, so the group is indecomposable. □

2. Groups of Finite Rank

REMARK 3.5. Let G be a group of finite rank, and assume $G = A \oplus B$. Then

(1) $\text{rank}(G) = \text{rank}(A) + \text{rank}(B)$
(2) If $\{a_1, ..., a_n\}$ is a basis for A and $\{b_1, ..., b_m\}$ is a basis for B, then $\{a_1, ..., a_n, b_1, ..., b_m\}$ is a basis for G with the following property:

$$\text{If there exists an element } g = \sum_{i=1}^{n} q_i a_i + \sum_{j=1}^{m} r_j b_j, \text{ then there exist}$$

$$\text{elements } g_A, g_B \text{ such that } g_A = \sum_{i=1}^{n} q_i a_i \text{ and } g_B = \sum_{j=1}^{m} r_j b_j$$

Conversely, if $\{a_1, ..., a_n, b_1, ...b_m\}$ is a basis for G with this property, then the pure subgroup generated by the a_i's and the pure subgroup generated by the b_j's give a decomposition of G. Thus, a group of finite rank is decomposable iff it has a basis with this property.

If we take the conjunction of the Π_2^0 formula $BASIS$ given is section 3 with a formula describing the property in Remark 3.5 we have the following Σ_3^0 formula for decomposable groups of finite rank:

$$(\exists\, \bar{a}, \bar{b} \in [G]^{<\omega})\ \{BASIS(\bar{a} \sqcup \bar{b}) \wedge \bar{a} \neq \emptyset \wedge \bar{b} \neq \emptyset \wedge (\forall y \in G)\,(\forall \bar{q} \in Q^{<\omega})$$
$$(\exists w \in G)[(|\bar{q}| = |\bar{a}| + |\bar{b}| \wedge y = \sum_i q_i a_i + \sum_j q_j b_j) \Rightarrow\ w = \sum_i q_i a_i]\}$$

Theorem 3.6. *The index set of computable decomposable groups of finite rank is* Σ_3^0*-complete.*

Proof. Recall that $Cof = \{n : W_n \text{ is cofinite}\}$ is Σ_3^0-complete. In order to prove our result, we describe a computable function from ω to groups of rank 2 such that G_n is decomposable iff W_n is cofinite.

Construction: We start with a group G generated by the following elements:

$$\langle g_1, g_2, \frac{g_1 + g_2}{2}, \frac{g_1}{3}, \frac{g_2}{5}, \frac{g_1}{7}, \frac{g_2}{11}, ...\rangle$$

(g_1 and g_2 are linearly independent).

The element g_1 is divisible by all odd-indexed primes, and g_2 is divisible by all even-indexed primes (except $p_0 = 2$), so they have incomparable (indeed, strictly maximal) types. Thus, like the example above, our initial group G is indecomposable.

The group G_n is generated by adding $\frac{g_2}{p_{2k+1}}$ for every k such that $\phi_n(k)\downarrow$.

In order for the group to be computable, we need the relation

$$d(k,g) \Leftrightarrow (\exists x \in G_n)\ p_k x = g$$

to be Σ_1^0 for all $k \in \omega$ and all $g \in G_n$. In G, this relation was already Σ_1^0, so the only concern would be determining whether $p_{2k+1}|g_2$ for some k. This will be true iff $\phi_n(k)\downarrow$, which is itself a Σ_1^0 relation. Thus, the group is computable.

Verification: If W_n is coinfinite, then g_1 is still divisible by infinitely many primes that do not divide g_2. Thus, the types remain incomparable, and the group remains indecomposable.

If W_n is cofinite, then the type of g_2 is strictly greater than the type of g_1. There are finitely many primes that divide g_1 but not g_2. Denote their product by m.

Lemma 3.7. *$G_n = A \oplus B$, where A is the pure subgroup generated by $a = \frac{g_1+mg_2}{2}$ and B the pure subgroup generated by g_2.*

Proof. We observe that $\frac{g_1+g_2}{2} = a - \frac{m-1}{2}g_2$ (Note that m is a product of odd primes).

Any element of the form $\frac{g_1}{p} \in G_n$ can be written

$$\frac{g_1}{p} = \frac{2}{p}a - \frac{m}{p}g_2$$

If $p \nmid m$, then $p \,|\, g_2$, so $\frac{m}{p}g_2 \in B$. Thus, every generating element of the group can be uniquely decomposed, so the group is decomposable. □

The group G_n is decomposable iff W_n is cofinite, so the theorem is proved. □

3. Groups of Infinite Rank

We can adapt the formula used for groups of finite rank to describe decomposable groups of infinite rank. However, this means the first existential quantifier is searching over infinite sets instead of finite sets, so the Σ^0_3 formula becomes a Σ^1_1 formula (here $BASIS$ is a Π^0_2-formula on infinite sets):

$$(\exists\, \bar{a}, \bar{b} \in [G]^{\leq \omega})\ [BASIS(\bar{a} \sqcup \bar{b}) \wedge \bar{a} \neq \emptyset \wedge \bar{b} \neq \emptyset \wedge (\forall y \in G)$$
$$(\forall \bar{q} \in Q^{<\omega})(\exists w \in G)(y = \sum_i q_i a_i + \sum_j q_j b_j) \Rightarrow\ w = \sum_i q_i a_i)]$$

Theorem 3.8. *1) The index set of computable decomposable groups of infinite rank is Σ^1_1-complete.*

2) The set of decomposable groups of infinite rank is $\mathbf{\Sigma^1_1}$-complete.

Proof. We will construct a function from trees in $\omega^{<\omega}$ to torsion-free abelian groups of infinite rank that takes a tree T and gives a group G_T that is decomposable iff T has an infinite path. (Recall that the set of trees in $\omega^{<\omega}$ which have an infinite path is $\mathbf{\Sigma^1_1}$-complete.)

Construction of the group G: We start with a countably infinite set of linearly independent elements: $x_1, x_2, ...$ and $\{x_\sigma\}_{\sigma \in \omega^{<\omega}}$ (which we denote as the x-elements), and $y_1, y_2, ...$ (the y-elements). These elements form a basis for our group. We will give them all strictly maximal type and introduce links connecting all the x-elements and separate links connecting all the y-elements.

The initial group G_0 is generated by the following elements:

- For $i, k > 0$ and $\sigma \in \omega^{<\omega}$,

$$\frac{x_i}{p^k_{\langle 0,i\rangle}}, \ \frac{y_i}{p^k_{\langle 1,i\rangle}}, \text{ and } \frac{x_\sigma}{p^k_{\langle 2,\sigma\rangle}}$$

- For $0 < i < j$,

$$\frac{x_i + x_j}{p_{\langle 3,\langle i,j\rangle\rangle}} \text{ and } \frac{y_i + y_j}{p_{\langle 4,\langle i,j\rangle\rangle}}$$

- For every $i \geq 0$ and $\sigma, \rho \in \omega^{<\omega}$,

$$\frac{x_i + x_\sigma}{p_{\langle 5,\langle i,\sigma\rangle\rangle}} \text{ and } \frac{x_\sigma + x_\rho}{p_{\langle 6,\langle \sigma,\rho\rangle\rangle}}$$

- For $n > 1$,

$$\frac{y_1 + y_2 + \ldots + y_n}{p_{\langle 7,n\rangle}}$$

All the x- and y-elements are elements of strictly maximal type, and due to the links, all the x-elements must be in the same direct summand of G_0 (as do the y-elements). Thus, G_0 can only be decomposed as $G_0 = A \oplus B$, where A is the pure subgroup containing all the x-elements, and B is the pure subgroup containing all the y-elements.

Now we add to G_0 links of the form

$$\frac{x_i + y_i}{p_{\langle 8,i\rangle}}$$

for $i \geq 0$, and denote by G the group generated by these elements. Now every x-element and every y-element are connected by a chain of links, so G is indecomposable.

Construction of G_T: Given a tree T in $\omega^{<\omega}$, we will add elements to G to form a group G_T that will be decomposable iff T has an infinite path through it. The idea is that if there is an infinite path π, then $G_T = A_T \oplus B_\pi$, where A_T is the pure subgroup containing the x-elements, and B_π is the pure subgroup of G_T containing all the elements of the form $y_i + x_{\pi \upharpoonright i}$ ($\pi \upharpoonright i$ is the prefix of π of length i). Note that if there is more than one infinite path through T, there will be more than one way to decompose G_T.

Enumerate T so that each string in T is enumerated after all of its initial segments. When we see $\sigma \in T$ with $|\sigma| = n$, we do the following: (It's worth noting that in each case, the introduction of the first element creates the second element. We list both simply to remind the reader that the second element also exists)

(1) For $i \leq n$, we add to the group the elements

$$\frac{y_i + x_{\sigma\restriction i}}{p^n_{\langle 1,i\rangle}} \text{ and } \frac{x_{\sigma\restriction i}}{p^n_{\langle 1,i\rangle}}$$

(2) For $i < n$, we add to the group the elements

$$\frac{(y_i + x_{\sigma\restriction i}) + (y_n + x_\sigma)}{p_{\langle 4,\langle i,n\rangle\rangle}} \text{ and } \frac{x_{\sigma\restriction i} + x_\sigma}{p_{\langle 4,\langle i,n\rangle\rangle}}$$

(3) We add to the group the elements

$$\frac{y_n + x_\sigma}{p_{\langle 8,n\rangle}} \text{ and } \frac{x_n - x_\sigma}{p_{\langle 8,n\rangle}}$$

(4) Finally, we add the elements

$$\frac{(y_1 + x_{\sigma\restriction 1}) + (y_2 + x_{\sigma\restriction 2}) + \ldots + (y_n + x_\sigma)}{p_{\langle 7,n\rangle}} \text{ and } \frac{x_{\sigma\restriction 1} + x_{\sigma\restriction 2} + \ldots + x_\sigma}{p_{\langle 7,n\rangle}}$$

Verification: If an infinite path π does exist, we shall see that $G_T = A_T \oplus B_\pi$, where A_T is the pure subgroup of G_T containing the x-elements, and B_π is the pure subgroup of G_T containing all elements of the form $y_i + x_{\pi\restriction i}$

Each x_i and x_σ is contained in A_T. We have $y_j = -x_{\pi\restriction j} + (y_j + x_{\pi\restriction j})$. Both of these elements are infinitely divisible by $p_{\langle 1,j\rangle}$ because $x_{\pi\restriction j}$ went through step (1) infinitely often. Thus, y_j does not have strictly maximal type in G_T.

For $0 < i < j$,

$$\frac{y_i + y_j}{p_{\langle 4,\langle i,j\rangle\rangle}} = \frac{(y_i + x_{\pi\restriction i}) + (y_j + x_{\pi\restriction j})}{p_{\langle 4,\langle i,j\rangle\rangle}} - \frac{x_{\pi\restriction i} + x_{\pi\restriction j}}{p_{\langle 4,\langle i,j\rangle\rangle}}$$

These elements were created during step (2) of some stage.

For $i > 0$,

$$\frac{x_i + y_i}{p_{\langle 8,i \rangle}} = \frac{x_i - x_{\pi \upharpoonright i}}{p_{\langle 8,i \rangle}} + \frac{y_i + x_{\pi \upharpoonright i}}{p_{\langle 8,i \rangle}}$$

These elements were created during step (3) of some stage.

For $n > 1$,

$$\frac{y_1 + y_2 + ... + y_n}{p_{\langle 7,n \rangle}} = -\frac{x_{\pi \upharpoonright 1} + x_{\pi \upharpoonright 2} + ... + x_{\pi \upharpoonright n}}{p_{\langle 7,n \rangle}} + \frac{(y_1 + x_{\pi \upharpoonright 1}) + (y_2 + x_{\pi \upharpoonright 2}) + ... + (y_n + x_{\pi \upharpoonright n})}{p_{\langle 7,n \rangle}}$$

These elements were created during step (4) of some stage.

We see that all the generating elements of G_T can be uniquely decomposed, so $G_T = A_T \oplus B_\pi$.

Now suppose G_T is decomposable as $G_T = A' \oplus B'$. All the x-elements still have strictly maximal type, so they must be in the same direct summand (A').

