The Thin Set Theorem for Pairs and Substructures of the Muchnik Lattice

By

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A dissertation submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

(Mathematics)

at the

UNIVERSITY OF WISCONSIN – MADISON

2014

Date of final oral examination: May 7, 2014

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Abstract

This thesis covers two major topics. First, answering a question in the reverse mathematics of infinitary combinatorics, we show that the Thin Set Theorem for Pairs TS(2), a very weak version of Ramsey's theorem, implies the Diagonally non-Computable Set Principle DNR, an important computability-theoretic principle, over the axiom system RCA_0. Second, we pursue several results concerning the lattice structure of the degrees of mass problems. Answering questions raised by Terwijn, we give a complete lattice-theoretic characterization of the intervals of the Muchnik lattice which do not have antichains of cardinality $2^{\aleph_2}$, and establish some further structural results in case the interval has no uncountable antichains. We also consider other degree structures associated with mass problems, obtaining some results on the lattice of finite mass problems as well as exploring some alternative notions of reducibility.

Acknowledgements

I would like to thank my advisor, Steffen Lempp, for his mathematical guidance, his confidence in my ability, and his patience and encouragement when my productivity waned. His high expectations for me combined with his kindness and patience when I did not always meet them have been invaluable to me. Thanks too go to the rest of the people in the logic group at UW-Madison and to my other friends in the mathematics department for many interesting conversations and a healthy intellectual atmosphere.

I would also like to thank my parents, David and Vickie, who encouraged my curiosity from the time I was small and have pushed me towards excellence and supported me throughout my mathematical career. Finally, I would like to thank my wife, Joyce, whose love, support, and encouragement have been a continual source of motivation and strength.

Contents

Chapter 1

Introduction

One conceptualization of computability theory is as the study of the difficulty of infinite problems. In this understanding, the computable functions are those problems which are easy; the noncomputability of the halting problem establishes that there is a problem which is not easy, and the partial order of Turing degrees captures the difficulty classes of all possible problems. This is a natural (and useful) way of thinking, but it misses an important subtlety: when we speak of Turing reductions, our domain of discourse is limited to those problems with a *single* solution. When a problem is a function $\phi : \mathbb{N} \to \mathbb{N}$ or a subset $A \subseteq \mathbb{N}$, computing the solution to that problem means producing exactly that function or that subset. But a real mathematical problem may not have only one solution: there are infinitely many different bases of a given infinite-dimensional vector space, and potentially many different prime ideals in a ring. The study of such problems can proceed in many different directions, all of which rely heavily on, but go beyond, the foundation of Turing reduction and the Turing degrees. To the extent to which this thesis has a unifying theme, it is the study of problems with multiple solutions, first from the perspective of reverse mathematics, and second from the perspective of mass problems.

The remainder of this chapter consists of background for, and a summary of, the results in this thesis. Section 1.1 summarizes the concepts and notation needed for the remainder of the work. Section 1.2 introduces reverse mathematics and provides the specific background for Chapter 2, while Section 1.3 summarizes the results of that chapter. Section 1.4 introduces mass problems and some notation and definitions needed for Chapters 3 and 4. Section 1.5 provides the basic definitions and notation from lattice theory needed for Chapters 3 and 4. Sections 1.6 and 1.7 summarize the results for Chapters 3 and 4, respectively.

1.1 Basic Concepts and Notation

Most of what follows assumes that the reader is familiar with the basics of computability theory. For the sake of completeness we will provide a summary of the common notation which is used in this thesis. Any reader for whom this material is not familiar is encouraged to consult a reference such as the first part of [18].

The symbol $\mathbb{N}$ refers to the set of natural numbers, including the number 0, while the symbol ω refers to the first infinite ordinal, or its order type. Since finite ordinals are identified with the natural numbers $0, 1, 2, \ldots$ by convention, these two are often used interchangeably when referring to the natural numbers, but when working in reverse mathematics we are forced to make the distinction due to the existence of nonstandard models of first-order arithmetic. In that case $\mathbb{N}$ will always refer to the first-order part of our model, and ω will refer to the true natural numbers (which have order type ω). Cardinalities always use $\aleph$-numbers. 2^ω is shorthand for the power set of ω, while ω^ω refers to the collection of functions from ω to ω.

If $A \subseteq \omega$, then $\overline{A}$ is the complement of A in ω and $|A|$ is the size of A. If σ is a string, then $|\sigma|$ is the length of σ.

A *problem* is an element either of 2^ω or of ω^ω; that is, either a subset of the natural numbers or a function from the natural numbers to itself. Mostly these are interchangeable; for the purposes of this thesis, we will usually use the definition of a problem as a function $f : \omega \to \omega$.

The e-th computable function is ϕ_e. We write $\phi_e(n) \downarrow= m$ if ϕ_e converges on input n and gives output m; we write $\phi_e(n) \uparrow$ if the computation does not converge. $\phi_{e,s}(n)$ refers to the first s steps of this computation, and has a value only if the computation converges within those first s steps, otherwise, it diverges.

The e-th oracle Turing machine with oracle f is Φ_e^f. Since $0 - -1$ valued functions and subsets of ω are interchangeable, we also write Φ_e^A when A is a set. The e-th Turing functional is Φ_e. If $f, g : \omega \to \omega$ are functions and (for some e) $\Phi_e^f(n) \downarrow= g(n)$ for every n, then g is *Turing reducible* to f and we write $g \leq_T f$. As shorthand for $\forall n(\Phi_e^f(n) \downarrow= g(n))$, we will sometimes write $\Phi_e(f) = g$ (commonly $\Phi_e(A) = B$ where $A, B \subseteq \omega$). As before, we write $\Phi_{e,s}^f(n)$ for the first s steps of the computation; this also restricts the *use* of the oracle f (the largest-index value requested of the oracle during the computation) to be at most s.

We write W_e for e-th computably enumerable set, which is the domain of the partial function ϕ_e. Similarly, W_e^f is the e-th c.e. set relative to f, and is the domain of Φ_e^f.

If $f \leq_T g$ and $g \leq_T f$, we say that f and g are *Turing equivalent* and write $f \equiv_T g$. The *Turing degrees* are the equivalence classes of $\equiv_T$. The partial order of Turing degrees is the quotient of the pre-partial-order $\leq_T$ by the equivalence relation $\equiv_T$, and is written $\mathscr{D}$. In a slight abuse of notation, we write $\leq_T$ for the partial order relation on the Turing degrees in $\mathscr{D}$ as well as the pre-partial order on functions and sets, and occasionally mix degrees and functions or sets when this is not ambiguous (e.g., writing

$f \leq_T \mathbf{d}$ where f is a function and $\mathbf{d}$ is a Turing degree).

If f and g are two functions, the join $f \oplus g$ is given by $(f \oplus g)(2n) = f(n)$ and $(f \oplus g)(2n+1) = g(n)$. In the partial order $\mathscr{D}$ of Turing degrees, the Turing degree of $f \oplus g$ is the least upper bound of the Turing degrees of f and g, so two Turing degrees $\mathbf{a}$ and $\mathbf{b}$ always have a join, written $\mathbf{a} \vee \mathbf{b}$, and $\mathscr{D}$ is an upper semilattice. A subset of $\mathscr{D}$ which is closed downwards and closed under join is a *Turing ideal*.

The relative halting set $\{e \mid \Phi_e^f(e)\downarrow\}$ is the *Turing jump* of f, written f'. In place of a function f, we may also write a Turing degree, in which case the halting set is defined by a representative of the Turing degree and the jump is taken to be the degree of that halting set. The Turing degree of computable functions is written $\mathbf{0}$, so the degree of the Halting problem K is $\mathbf{0}'$.

A few conventions are as follows. The Greek letters ϕ and ψ refer to partial computable functions, while Φ and Ψ refer to Turing functionals. Capital Roman letters are generally sets, though they are sometimes functions, P is often a partial order, and L and M are lattices. Lowercase letters e, n, m, s, t, etc. are natural numbers, as is capital N. Depending on context, lowercase letters a, b, c, x, y, z may be natural numbers or generic elements of sets. The letters f, g, h represent functions. The lowercase Greek letters σ and τ are strings. Boldface lowercase Roman letters are Turing degrees. Calligraphic letters like $\mathcal{A}$ and $\mathcal{X}$ are mass problems, degrees of mass problems, and other elements of lattices with underlying structure. Inequalities like $\leq$ without a subscript refer to the natural numbers or to *generic* partial orders and lattices; specific orders (such as the various reducibilities) will have subscripts to avoid confusion.

1.2 Reverse Mathematics of Infinite Combinatorics

1.2.1 History and Background

Reverse mathematics is a field residing at the intersection between computability theory and proof theory. The main goal of the program is to take various theorems from mathematics and determine their strength; the idea behind the name is that it is reversed mathematics because we begin with theorems and determine which axioms were needed to prove them. Since, presumably, the theorems of mathematics are all *true* and thus all equivalent over the usual strong axiom systems such as ZFC or full second-order arithmetic, one needs to work over a fairly limited base system of axioms. The usual set of axioms used is called RCA_0, which stands for *recursive comprehension axiom.* RCA_0 is a subsystem of the full set of axioms for second-order arithmetic. It consists of the axioms for a discretely ordered semiring, together with the induction schema for Σ^0_1 statements (with parameters) and set comprehension schema for Δ^0_1 statements (with parameters). Since in the standard model of arithmetic, Δ^0_1-definable subsets of $\mathbb{N}$ are exactly the computable sets, RCA_0 is often thought of as stating that "computable sets exist."

Starting with RCA_0 and working upwards in strength is a linear hierarchy of five historically important systems often called the "big five." While we will not be concerned with the two highest of these, the others will be relevant. Above RCA_0 lies WKL_0, which consists of the axioms of RCA_0 plus *Weak König's Lemma*, which is equivalent to the statement that every infinite binary tree has an infinite path. Above WKL_0 is ACA_0, which stands for *arithmetic comprehension axiom* and consists of RCA_0 plus the comprehension schema for all arithmetic statements (i.e. Σ^0_n statements with parameters for all n). More about these systems can be found in Simpson [17], the usual reference for the project as a whole.

Many classic theorems of mathematics are equivalent, over RCA_0, to one of the big five. For instance, the statement that every countable commutative ring has a prime ideal is equivalent to WKL_0. The statement that every countable commutative ring has a maximal ideal is equivalent to ACA_0, as is the statement that every countable vector space has a basis. For a long time, it was widely assumed that most theorems of ordinary mathematics were equivalent to one of the big five.

Then came Ramsey's Theorem.

Definition 1.1. Let n and k be positive integers. Then *Ramsey's Theorem for n-tuples and k colors*, written RT^n_k, is the following statement: For any function $f : [\mathbb{N}]^n \to \{c \in \mathbb{N} \mid c < k\}$, there is an infinite set $A \subseteq \mathbb{N}$, called a *homogeneous set*, such that $|f([A]^n)| = 1$.

Here the notation $[S]^n$, for a set S, means the set of unordered n-tuples with elements taken from S, and $f([A]^n)$ refers as usual to the image of $[A]^n$ under f. In informal terms, RT^n_k says that if the n-tuples of natural numbers are colored with k colors, there is some infinite subset $A \subseteq \mathbb{N}$ such that all n-tuples taken from A have the same color. When $n = k = 2$, this is the infinite version of the usual finite Ramsey's Theorem about 2-coloring the edges of a complete graph.

In some cases Ramsey's Theorem is simple. The principles RT^1_k and RT^n_1 are all provable in RCA_0. It is also not too difficult to show that, over RCA_0, RT^n_2 and RT^n_c are equivalent for n and any $c \geq 2$. Moreover, we have the following result due to Jockusch [13]:

Theorem 1.2 (Jockusch, 1972)**.** *For every* $n \geq 3$*, we have* $\mathrm{RCA}_0 \vdash \mathrm{RT}^n_2 \leftrightarrow \mathrm{ACA}_0$*.*

The case $n = 2$, the so-called *Ramsey's Theorem for Pairs*, was not so easy to classify. From the same paper of Jockusch we know some bounds:

Theorem 1.3 (Jockusch, 1972)**.** *We have* $\mathrm{ACA}_0 \vdash \mathrm{RT}^2_2$ *and* $\mathrm{WKL}_0 \nvdash \mathrm{RT}^2_2$*.*

But nothing more was proved for many years, until a breakthrough result by Seetapun [16]:

Theorem 1.4 (Seetapun, 1995)**.** $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \nvdash \mathrm{ACA}_0$*.*

Ramsey's Theorem for Pairs is *not* equivalent to one of the big five! Even more, it does not even lie in the same linear order [14]:

Theorem 1.5 (Liu, 2012)**.** $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \nvdash \mathrm{WKL}_0$*.*

The last twenty years have seen a great deal of investigation into Ramsey's Theorem for Pairs and many similar theorems and principles of infinitary combinatorics. Some of these results are outlined further in Chapter 2. One of the weak combinatorial principles arising from this study is the Thin Set Theorem, in particular the Thin Set Theorem for Pairs:

Definition 1.6. The *Thin Set Theorem for Pairs*, TS(2), states that for any function $f : [\mathbb{N}]^2 \to \mathbb{N}$, there is an infinite set A, called a *thin set*, such that $f([A]^2) \neq \mathbb{N}$.

Informally, TS(2) says that if we color the pairs of natural numbers with infinitely many colors, we can find some infinite subset on which at least one of those colors is omitted. This is clearly implied by RT^2_2, but it seems much weaker; indeed, while it is strictly stronger than RCA_0, until the results presented in Chapter 2 it was not known to imply anything of note. Wang [22] recently showed that Thin Set is weaker than Ramsey's Theorem in another sense: In contrast to Theorem 1.2, even arbitrarily high-dimensional forms of Thin Set do not imply ACA_0 over RCA_0. Nevertheless, TS(2) does have some interesting consequences; this is the focus of Chapter 2.

1.2.2 Computability and ω-models

It may seem from the presentation above that reverse mathematics has much more to do with proof theory than with computability. This impression is somewhat misleading. In actual practice, a great deal of the work done in reverse mathematics is computability theory.

We alluded in Section 1.1 to the necessity of distinguishing in reverse mathematics between ω, the order type of the actual natural numbers (which is often identified with them), and $\mathbb{N}$, the natural numbers within our model. Let us unpack this a bit more. Because the language in reverse mathematics is that of *second*-order arithmetic, any model of RCA_0 can be captured by two pieces of information: the first-order part, which is the structure of the natural numbers in that model, and the second-order part, which says which *subsets* of the natural numbers exist. The first-order part may be standard—that is, the natural numbers may have order type ω—or it may not. What happens if the first-order part is standard? In that case, we get something called an ω-model.

Once we have established that we are working in an ω-model, all that remains to describe a model of RCA_0 is to determine which subsets of ω exist in the model. Because of this, we often identify these models with the set of subsets of ω that are present. It turns out that the recursive comprehension schema is exactly enough to prove that the join of any two sets exists, and that if a set A exists, then so does every set that is Turing reducible to A. Therefore, the ω-models of RCA_0 can be identified with the Turing ideals.

Moreover, many of the theorems and principles that we work with in reverse mathematics assert the existence of a set with some properties, given set and number parameters. One way to think of this is as a problem with many possible solutions: given a collection of parameters, any set satisfying the properties in the statement is a solution to the problem. If we wish to describe the class of ω-models of RCA_0 which are models of some principle, this can often be done by describing how much computational power is needed to solve any instance of the problem. As an example of this, the ω-models of ACA_0 are exactly those Turing ideals in which the jump of every set exists. Similarly, since weak König's Lemma also has computational content, one can use this to deduce information about ω-models of WKL_0; for instance, using the low basis theorem and a little subtlety, one can show that there is an ω-model of WKL_0 consisting only of low sets and indeed one where every set is bounded by a single low degree (see [17] for a proof).

The presence of computability theory in the study of ω-models carries over to the general case. Since all the statements of computability theory can be expressed in second-order arithmetic, a lot of useful work can be translated over and proved in RCA_0, without the assumption that the first-order part is standard. One has to ensure that one does not use too much induction or rely on the existence of sets which may not be present in the underlying model, but much work does transfer over.

In practice, a common approach to showing that some statement of the form $\mathrm{RCA}_0 \vdash P \to Q$ is to show that if we have an instance of a Q-problem A coming from a set X

of parameters for Q, we can X-computably extract parameters Y for a P-problem B such that if we have a solution to B, we can X-computably extract a solution to A. This provides a proof that ω-models of $\mathrm{RCA}_0 + P$ are also models of Q. The proof is then analyzed and re-written to appeal only to the actual axioms of RCA_0, if possible, producing a proof that $\mathrm{RCA}_0 \vdash P \to Q$. This is in fact the approach we take in Chapter 2.

1.3 Main Theorem of Chapter 2

The main result of Chapter 2 is the following theorem:

Theorem 1.7. $\mathrm{RCA}_0 \models \mathrm{TS}(2) \to \mathrm{DNR}$.

Here, DNR is the *Diagonally non-Computable Set principle*, which states that for any set A, there exists a function f which is diagonally non-computable relative to A; that is, for every e, $f(e) \neq \Phi_e^A(e)$.

1.4 Mass Problems

One perspective on theorems like RT^2_2 is that they describe a class of *problems*: in the case of RT^2_2, the problem of constructing a homogeneous set for a given 2-coloring of $[\mathbb{N}]^2$. Since there may be many solutions to such a problem, with varying computational content, the Turing degrees do not give us an immediate way of classifying the difficulty of such problems. One way to approach this task is to abstract away the logical content of such a problem (e.g., as an instance of Ramsey's Theorem) and identify a problem with its set of solutions, developing a theory of what are called *mass problems*.

Definition 1.8. A *mass problem* $\mathcal{S}$ is a subset of ω^ω. The elements of $\mathcal{S}$ are the *solutions* to $\mathcal{S}$.

How should we compare two mass problems $\mathcal{S}$ and $\mathcal{T}$? One natural idea is to say that $\mathcal{S}$ is harder than $\mathcal{T}$ if being able to solve $\mathcal{S}$ means that we can solve $\mathcal{T}$; that is, if given any solution to $\mathcal{S}$ we can find a solution to $\mathcal{T}$. As it turns out there are two different ways to cash this out:

Definition 1.9. Let $\mathcal{S}$ and $\mathcal{T}$ be mass problems. We say that $\mathcal{T}$ is *Medvedev*, or *strongly*, *reducible* to $\mathcal{S}$, written $\mathcal{T} \leq_s \mathcal{S}$, if there is a Turing functional Φ such that, for every $A \in \mathcal{S}$, $\Phi(A) \in \mathcal{T}$.

Informally speaking, $\mathcal{T} \leq_M \mathcal{S}$ if there is a single algorithm which, given a solution to $\mathcal{S}$, produces a solution to $\mathcal{T}$.

If we reverse the quantifiers above we come up with another reducibility notion:

Definition 1.10. Let $\mathcal{S}$ and $\mathcal{T}$ be mass problems. We say that $\mathcal{T}$ is *Muchnik*, or *weakly*, *reducible* to $\mathcal{S}$, written $\mathcal{T} \leq_w \mathcal{S}$, if for every $A \in \mathcal{S}$, there is a Turing functional Φ such that $\Phi(A) \in \mathcal{T}$.

Muchnik reducibility is nonuniform; it says that every solution to $\mathcal{S}$ has enough computational information to yield a solution to $\mathcal{T}$, but the way to extract that information may be different each time.

It is clear that Medvedev reducibility implies Muchnik reducibility, but the other implication does not hold: Medvedev reducibility is, in fact, a stronger notion. It is straightforward to verify that both of these reducibilities are transitive and reflexive, so we can define equivalence relations in the natural way.