Each y_j can be decomposed $y_j = a_j + b_j$, where $a_j \in A'$ and $b_j \in B'$. We know a_j and b_j are infinitely divisible by $p_{\langle 1,j \rangle}$ because y_j is (see Proposition 3.1). The only other basis elements that could be infinitely divisible by this prime are the elements x_σ with $|\sigma| = j$.

LEMMA 3.9. *If there is any $y_j \in A'$, then $G_T = A'$ (and $B' = 0$).*

PROOF. Suppose $y_j \in A'$ ($y_j = a_j$), and that another element $y_i \notin A'$. We let $q = p_{\langle 4, \langle j,i \rangle \rangle}$ (we can assume that $j < i$). By the construction of G, q must divide $y_i + y_j$. Thus, q also divides $b_i + b_j$ (which is just b_i). There is some finite sum such that

$$k_0 b_i = k_1 y_i + \sum_{|\sigma|=i} k_\sigma x_\sigma$$

with each $k_\sigma, k_0, k_1 \in Z$. (Recall that the only other basis elements that could be infinitely divisible by $p_{\langle 1,i \rangle}$ are the elements x_σ with $|\sigma| = i$.) We can also write

$$k_0 a_i = (k_0 - k_1) y_i - \sum_{|\sigma|=i} k_\sigma x_\sigma$$

Note that if $k_0 \neq k_1$, then $y_i \in A'$. Thus, $k_1 = k_0$, so

$$b_i = y_i + \frac{1}{k_0} \sum_{|\sigma|=i} k_\sigma x_\sigma$$

and this must be divisible by q. However, q does not divide y_i, nor any nontrivial linear combination of y_i with x-elements (though q does divide $y_i + y_j$). Therefore, $y_i \in A'$. This is true for every y_i, so $A' = G_T$. □

We assume $B' \neq 0$, so there is no $y_j \in A'$. There is no $y_j \in B'$, either. This is because there are no elements

$$\frac{x_j}{p_{<8,j>}}, \frac{y_j}{p_{<8,j>}}$$

So we see that every y_j decomposes as $y_j = a_j + b_j$, with both components being nonzero.

Now suppose y_1, y_2 decompose as

$$y_1 = \sum_{|\sigma|=1} k_\sigma x_\sigma + b_1 \text{ and } y_2 = \sum_{|\rho|=2} l_\rho x_\rho + b_2$$

We shall denote $p_{\langle 7,2 \rangle}$ by r. $r|(y_1 + y_2)$, so it must also divide

$$a_1 + a_2 = \sum_{|\sigma|=1} k_\sigma x_\sigma + \sum_{|\rho|=2} l_\rho x_\rho$$

Although r does not divide any x_σ, from step (4) of the construction we see that r divides elements of the form $x_\sigma + x_{\sigma\hat{\ }m}$ where $|\sigma| = 1$ (and $\sigma\hat{\ }m \in T$). Thus, r also divides elements of the form

$$\sum_m l_m (x_\sigma + x_{\sigma\hat{\ }m}) = k x_\sigma + \sum_m l_m x_{\sigma\hat{\ }m}$$

where $k = \sum_m l_m$ (and $\sigma\hat{\ }m \in T$).

From this we see that

$$r|\ (\sum_{|\sigma|=1} k_\sigma x_\sigma + \sum_{|\rho|=2} l_\rho x_\rho) \text{ iff } k_\sigma \equiv \sum_{\rho \succ \sigma} l_\rho (\text{mod } r)$$

for each σ with $|\sigma| = 1$.

Similarly, $p_{\langle 7,3\rangle} \mid (y_1 + y_2 + y_3)$, so for each $\tau \in T$ with $|\tau| = 3$, $p_{\langle 7,3\rangle} \mid (x_\sigma + x_\rho + x_\tau)$, where $\sigma \prec \rho \prec \tau$.

By the same reasoning, we see that if $y_3 = \sum_{|\tau|=3} m_\tau x_\tau + b_3$, then for each $\sigma \in T$ with $|\sigma| = 1$,

$$k_\sigma \equiv \sum_{\rho \succ \sigma} l_\rho \equiv \sum_{\tau \succ \sigma} m_\tau \ (\text{mod } p_{\langle 7,3\rangle})$$

There are infinitely many such equivalences, so we see that

$$k_\sigma = \sum_{\rho \succ \sigma} l_\rho = \sum_{\tau \succ \sigma} m_\tau = ...$$

It is also true that for each $\rho \in T$ with $|\rho| = 2$,

$$l_\rho \equiv \sum_{\tau \succ \rho} m_\tau \ (\text{mod } p_{\langle 7,3\rangle})$$

Continuing this process, we see that the following also holds:

$$l_\rho = \sum_{\tau \succ \rho} m_\tau = ...$$

Thus, if we choose a σ of length 1 such that $k_\sigma \neq 0$ (which we are guaranteed by the fact that $y_1 \notin B'$), there must be a ρ of length 2 such that $\sigma \prec \rho$ and $l_\rho \neq 0$, and a τ of length 3 such that $\rho \prec \tau$ and $m_\tau \neq 0$. By repeating this process, we find an infinite path through T.

Thus, G_T is decomposable iff T has an infinite path. $\square$

If we start with a computable tree T, then by definition the set of strings σ with $\sigma \in T$ is computable. This means determining whether certain elements are divisible by certain primes in the computable group G_T is itself a computable relation. For example, if $|\sigma| = n$, then relations such as

$$p_{\langle 1,n \rangle} | x_\sigma \text{ and } p_{\langle 8,n \rangle} | (x_n + y_n)$$

will be determined computably.

For $k \in \omega$, determining if there are at least k strings $\tau_1, \tau_2, ..., \tau_k \in T$ with $\sigma \preceq \tau_i$ is a Σ^0_1 relation. Thus, the relation

$$p^k_{\langle 1,n \rangle} | x_\sigma$$

will be Σ^0_1, as desired.

CHAPTER 4

Completely Decomposable Groups

We now turn our attention to the class of completely decomposable groups, specifically the problem of determining which of these groups have computable presentations. The approach we take is to code sets $S \subseteq \omega$ into this class of groups so that no two sets give isomorphic structures ($S_1 \neq S_2 \Rightarrow G_{S_1} \not\cong G_{S_2}$). If we can determine the arithmetical complexity of the sets S for which G_S has a computable presentation, then this indicates to us the difficulty of determining the isomorphism class of these groups.

We do not presume to consider the entire class of completely decomposable groups. We will only discuss groups in which each nonzero element is divisible by only finitely many distinct primes. We give an exact classification for these groups, with the loftier goal of developing ideas that may be applied to a larger class of groups.

1. Background

In [**2**] Downey, Kach, Goncharov, Knight, Kudinov, Melnikov, and Turetsky consider a specific subclass of completely decomposable groups.

Definition 4.1. Given a prime p, let Q_p be the subgroup of Q generated by

$$\langle \frac{1}{p^k} : k > 0 \rangle$$

For a set $S \subset \omega$, let

$$G_S = \bigoplus_{n \in S} Q_{p_n}$$

Suppose G_S has a computable copy. It is clear that $n \in S$ iff there is an element in G_S that is infinitely divisible by p_n. This gives us the Σ^0_3 formula

$$n \in S \Leftrightarrow (\exists x \in G)(\forall k \in \omega)(\exists y \in G)\ p_n^k y = x$$

Thus, if G_S has a computable copy, S must be Σ^0_3. In fact, the converse of this statement holds as well.

THEOREM 4.2. *(Downey, Kach, Goncharov, Knight, Kudinov, Melnikov, Turetsky) G_S has a computable presentation iff S is Σ^0_3.*

We will consider a larger subclass of completely decomposable groups.

DEFINITION 4.3. There is an effective listing of all finite subsets $\{D_0, D_1, ...\}$ of the set of primes P defined by

$$m = \sum_{p_n \in D_m} 2^n$$

Thus,

$$D_0 = \emptyset$$

$$D_1 = \{2\}$$

$$D_2 = \{3\}$$

$$D_3 = \{2, 3\}$$

$$D_4 = \{5\}$$

$$...$$

DEFINITION 4.4. Given a finite set $D \subset P$, let Q_D be the subgroup of Q generated by

$$\langle \frac{1}{p^k} : p \in D, k > 0 \rangle$$

For a set $S \subset \omega$, let

$$H_S = \bigoplus_{m \in S} Q_{D_m}$$

The question we seek to answer is: for which sets S is there a computable presentation of H_S?

2. Prime Sets

Definition 4.5. In a group of the form H_S we refer to an element's *prime set* to denote the set of primes by which it is infinitely divisible.

Definition 4.6. In a completely decomposable group

$$G = \oplus_i G_i$$

if an element $y \in G_i$ for some i, we call y a *true element.* For an arbitrary element x, we can write

$$x = \sum_i g_i$$

with each $g_i \in G_i$. We call this expression the *true decomposition* of x.

Remark 4.7. Suppose y_i and y_j are true elements in a complete decomposition of a group G, and the prime set of y_i is a subset of the prime set of y_j. Then it may be possible to write a complete decomposition of G in which the element $g = y_i + y_j$ is true and y_i is not true. Because y_i and $y_i + y_j$ have the same prime set, the summand containing $y_i + y_j$ will be isomorphic to the one it replaces.

It is true that there is, up to isomorphism, only one way to write a completely decomposable group as a direct sum of rank 1 groups. However, unless the prime sets of true elements form an anti-chain, there is more than one way to choose the summands. This means that the property of being a true element may depend on the choice of summands. We will use true elements in some of the proofs that follow. When we do, we will beforehand fix a complete decomposition of the group under consideration.

Remark 4.8. In a completely decomposable group G, the prime set of any element is the intersection of the prime sets of the elements in its true decomposition (in any complete decomposition of G).

Proof. Fix a complete decomposition of G. If the true decomposition of an element x is given by

$$x = \sum_i g_i$$

then a prime p divides x if and only if p divides each g_i. □

In G_S it is enough to say that $n \in S$ iff there is an element in G_S that is infinitely divisible by p_n. In H_S it is true that if $m \in S$ then there is an element in H_S with prime set D_m, but the converse does not hold.

Proposition 4.9. *$m \in S$ iff there is some $x \in H_S$ with prime set D_m, such that there is no finite set of elements $\{g_1, ...g_n\}$ with $x = \sum\limits_i g_i$ and the prime set of x is a proper subset of the prime set of each g_i. We call such an element x* indecomposable.

Proof. Recall that $H_S = \bigoplus\limits_{m \in S} Q_{D_m}$. If $m \in S$, then there is an element $x \in Q_{D_m}$ with prime set D_m. For any finite sum $x = \Sigma_i g_i$, if we project both sides onto the summand Q_{D_m} we get

$$\bar{x} = \sum_i \bar{g}_i$$

The left side is nonzero, so at least one $\bar{g}_i$ must be nonzero. Thus, its prime set must be a subset of D_m.

If $m \notin S$, then the only way an element of H_S can have prime set D_m is if it is the sum of elements whose prime sets intersect at D_m. Since none of the true elements have prime set D_m, they must all be proper supersets of it. □

We can now characterize membership in S with an arithmetical statement, but what is its complexity? First, let us examine the statement "there is an element in H_S

with prime set D_m."

$$(\exists x \in H_S)[(\forall p \in D_m)(\forall k \in N)(\exists y \in H_S)\, p^k\, y = x$$
$$\wedge(\forall p \notin D_m)(\exists k \in N)(\forall z \in H_S)p^k z \neq x]$$

This is a Σ^0_4 formula. However, because each element is only divisible by finitely many distinct primes, this means we can find an element which is infinitely divisible by every prime $p \in D_m$ and not divisible by all other primes.

$$(\exists x \in H_S)[(\forall p \in D_m)(\forall k \in N)(\exists y \in H_S)\, p^k\, y = x$$
$$\wedge(\forall p \notin D_m)(\forall z \in H_S)pz \neq x]$$

This formula is Σ^0_3, so it is no harder to determine whether there is an element in H_S with prime set D_m than it is to determine whether there is an element in G_S that is infinitely divisible by p_n.

Now let us add to this the clause stating that the element must also be indecomposable.

$$m \in S \Leftrightarrow (\exists x \in H_S)[(\forall p \in D_m)(\forall k \in N)(\exists y \in H_S)\, p^k\, y = x$$
$$\wedge(\forall p \notin D_m)(\forall z \in H_S)pz \neq x$$
$$\wedge(\forall \bar{x} \in H_S^{<\omega})\,(\sum_i x_i = x \Rightarrow \bigvee_i(\forall p \notin D_m)(\exists k \in N)(\forall z \in H_S)p^k\, z \neq x_i]$$

This is a Σ^0_4 formula. From this we have learned that determining the existence of an element with a given prime set is Σ^0_3, but determining whether that element is indecomposable is Π^0_3. This means that if we are to build H_S for an arbitrary Σ^0_4 set S, then a key part of the construction will involve making seemingly indecomposable elements decomposable.

3. $H_S \oplus H_\omega$

If we are to take an element with prime set D_m and make it decomposable, we need two or more elements whose prime sets intersect at D_m. How can we be guaranteed to find such elements? One solution is to add extra summands to the group in a "controlled" way.

We denote by H_ω the completely decomposable group in which every summand has a different prime set, and every finite set of primes is the prime set of some summand in H_ω.

THEOREM 4.10. *There is a computable copy of $H_S \oplus H_\omega$ iff S is Σ^0_4.*

Sketch: Let S be a Σ^0_4 set. There exists a Σ^0_2 relation T such that for all m

$$m \in S \Longleftrightarrow \exists y \forall x \langle m, y, x\rangle \in T$$

and the witnessing y is unique. We use a computable approximation $\{T_s\}_s$ for T so that for all k, $k \in T$ iff $k \in T_s$ for all but finitely many s. We assume the 0^{th} existential witness for membership of any number k in T fails to witness $k \in T$.

For every pair m and y, we create an element $g_{m,y}$ (which will be made infinitely divisible by p for every $p \in D_m$). If y witnesses that $m \in S$, then $g_{m,y}$ will span the direct summand in H_S that is isomorphic to Q_{D_m}. Otherwise, we will decompose $g_{m,y}$ by writing it as the sum of two elements of H_ω in different direct summands. Thus, there will be a (second) direct summand isomorphic to Q_{D_m} if $m \in S$. If $m \notin S$, then no y will witness its membership, so each $g_{m,y}$ will be decomposed.

In order to decompose $g_{m,y}$, we introduce two ω-elements $a_{m,y,x}$ and $b_{m,y,x}$ for the least x such that $\langle m, y, x\rangle \notin T_s$ (we will use ω-elements to construct our copy of H_ω). If there is an x such that $\langle m, y, x\rangle \notin T$, then we will permanently assign

$$g_{m,y} = a_{m,y,x_0} + b_{m,y,x_0}$$

where x_0 is the least such x. If there is a y such that for every x, $\langle m, y, x\rangle \in T$, then every attempt at a decomposition will eventually fail, and we are left with a direct summand isomorphic to Q_{D_m}.

For every element $g_{m,y}$, if the least x such that $\langle m, y, x\rangle \in T_s$ changes from one stage to another, we either trash our current ω-elements (if we want them permanently deleted) or recycle them (if we may want to come back to them later). We recycle ω-elements $a_{m,y,x}$ and $b_{m,y,x}$ by creating a third ω-element $w_{m,y,x}$ and writing the former two as a linear combination of $g_{m,y}$ and the latter. If we wish to continue working on $a_{m,y,x}$ and $b_{m,y,x}$ at a later stage, we simply trash $w_{m,y,x}$.

There is an important distinction to be made. Every ω-element will have a finite set assigned to it (and every finite set will eventually be permanently assigned to an ω-element, giving us the subgroup H_ω. Similarly, each element $g_{\omega,y}$ will be made infinitely divisible by p for each $p \in D_m$, but we do not consider D_m to be "assigned" to $g_{m,y}$, as it is not an ω-element.

Construction: At stage s, create elements $g_{m,s}$ for every $m \leq s$ and $g_{s,s'}$ for $s' < s$ (the latter is just to ensure that we have a $g_{m,y}$ for every pair m and y).

For every element $g_{m,y}$, we find the least x such that $\langle m, y, x\rangle \notin T$. Because of our initial assumption, such an x will always exist.

If there exist no ω-elements $a_{m,y,x}$ and $b_{m,y,x}$, we create them and assign to them the two finite sets of least canonical index, $A_{m,y,x}$ and $B_{m,y,x}$, that are unassigned and such that $A_{m,y,x} \cap B_{m,y,x} = D_m$ (and neither set is D_m). If there exist ω-elements $a_{m,y,x'}$ and $b_{m,y,x'}$ for any $x' > x$, we trash them. If there exist unrecycled ω-elements $a_{m,y,x''}$ and $b_{m,y,x''}$ for any $x'' < x$, we recycle them. These processes are described below.

If $a_{m,y,x}$ and $b_{m,y,x}$ already exist and are currently being recycled, then we trash the elements $w_{m,y,x}, c_{m,y,x}$, and $d_{m,y,x}$ and consider them unrecycled (these are elements created during the recycling process, as described below).

Recycling: To recycle $a_{m,y,x}$ and $b_{m,y,x}$, we do the following. We wish to keep these elements divisible by the primes that already divide them, and not introduce any new infinite divisibilities.

1) We introduce a new ω-element $w_{m,y,x}$ and assign to it the least unassigned set $C_{m,y,x}$ (ordered by canonical index) such that $C_{m,y,x} \cap (A_{m,y,x} \cup B_{m,y,x}) = D_m$.

2) Let $k = \sup\{z : p^z | a_{m,y,x} \text{ for } p \in D_m\}$ (this is the number of stages we have "worked on" $a_{m,y,x}$). We make $w_{m,y,x}$ divisible by p^k for every $p \in C_{m,y,x}$.

3) Let q be the product of all the primes $p \in (A_{m,y,x} \cup B_{m,y,x}) - D_m$. We find two integers α, β, such that:

a) $p^k | \alpha$ for every $p \in A_{m,y,x} - D_m$,

b) $p^k | \beta$ for every $p \in B_{m,y,x} - D_m$,

c) $p \nmid \alpha\beta$ for every $p \in C_{m,y,x} - D_m$, and

d) $\alpha + \beta = 1$.

4) We then declare

$$a_{m,y,x} = \alpha g_{m,y} + q^k w_{m,y,x}$$

$$b_{m,y,x} = \beta g_{m,y} - q^k w_{m,y,x}$$

5) We create "false" ω-elements $c_{m,y,x}$ and $d_{m,y,x}$ which will take the place of $a_{m,y,x}$ and $b_{m,y,x}$ (respectively) in H_ω as long as they are being recycled. We do not consider $a_{m,y,x}$ and $b_{m,y,x}$ to be ω-elements until they are unrecycled, and we reassign their assigned sets to the new elements.

Note that we still have the relation $a_{m,y,x} + b_{m,y,x} = g_{m,y}$, and we can trash the false elements and $w_{m,y,x}$ if we wish to unrecycle $a_{m,y,x}$ and $b_{m,y,x}$. Also, the greatest common divisor of $\alpha g_{m,y}$ and $q^k w_{m,y,x}$ is the product

$$\prod_{p \in A_{m,y,x}} p^k$$

Thus, $a_{m,y,x}$ is not divisible by any new primes (the same is true for $b_{m,y,x}$).

Trashing: To trash $a_{m,y,x}$ and $b_{m,y,x}$, we define $k = \sup\{z : p^z | a_{n,y,x} \text{ for } p \in D_m\}$. Then we find two integers α, β such that:

a) $|\alpha|, |\beta| \geq s$

b) $p^k | \alpha$ for every $p \in A_{m,y,x} - D_m$,

c) $p^k | \beta$ for every $p \in B_{m,y,x} - D_m$,

d) $\alpha + \beta = 1$.

We then declare

$$a_{m,y,x} = \alpha g_{m,y} \quad \text{and} \quad b_{m,y,x} = \beta g_{m,y}$$

To trash $c_{m,y,x}$ and $d_{m,y,x}$, we remove their labels, declare them both to be new ω-elements, and assign them new unassigned sets. Their previously assigned sets are now assigned to $a_{m,y,x}$ and $b_{m,y,x}$ (again).

To trash $w_{m,y,x}$, remove its label and unassign its assigned set. It is no longer considered an ω-element.

At the end of every stage, we introduce a new ω-element and assign to it the least unassigned finite prime set (ordered by index). This is to ensure we process all of ω.

Finally, we make each $g_{m,y}$ divisible by another power of p for every $p \in D_m$. We also make every ω-element divisible by another power of each prime with index in its assigned set. We also add to the group the sum of any two elements already in the group (if it doesn't already exist) and the inverse of every element already in the group (if it doesn't already exist).

Verification: We will see that the group we have constructed is isomorphic to H_S.

LEMMA 4.11. *$g_{m,y}$ is a sum of ω-elements iff $\exists x\langle m, y, x\rangle \notin T$.*

PROOF. Suppose $\exists x\langle m, y, x\rangle \notin T$. Let x_0 be the least such x. Because T is Σ^0_2, there will be a stage s when $\langle m, y, x\rangle \in T_t$ for every $x < x_0$ and $t \geq s$. At this point, there are two elements a_{m,y,x_0}, b_{m,y,x_0} that will never be trashed. They may be recycled, but each time this happens they will subsequently be unrecycled. This means that the two sets assigned to these elements will be permanently assigned to them, and at each stage when they aren't being recycled (of which there will be infinitely many), the elements are made divisible by an additional power of the primes in the sets.

Thus, we can write $g_{m,y}$ as the sum of two ω-elements, a_{m,y,x_0} and b_{m,y,x_0}. For each $x < x_0$, the elements $a_{m,y,x}$ and $b_{m,y,x}$ will be have been recycled, so they will be linear combinations of $g_{m,y}$ and $w_{m,y,x}$.

If $\forall x\langle m, y, x\rangle \in T$, then for all x the elements $a_{m,y,x}$ and $b_{m,y,x}$ will eventually be trashed or recycled without being unrecycled. Thus, there are no permanent ω-elements a and b such that $g_{m,y} = a + b$. □

The ω-elements give us the subgroup H_ω. The only other elements introduced in the construction were the $g_{m,y}$. As we have seen, for every m and y, $g_{m,y}$ is the sum of ω-elements iff $\exists x\langle m, y, x\rangle \notin T$. By the definition of S we have:

$$m \notin S \Leftrightarrow \forall y \exists x\langle m, y, x\rangle \notin T$$

This means if $m \notin S$, then for every y, $g_{m,y}$ is the sum of ω-elements.

By assumption,

$$m \in S \Leftrightarrow \text{ there is a unique } y \text{ such that } \forall x \langle m, y, x\rangle \in T$$

Thus, $g_{m,y}$ is the sum of ω-elements for all but one y. The pure subgroup generated by these elements (for every $m \in S$) is the subgroup H_S.

We have seen that for every y, $g_{m,y}$ is an element of H_S or the sum of two unique ω-elements. Every trashed former ω-element can be uniquely written as the linear combination of a certain $g_{m,y}$ and other ω-elements. The same is true for any permanently recycled ω-element. Any ω-element that is neither trashed nor permanently recycled is an element in H_ω. These are all the elements introduced in the construction, so the group we have constructed is indeed isomorphic to $H_S \oplus H_\omega$. □

The question remains: does every Σ^0_4 set S have a computable presentation of H_S? It seems that without knowing how the sets whose indices are in S relate to one another, we require extra summands to "catch" the elements we wish to make decomposable. In the next chapter we will see that the extra group H_ω (or something similar to it) is necessary.

CHAPTER 5

Limitwise Monotonic Functions

In our quest to find the prime sets of indecomposable elements, we may be temporarily fooled by a decomposable element. If we were to discover that an element is decomposable, then we would know there must be an indecomposable element with a larger prime set (in fact, there must be at least two). For $m, n \in \omega$, if $D_m \subseteq D_n$ then $m \leq n$ (though the converse does not hold). Using a $\mathbf{0}''$oracle to tell us the prime set of any given element, we can find a monotonically increasing approximation of the index of a prime set of an indecomposable element.