Definition 1.11. If $\mathcal{S}$ and $\mathcal{T}$ are mass problems, we say that $\mathcal{S}$ and $\mathcal{T}$ are *Medvedev equivalent*, written $\mathcal{S} \equiv_s \mathcal{T}$ (or *Muchnik equivalent*, $\mathcal{S} \equiv_w \mathcal{T}$), if both $\mathcal{S} \leq_s \mathcal{T}$ and $\mathcal{T} \leq_s \mathcal{S}$ ($\mathcal{S} \leq_w \mathcal{T}$ and $\mathcal{T} \leq_w \mathcal{S}$, respectively). The equivalence classes of these relations are the *Medvedev degrees* and *Muchnik degrees*, respectively.

The pre-partial orders $\leq_s$ and $\leq_w$ on mass problems induce natural partial orders, written the same way, on the Medvedev and Muchnik degrees respectively. Both of these partial orders have some nice structure:

Theorem 1.12. *The partial orders $\leq_s$ and $\leq_w$ on the Medvedev and Muchnik degrees form distributive lattices with least element (the equivalence class of the set containing the zero function) and greatest element (the equivalence class of the empty set).*

We write $\mathscr{M}$ for the Medvedev lattice and $\mathscr{M}_w$ for the Muchnik lattice.

Terwijn has studied the structures of these lattices, in particular, he has shown [20] that the finite intervals of the Medvedev lattice are exactly the finite Boolean algebras, and the infinite intervals all have antichains of cardinality $2^{2^{\aleph_0}}$. Additionally, he has given [21] a characterization of the finite intervals of the Muchnik lattice, which admit a lot more variety, and tantalizingly noted that the situation is more complicated for infinite intervals; in particular, that some countable linear orders (but not others) are intervals of the Muchnik lattice, and that there are intervals with width $\aleph_0$, intervals with antichains of size $2^{\aleph_0}$ but not $2^{2^{\aleph_0}}$, and intervals (in particular the whole lattice) with antichains of size $2^{2^{\aleph_0}}$.

Inspired by these hints, we give in Chapter 3 a lattice-theoretic characterization that encompasses Terwijn's characterization of the finite intervals of the Muchnik lattice and also applies to many infinite intervals. In particular, we characterize all such intervals with no antichains of cardinality $2^{\aleph_2}$, and explore in greater depth the structure of the Muchnik lattice. Chapter 4 deals with various suborders of the Muchnik and Medvedev lattices.

1.5 Some Lattice Theory

Much of the work in Chapter 3 relies on a number of results from lattice theory. This section contains the essential definitions from lattice theory used in Chapters 3 and 4, though many of the more advanced definitions are repeated when they become relevant.

Definition 1.13. A *lattice* L is a partially ordered set with the property that for any $x, y \in L$, x and y have a least upper bound (*join*, written $x \vee y$) and greatest lower bound (*meet*, written $x \wedge y$) in L.

Lattices include all linear orders, the Muchnik and Medvedev degrees under their respective partial orders, and the subsets of ω under the subset relation, but not the Turing degrees, since not every pair of Turing degrees has a meet. The Turing degrees form an *upper semilattice*, with joins but not necessarily meets.

Definition 1.14. If P and Q are two partial orders, we say that an injective map $f : P \to Q$ is an *embedding* (and that P *embeds into* Q) if f preserves the partial order: $a \leq_P b$ implies that $f(a) \leq_Q f(b)$. If P and Q are upper semilattices, then f is an embedding of upper semilattices if it also preserves joins ($f(a \vee_P b) = f(a) \vee_Q f(b)$), and if P and Q are lattices, then f is a lattice embedding if if preserves both joins and meets.

Definition 1.15. A lattice L is *distributive* if, for every $x, y, z \in L$, $(x \vee y) \wedge z =$

$(x \wedge z) \vee (y \wedge z)$ and $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$. That is, meets and joins distribute over each other.

The lattices mentioned above are all distributive; an example of a non-distributive lattice is the lattice of subfields of $\mathbb{Q}(i, \sqrt{2})$ ordered under inclusion: for $x = \mathbb{Q}(i)$, $y = \mathbb{Q}(\sqrt{2})$, and $z = \mathbb{Q}(i\sqrt{2})$, we have $(x \vee y) \wedge z = z$ while $(x \wedge z) \vee (y \wedge z) = \mathbb{Q}$, and $(x \wedge y) \vee z = z$ but $(x \vee z) \wedge (y \vee z) = \mathbb{Q}(i, \sqrt{2})$.

Definition 1.16. A lattice L is *complete* if for every subset $S \subseteq L$, S has a supremum in L. The supremum of S is often called the *join* of S and is written $\bigvee S$.

Note that a complete lattice must have a largest element (the supremum of L) and a smallest element (the supremum of $\emptyset$); this is not true of infinite lattices in general.

Definition 1.17. An *interval* of a lattice L is the set $\{x \in L \mid a \leq x \leq b\}$ for some (not necessarily distinct) $a \leq b \in L$ and is often written $[a, b]$.

Many properties of lattices are preserved by taking intervals.

Proposition 1.18. *If L is a distributive lattice and M is an interval in L, then M is also a distributive lattice. The same is true when "distributive" is replaced by "complete."*

Since many elements of a lattice can be obtained as the joins of other elements, we may ask which elements cannot be thus obtained.

Definition 1.19. An element x of a lattice L is *join-irreducible* if it is not the least element of L, and if for any $y, z \in L$, if $y \vee z = x$, then $y = x$ or $z = x$. The notion of *meet-irreducible* can be defined in a similar way.

For finite lattices, we can think of the lattice as being generated (in a sense) by taking all possible joins of its join-irreducible elements. For infinite lattices, we have no such luck. However, there is another notion which coincides with join-irreducibility in the case of finite lattices and is very helpful for understanding some infinite lattices, including the Muchnik lattice.

Definition 1.20. Let L be a complete lattice. An element x of L is *completely join-prime* if, for any subset $S \subseteq L$, $x \leq \bigvee S$ implies that there is some $y \in S$ such that $x \leq y$.

We are particularly interested in the partial order formed by the completely join-prime elements of a lattice.

Definition 1.21. Let L be a complete lattice. The partial order of completely join-prime elements of L is $\mathcal{J}_{\mathcal{P}}(L)$.

A few more definitions are needed in order to state the main results of Chapter 3.

Definition 1.22. A complete lattice L is *completely distributive* if it satisfies, for every doubly indexed subset $\{x_{ij}\}_{i \in I, j \in J}$ of L:

$$\bigwedge_{i \in I} \left(\bigvee_{j \in J} x_{ij} \right) = \bigvee_{\alpha : I \to J} \left(\bigwedge_{i \in I} x_{i\alpha(i)} \right).$$

In particular, meets distribute over arbitrary joins and joins distribute over arbitrary meets.

Definition 1.23. An element k of a complete lattice L is *compact* if, for every subset $S \subseteq L$, $k \leq \bigvee S$ implies that there is some finite $T \subseteq S$ such that $k \leq \bigvee T$. A complete lattice L is *algebraic* if, for each $a \in L$, $a = \bigvee \{k \in L \mid k \text{ compact and } k \leq a\}$.

These definitions, and discussion of them, can be found in [7].

Definition 1.24. A lattice is *superalgebraic* if it is algebraic and completely distributive.

1.6 Results of Chapter 3

The main result of Chapter 3 is a characterization of the intervals of the Muchnik that have no antichains of cardinality $2^{\aleph_2}$.

Theorem 1.25. *A lattice L with no antichains of cardinality $2^{\aleph_2}$ is isomorphic to an interval of the Muchnik lattice $\mathscr{M}_w$ if and only if the following hold:*

1. *L is superalgebraic,*
2. *$\mathcal{J}_{\mathcal{P}}(L)$ is an initial segment of an upper semilattice, and*
3. *$\mathcal{J}_{\mathcal{P}}(L)$ has the countable predecessor property.*

In addition to this characterization, we also give, via proving a more general result about superalgebraic lattices, some interesting information about the structure of those intervals of $\mathscr{M}_w$ with no uncountable antichains:

Theorem 1.26. *Let L be an interval of the Muchnik lattice $\mathscr{M}_w$ with no uncountable antichain. Then every element of L is the join of finitely many join-irreducible elements of L.*

1.7 Results of Chapter 4

The main results of this section concern the lattice of finite mass problems (mass problems which are finite as sets) and other sublattices of $\mathscr{M}_w$. We prove that intervals in this sublattice are as large as possible.

Theorem 1.27. *Let* $\mathbf{B} <_w \mathbf{A}$ *be Muchnik degrees of finite mass problems. Then there is a set of cardinality* $2^{\aleph_0}$ *of pairwise incomparable Muchnik degrees of finite mass problems* $\{\mathbf{X}_\alpha\}_{\alpha<\in 2^\omega}$ *such that* $\mathbf{B} < \mathbf{X}_\alpha < \mathbf{A}$ *for all* $\alpha \in 2^\omega$*.*

We show a similar result for countable mass problems, discuss co-countable mass problems, and explore several alternative reducibilities for mass problems and the structures they generate.

Chapter 2

Thin Set for Pairs Implies DNR

2.1 Background and Definitions

The reverse mathematics of combinatorial principles has generated a good amount of interest in recent years. In particular, a lot of principles have emerged with reverse mathematical strength between RCA_0 (Recursive Comprehension) and ACA_0 (Aritheoremetic Comprehension) which are not equivalent to either of these axiom sets or to the other member of the "big five" sitting between them, WKL_0 (Weak König's Lemma). The picture in reverse mathematics has been greatly expanded by a large web of these combinatorial results, and it has been an ongoing and interesting project to understand this web, adding new principles and establishing implications (and non-implications) among them.

Some illustrations and descriptions of this fascinating area can be found in [11] and [12]; here, we will be content with a brief outline of some of the more important results in order to set the context for our results.

We first recall the following useful bit of notation: $[X]^n$ refers to the set of all *unordered* n-tuples whose elements are elements of the set X. Thus, a function $f : [X]^n \to \alpha$ should be interpreted as a coloring of the unordered n-tuples from X with colors taken from the set α.

- **Ramsey's Theorem** (RT^n_k). Ramsey's Theorem is generally divided into a family of principles, referred to as RT^n_k for positive integers n and k, and corresponding to k-colorings of n-tuples. In particular, RT^n_k states that for any function $f : [\mathbb{N}]^n \to \{a \in \mathbb{N} \mid a < k\}$ there is an infinite homogeneous set A; that is, an infinite set $A \subseteq \mathbb{N}$ such that f restricted to domain $[A]^n$ takes only one value. It is easy to see that RT^n_1 and RT^1_k are all provable in RCA_0; moreover, it is also not too difficult to show by induction that RT^n_k and RT^n_c are equivalent for all $c, k \geq 2$. Not as obvious, but also known (due to Jockusch [13]), is the fact that RT^n_2 is equivalent over RCA_0 to ACA_0 for all $n \geq 3$. This leaves RT^2_2 as the only remaining case.

- **Ramsey's Theorem for pairs** (RT^2_2). Ramsey's Theorem for pairs lies strictly between RCA_0 and ACA_0 (the fact that it lies strictly below ACA_0 is due to Seetapun [16]). It is known that RT^2_2 neither implies nor is implied by WKL_0 (the latter following from a result of Jockusch [13] and the former due to Liu [14]), and it is RT^2_2 and principles weaker than it which make up much of the web of principles alluded to above.

- **Stable Ramsey's Theorem for pairs** (SRT^2_2). Stable Ramsey's Theorem states that if $f : [\mathbb{N}]^2 \to \{0, 1\}$ is a function with the property that, for every n, there is some M such that for all $m \geq M$, $f(\{n, m\}) = f(\{n, M\})$ (this is sometimes stated as "for all n, $\lim_m f(\{n, m\})$ exists"), then there is an infinite homogeneous set A. The idea is that the coloring stabilizes; in the language of graphs, every vertex is connected either to cofinitely many other vertices by a color-0 edge, or cofinitely many other vertices by a color-1 edge. For a while it was an open question whether SRT^2_2 was strictly weaker than RT^2_2 or whether they were equivalent; this has been resolved by Chong, Slaman, and Yang [5] in favor of the former.

- **Cohesive Principle** (COH). COH states that for any infinite sequence of sets $(R_i)_{i\in\mathbb{N}}$, there is an infinite set A such that for each i, either $A \subseteq^* R_i$ or $A \subseteq^* \overline{R_i}$. (Here, $\subseteq^*$ means that all but finitely many elements of the left hand side are contained in the right hand side.) As originally proved by Cholak, Jockusch, and Slaman [4], RT^2_2 is equivalent over RCA_0 to the conjunction of SRT^2_2 and COH.

- **Chain-Antichain, Ascending or Descending Sequence**, and friends (CAC, ADS, etc.). Chain-Antichain (CAC) states that every infinite partial order has an infinite subset that is either a chain or an antichain. Ascending or Descending Sequence (ADS) states that every infinite linear order has either an infinite ascending sequence or an infinite descending sequence. These principles were explored by Hirschfeldt and Shore in [11], who showed that they lie strictly below SRT^2_2. These principles themselves, like RT^2_2, split into stable and cohesive versions: SCAC, CCAC (which is equivalent to ADS), SADS, and CADS. All of these splittings are strict.

- **Diagonally Non-Computable Set Principle** (DNR). A computability principle more measure-theoretic than combinatorial in flavor, DNR states that for every set A, there is a set which is diagonally non-computable relative to A. DNR has proved to be important in that some, but not all, of the various combinatorial principles of interest imply it. For instance, SRT^2_2 is known to imply DNR (proved by Hirschfeldt et al. [10]), but Hirschfeldt and Shore proved [11] that CAC does not imply DNR, immediately giving the result that CAC does not imply SRT^2_2. Other, similar results have also been achieved.

- **Free Set and Thin Set Theorems** (FS(n) and TS(n)). The Free Set and Thin Set Theorems represent another kind of weakening of Ramsey's Theorem, dealing with colorings with infinitely many colors. The principle FS(n) states that for any function $f : [\omega]^n \to \omega$, there is an infinite *free set* A, such that $f(X) \in X \cup (\omega \setminus A)$ for every n-tuple X of elements from A. The principle TS(n) states that for every such function, there is an infinite *thin set* A, with the property that f restricted to $[A]^n$ omits some color; that is, $f([A]^n) \subsetneq \omega$. For every n, RT^n_2 implies FS(n), which implies TS(n). (These facts, and others, are proved by Cholak, Giusto, Hirst, and Jockusch in [3] where they establish a number of results about Free Set and Thin Set.) These theorems are much weaker than Ramsey's theorem; while RT^3_2 implies ACA_0 already, Wang has recently shown ([22]) that even the conjunction of FS(n) (and TS(n)) over all n is not enough to imply ACA_0.

The focus of this chapter is on the Thin Set Theorem for Pairs, or TS(2). This principle should be regarded as a substantial weakening of RT^2_2. TS(2) has something of a reputation for being almost uselessly weak; while it lies strictly above RCA_0, it was known to imply almost nothing of note. The goal of this paper is to vindicate this little flower of the reverse math jungle and show that it is not, in fact, uselessly weak but represents a notably different direction of weakening of RT^2_2 than the collection of principles living below CAC. Our goal is to prove the following theorem:

Theorem 2.1 (Main Theorem)**.** $\mathrm{RCA}_0 \vdash \mathrm{TS}(2) \to \mathrm{DNR}$.

We will approach the proof of Theorem 2.1 by first proving that it holds in ω-models of RCA_0 using techniques from computability theory, and then to show how to modify the proof to hold in the general setting. The interested reader may wish to compare this proof to the simpler proof that SRT^2_2 implies DNR in [10] from which it takes its inspiration.

2.2 ω-models of TS(2) are models of DNR

Our proof that ω-models of TS(2) are models of DNR begins with two lemmas.

Lemma 2.2. *There are sets $\{A_i\}_{i\in\omega}$, uniformly $\leq_T 0'$, a partition of ω, and a computable function f, such that for every e, if $W_e \subseteq \overline{A_i}$, then $|W_e| < f(e,i)$. In fact we may take $f(e,i) = (k+1)(k+2)$, where $k = \max(e,i)$.*

Proof. Let $R(e,i)$ be the requirement "if $W_e \subseteq \overline{A_i}$, then $|W_e| < f(e,i)$."

We will prove this by $0'$-computable construction in stages.

Begin with all A_i empty. For each $n \geq 0$, at stage n, we have already decided in previous stages $y \in A_i$ (for any i) for at most $n(n+1)$ numbers (by induction), and will decide $y \in A_i$ for at most $2n+2$ new numbers y in stage n, giving at most $(n+1)(n+2)$ numbers in some A_i (call these numbers *used*) by the end. We do this in three steps.

First, for each $0 \leq e < n$, we ensure that $R(e,n)$ is satisfied. Check using $0'$ if W_e has at least $(n+1)(n+2)$ elements. We are done with those W_e which do not; they are small and $R(e,n)$ is already satisfied. For those which do, take one element from each which has not been used. Since only $n(n+1)$ numbers have been used, each such W_e will contain such an element (in fact, n such elements, so we can choose them all distinct if we like). Put all these elements into A_n, satisfying $R(e,n)$ by negating the hypothesis. This uses at most n new numbers.

Next, we ensure that $R(n,i)$ is satisfied for each $0 \leq i \leq n$. Check if W_n has at least $(n+1)(n+2)$ elements; if not, $R(n,i)$ is already satisfied for every i. If so, since we have used at most $n(n+1)+n$ numbers so far, W_n contains at least $n+1$ elements which have not been used, say $\{x_i\}_{0\leq i\leq n}$. Put $x_i \in A_i$ for each $0 \leq i \leq n$, satisfying $R(n,i)$ for each $0 \leq i \leq n$. This uses at most $n+1$ new numbers.

Finally, if n is not yet in some A_i, put $n \in A_0$.

Thus at the end of stage n, we have ensured that $R(e,i)$ is satisfied for each $0 \leq e, i \leq n$, have used at most $n(n+1)+n+(n+1)+1 = (n+1)(n+2)$ numbers total, and ensured that all numbers up to n have been used.

The collection $\{A_i\}_{i\in\omega}$ constructed in the end are uniformly $0'$ computable, since it takes only to stage n to find out for which A_i we have $n \in A_i$, and there is a unique such A_i, so this is a partition of ω. Finally, all the requirements $R(e,i)$ are satisfied (each by stage $\max(e,i)$). □

Observe that the proof, and thus the result, relativizes to W_e^X and $\{A_i\}_{i\in\omega}$ uniformly $\leq_t X'$.

Lemma 2.3. *Let Turing Ideal $\mathcal{I}$ model TS(2). Then for all $\{A_i\}_{i\in\omega}$ uniformly $\leq_T C'$, where $C \in \mathcal{I}$, there is an infinite $B \in \mathcal{I}$ and an n so that $B \subseteq \bar{A}_n$.*

Proof. By the limit lemma, there is a function $f : \omega^2 \to \omega$, $f \leq_T C$, such that $A_i = \{x \in \omega : \lim_{m\to\infty} f(x,m) = i\}$. Taking $f : [\omega]^2 \to \omega$ by ignoring (x,m) with $x \geq m$ does not change these limits.

Then by TS(2), since $f \leq_T C \in \mathcal{I}$,

$$\exists n \exists B \in \mathcal{I} \ \left[f([B]^2) \subseteq \omega \setminus \{n\}\right].$$

So $\forall x \in B$ $(\lim_{m\to\infty} f(x,m) \neq n)$, and hence $B \subseteq \overline{A_n}$. □

Theorem 2.4. *Every ω-model of TS(2) is a model of DNR.*

Proof. Let $\mathcal{I}$ be a Turing ideal that is an ω-model of TS(2). We'll show that $\mathcal{I}$ contains a diagonally non-computable function, but everything relativizes naturally to find a DNR relative to any $X \in \mathcal{I}$.