DEFINITION 5.1. For a Turing degree $\mathbf{d}$, $F : \omega \to \omega$ is a $\mathbf{d}$-*limitwise monotonic function* if there is a $\mathbf{d}$-computable function $f : \omega \times \omega \to \omega$ such that:

1) $F(n) = \lim_s f(n, s)$ for all n

2) $f(n, s) \leq f(n, s + 1)$ for all n, s

If $\boldsymbol{d} = \mathbf{0}$, we simply call F a limitwise monotonic function.

PROPOSITION 5.2. *If S is the range of a $\mathbf{d}$-limitwise monotonic function, then S is $\Sigma_2^{\mathbf{d}}$ (Σ_2^0 relative to $\mathbf{d}$).*

PROOF. This holds simply because

$$m \in S \Leftrightarrow (\exists k \in \omega)(\exists t \in \omega)(\forall s > t) f(k, s) = m$$

and the equation $f(k, s) = m$ is a $\mathbf{d}$-computable statement. □

PROPOSITION 5.3. *(Khoussainov, Nies, and Shore, [**3**]) There is a $\Delta_2^{\mathbf{d}}$ set which is not the range of a $\mathbf{d}$-limitwise monotonic function.*

PROOF. See [**3**]. □

THEOREM 5.4. *Given a computable presentation of the group H_S, there is a $\mathbf{0}''$-limitwise monotonic function whose range is S.*

COROLLARY 5.5. *There exists a Δ^0_4 set S for which there is no computable presentation of H_S.*

PROOF OF THEOREM 5.4. This proof is done using a $\mathbf{0}''$oracle. This means that we assume that we can computably determine any Δ^0_3 question. We can use this to find an element's prime set. Given an element x, we ask for each prime p

$$(\exists q \in P)(\exists y \in H_S)[q > p \wedge qy = x]$$

until we get a negative answer. This will happen eventually because there are only finitely many distinct primes that divide x. Once we have established the largest prime p which divides x, we can ask for each $q \leq p$

$$(\forall k \in \omega)(\exists y \in H_S)q^k y = x$$

This process tells us the prime set of x.

We enumerate every element of $H_S = \{g_0, g_1, ...\}$. We describe a $\mathbf{0}''$-computable function $f(n, s)$ which is increasing with respect to s. $f(n, s)$ will give the index of the prime set D of g_n. If we find that g_n is decomposable, we shift $f(n, s)$ to the index of the prime set of an element in its decomposition. If that element turns out to be decomposable, we add new elements to the decomposition of g_n and shift our focus once again.

At the beginning of stage s, we set $f(k, s) = 0$ for all $k > s$.

For all elements g_n with $n \leq s$, we determine the prime set of g_n. If g_n is not decomposed already, we check if g_n can be written as a sum of elements $g_{m_1}, g_{m_2}, ..., g_{m_l}$ with each $m_i \leq s$ where each element in the sum has a prime set properly containing the prime set of g_n. If not, we set $f(n, s)$ to be the index of the prime set of g_n.

If we do find such a sum, then g_n is decomposable. For any subsequent stage t, we set $f(n,t)$ to be the minimum of the indices of the prime sets of the elements in the decomposition. If we later find that we an element in the decomposition of g_n is decomposable, we remove this element from the decomposition of g_n and add the elements in its decomposition. (Note that this will not cause f to decrease because it will be assigned to the minimal index of prime sets in the decomposition, and each element that was just added has index greater than the one that was replaced.)

We will see that $\lim_s f(n,s)$ must exist. Fix a complete decomposition of H_S. For any n, let $g_n = \Sigma_j y_j$ be its true decomposition. Then at any stage s, $f(n,s)$ is bounded by the minimum index of the prime sets of the y_j's because in any decomposition $g_n = \Sigma_i g_i$, every y_j (or a rational multiple thereof) must be in the true decomposition of some g_i. Because $f(n,s)$ is monotonically increasing, there is some $F(n) = \lim_s f(n,s)$.

This means that there is an element in the decomposition of g_n whose prime set has index $F(n)$, and this element is never found to be decomposable (because whenever we add new elements to the decomposition, their prime sets must have a greater index than the element they replaced, so there can only be finitely many with index $= F(n)$). Thus, it is an indecomposable element, so $F(n) \in S$. Because we iterate every element of H_S, we will find every $n \in S$. □

Corollary 5.5 tells us that the arithmetical hierarchy does an insufficient job of characterizing the sets we seek. This prompts us to ask: does the converse of Theorem 5.4 hold? Does every set S which is the range of a $\mathbf{0}''$-limitwise monotonic function have a computable presentation of H_S? The answer is quite clearly no. To see why this is the case, we must add to the language of the group structure. Because the construction above was done with a $\mathbf{0}''$oracle, we must answer any Δ^0_3 questions as if they were computable.

Definition 5.6. An *expanded group* $(G, +, \pi)$ is a group $(G, +)$ with additional unary relations $\{R_m\}_m$ which state that an element has a prime set of index m and for every n an n-ary relation Λ_n which consists of all linearly independent n-tuples in G. In any computable presentation of G, these relations are computable (as well as $+$).

For any $g \in G$, $R_m(x)$ will hold for exactly one m, so we will represent these relations as a function $\pi : G \to \omega$ which gives the index of the prime set of an element. We can also combine the relations Λ_n into $\Lambda \subset G^{[<\omega]}$, a relation on finite subsets of G.

The first step in building a computable presentation of H_S for an S which is the range of a $\mathbf{0}''$-limitwise monotonic function is to build a computable presentation of an expanded H_S for any S which is the range of a limitwise monotonic function.

Let $F(*) = \lim_s f(*, s)$ be a limitwise monotonic function. Let us suppose that $f(0, 0) = 1$, which is the index of the prime set $\{2\}$. We have little choice but to create an element and declare it to be infinitely divisible by 2. Suppose further that $f(0, 1) = 2$, the index of the prime set $\{3\}$, and that no other index $f(k, s)$ for $s > 0$ gives us a prime set containing 2. Then there should be no nonzero element infinitely divisible by 2 in the final group we have constructed, but there is nowhere for us to "hide" this element we have created.

From this, we see that a problem could occur whenever $D_{f(k,s)} \nsubseteq D_{f(k,s+1)}$. The natural solution to this is to consider the prime sets themselves, rather than their indices. A new definition may be necessary.

1. Limitwise Set-Monotonic Functions

In order to avoid the problems described at the end of the previous section, we must force the limitwise monotonic functions to grow only along subsets. This makes the idea of referring to the index of the prime set unnecessary, so we will deal directly with the prime sets themselves.

Definition 5.7. For a Turing degree $\mathbf{d}$, $F : \omega \to P(\omega)$ is a $\mathbf{d}$-*limitwise set-monotonic function* if there is a $\mathbf{d}$-computable function $f : \omega \times \omega \to P(\omega)$ such that:

1) $F(n) = \lim_s f(n, s)$ for all n

2) $f(n, s) \subseteq f(n, s + 1)$ for all n, s

We also need to alter how we define the group we wish to construct.

Definition 5.8. Let T be a collection of distinct finite sets of primes. For $D \in T$, let Q_D be the subgroup of Q generated by $\langle \frac{1}{p^k} : p \in D, k > 0 \rangle$. Let

$$H_T = \bigoplus_{D \in T} Q_D$$

Using a limitwise set-monotonic function fixes the issue mentioned above, but a new problem arises. Suppose we try to construct such a function using a construction similar to the one given in the proof of Theorem 5.4. Suppose further that in our construction we come across an element g_n with the following true decomposition.

$$\underset{2}{g_n} = \underset{2,5}{y_1} + \underset{2,7}{y_2}$$

The primes listed below an element denote its prime set. g_n is decomposable, and we will eventually discover this as we enumerate the elements of the group. However, we cannot be guaranteed to find the decomposition shown above. Suppose that $\underset{2,3,5}{y_3}, \underset{2,3,7}{y_4}$ are also true elements, and let

$$\underset{2,5}{z_1} = \underset{2,5}{y_1} + \underset{2,3,5}{y_3}$$

$$\underset{2,7}{z_2} = \underset{2,7}{y_2} + \underset{2,3,7}{y_4}$$

$$\underset{2,3}{z_3} = \underset{2,3,5}{y_3} + \underset{2,3,7}{y_4}$$

Suppose that in our construction z_1, z_2, z_3 are all enumerated before y_1 and y_2. Then we will see g_n decompose as

$$\underset{2}{g_n} = \underset{2,5}{z_1} + \underset{2,7}{z_2} - \underset{2,3}{z_3}$$

We can assume that we shift our focus from g_n to z_3 (if for no other reason than that it has the prime set of least index). An important point in the proof of Theorem 5.4 was the fact that the limitwise monotonic function would also be bounded above by some element in the true decomposition of g_n. However, the nonlinear nature of the subset relation means that in a case such as this, we may lose that bound. Therefore, it is possible that $\lim_s f(n,s)$ may be infinite.

THEOREM 5.9. *There is a group H_T with a computable presentation such that T is not the range of a $\mathbf{0}''$-limitwise set-monotonic function.*

PROOF. We will show that there is an expanded group H_T with a computable presentation such that T is not the range of a limitwise set-monotonic function.

Let $\{\phi_e\}_e$ be an effective listing of the computable functions with domain $\omega \times \omega$.

Construction: For every e there will be an element $x_e \in H_T$ which witnesses that ϕ_e is not a limitwise monotonic function with range T. Each x_e will have prime set $\{p_{2e}\}$, and if for some n, s we see that $\phi_e(n,s) = \{p_{2e}\}$, we make x_e decomposable. We call n a witness for e. If at any future stage t, $\phi_e(n_e, t)$ gives the prime set of an indecomposable element, we will make that element decomposable as well.

Stage s: We create an element x_s that is linearly independent from all other elements $x_{e'}$ and give it prime set $\{p_s\}$. For every $e \leq s$ which does not already have a witness, we determine if there is an $n_e \leq s$ such that $\phi_e(n_e, s) = \{p_{2e}\}$. If we find such an n_e, we declare it to be a witness for e. We then create three new elements: a_e, b_e, c_e, and declare

$$\underset{p_{2e}}{x_e} = \underset{p_{2e},p_1}{a_e} + \underset{p_{2e},p_3}{b_e} + \underset{p_{2e},p_5}{c_e}$$

The three new elements form a linearly independent set, and their prime sets are denoted in the above equation.

If at a future stage t we see that $\phi_e(n_e, t) \nsubseteq \phi_e(n_e, t+1)$, then we disregard ϕ_e henceforth because it does not give a limitwise monotonic function.

If at a future stage t we see that $\phi_e(n_e, t)$ gives the prime set for a_e, b_e, or c_e, then we make this element decomposable as well. For example, if $\phi_e(n_e, t) = \{p_{2e}, p_1\}$, then we create four new elements $\alpha_e, \beta_e, \gamma_e, \delta_e$ which are linearly independent from one another, and declare

$$\begin{aligned}
\underset{p_{2e},p_1}{a_e} &= \underset{p_{2e},p_1,p_3}{\gamma_e} + \underset{p_{2e},p_1,p_5}{\delta_e} \\
\underset{p_{2e},p_3}{b_e} &= \underset{p_{2e},p_3}{\alpha_e} - \underset{p_{2e},p_1,p_3}{\gamma_e} \\
\underset{p_{2e},p_5}{c_e} &= \underset{p_{2e},p_5}{\beta_e} - \underset{p_{2e},p_1,p_5}{\delta_e}
\end{aligned}$$

Then

$$\underset{p_{2e}}{x_e} = \underset{p_{2e},p_1}{a_e} + \underset{p_{2e},p_3}{b_e} + \underset{p_{2e},p_5}{c_e} = \underset{p_{2e},p_3}{\alpha_e} + \underset{p_{2e},p_5}{\beta_e}$$

We see that $\phi_e(n_e, t)$ is the prime set of a_e, which is now decomposable. We will have no reason to make α_e or β_e decomposable, so they will be true elements in H_T, and the true decomposition of x_e is fixed. If $\phi_e(n_e, t')$ gives the prime set of γ_e or δ_e at a future stage t', we will treat that element as we did x_e, using three new elements and three different odd-indexed primes.