Let $\{A_i\}_{i\in\omega}$ be as in Lemma 2.2. By Lemma 2.3, there is an infinite $B \in \mathcal{I}$ and an $n \in \omega$ such that $B \subseteq \overline{A_n}$. Then for all e, if $W_e \subseteq B$ then $|W_e| < (k+1)(k+2)$, where $k = \max(e, n)$. Call this quantity $f_n(e)$.

Let g be such that $W_{g(e)}$ is the set consisting of the first $f_n(e)$ many elements of B. (g is not computable, but is computable in B.) For any e, if $W_e = W_{g(e)}$, then $W_e \subseteq B \subseteq \overline{A_n}$, and so $|W_e| < f_n(e)$ by construction of A_n. But $|W_{g(e)}| = f_n(e)$, a contradiction. So $\forall e(W_e \neq W_{g(e)})$.

We can now finish with a standard argument giving a diagonally non-computable function from a fixed-point free function. Let f be computable such that $W_{f(e)} = W_{\Phi_e(e)}$ if $\Phi_e(e)\downarrow$, and $W_{f(e)} = \emptyset$ otherwise, and consider $h = g \circ f$. Now if $\Phi_e(e)\downarrow$, then $W_{h(e)} = W_{g(f(e))} \neq W_{f(e)} = W_{\Phi_e(e)}$, so it follows that $h(e) \neq \Phi_e(e)$. As h is total (since both f and g are), this means that h is a diagonally non-computable function. But $h \leq_T B \in \mathcal{I}$, so h is the function in $\mathcal{I}$ that we wanted. $\square$

2.3 TS(2) Implies DNR

In order to check that $\mathrm{RCA}_0 \vdash \mathrm{TS}(2) \to \mathrm{DNR}$, it suffices to show that the above proof can be carried out in RCA_0. To do this we need to, first, eliminate all references to jumps (we can only talk about functions that exist in the model) and then check that the proof only requires Σ^0_1-induction.

So, for instance, Lemma 2.2 becomes, recalling that there is an enumeration within a model $\mathcal{M}$ of all the $\mathcal{M}$-ce sets W_e (where indices range over the first-order part of $\mathcal{M}$):

Lemma 2.5. *There is a function $A(x,m)$ and a function $f(e,i)$ such that*

- $\lim_m A(x,m)$ *exists for each x,*
- *For each e, if $\forall x \in W_e(\lim_m A(x,m) \neq i)$, then $|W_e| < f(e,i)$.*

Similarly, Lemma 2.3 becomes:

Lemma 2.6. *For every function $A(x,m)$ such that $\lim_m A(x,m)$ exists for each x, there is an infinite B and an n such that $\forall x \in B[\lim_m A(x,m) \neq n]$.*

The proof of Lemma 2.6 is exactly the same as that of Lemma 2.3, except it is even easier: there is no need to apply the limit lemma, because we are using the limit notion in the first place as we don't have access to the jump. (Also, all instances of ω are replaced by $\mathbb{N}$, referring instead to the first-order part of $\mathcal{M}$.)

The proof of Lemma 2.5 is somewhat more subtle, since we don't have access to any oracle with which to determine the size of W_e.

Proof. Let $R(i,e)$ be the requirement that if $\forall x \in W_e(\lim_m A(x,m) \neq i)$, then $|W_e| < f(e,i)$. Let $\langle e,i \rangle$ be the pairing function that orders $(e_1,i_1) < (e_2,i_2)$ if $\max e_1, i_1 < \max e_2, i_2$ or these are equal and i_1e_1 precedes i_2e_2 in lexicographic order. At stage 0, we define $A(x,0) = 0$. At stage $m > 0$, we do the following:

First, run stages $0 \leq n < m$ from the proof of Lemma 2.2 as substages of stage m, except that, first, every time we need information about W_e or its size, we use instead the corresponding information about $W_{e,m}$ (which we can know), and second, whenever we would put $x \in A_i$, instead define $A(x,m) = i$. Also, omit the last step of each substage n (where we would put $A(n,m) = 0$ if it is not yet defined); it will not be necessary. Observe that for each $\langle e,i\rangle \leq \langle m,m\rangle$, this attempts to satisfy $R(e,i)$ (based on the assumption that $W_e = W_{e,m}$) in order. Say that the requirement $R(e,i)$ assigns x at stage m.

Then, write $A(x,m) = 0$ for all x for which $A(x,m)$ is not yet defined.

There is a subtle problem here, in that in the original proof of Lemma 2.2, we on several occasions made an arbitrary choice from elements of W_e. If we happened to make different choices at different stages m in the above construction, we could possibly ruin the existence of $\lim_m A(x,m)$. On the other hand, we can't mandate that we always make the same choices, because if we chose an element from W_{e_1}, say, to put into A_{i_1} (i.e., $A(x,m_0) = i_1$, using x to satisfy $R(e_1,i_1)$), then discovered at a later stage m_0 that W_{e_2} was large enough that it had to have an intersection with A_{i_2}, we might need to put $A(x,m_1) = i_2$. To solve this, we keep track of which requirement $R(e,i)$ assigns x at stage m, and only allow this to change to a higher priority requirement. Since we deal with requirements within each stage in their priority order, making this restriction does not hamper us at all.

We have to verify that, first, the construction can be carried out in RCA_0, and second, RCA_0 can verify that $\lim_m A(x,m)$ exists for all x and satisfies the stated requirements.

It is clear that each of the things we wish to do can be carried out in RCA_0 provided that we can show in RCA_0 that, as in the proof of Lemma 2.2, when we begin substage n of stage m we have defined $A(x,m)$ for at most $n(n+1)$ many elements x. This can be shown by Δ^0_1 induction (on n), so holds in RCA_0.

Next, we verify that $\lim_m A(x,m)$ exists for each x. Suppose there is a stage m_0 at which $A(x,m) \neq 0$ (if not, the limit exists and is 0). Then x is assigned by some $R(e,i)$

at some stage m_0. Since $A(x, m_1)$ can only be defined differently if x is assigned by a higher-priority requirement at stage m_1 than at stage m_0, it follows that there are at most $\langle e, i\rangle + 1$ many m such that $A(x, m) \neq A(x, m+1)$. Hence $\lim_m A(x, m)$ exists.

Finally, we need to check that each $R(e, i)$ is satisfied. It follows by induction that for each (e, i), there are at most $\langle e, i\rangle$ many numbers x such that $A(x, m)$ is ever assigned by any requirement $R(e', i')$ with $\langle e', i'\rangle < \langle e, i\rangle$. This is by Π^0_1 induction (which holds in RCA_0), via a formula stating that for all finite sequences of length $\langle e, i\rangle + 1$ and all m, it is not the case that each element of the sequence has been assigned by some $R(e', i')$ with $\langle e', i'\rangle < \langle e, i\rangle$ by stage m. So there is some stage $m(e, i)$ after which no new elements are ever assigned by requirements $R(e', i')$ with $\langle e', i'\rangle < \langle e, i\rangle$. So if $|W_e| \geq (k+1)(k+2)$, where $n = \max(e, i)$, then picking a stage $m \geq m(e, i)$ such that $|W_{e,m}| \geq (k+1)(k+2)$, we know that $A(x, m) = i$ for some $x \in W_{e,m}$, and furthermore, x can never be assigned by some $R(e', i')$ with $\langle e', i'\rangle < \langle e, i\rangle$. (It can't have been before, since if it were, it could never have been assigned by $R(e, i)$ as which requirement assigns an element can change only to requirements of higher priority. And since it has never been, it can't in the future by our definition of $m(e, i)$.) Hence, since x is never assigned later by a requirement with higher priority, it will always be assigned by $R(e, i)$. Thus $A(x, m') = i$ for all $m' \geq m$; that is, $\lim_m(x, m) = i$. □

Since our two lemmas are both provable in $\mathrm{RCA}_0 + \mathrm{TS}(2)$, it remains to check the construction of the function in the proof of Theorem 2.4. Everything in the construction goes through in RCA_0 (pretty much verbatim, in fact), so this completes the proof of Theorem 2.1.

2.4 Remarks

It is worth noting that we don't require even the full strength of TS(2) for this proof. Even a "stable" version of TS(2) suffices, where we assume (naturally enough) that for each x, $\lim_y c(\{x, y\})$ exists (where c is the coloring). This does not seem terribly interesting, however, if for no other reason than that the corresponding "cohesive" version of TS(2) is, of course, false.

The fact that TS(2) implies DNR, but other weakenings of RT^2_2, such as CAC, do not, suggests that DNR may prove to be a very useful tool for classifying various combinatorial principles. It has already been used to great effect to show several non-implications among combinatorial principles, and the current result suggests that such use may be fruitfully expanded. It may be that DNR is a much more interesting point of comparison for the reverse mathematics of combinatorial principles than something like WKL_0.

Chapter 3

Intervals of the Muchnik Lattice

In a recent paper, Terwijn [21] gave a characterization of the finite intervals of the Muchnik lattice $\mathscr{M}_w$. Contrasting with his results on the Medvedev lattice in [20], where the finite intervals are exactly the boolean algebras and all infinite intervals have antichains of cardinality $2^{2^{\aleph_0}}$, there is quite a lot of variety in the lattices that can appear as intervals of $\mathscr{M}_w$. Our goal in this chapter is to give a characterization of as large a class of these lattices as possible. We give a characterization of those intervals of $\mathscr{M}_w$ which have no antichains of cardinality $2^{\aleph_2}$.

An overview of mass problems and Muchnik reducibility can be found in Rogers [15], while Grätzer [8] and Davey and Priestley [7] contain background in lattice theory. However, all of the definitions and results necessary for this chapter can be found either in Sections 1.4 and 1.5 or below.

3.1 Ways to Think About Muchnik Degrees

It is the goal of this chapter to describe, as much as possible, the intervals of the lattice of Muchnik degrees. To do this, we will first need to develop an easier way of working with Muchnik degrees. In particular, we would like to have something of a canonical representative for each Muchnik degree, and an easy way of delving into their internal structures and comparing them. Most of this simplifying work is folklore, but it is worth making it explicit, because by the end our way of looking at Muchnik degrees will be much different than the standard definition.

Lemma 3.1. *The Muchnik degree of a mass problem depends only on the Turing degrees of its members. As a consequence, Muchnik reducibility has the same property.*

Proof. Suppose that the members of the mass problems $\mathcal{S}$ and $\mathcal{T}$ yield the same sets of Turing degrees. Then, take any $A \in \mathcal{T}$. There is some $B \in \mathcal{S}$ such that $B \equiv_T A$. In particular, $B \leq_T A$. Thus, $\mathcal{S} \leq_w \mathcal{T}$. By the same argument, $\mathcal{T} \leq_w \mathcal{S}$. Hence $\mathcal{S}$ and $\mathcal{T}$ have the same Muchnik degree. □

In light of Lemma 3.1, we can stop thinking about Muchnik degrees as collections of mass problems, and instead as collections of sets of Turing degrees. It is worth noting that this does not work in the case of Medvedev reducibility. Now, we would like to pick a canonical member of a Muchnik degree.

Lemma 3.2. *If $\mathcal{S}$ is a collection of Turing degrees, then it is Muchnik equivalent to* $\mathrm{ucl}(\mathcal{S})$*, where* $\mathrm{ucl}()$ *denotes the upward closure in the set of Turing degrees; that is,* $\mathrm{ucl}(\mathcal{S})$ *is the set of all Turing degrees in upper cones of elements of $\mathcal{S}$.*

Proof. It is certain that $\mathrm{ucl}(\mathcal{S}) \leq_w \mathcal{S}$, since every element of $\mathcal{S}$ is also an element of $\mathrm{ucl}(\mathcal{S})$, and computes itself. On the other hand, every $\mathbf{a} \in \mathrm{ucl}(\mathcal{S})$ is in the upper cone of some $\mathbf{b} \in \mathcal{S}$, and we have $\mathbf{b} \leq_T \mathbf{a}$. Hence $\mathcal{S} \leq_w \mathrm{ucl}(\mathcal{S})$ also. □

So we can restrict our attention to upward closed sets of Turing degrees. Does this give us our canonical representative? It turns out that it does.

Lemma 3.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be different upward closed sets of Turing degrees. Then they are not Muchnik equivalent.*

Proof. Since $\mathcal{S}$ and $\mathcal{T}$ are different, it follows that one of them (say $\mathcal{S}$) contains a Turing degree $\mathbf{a}$ that is not in the other. Moreover, since $\mathcal{T}$ is upward closed, $\mathbf{a}$ cannot be in the upper cone of any Turing degree in $\mathcal{T}$ (otherwise $\mathcal{T}$ would also contain $\mathbf{a}$). Hence there is no $\mathbf{b} \in \mathcal{T}$ such that $\mathbf{b} \leq_T \mathbf{a}$, and hence $\mathcal{T} \not\leq_w \mathcal{S}$. Thus they are not equivalent. □

Since we now have a canonical representative for the Muchnik degrees, we would like to be able to give a nice characterization of the ordering in terms of those canonical representatives. We can.

Lemma 3.4. *The ordering on the Muchnik degrees is given by reverse inclusion of their representatives. That is, if $\mathcal{S}$ and $\mathcal{T}$ are upward closed sets of Turing degrees, then $\mathcal{S} \leq_w \mathcal{T}$ if and only if $\mathcal{T} \subseteq \mathcal{S}$.*

Proof. First, suppose that $\mathcal{T} \not\subseteq \mathcal{S}$. Then we can repeat the proof of Lemma 3.3 to find that $\mathcal{S} \not\leq_w \mathcal{T}$. Conversely, suppose that $\mathcal{T} \subseteq \mathcal{S}$. Then for any $\mathbf{a} \in \mathcal{T}$, we also have $\mathbf{a} \in \mathcal{S}$, and of course $\mathbf{a} \leq_T \mathbf{a}$. Hence $\mathcal{S} \leq_w \mathcal{T}$. □

Now we have, for each Muchnik degree, a canonical representative which is an upward closed set of Turing degrees, and the Muchnik ordering is given by reverse inclusion on those representatives. However, these are not the representatives we are actually going to use. Mostly, they are not really that convenient: upward closed sets of Turing degrees are enormous and unwieldy, and navigating the fact that the ordering is *reverse* inclusion is at best annoying and at worst a continual source of confusion. Luckily, we have an easy remedy. We can take complements. The complements of upward closed sets of Turing degrees are downward closed sets of Turing degrees, and the corresponding ordering is by inclusion rather than reverse inclusion. Thus, we have the following:

Lemma 3.5. *The Muchnik degrees are in one to one correspondence with the downward closed subsets of the Turing degrees, and the ordering on Muchnik degrees corresponds to the subset ordering on these sets. Meet and join in the lattice of Muchnik degrees then correspond to intersection and union of these subsets.*

We will take the liberty, from here on, of treating these corresponding downward closed sets of Turing degrees as identical to the Muchnik degrees they represent, since the lattice structure they form is identical to the Muchnik lattice we are concerned with. This has the potential for some confusion, but in the author's opinion it makes the proofs themselves much more intuitive and simpler to present and is well worth it.

3.2 Some Lattice Theory

We also need a number of ideas from lattice theory.

Definition 3.6. A lattice L is *complete* if, for every subset $S \subseteq L$, S has a supremum in L.

If L is a complete lattice, the supremum of a subset $S \subseteq L$ in L is sometimes called the join of S and will be written $\bigvee_L S$ (or $\bigvee S$ if the lattice L is clear). In a complete lattice, every subset S also has an infimum (or meet): the supremum of the set of lower bounds of S. Every finite lattice is necessarily complete, and every complete lattice has both a least and greatest element.

Definition 3.7. An *interval* of a lattice L is the set $\{x \in L \mid a \leq x \leq b\}$ for some (not necessarily distinct) $a \leq b \in L$ and is often written $[a, b]$.

Proposition 3.8. *If L is a complete lattice and M is an interval of L, then M is also a complete lattice.*

Proof. Let $S \subseteq M$ and let $\mathcal{A}$ and $\mathcal{B}$ be the least and greatest elements of M, respectively. If S is empty, then $\bigvee_M S$ is $\mathcal{A}$, so suppose that S is nonempty. Then $S \subseteq L$; certainly $\bigvee_L S \leq \mathcal{B}$, since $\mathcal{B} \in L$ and $\mathcal{B}$ is an upper bound for all the elements of S. Further, for $\mathcal{X} \in S$, we have $\mathcal{A} \leq \mathcal{X} \leq \bigvee_L S \leq \mathcal{B}$, so it follows that $\bigvee_L S \in M$. Since every element of M is also in L, $\bigvee_L S$ must be the supremum of S in M as well. $\square$

For a general partial order, we have the notion of a convex subset.

Definition 3.9. Let P be a partial order. A subset $S \subseteq P$ is *convex* if for every $x, y \in S$ and $a \in P$, $x \leq a \leq y$ implies that $a \in S$.

Observe that, unlike intervals of lattices, convex subsets of partial orders need not have endpoints.

Filling a similar role to the join-irreducible elements in Terwijn's characterization in [21] of the finite intervals of $\mathscr{M}_w$, we have the completely join-prime elements and the suborder they form.

Definition 3.10. Let L be a complete lattice. An element $\mathcal{X}$ is called *completely join-prime* if for any $S \subseteq L$, $\mathcal{X} \leq \bigvee S$ implies that there is some $\mathcal{Y} \in S$ such that $\mathcal{X} \leq \mathcal{Y}$.

Definition 3.11. Let L be a complete lattice. The partial order of completely join-prime elements of L is $\mathcal{J}_{\mathcal{P}}(L)$.

The completely join-prime elements of the Muchnik lattice are particularly nice:

Lemma 3.12. *The completely join-prime elements of $\mathscr{M}_w$ are the downward closures of single Turing degrees. Therefore, $\mathcal{J}_{\mathcal{P}}(\mathscr{M}_w) \cong \mathscr{D}$.*

Proof. First we show that the downward closure $\mathcal{A}$ of a Turing degree $\mathbf{a}$ is completely join-prime. Let $S \subseteq \mathscr{M}_w$ and suppose that $\mathcal{A} \leq \bigvee S$. Then $\mathbf{a} \in \mathcal{X}$ for some $\mathcal{X} \in S$, and hence $\mathcal{A} \leq \mathcal{X}$. This shows that $\mathcal{A}$ is completely join-prime.

Second, suppose that $\mathcal{A} \in \mathscr{M}_w$ is completely join-prime. Let $S = \{\mathrm{dcl}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{A}\}$. Then $\mathcal{A} \leq \bigvee S$, so it follows that $\mathcal{A} \leq \mathrm{dcl}(\mathbf{a})$ for some $\mathbf{a} \in A$. Since $\mathrm{dcl}(\mathbf{a}) \leq \mathcal{A}$ for any $\mathbf{a} \in \mathcal{A}$, it follows that $\mathcal{A}$ must be the downward closure of the single Turing degree $\mathbf{a}$. $\square$

We also have a way of extracting a lattice from a partial order.

Definition 3.13. Let P be a partial order. Then $\mathcal{O}(P)$ is the lattice of downward closed subsets of P, ordered by inclusion.

Terwijn [21] calls this $H(P)$, but we will follow Davey and Priestley [7] in calling it $\mathcal{O}(P)$. In light of Definition 3.13, we can restate Lemma 3.5 as $\mathscr{M}_w \cong \mathcal{O}(\mathscr{D})$.