It will be important to note that every element we introduce to the group this way has a distinct prime set. This is because every element we introduce is infinitely divisible by precisely one even-indexed prime. This also means we do not have to worry about interference by another witness $n_{e'}$ because $\phi_{e'}(n_{e'}, s)$ must contain $p_{2e'}$.

At the end of every stage, we make each element divisible by another power of every prime in its prime set. We also add to the group the sum of any two elements already in the group (if it doesn't already exist) and the inverse of every element already in the group (if it doesn't already exist). This concludes the construction.

Verification: For each e, one of four things can happen.

1) We never find a witness n_e such that $\phi_e(n_e, s) = \{p_{2e}\}$ for some s.

2) We find an n_e, and for some s, $\phi_e(n_e, s) \nsubseteq \phi_e(n_e, s+1)$

3) $\phi_e(n_e, s) \subseteq \phi_e(n_e, s+1)$ for every s, and $\Phi_e(n_e) = \lim_s \phi_e(n_e, s)$ converges

4) $\phi_e(n_e, s) \subseteq \phi_e(n_e, s+1)$ for every s, and $\Phi_e(n_e) = \lim_s \phi_e(n_e, s)$ diverges

Case 1): Φ_e is not onto T.

Case 2): ϕ_e does not give us a limitwise monotonic function.

Case 3): Whenever $\phi_e(n_e, s)$ changes, we take steps to make sure it does not give the prime set of an indecomposable element. Thus, $\Phi_e(n_e) \notin T$.

Case 4): Φ_e is not total.

It also must be stated that even if $\Phi_e(n_e)$ diverges, every element introduced to the group along the way is a finite linear combination of true elements, and these true elements have distinct finite prime sets. Thus, the final group is isomorphic to H_T for some T. $\square$

2. Partial Limitwise Set-Monotonic Functions

It is clear that if we wish to proceed with limitwise set-monotonic functions, we need to allow our function to diverge. This will indicate that the element we were considering has decomposed, and new elements must be introduced to the group. We may have an infinite chain of elements with increasing prime sets, but this way we avoid making a single element divisible by infinitely many distinct primes.

Definition 5.10. For a Turing degree $\mathbf{d}$, $F : \omega \rightharpoonup P(\omega)$ is a *partial limitwise set-monotonic* $\mathbf{d}$*-function* if there is a partial $\mathbf{d}$-computable function $f : \omega \times \omega \rightharpoonup P(\omega)$ such that:

1) $F(n) = \lim_s f(n, s)$ for all n

2) $f(n, s) \subseteq f(n, s + 1)$ for all n, s such that $f(n, s + 1) \downarrow$

3) For all n, s, if $f(n, s) \uparrow$, then $f(n, s + 1) \uparrow$

Suppose we have $f(n, s)$ give us the prime set of some element g at stage s. If we see g decompose at this stage, we can have $f(n, s) \uparrow$ and have another index $n' > n$ pick up where we left off. Any index k such that $f(k, s) \downarrow$ for all s will eventually give us the prime set of an indecomposable element. This process means that the partial limitwise set-monotonic function alone will not be sufficient information to construct the group. We will also need some way of coding this transition from n to n'.

Definition 5.11. If $F(*) = \lim_s f(*, s)$ is a partial $\mathbf{d}$-limitwise set-monotonic function, than a *decomposing function* for F consists of two parts:

1) A computable function Φ with domain $\omega \times \omega$ that will output the current expression for an element as a linear combination of elements which are indecomposable at a given stage, as well as the prime sets of all elements involved. For example, the output of $\Phi(n, s)$ will be

$$\underset{Q_n}{g_n} = \sum_l q_l \underset{P_{l,s}}{x_l}$$

Here Q_n is the prime set of g_n, $P_{l,s}$ is the prime set of x_l at stage s, and the coefficients on the right side are rationals.

2) A computable function I with domain $\omega \times \omega$ which will detail how we rewrite existing elements as linear combinations of new elements at each stage. For example, $I(k, s)$ will be of the form

$$\underset{P_{k,s-1}}{x_k} \rightarrow \sum_l q_l \underset{P_{l,s}}{x_l}$$

A decomposing function has the following properties:

1) For any n, $\Phi(n,s)$ will converge as $s \to \infty$, and for any linear combination of elements

$$\sum_l m_l x_l$$

with each $m_l \in Z$ such that $f(l,s) \downarrow$ for each l and every s, there is some n such that $\Phi(n,s)$ converges to that sum.

2) For any n and any $s > 0$, $\Phi(n,s)$ can be derived by taking $\Phi(n,s-1)$ and replacing each element x_k with the sum given by $I(k,s)$.

3) For any k and any $s > 0$, if $I(k,s)$ states that

$$x_k \to \sum_l q_l x_l$$

then $f(k,s-1) = \bigcap_{\{l:q_l \neq 0\}} f(l,s)$. If we replace each element x_l in the sum above with the sum given by $I(l,s+1)$

$$x_l \to \sum_j r_j x_j$$

and rewrite the expression as

$$\sum_l q_l (\sum_j r_j x_j) = \sum_i \eta_i x_i$$

then $f(k,s-1) = \bigcap_{\{i:\eta_i \neq 0\}} f(i,s+1)$. We can repeat this process any number of times, and the intersection of the prime sets of any elements with nonzero coefficients will still be $f(k,s-1)$.

4) There are only finitely many stages s such that $I(k,s)$ indicates that x_k has moved.

5) For any k and any $s > 0$, $f(k,s-1) \downarrow$ and $f(k,s) \uparrow$ iff $I(k,s)$ says

$$x_k \to \sum_l q_l x_l$$

with $q_k = 0$. Furthermore, the label x_k will never appear in an expression from this stage forward and $f(k,t)\uparrow$ for all $t > s$.

6) For any $s > 0$, any k, if $f(k,s) \neq f(k,s-1)$, then $I(k,s)$ states that

$$x_k \to \sum_l q_l x_l$$

with $q_k \neq 0$ and $q_l \neq 0$ for at least one $l \neq k$. Also, the sum $x_k - \sum_{l \neq k} q_l x_l$ cannot be a scalar multiple of any expression $\Phi(n,s)$ with $n < s$.

7) For any n and any $s > n$,

$$\bigcap_{\{l:\, q_l \neq 0 \text{ in } \Phi(n,s-1)\}} f(l,s-1) = \bigcap_{\{l:\, q_l \neq 0 \text{ in } \Phi(n,s)\}} f(l,s)$$

8) For any s, the sums listed in $I(k,s)$ for all k form a linearly independent set.

9) For any k, there is a prime p such that for any n,s, if $q_k \neq 0$ is the coefficient of x_k in $\Phi(n,s)$, then the denominator of q_k is not divisible by any prime $q > p$. Also, for any j,t, if $r_k \neq 0$ is the coefficient of x_k in $I(j,t)$, then the denominator of r_k is not divisible by any prime $q > p$.

10) For any k and any prime p, either $p \in f(k,s)$ for some s or there is some $K \in \omega$ such that for any n,s, if $q_k \neq 0$ is the coefficient of x_k in $\Phi(n,s)$ or $I(n,s,)$, then the denominator of q_k is not divisible by p^K.

11) For any k, there is a pair (n,s) such that $\Phi(n,s)$ states

$$g_n = x_k$$

12) For any s and any n,m, if $\Phi(n,s) = \Phi(m,s)$, then $n = m$.

We will use the decomposing function to build a computable copy of the expanded group H_T. The function Φ serves two purposes: first, it allows us to ensure that the basis we will build is indeed a basis. Secondly, it guarantees that the prime set of each element in the group does not change throughout the construction. The function I tells us how to move the labels.

THEOREM 5.12. *If there is a computable copy of H_T, there is an injective partial $\mathbf{0}''$-limitwise set-monotonic function F whose range is T, and F has a decomposing function computable in $\mathbf{0}''$.*

PROOF. Because we are using a $\mathbf{0}''$oracle, we can determine whether a finite set of elements is linearly independent as well as the prime set of any given element.

We want to find a basis $\{x_k\}_{k\in\omega}$ for the group such that the prime sets of the basis elements are in one-to-one correspondence with the sets in T. We will create labels $\{x_k\}_{k\in\omega}$ which will approximate this basis. The labels will be initially assigned to an element, but can move from one element to another in order to give us a better approximation. Each label will only move finitely many times. The elements that have labels assigned to them permanently will constitute the basis we seek. We will delete a label x_k at stage s if $f(k,s)\uparrow$ and create a new label (so that the span of the basis does not shrink). Any time a label moves, this will be reflected in the decomposing function.

We will enumerate the nonzero elements of the group $\{g_0, g_1, g_2, ...\}$. At stage s, we want to ensure that every g_t with $t \leq s$ can be written as a sum of basis elements, each of which has a prime set containing the prime set of g_t. We do this by asking at stage s if g_s is linearly independent from the basis so far. If so, we make it a new basis element. If not, we determine its prime set D and how it can be expressed as a sum of basis elements

$$\underset{D}{g_s} = \sum_k q_k \underset{P_{k,s}}{x_k}$$

If for each k with $q_k \neq 0$, $D \subseteq P_{k,s}$, we do nothing. If $P_{k,s} \subsetneq D$ for some k, we remove one of these elements from the basis and replace it with g_s. It is also possible that neither of these cases hold, and there exists some basis element whose prime set is incomparable with D. This means that we will have to search through sums of elements to find a way to satisfy our goal, but eventually a solution will be found.

Definition 5.13. At every stage of the construction, each element g_n with $n \leq s$ will be a linear combination of elements with labels, and its prime set should be the intersection of the prime sets of these labeled elements. If this is not the case and there is a labeled element of least prime set, that element is decomposable. If there are two or more labeled elements of least prime set, we say that we have found a *hidden divisibilty.*

Example 5.14. To illustrate this phenomenon, suppose that T is the set of all initial segments of the primes:

$$T = \{\{2\}, \{2,3\}, \{2,3,5\}, ...\}$$

and that $y_0, y_1, y_2, ...$ is a basis of true elements in H_T where each y_j has prime set $\{p_0 = 2, p_1, ..., p_j\}$. Then there is an alternate basis $g_0, g_1, g_2, ...$ of H_T where

$$g_0 = y_0$$

$$g_1 = y_0 + y_1$$

$$g_2 = y_0 + y_1 + y_2$$

$$...$$

Every element in this basis has prime set $\{2\}$, but our goal is to construct a basis of H_T where the prime sets of the basis elements are in one-to-one correspondence with the sets in T.

We can fix this by replacing g_1 in the basis with $g_1 - g_0 = y_1$. The set $\{g_0, y_1\}$ has the same span as $\{g_0, g_1\}$, but does a better job of representing the sets in T. During the construction, we will continuously look for linear combinations of basis elements that have more infinite divisibilities than the elements in the sum. This will uncover these hidden divisibilities and make sure we get the one-to-one correspondence with T.

We would also like to make the true decomposition of every g_s as short as possible to ensure that every element in the group is a finite sum of basis elements. If we see a sum of the form

$$g_s = x_i + x_j + x_k + x_l$$

where the prime set of x_j is properly contained in the prime sets of x_k and x_l, we can shorten this expression for g_s by moving the label x_j to the element $x_j + x_k + x_l$. If we do, the sum is now

$$g_s = x_i + x_j$$

and the prime set of x_j remains the same. We don't want to move the label if doing so would "injure" some g_t with $t < s$ by making its expression longer.

If at stage s of the construction we determine that a basis element x_k is decomposable, we will have to remove it from the basis and set $f(k, s) \uparrow$. We will also add elements to the basis until the decomposed element is in the span of the basis again.

When we want to remove an element from the basis, we will have to add new elements to the basis so that its span does not shrink. The decomposing function will track how this happens. This will allow us to reconstruct the group from the partial limitwise set-monotonic function.