Terwijn's characterization of the finite intervals in $\mathscr{M}_w$ rests on the duality between a finite distributive lattice L and the partially ordered set $\mathcal{J}(L)$ of its join-irreducible elements, with $L \cong \mathcal{O}(\mathcal{J}(L))$. This duality does not hold for all infinite distributive lattices, but for a particular class of lattices, which happens to include the intervals of $\mathscr{M}_w$, we can replace $\mathcal{J}(L)$ with $\mathcal{J}_{\mathcal{P}}(L)$ and still get this duality.

Definition 3.14. A complete lattice L is *completely distributive* if it satisfies, for every doubly indexed subset $\{x_{ij}\}_{i\in I, j\in J}$ of L:

$$\bigwedge_{i\in I}\left(\bigvee_{j\in J} x_{ij}\right) = \bigvee_{\alpha: I\to J}\left(\bigwedge_{i\in I} x_{i\alpha(i)}\right).$$

In particular, meets distribute over arbitrary joins and joins distribute over arbitrary meets.

Definition 3.15. An element k of a complete lattice L is *compact* if, for every subset $S \subseteq L$, $k \leq \bigvee S$ implies that there is some finite $T \subseteq S$ such that $k \leq \bigvee T$. A complete lattice L is *algebraic* if, for each $a \in L$, $a = \bigvee \{k \in L \mid k \text{ compact and } k \leq a\}$.

These definitions, and discussion of them, can be found in [7].

Definition 3.16. A lattice is *superalgebraic* if it is algebraic and completely distributive.

As it turns out, there are several equivalent conditions for a lattice being superalgebraic, and it is these equivalent conditions which we are interested in.

Theorem 3.17 ([7], 10.29 and [6], 2.5)**.** *Let L be a lattice. Then the following are equivalent:*

- *L is superalgebraic.*
- *$L \cong \mathcal{O}(P)$ for some partially ordered set P.*
- *Every element of L is the join of a set of completely join-prime elements of L.*
- *(Duality) $L \cong \mathcal{O}(\mathcal{J}_{\mathcal{P}}(L))$ via the isomorphism $a \mapsto \{x \in \mathcal{J}_{\mathcal{P}}(L) \mid x \leq a\}$.*

Corollary 3.18. *$\mathscr{M}_w$ is a superalgebraic lattice.*

Proof. By Lemma 3.5 we know that $\mathscr{M}_w \cong \mathcal{O}(\mathscr{D})$. □

Just as distributivity and completeness are preserved by taking intervals, so too is the property of being superalgebraic.

Lemma 3.19. *Every interval of a superalgebraic lattice is itself a superalgebraic lattice.*

Proof. Let L be a superalgebraic lattice and M be an interval of L, and let the least element of M be $\mathcal{A}$. If $\mathcal{X} \vee \mathcal{A}$ is an element of M and $\mathcal{X}$ is completely join-prime in L, then either $\mathcal{X} \vee \mathcal{A}$ is completely join-prime in M or $\mathcal{X} \vee \mathcal{A} = \mathcal{A}$. For suppose that $S \subseteq M$ and $\mathcal{X} \vee \mathcal{A} \leq \bigvee S$. If S is empty, then $\mathcal{X} \vee \mathcal{A} = \mathcal{A}$; otherwise, $\bigvee S$ is the same in both M and L. In that case, certainly $S \subseteq L$, so by the fact that $\mathcal{X}$ is completely join-prime in L, we see that for some $\mathcal{Y} \in S$, $\mathcal{X} \leq \mathcal{Y}$, and hence $\mathcal{X} \vee \mathcal{A} \leq \mathcal{Y} \vee \mathcal{A} = \mathcal{Y}$, with the last equality because $\mathcal{A} \leq \mathcal{Y}$. This implies that $\mathcal{X} \vee \mathcal{A}$ is completely join-prime in M.

Now let $\mathcal{X}$ be any element of M. If $\mathcal{X}$ is the least element of M, it is the supremum of the empty set and we are done. Otherwise, let $\mathcal{A}$ be the least element of M. Since L is superalgebraic, there is some set S of completely join-prime elements of L such that $\mathcal{X} = \bigvee S$; the set is nonempty since $\mathcal{X}$ is not the least element of M and thus not the least element of L; indeed, S must contain some element $\mathcal{Y} \not\leq \mathcal{A}$. For every element $\mathcal{Y} \in S$, observe that $\mathcal{A} \leq \mathcal{Y} \vee \mathcal{A} \leq \mathcal{X} \vee \mathcal{A} = \mathcal{X}$, so that $\mathcal{Y} \vee \mathcal{A} \in M$. Letting $S_{\mathcal{A}} = \{\mathcal{Y} \vee \mathcal{A} \mid \mathcal{Y} \in S\}$, we see by the preceding paragraph that every element of $S_{\mathcal{A}}$ is either the least element $\mathcal{A}$ of M or a completely join-prime element of M, and that there is at least one of the latter. Letting $T = S_{\mathcal{A}} \setminus \mathcal{A}$, we have that $\bigvee T = \bigvee S_{\mathcal{A}} = \mathcal{A} \vee (\bigvee S) = \mathcal{A} \vee \mathcal{X} = \mathcal{X}$, so that $\mathcal{X}$ is the join of a set of completely join-prime elements of M. It follows that M is superalgebraic. □

Taking intervals also has a nice relationship with the duality condition.

Lemma 3.20. *Let L and M be superalgebraic lattices. Then M is isomorphic to an interval of L if and only if $\mathcal{J}_{\mathcal{P}}(M)$ is isomorphic to a convex subset of $\mathcal{J}_{\mathcal{P}}(L)$.*

Proof. First let M be an interval of L, and let $\mathcal{A}$ and $\mathcal{B}$ be the least and greatest elements of M, respectively. Let R be the set of completely join-prime elements $\mathcal{X} \in L$ such that $\mathcal{X} \not\leq \mathcal{A}$ and $\mathcal{X} \leq \mathcal{B}$. R is clearly a convex subset of $\mathcal{J}_{\mathcal{P}}(L)$. By the proof of Lemma 3.19 we know that for such $\mathcal{X}$, $\mathcal{X} \vee \mathcal{A}$ is completely join-prime in M. Conversely, if $\mathcal{Y}$ is completely join-prime in M, then there is (by complete distributivity) a least $\mathcal{X} \in L$ such that $\mathcal{X} \vee \mathcal{A} = \mathcal{Y}$. This $\mathcal{X}$ is completely join-prime in L. For if not, take a set $S \subseteq L$ with $\mathcal{X} \leq \bigvee S$ and $\mathcal{X} \not\leq \mathcal{Z}$ for any $\mathcal{Z} \in S$. Then $S' = \{(\mathcal{Z} \wedge \mathcal{B}) \vee \mathcal{A} \mid \mathcal{Z} \in S\}$ is a subset of M with the same property for $\mathcal{Y}$, contradicting that $\mathcal{Y}$ is completely join-prime in M. It follows that $\mathcal{X} \in R$. Finally, if $\mathcal{X}_1$ and $\mathcal{X}_2$ are two elements of R (without loss of generality, $\mathcal{X}_2 \not\leq \mathcal{X}_1$), then $\mathcal{X}_1 \vee \mathcal{A} = \mathcal{X}_2 \vee \mathcal{A}$ implies that $\mathcal{X}_2 \leq \mathcal{X}_1 \vee \mathcal{A}$ and thus that $\mathcal{X}_2$ is not join-prime, a contradiction. Hence we have an isomorphism between R and $\mathcal{J}_{\mathcal{P}}(M)$.

Second, suppose that $\mathcal{J}_{\mathcal{P}}(M)$ is isomorphic to a convex subset of $\mathcal{J}_{\mathcal{P}}(L)$. We will show that every such convex subset is of the form above: it is the set of elements $\mathcal{X} \in \mathcal{J}_{\mathcal{P}}(L)$ such that $\mathcal{X} \not\leq \mathcal{A}$ and $\mathcal{X} \leq \mathcal{B}$ for some $\mathcal{A} \leq \mathcal{B}$. To see this, let R be a convex subset of $\mathcal{J}_{\mathcal{P}}(L)$. Let R_L be the set of elements of $\mathcal{J}_{\mathcal{P}}(L)$ contained in the downward closure of R but not contained in R. Let $\mathcal{A} = \bigvee R_L$ and $\mathcal{B} = \bigvee R$. Certainly we have $\mathcal{X} \leq \mathcal{B}$ for every $\mathcal{X} \in R$. Additionally, if $\mathcal{X} \in R$, then $\mathcal{X} \not\leq \mathcal{A}$, since $\mathcal{X} \not\leq \mathcal{Y}$ for any $\mathcal{Y} \in R_L$. ($\mathcal{Y}$ is in the downward closure of R, and if some element of R were below $\mathcal{Y}$, then by convexity of R, $\mathcal{Y}$ would be in R.) On the other hand, if $\mathcal{X} \in \mathcal{J}_{\mathcal{P}}(L)$ and $\mathcal{X} \leq \mathcal{B} = \bigvee R$, then $\mathcal{X}$ lies below some element of R (and thus it lies in the downward closure of R) because $\mathcal{X}$ is completely join-prime, and if $\mathcal{X} \not\leq \mathcal{A}$, then it does not lie in R_L. So any $\mathcal{X} \in \mathcal{J}_{\mathcal{P}}(L)$ with $\mathcal{X} \not\leq \mathcal{A}$ and $\mathcal{X} \leq \mathcal{B}$ must be in R. Now we observe that the first direction shows that when R is of this form, it is isomorphic to $\mathcal{J}_{\mathcal{P}}(T)$ for the interval

T in L with least element $\mathcal{A}$ and greatest element $\mathcal{B}$. By the duality condition for superalgebraic lattices and the fact that T is superalgebraic (by Lemma 3.19), we have $T \cong \mathcal{O}(\mathcal{J}_{\mathcal{P}}(T)) \cong \mathcal{O}(\mathcal{J}_{\mathcal{P}}(M)) \cong M$; that is, M is isomorphic to an interval of L. □

3.3 The Main Theorem

We are now ready to state our characterization of intervals in the Muchnik lattice.