Construction: We computably enumerate the nonzero elements of the group $g_0, g_1, g_2, \ldots$. We will construct a basis for the group and give every basis element a label x_k. If x_k and x_l are basis elements and $k < l$, then x_k has higher priority than x_l. We denote the prime set of x_k at stage s by $P_{k,s}$.

Stage 0: We add g_0 to the basis and give it the label x_0. We determine the prime set D of g_0 and set $f(0, 0) = D$.

Stage $s > 0$: We check to see if g_s is linearly independent from the basis elements. If it is, we create a new label and apply it to g_s. If not, we determine the expression for g_s as a linear combination of basis elements.

$$\underset{D}{g_s} = \sum_k q_k \underset{P_{k,s}}{x_k}$$

If $D \subseteq \cap_{\{k:q_k \neq 0\}} P_{k,s}$, then we do nothing, and this stage is finished. If not, we do the following.

If there is an element x_k such that $P_{k,s} \subsetneq D$ and $P_{k,s} \subsetneq P_{j,s}$ for every other x_j with $q_j \neq 0$, we know that element is decomposable. We remove it from the basis and permanently delete its label (never to be used again). We then add g_s to the basis and give it a new label.

If there are two or more x_k's such that $P_{k,s} \subsetneq D$ and $P_{k,s} \subseteq P_{j,s}$ for every other x_j with $q_j \neq 0$, we may have found a hidden divisibility. We find the x_k in the sum of lowest priority with this property and move its label to g_s.

It is possible that there is no x_k with minimal prime set which is also a proper subset of D. In this case, we consider the elements in the sum whose prime sets do not contain D. We search through sums of group elements to see if any of them are decomposable, and for any of these elements with the same prime set, we search through linear combinations of them for hidden divisibilities. We want to find new elements for some of the labels that allow us to write each old basis element in the sum as a sum of new basis elements whose prime sets contain its own, such that when we combine the old basis elements to sum to g_s, we see that g_s is a sum of new basis elements, each of which having a prime set containing D.

The process for doing this is as follows. We will simultaneously perform two types of searches.

1) If there are two or more basis elements $\{x_{k_1}, x_{k_2}, ...x_{k_n}\}$ in the sum for g_s with the same prime set, we search through all linear combinations $\sum_i q_i x_{k_i}$ for a hidden

divisibility. If we find one, we reassign the lowest priority label with $q_i \neq 0$ to this sum. This will change the expression for g_s as a sum of basis elements.

2) We also search through all sums of all group elements for evidence that one of the basis elements in the sum of g_s is decomposable. If we find a decomposition

$$x_k = \sum_j g_j$$

then for each g_j we ask if it is linearly independent from the other basis elements (including any temporary labels, but excluding x_k or any other element we now know to be decomposable). If it is not, we find how to write it as a linear combination of these basis elements. If it is linearly independent, we give it a temporary label that is different from all other labels.

After doing this for each g_j, we check the expression for g_s with the new temporary labels instead of the labels attached to decomposable elements. If the expression does not meet the requirement for this stage, we remove the temporary labels and resume our search. If the expression does meet the requirement, we are done. We move the label from each decomposable element to one in its decomposition with a temporary label. After doing this, if there are any elements with temporary labels but not permanent labels, we give these elements brand new labels.

Here are some examples to illustrate this process. We take $y_0, y_1, ...$ to be true elements in a fixed decomposition of H_T.

EXAMPLE 5.15. At stage 0, we apply the label x_0 to the element

$$\underset{2}{g_0} = \underset{2}{y_0}$$

At stage 1, we apply the label x_1 to the element

$$\underset{2}{g_1} = \underset{2}{y_0} + \underset{2,3}{y_1}$$

At stage 2, we apply the label x_2 to the element

$$\underset{5}{g_2} = \underset{5}{y_2}$$

At stage 3, we apply the label x_3 to the element

$$\underset{5}{g_3} = \underset{5}{y_2} + \underset{3,5}{y_3}$$

At stage 4, we discover that the element

$$\underset{3}{g_4} = \underset{2,3}{y_1} + \underset{3,5}{y_3}$$

can be written as

$$\underset{3}{g_4} = \underset{2}{x_1} - \underset{2}{x_0} + \underset{5}{x_3} - \underset{5}{x_2}.$$

We search and discover that there are two hidden divisibilities:

$$\underset{2}{x_1} - \underset{2}{x_0} = \underset{2,3}{y_1}$$

and

$$\underset{5}{x_3} - \underset{5}{x_2} = \underset{3,5}{y_3}$$

so we move the label x_1 to the element y_1 and the label x_3 to the element y_3. Now we have

$$\underset{3}{g_4} = \underset{2,3}{x_1} + \underset{3,5}{x_3}$$

and we have met the requirement for g_4 at this stage.

EXAMPLE 5.16. At stage 0, we apply the label x_0 to the element

$$\underset{2}{g_0} = \underset{2,3}{y_0} + \underset{2,5}{y_1}$$

At stage 1, we apply the label x_1 to the element

$$\underset{3}{g_1} = \underset{2,3}{y_0} + \underset{3,5}{y_2}$$

At stage 2, we discover the element

$$\underset{5}{g_2} = \underset{2,5}{y_1} - \underset{3,5}{y_2} = \underset{2}{x_0} - \underset{3}{x_1}$$

There is no hidden divisibility here, but we can search and discover that both x_0 and x_1 are decomposable. In fact, we will come across equations of the form

$$x_0 = y_0 + \sum_i g_i$$

and

$$x_1 = y_0 + \sum_j g_j$$

where each g_i is infinitely divisible by 2 and 5, and g_j is infinitely divisible by 3 and 5. Since we have found x_0 and x_1 to be decomposable, we give y_0 a temporary label, as well as the other elements in the equations. We find that when we combine these sums as described by the expression $x_0 - x_1$, the y_0's cancel, leaving us with only basis elements whose prime sets contain 5.

This means we delete the labels x_0 and x_1 and give each element with a temporary label a brand new label.

Example 5.17. At stage 0, we apply the label x_0 to the element

$$\underset{2}{g_0} = \underset{2}{y_0} + \underset{2,5,7,11}{y_1}$$

At stage 1, we apply the label x_1 to the element

$$\underset{3,5}{g_1} = \underset{3,5}{y_2}$$

At stage 2, we apply the label x_2 to the element

$$\underset{5,7}{g_2} = \underset{2,5,7,11}{y_1} + \underset{3,5,7}{y_3}$$

At stage 3, we apply the label x_3 to the element

$$\underset{3,5}{g_3} = \underset{3,5}{y_2} - \underset{3,5,7}{y_3}$$

At stage 4, we discover the element

$$\underset{2}{g_4} = \underset{2}{y_0} = \underset{2}{x_0} + \underset{3,5}{x_1} - \underset{5,7}{x_2} - \underset{3,5}{x_3}$$

g_4 and x_0 have the same prime set, but none of the other basis elements have 2 in their prime sets. x_2 is decomposable, and this is clear from the expression above because no other element has a prime set which is a subset of $\{5, 7\}$. x_1 and x_3 are indecomposable, so we will not be able to find a decomposition of either element. Because they have the same prime set, we search their span for a hidden divisibility. We eventually discover that $y_3 = x_1 - x_3$ has prime set $\{3, 5, 7\}$, so we move the label x_3 to the element y_3.

We will eventually discover a decomposition for x_2 of the form

$$y_3 + \sum_i g_i$$

where each g_i is infinitely divisible by 2, 5, and 7. y_3 already has a label x_3, so we give each g_i a temporary label.

We can now write

$$\underset{5,7}{g_2} = \underset{3,5,7}{x_3} + \underset{2,5,7,\ldots}{\sum_i g_i}$$

$$\underset{3,5}{g_3} = \underset{3,5}{x_1} - \underset{3,5,7}{x_3}$$

$$\underset{2}{g_4} = \underset{2}{x_0} + \underset{2,5,7,\ldots}{\sum_i g_i}$$

Note that if we find the decomposition of x_2 before the hidden divisibility among x_1 and x_3, we will be unable to satisfy our requirement for g_4 and keep searching. Once we find the hidden divisibility we can check the decomposition again and see that our goal is met.

We resume the construction. The next step is to minimize the length of the expression for every g_t with $t \leq s$. Starting with g_0, we check its current expression

$$g_0 = \sum_k q_k x_k$$

to see if there are two basis elements in the sum, x_i, x_j with $i < j$ such that $P_{i,s} \subsetneq P_{j,s}$. If there are, we take the x_i of lowest priority with this property, and we move the label x_i to the element

$$q_i x_i + \sum_{\{j:\, i<j \text{ and } P_{i,s} \subsetneq P_{j,s}\}} q_j x_j$$

We have replaced all these terms in the expression with the term x_i, and we say that x_i has *absorbed* the x_j's. We check the prime set of this new basis element. If it is strictly larger than $P_{i,s}$ (the prime set of the former basis element), then the previous element was decomposable. In this case, we delete the label x_i and replace it with a new label $x_{i'}$.

We repeat this until no more such pairs remain in the expression. We then repeat this process for $g_1, g_2, ... g_s$, with the added condition that we do not choose an x_i which is currently in the expression for a higher priority element g_t.

EXAMPLE 5.18. At stage 0, we apply the label x_0 to the element

$$\underset{2}{g_0} = \underset{2}{y_0}$$

At stage 1, we apply the label x_1 to the element

$$\underset{2}{g_1} = \underset{2}{y_0} + \underset{2,3,5}{y_1} + \underset{2,3,7}{y_2}$$

At stage 2, we apply the label x_2 to the element

$$\underset{2,3,5}{g_2} = \underset{2,3,5}{y_1}$$

At stage 3, we discover the element

$$\underset{2,3}{g_3} = \underset{2,3,5}{y_1} + \underset{2,3,7}{y_2} = \underset{2}{x_1} - \underset{2}{x_0}$$

so we reassign the label x_1 to g_3.

We now write $\underset{2}{g_1} = \underset{2}{x_0} + \underset{2,3}{x_1}$. We could shorten this expression by moving the label x_0 to g_1, but that would lengthen the expression for g_0. Thus, the label remains where it is.

Suppose at a later stage we find that x_1 is decomposable as

$$\underset{2,3}{x_1} = \underset{2,3,5}{z_1} + \underset{2,3,7}{z_2}$$

where

$$\underset{2,3,5}{z_1} = \underset{2,3,5}{y_1} + \underset{2,3,5,7,11}{y_3} + \underset{2,3,5,7,13}{y_4}$$

and

$$\underset{2,3,7}{z_2} = \underset{2,3,7}{y_2} - \underset{2,3,5,7,11}{y_3} - \underset{2,3,5,7,13}{y_4}$$

We delete the label x_1 and assign the labels x_3 and x_4 to z_1 and z_2 (respectively). We now write $g_1 = x_0 + x_3 + x_4$. Again, we cannot shorten this without injuring g_0.

At a later stage s, we apply the label x_5 to the element

$$\underset{2,3,5,7,11}{g_s} = \underset{2,3,5,7,11}{y_3}$$

At a later stage, we find the hidden divisibility

$$\underset{2,3,5,7}{g_s} = \underset{2,3,5,7,11}{y_3} + \underset{2,3,5,7,13}{y_4} = \underset{2,3,5}{x_3} - \underset{2,3,5}{x_2}$$

We reassign x_3 to g_s. We now write

$$\underset{2}{g_1} = \underset{2}{x_0} + \underset{2,3,5}{x_2} + \underset{2,3,5,7}{x_3} + \underset{2,3,7}{x_4}$$

This means we can shorten the expression for g_1 by moving the label x_2 to the element $x_2 + x_3$ (we can also say that x_2 absorbed x_3). Now we write

$$\underset{2}{g_1} = \underset{2}{x_0} + \underset{2,3,5}{x_2} + \underset{2,3,7}{x_4}$$

Now we determine the values of $f(l,s)$. If there is an element labeled x_l, set $f(l,s)$ to be its prime set. If there was an element x_l that decomposed at a previous stage, we have $f(l,s)$ diverge. Otherwise, we set $f(l,s) = \emptyset$.

At the end of the stage, for each $n \leq s$ we set $\Phi(n,s)$ be the current expression for g_n, including the prime sets of all the elements involved. If a brand new label x_k has been assigned at this stage to an element g_n with $n > s$, we set $\Phi(n,s)$ to state that

$$g_n = x_k$$

We set I to show how each previous basis element is written as the sum of new basis elements as described in Definition 5.11. If a label x_k doesn't move (or was deleted at a previous stage), then $I(k,s)$ simply says

$$x_k \to x_k$$

If x_l is a label that was created at this stage or has yet to be created, $I(l,s)$ is undefined.

Verification: We need to show that the basis we have constructed is indeed a basis for H_T, and that the prime sets of the basis elements are in one-to-one correspondence with the sets in T. We fix a complete decomposition of H_T.

Definition 5.19. During any stage of the construction, we say that a label x_k is *active* if it has not been deleted. We say x_k is permanent if it is active at cofinitely many stages. We say a label x_k has *settled* if one of the following two things have happened:

1) x_k has been (permanently) deleted

2) x_k is active and assigned to an element from which it will never be removed

Definition 5.20. In any complete decomposition of H_T, an indecomposable element y_j has an element in its true decomposition whose prime set is a proper subset of the prime set of every other element in the true decomposition of y_j. We call the summand containing this element the *least summand* of y_j.

The first thing we need to show is that every label eventually settles. For that, we need the following lemma.

Lemma 5.21. *If two indecomposable basis elements have the same least summand in a decomposition, then one of their labels will be removed eventually.*

Proof. Assume x_j and x_k (with $j < k$) are indecomposable and share a least summand in a decomposition. We write

$$x_j = q_i y_i + \sum_l q_l y_l \text{ and } x_k = r_i y_i + \sum_l r_l y_l$$

with the prime set of y_i properly contained in the prime set of each y_l. Denote

$$\Lambda = \{y_l \neq y_i |\, q_l \neq 0 \text{ or } r_l \neq 0\}.$$

We note that $\{y_i\} \cup \Lambda$ is a linearly independent set, while $\{x_j, x_k\} \cup \Lambda$ is not.

Let y_λ be the last $y_l \in \Lambda$ to be enumerated. Suppose the labels x_j and x_k have not moved by the stage when we meet the requirement for y_λ. Because x_j, x_k, and every other $y_l \in \Lambda$ are already in the span of the basis, y_λ must also be in the span.

In fact, from the above expressions for x_j and x_k, we get an expression of the form

$$y_\lambda = r_i x_j - q_i x_k + (\quad)$$

where the last part of the sum is comprised of basis elements whose prime sets all properly contain the prime set of y_i (which is also the prime set of x_j and x_k).

Thus, we have discovered a hidden divisibility, and the label x_k is moved to y_λ. $\square$

COROLLARY 5.22. *F is injective.*

LEMMA 5.23. *If an indecomposable basis element x_k is moved via a hidden divisibility, there must be an indecomposable basis element with higher priority in the sum whose true decomposition contains an element in the least summand of x_k.*

PROOF. This means that

$$g = \sum_l q_l x_l$$

with $q_k \neq 0$ and the prime set of g properly containing the prime set of x_k.

If we project both sides of this equation onto the least summand of x_k, we get

$$0 = \sum_l q_l \bar{x}_l$$

where $\bar{x}_l$ is the projection of x_l. Because the prime set of g is larger than the prime set of x_k, its projection is 0. However, $\bar{x}_k$ is nonzero, so there must be at least one other $\bar{x}_l \neq 0$. If $l > k$ or if $P_{l,s} \subsetneq P_{k,s}$, then x_l would be moved (or deleted) instead of x_k. Thus, x_l is an indecomposable element of higher priority, and it shares the same least summand as x_k. $\square$

For the next proof, it will be key to note that, though there are 3 different mechanisms in the construction that can remove a label from an element (decomposability, hidden divisibility, and absorption), only hidden divisibility increases the prime set associated with the label. If the prime set grows via absorption, then the original element was decomposable, so we delete the label.

LEMMA 5.24. *Each label can only be moved finitely many times via hidden divisibility.*

PROOF. We prove this by induction. x_0 can never be moved by hidden divisibility because there is no higher priority basis element to move it.

Assume we have reached a stage s where each x_j with $j < k$ will no longer be moved by hidden divisibility. This means that $f(j,t) = f(j,s)$ for every stage $t > s$ where the label x_j still exists. If x_j moves x_k via hidden divisibility at stage t, then after x_k has moved it is clear that $f(j,t) \subsetneq f(k,t)$. Thus, $f(j,t') \neq f(k,t')$ for all subsequent stages t'. This means that x_j can never move x_k again.

Therefore, for each $j < k$, x_j can only move x_k at most once after stage s. □

LEMMA 5.25. *The expression for every group element converges.*

PROOF. We prove this by induction as well. If g_0 is indecomposable, then its expression will be

$$g_0 = x_0$$

at every stage.

If g_0 is decomposable, then let

$$g_0 = \sum_j y_j$$

be the true decomposition of g_0. Once we have reached a stage s where every y_j in the true decomposition of g_0 has been enumerated, then each y_j is written as the sum of basis elements whose prime sets contain its own:

$$y_j = \sum_k q_k x_k$$

Let Q_j be the prime set of y_j. Because y_j is indecomposable, there must be at least one x_k assigned to an indecomposable element with $P_{k,s} = Q_j$. At this point, the only basis elements in the expression of g_0 are those with prime set Q_j for some j, and possibly some higher priority basis elements. This is because any lower priority basis elements with larger prime sets would be absorbed.

At this point, there are only two ways new elements may be introduced into the expression for g_0: a basis element may decompose or a label may be moved via hidden divisibility. If a decomposition occurs, then a basis element is replaced with basis elements of greater prime sets and its label is deleted. This can only be happen finitely many times before every new element is absorbed into some y_j.

We know that a basis element x_k can only be moved finitely many times via hidden divisibility. Though new basis elements x_l may be introduced to the expression for g_0 in this manner, this can only happen if $l < k$. Thus, only finitely many new elements may be introduced to the expression for g_0.

Now assume that the expression for every g_t with $t < s$ has converged. For each $t < s$, each basis element in the expression for g_t has settled. We would like to use a proof similar to the one for g_0, but we cannot allow basis elements in the expression for any g_t with $t < s$ to absorb lower priority basis elements.

Again, the only ways new elements can be introduced into the expression for g_s are if a basis element decomposes or a label is moved via hidden divisibility. It is possible that this process may introduce a basis element that is in the expression for some g_t with $t < s$. However, it is key to note that a decomposing basis element introduces one or more basis elements with larger prime sets, and that moving a label via hidden divisibility may introduce a higher priority basis element with the same prime set, but it also increases the prime set of the element associated with the label that was moved. Because the basis elements in the expression for each g_t with $t < s$

have settled, their prime sets are fixed from this point on. Thus, after a certain point none of these elements will be introduced into the expression for g_s.

Once we have reached this stage, we can rewrite the expression for g_s as

$$g_s = \sum_{i \in I} q_i x_i + \sum_{j \in J} q_j x_j$$

where I is the set of indices of basis elements in the expression of g_s that are also in the expression for some g_t with $t < s$, and J is the set of indices of basis elements in the expression of g_s that are not in the expression for any g_t with $t < s$. We can use the same proof above with the element

$$g'_s = \sum_{j \in J} q_j x_j$$

in the place of g_0. We know that no basis elements in the expression for any g_t with $t < s$ will appear at future stages, so there is nothing to stop basis elements in the expression for g_s from absorbing lower priority basis elements. If this elements is 0, then g_s is in the span of basis elements with permanent labels, and its expression cannot change from this point on. □

Lemma 5.26. *If the label x_k is permanent, then there is an element that has x_k in its expression at cofinitely many stages.*

Proof. We can assume we have reached a stage where every label x_j for $j < k$ has settled, and that x_k will never again be moved by hidden divisibility. This means that any of these labels which are still active are assigned to indecomposable elements, no two of which share a least summand. Let g_s be the element assigned the label x_k at this stage.

At future stages, the label x_k can only move by absorbing lower priority basis elements with larger prime sets, so as long as x_k is active, the expression for g_s will be

$$g_s = q_k x_k + \sum_l q_l x_l$$

where each x_l has a strictly larger prime set than that of x_k.

There are 3 ways in which x_k can be removed from the expression for g_s:

1) The label x_k is deleted.

2) The element x_k is "absorbed" by a higher priority basis element

3) The element x_k moves a lower priority basis element via a hidden divisibility (as in the expression for g_4 in Example 5.15).

2) is impossible because all the higher priority basis elements have settled. 3) is impossible for the element g_s because that would require another basis element in the expression with the same prime set as x_k. Thus, as long as x_k remains active, it cannot be removed from the expression for g_s. □

Thus, there is an element g_t which is the highest priority element with x_k in its expression at cofinitely many stages.

LEMMA 5.27. *Every label eventually settles.*

PROOF. This follows from Lemmas 5.24, 5.25, and 5.26. For any k, x_k can only be moved finitely many times via hidden divisibility. Also, there must be a highest priority element g_s that has x_k in its expression at cofinitely many stages. The expression for g_s must converge, so x_k can only absorb finitely many elements. Thus, the label x_k moves finitely many times. □

LEMMA 5.28. *A permanent label cannot settle on a decomposable element.*

PROOF. Suppose the label x_k is assigned to a decomposable element and we are at a stage in the construction where x_k can be written

$$x_k = \sum_j y_j$$

where the prime set of each y_j is larger than that of x_k and each y_j has been enumerated by the construction. Then each y_j is written as the linear combination of basis elements whose prime sets contain its own, yet at least one of them must have x_k in its expression for the above sum to be true. From this contradiction we see that the label x_k must be moved. □

LEMMA 5.29. *F is surjective onto T.*

PROOF. To see that the range of F is contained in T, we note that a permanent label must settle on an indecomposable element, whose prime set must be in T.

To see that F is surjective, let $Q \in T$ be a set of primes. Then there is an indecomposable element $y_j \in H_T$ with prime set Q. Once we have reached a stage in the construction where y_j has been enumerated, there must be an indecomposable basis element which shares a least summand with y_j. The only way such a label can move is via hidden divisibility by a indecomposable basis element of higher priority with the same least summand. Thus, there must be a basis element whose prime set is Q at cofinitely many stages, so the limit of the function f at that index will be Q. □

Now it remains to be shown that our decomposing function has all the properties listed in Definition 5.11.

LEMMA 5.30. *Properties 1) - 12) given in Definition 5.11 hold.*

PROOF. 1) This is true by Lemma 5.25.

2) This is obvious.

3) If $x_k \to \sum\limits_l q_l x_l$ in our construction, then any prime that infinitely divides each x_l also infinitely divides the previous x_k, so $f(k, s-1) \supseteq \bigcap_{\{l: q_l \neq 0\}} f(l, s)$.

Also, each former basis element is rewritten as the sum of new basis elements with greater or equal current prime sets, so we have that $f(k, s-1) \subseteq \bigcap_{\{l: q_l \neq 0\}} f(l, s)$.

The rest of property 3) merely states that no matter how the former basis element is written as a linear combination of basis elements at future stages, its prime set will remain constant.

4) Every label eventually settles.

5) If $q_k = 0$, this means that x_k decomposed, and the label was deleted. We created a new label to take its place. For every label that we delete, we must create at least one new label.

6) The only way that the prime set of a basis element can change is if the label moves (and the label is not deleted). We know that the new basis element cannot be any element g_n with $n < s$ because those elements were already written as a linear combination of basis elements with equal or larger prime sets.

7) The prime set of g_n is given by the intersection of the prime sets of the basis elements in its expression, and this should be constant at all stages.

8) This is simply because the basis elements at stage $s-1$ are linearly independent. When we rewrite them as linear combinations of new basis elements, they are still linearly independent. Any label that is deleted at stage s is rewritten as a linear combination of basis elements, at least one of which must have a new label. If x_l is a newly created label at stage s, then $I(l, s)$ is undefined, so this is not an issue.

9) Each element is only divisible by finitely many distinct primes. This property holds if we have constructed a true basis because then the existence of an element

$$g = \sum_l q_l x_l$$

implies the existence of each element $q_l x_l$. A basis element x_k may not be a true element in any decomposition of H_T. However, if we fix a true basis $y_0, y_1, y_2, ...$ of H_T, then there are only finitely many elements y_j which, when written as a sum of our basis elements, contain x_k in their expressions (because each such y_j must have a prime set contained in that of x_k). Thus, there is a prime p such that no prime greater than p divides any y_j that has x_k in its expression. If we write an element as a linear combination of the true elements and then rewrite these elements using our basis, we see that this property holds.

10) If x_k is not a permanent label, then there are only finitely many expressions $\Phi(n, s)$ containing x_k. If x_k is a permanent label, then any prime p not in its prime set only divides it finitely many times. Again, this property holds if we have constructed a true basis, so it holds for our basis by the same argument stated in the proof of Property 9).

11) If a new label x_k is created at stage s, there is some n such that $g_n = x_k$, and this is reflected in $\Phi(n, s)$.

12) At stage s, $\Phi(n, s)$ describes how the element g_n is written as a linear combination of basis elements, so this must be unique for each element.

□

This completes the proof for Theorem 5.12 □

3. Constructing H_T

THEOREM 5.31. *If T is the range of an injective partial limitwise set-monotonic function with a computable decomposing function, then there is a computable copy of the expanded group H_T*

PROOF. We will construct the group using the decomposition function, using the limitwise set-monotonic function $F(*) = \lim_s f(*, s)$ to tell us what the prime sets of elements should be.

Construction:

Stage 0: We create an element and label it x_0. We declare that its prime set is $f(0, 0)$.

Stage $s > 0$: If $I(k, s)$ states that

$$x_k \to \sum_l q_l x_l$$

with $q_k = 0$, then for every new label in the expression we create a new element and give it that label. We then declare that the above equation holds, and delete the label x_k.

If $q_k \neq 0$ then we check to see if there already exists an element g such that

$$q_k g = x_k - \sum_{l \neq k} q_l x_l$$

if so, we move the label x_k to this element. If not, we create such a g, declare the above equation holds, and move the label x_k to it.

After doing this for every k, we declare that the prime set of each x_k is $f(k, s)$. Then for every k and every prime in $f(k, s)$ we add another element to make it divisible by a higher power of that prime.

We declare all labeled elements to be linearly independent from one another, and linear independence of other elements can be determined by their expression as linear combinations of labeled elements.

For each n, if $\Phi(n, s)$ states that

$$g_n = \sum_l q_l x_l$$

then for each l we create an element $q_l x_l$, if it does not already exist.

For any pair of elements x, y in the group, if $x + y$ is not in the group, but there is a scalar multiple of $x + y$ already in the group, then we add $x + y$ to the group. We then add the inverse of every element to the group, if it does not already exist.

Verification: We denote the finished group by $\mathcal{G}$.

Lemma 5.32. *Every label settles.*

Proof. This is clear by property 5) □

Lemma 5.33. *Linear independence is a computable relation of $\mathcal{G}$.*

Proof. At every stage the elements with active labels form a linearly independent set. The linear independence of all other group elements derives from their expression as a linear combination of these elements. Property 8) guarantees that the basis elements at stage $s-1$ are still linearly independent at stage s. □

Lemma 5.34. *Every element in $\mathcal{G}$ can be uniquely written as a finite linear combination of elements with permanent labels.*

Proof. We know this because of properties 1) and 2), and the previous lemma. By induction, it is clear that every element introduced in the group is a linear combination of elements to which the labels were originally applied. Lemma 11) means that Φ gives us the expression for each of these elements at every subsequent stage of the construction, and these expressions all converge. Thus, the expression for every element introduced into the group converges. Uniqueness comes from the fact that at any stage of the construction, the active labeled elements are linearly independent. □

Lemma 5.35. *The prime sets of the elements with permanent labels are precisely the sets in T (with no repetitions).*

Proof. F is an injective limitwise set-monotonic function with range T. Thus, if x_k is a permanent label, then there was no s such that $I(k, s)$ showed x_k decomposing. Property 5) tells us that $f(k, s) \downarrow$ for all s.

The prime set of x_k at stage s is always equal to $f(k, s)$, so the prime set of x_k is $F(k)$ at cofinitely many stages. Property 9) guarantees that x_k is only divisible by finitely many primes. Property 10) guarantees that no prime outside of $F(k)$ infinitely divides x_k. Thus, the prime set of x_k in the final group is $F(k)$.

The function F is injective, so no prime sets are repeated. □

Lemma 5.36. *$\mathcal{G}$ is completely decomposable.*

Proof. $\mathcal{G}$ is generated by elements with newly created labels, and elements given by $\Phi(n, s)$ by some n, s or $I(j, t)$ for some j, t. The basis we have constructed is a true basis because for every linear combination $\sum_l q_l x_l \in \mathcal{G}$, there is an element $q_l x_l \in \mathcal{G}$ for each l. □

Lemma 5.37. *In $\mathcal{G}$, infinite divisibility is a computable relation.*

Proof. Property 3) guarantees that each former basis element still has a constant prime set throughout the construction. Property 6) guarantees that if for any index k and stage s, $f(k, s) \neq f(k, s+1)$, then the label x_k moves at stage s. This means that as long as a label remains stationary, the prime set of its labeled element will not change. Property 7) guarantees that each element represented by $\Phi(n, s)$, once introduced to the group, has a constant prime set. □

This ends the proof of Theorem 5.31. □

4. Conclusion

In the proof of Theorem 5.12 we have taken an arbitrary computable presentation of the group H_T and constructed a Δ^0_4 basis of indecomposable elements whose prime sets are precisely the sets in T (without repetition). If we wish to build H_T, the decomposing function allows us to build this basis, while the partial limitwise set-monotonic function gives us the sets in T.

We can also use the same construction but remove the process of absorbing lower priority basis elements to shorten expressions. If we do, we can build a Π^0_3 linearly independent set of indecomposable elements whose prime sets are precisely the sets in T (without repetition), but this set need not be a basis for H_T. Using this method, it may be possible to prove something similar to Theorem 5.31, but with an alternate definition for the decomposing function.

It should be possible to extend Theorem 5.31 to groups H_T where T is a collection of (not necessarily distinct) finite sets of primes, and the number of basis elements with prime set D corresponds to the multiplicity of D in T. We should also be able to allow for infinite sets of primes by approximating the prime set of each element up to p_s at stage s. If we intend to gain a better understanding of completely decomposable groups in general, then working to expand Theorem 5.31 will be a worthwhile task.

We may also be able use the fact that T is a collection of distinct finite sets of primes to find a more concise version of Theorem 5.12. Because we can precisely determine the prime set of any element, any time we have two labeled elements with the same prime set, we know at least one of them must be decomposable or they have the same least summand. In the latter case, there is a linear combination of the two elements which has a larger prime set or is the sum of larger prime sets (and hence, decomposable). This may allow the basis we find to be Π^0_3 instead of merely Δ^0_4.

I attempted to replace the equations coded in the decomposing function with a description of the linear dependence relation (as it pertains to formerly and currently labeled elements at a given stage). At stage s, rather than code the equation

$$x_k \to \sum_l q_l x_l$$

into $I(k,s)$, we could declare these elements linearly dependent:

$$\Lambda(x_{k,s-1}, x_{l_1,s}, x_{l_2,s}, ...x_{l_m,s})$$

This would be a simpler description of the group, but, when reconstructing the expanded group we would be tasked with choosing coefficients for these linear combinations that would make the linear dependence relation Λ consistent. The following example indicates that this is likely impossible in some cases.

EXAMPLE 5.38. Suppose we are attempting to build the decomposing function using only the linear dependence relation Λ (we will ignore prime sets in this example, as they are irrelevant). Assume the first four elements g_0, g_1, g_2, g_3 are a linearly independent set, so that they are initially given the labels x_0, x_1, x_2, x_3, respectively (at stages 0,1,2,3, respectively). As in the construction for the proof of Theorem 5.12, we want to track how these elements are to be written as linear combinations of labeled elements at future stages. The difference here is that we will only be listing the elements used in the equations, not the equations themselves. To properly define Λ, it will be necessary to note the location of a label x_k at a stage t, and we denote this element $x_{k,t}$. Thus, $x_{0,0} = g_0$, $x_{1,1} = g_1$, $x_{2,2} = g_2$, and $x_{3,3} = g_3$.

Suppose by stage s we have the following expressions for g_0, g_1, g_2, g_3:

$$g_0 = x_4 + x_5 + 2x_6 + x_7$$

$$g_1 = x_4 + 2x_5 + 3x_6 + 3x_7$$

$$g_2 = 3x_4 + 5x_5 + x_6 + 2x_7$$

$$g_3 = 3x_4 + 2x_5 + 2x_6 - x_7$$

Then we have $\Lambda(x_{n,n}, x_{4,s}, x_{5,s}, x_{6,s}, x_{7,s})$ for $n = 0, 1, 2, 3$. In fact, any 5-element subset of $\{x_{0,0}, x_{1,1}, x_{2,2}, x_{3,3}, x_{4,s}, x_{5,s}, x_{6,s}, x_{7,s}\}$ is linearly dependent, but no 4-element subset is. This means that if we were given this information and attempting to build a computable copy of the expanded group, we could choose any coefficients we wanted for the above equations, as long as no set of 4 of the listed elements are linearly dependent.

Now suppose that at stage $s+1$ of the construction of the decomposing function, we see that x_7 is actually decomposable:

$$x_7 = x_4 + x_8$$

(where x_8 is a new label in the construction). This means that the new expressions for g_0, g_1, g_2, g_3 are

$$g_0 = x_4 + x_5 + 2x_6 + (x_4 + x_8) = 2x_4 + x_5 + 2x_6 + x_8$$

$$g_1 = x_4 + 2x_5 + 3x_6 + 3(x_4 + x_8) = 4x_4 + 2x_5 + 3x_6 + 3x_8$$

$$g_2 = 3x_4 + 5x_5 + x_6 + 2(x_4 + x_8) = 5x_4 + 5x_5 + x_6 + 2x_8$$

$$g_3 = 3x_4 + 2x_5 + 2x_6 - (x_4 + x_8) = 2x_4 + 2x_5 + 2x_6 - x_8$$

The only differences between the new expressions and the old expressions are the coefficients of x_4 and the substitution of x_8 for x_7. However, the new coefficients allow us to write

$$2g_0 - g_1 = x_6 - x_8$$

$$2g_2 - 5g_3 = -8x_6 + 9x_8$$

Now we must declare $\Lambda(x_{4,s}, x_{4,s+1}), \Lambda(x_{5,s}, x_{5,s+1}), \Lambda(x_{6,s}, x_{6,s+1})$, which merely indicate that the labels x_4, x_5, and x_6 did not move. We must also declare

$$\Lambda(x_{7,s}, x_{4,s+1}, x_{8,s+1})$$

$$\Lambda(x_{0,0}, x_{1,1}, x_{6,s+1}, x_{8,s+1})$$

$$\Lambda(x_{2,2}, x_{3,3}, x_{6,s+1}, x_{8,s+1})$$

If we are given this information to build a computable copy of the group, then we get to create a new element with the label x_8 and choose rationals q_4, q_8 so that

$$x_7 = q_4 x_4 + q_8 x_8$$

However, the fact that $\{g_0, g_1, x_6, x_8\}$ form a linearly dependent set means that q_4 can only have one possible value in order to satisfy this requirement, which is dependent entirely on the expressions for g_0 and g_1 at stage s. The fact that $\{g_2, g_3, x_6, x_8\}$ are linearly dependent puts a similar requirement on the value of q_4. There is no way to guarantee that these requirements do not conflict with one another with the information we had at stage s.

If we want to build a computable copy of the expanded group H_T so that the linear dependencies at any given stage are consistent with the linear dependencies at the same stage of the construction of the decomposing function, then the decomposing function must give the exact equations used in moving labels. One idea to remedy this is to forgo this consistency via a finite injury construction, but it is not at all clear that this is possible.

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