Theorem 3.21 (Main Theorem). *A lattice L with no antichains of cardinality $2^{\aleph_2}$ is isomorphic to an interval of the Muchnik lattice $\mathscr{M}_w$ if and only if the following hold:*

1. *L is superalgebraic,*
2. *$\mathcal{J}_{\mathcal{P}}(L)$ is an initial segment of an upper semilattice, and*
3. *$\mathcal{J}_{\mathcal{P}}(L)$ has the countable predecessor property.*

Proof that the conditions are necessary. Let L be an interval of the Muchnik lattic M_w. By Corollary 3.18, $\mathscr{M}_w$ is superalgebraic. By Lemma 3.19, L is also superalgebraic. By Lemma 3.12, $\mathcal{J}_{\mathcal{P}}(\mathscr{M}_w) \cong \mathscr{D}$. By Lemma 3.20, it follows that $\mathcal{J}_{\mathcal{P}}(L)$ is isomorphic to a convex subset of the Turing degrees $\mathscr{D}$. Because $\mathscr{D}$ is an upper semilattice, every convex subset $\mathscr{D}$ is an initial segment of an upper semilattice, and because $\mathscr{D}$ has the countable predecessor property, so does every convex subset of $\mathscr{D}$. Hence $\mathcal{J}_{\mathcal{P}}(L)$ has these properties as well. □

The proof that these conditions are sufficient is a little more involved.

Proof that the conditions are sufficient. Let L be a superalgebraic lattice such that $\mathcal{J}_{\mathcal{P}}(L)$ is the initial segment of an upper semilattice and has the countable predecessor property.

First, we need a minor set-theoretic lemma.

Lemma 3.22. *For any infinite cardinal κ, the powerset $\mathscr{P}(\kappa)$ has an antichain of cardinality 2^κ.*

Proof. For any subset $X \subseteq \kappa$, define $A_X \subset \kappa + \kappa$ (where the summation is ordinal arithmetic) by $\alpha \in A_X$ if and only if $\alpha \in X$, and $\kappa + \alpha \in A_X$ if and only if $\alpha \notin X$, where α ranges over ordinals less than κ. Then if $X \neq Y$, A_X and A_Y are incomparable. It follows that $\mathscr{P}(\kappa + \kappa)$ has an antichain of cardinality 2^κ. But taking a bijection $f : \kappa + \kappa \to \kappa$ (which exists because κ is infinite) we get that $\mathscr{P}(\kappa) \cong \mathscr{P}(\kappa + \kappa)$, so that $\mathscr{P}(\kappa)$ must have an antichain of cardinality 2^κ. □

We can use this to put an upper bound on the size of antichains in $\mathcal{J}_\mathcal{P}(L)$.

Lemma 3.23. *$\mathcal{J}_\mathcal{P}(L)$ has no antichains of cardinality $\aleph_2$.*

Proof. Suppose that $\{\mathcal{A}_n\}_{n \in \omega_2}$ was an antichain in $\mathcal{J}_\mathcal{P}(L)$. Let $\{S_\alpha\}_{\alpha < 2^{\aleph_2}} \subset \mathscr{P}(\omega_2)$ be an antichain (under $\subseteq$) of subsets of ω_2; such an antichain exists by Lemma 3.22. Define $\mathcal{X}_\alpha = \sup\{\mathcal{A}_n\}_{n \in S_\alpha}$. We will show that $\{\mathcal{X}_\alpha\}_{\alpha < 2^{\aleph_2}}$ is an antichain in L.

Let $\alpha \neq \beta$. Suppose for contradiction that $\mathcal{X}_\alpha \leq \mathcal{X}_\beta$ (the other ordering works the same way). Then there is some $p \in S_\alpha \setminus S_\beta$, and $\mathcal{A}_p \leq \mathcal{X}_\alpha \leq \mathcal{X}_\beta$. Since $\mathcal{A}_p$ is completely join-prime, there is some $q \in S_\beta$ such that $\mathcal{A}_p \leq \mathcal{A}_q$. But $p \neq q$, since $p \notin S_\beta$, and thus this is a contradiction, since the $\mathcal{A}_n$ were incomparable. Thus $\{\mathcal{X}_\alpha\}_{\alpha < 2^{\aleph_2}}$ is an antichain in L.

But L has no antichains of cardinality $2^{\aleph_2}$, so this is a contradiction. Hence $\mathcal{J}_\mathcal{P}(L)$ has no antichains of cardinality $\aleph_2$. □

Lemma 3.24. *Suppose that P is a partial order with no antichains of cardinality $\aleph_2$ and the countable predecessor property. Then P has cardinality at most $\aleph_1$.*

Proof. We will show that P is the union of a chain of length at most ω_1 of subsets of P which have cardinality at most $\aleph_1$. Define subsets P_α of P in the following way:

P_0: Let X_0 be any maximal antichain in P. By hypothesis, it has at most $\aleph_1$ elements. Below each element of X there are at most countably many elements of P. Hence the downward closure of X_0 in P is a union of at most $\aleph_1$ sets, each with cardinality at most $\aleph_0$, hence it has cardinality at most $\aleph_1$. Let P_0 be the downward closure of X.

Successor stages $\alpha + 1$: If $P = P_\alpha$ stop, as P thus has cardinality at most $\aleph_1$. Otherwise, take a maximal antichain $X_{\alpha+1}$ in $P \setminus P_\alpha$. It is an antichain in P so it is of cardinality at most $\aleph_1$; also, it is nonempty. By the countable predecessor property, the downward closure of $X_{\alpha+1}$ in P has cardinality at most $\aleph_1$. Let $P_{\alpha+1}$ be the union of P_α and the downward closure of $X_{\alpha+1}$; it has cardinality at most $\aleph_1$.

Countable limit stages α: Define $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$. It is a countable union of sets of cardinality at most $\aleph_1$, so it has cardinality at most $\aleph_1$.

It remains to prove that $P = \bigcup_{\alpha<\omega_1} P_\alpha$. Suppose not. Then there is some $x \in P$ such that $x \notin \bigcup_{\alpha<\omega_1} P_\alpha$. At every successor stage $\alpha + 1$, $x \notin P_{\alpha+1}$, so it follows that x is not in P_α, nor is it below any element of $X_{\alpha+1}$. By maximality of $X_{\alpha+1}$, it follows that x must be above some element of $X_{\alpha+1}$. It follows that, for each countable α, some element of $X_{\alpha+1}$ is a predecessor of x. But these are all distinct, and there are uncountably many countable ordinals α, so it follows that x has uncountably many predecessors, contradicting the fact that P has the countable predecessor property. Hence there is no such x and $P = \bigcup_{\alpha<\omega_1} P_\alpha$.

This expresses P as a union of a collection of size $\aleph_1$ of sets with cardinality at most $\aleph_1$, so it follows that P is of cardinality at most $\aleph_1$. $\square$

Together, these give us an important fact about $\mathcal{J}_{\mathcal{P}}(L)$.

Lemma 3.25. *$\mathcal{J}_{\mathcal{P}}(L)$ has cardinality at most $\aleph_1$.*

Proof. Lemma 3.24 applies to $\mathcal{J}_{\mathcal{P}}(L)$ by Lemma 3.23. $\square$

We can now use a powerful result of Abraham and Shore [1] to map $\mathcal{J}_{\mathcal{P}}(L)$ to a nice part of $\mathscr{D}$.

Proposition 3.26. *Let P_0 be the result of adding a smallest element 0 to $\mathcal{J}_{\mathcal{P}}(L)$. Then P_0 is isomorphic to an initial segment of the upper semilattice $\mathscr{D}$ of Turing degrees.*

Proof. By assumption, $\mathcal{J}_{\mathcal{P}}(L)$, and thus P_0, has the countable predecessor property and is an initial segment of an upper semilattice. By Lemma 3.25, P_0 has cardinality at most $\aleph_1$, and by construction, P_0 has a least element. Abraham and Shore proved [1, Theorem 3.12] that every such partial order is isomorphic to an initial segment of $\mathscr{D}$. $\square$

It follows by removing the least element of P_0 that $\mathcal{J}_{\mathcal{P}}(L)$ is isomorphic to a convex subset of $\mathscr{D}$. By Lemma 3.12, we have that $\mathcal{J}_{\mathcal{P}}(L)$ is isomorphic to a convex subset of $\mathcal{J}_{\mathcal{P}}(\mathscr{M}_w)$, and thus by Lemma 3.20, it follows that L is isomorphic to an interval of $\mathscr{M}_w$. $\square$

As corollaries to Theorem 3.21 we can get interesting characterizations in several special cases:

Corollary 3.27 (Terwijn [21], 3.14)**.** *A finite distributive lattice L is isomorphic to an interval of $\mathscr{M}_w$ if and only if the join-irreducible elements $\mathcal{J}(L)$ of L form an initial segment of an upper semilattice.*

Proof. Such a lattice is automatically superalgebraic, and for finite lattices the join-irreducible elements are necessarily completely join-prime, so $\mathcal{J}_{\mathcal{P}}(L) = \mathcal{J}(L)$. Finally, $\mathcal{J}_{\mathcal{P}}(L)$ is finite and therefore necessarily has the countable predecessor property. □

The reader will observe that Terwijn also uses the condition of not being "double-diamond-like"—that is, not having a pair of elements of $\mathcal{J}(L)$ with two minimal mutual upper bounds—as an equivalent of forming an initial segment of an upper semilattice. In the finite case, these coincide just as the notions of join-irreducible and completely join-prime do. As soon as one steps into the infinite case, both of these simplifications break down. For instance, the linear order $\omega+1$ occurs as an interval of $\mathscr{M}_w$; its greatest element is join-irreducible but not completely join-prime. Similarly, one has a partial order P with two minimal elements and an infinite descending chain of mutual upper bounds for these two elements; the lattice L with $\mathcal{J}_{\mathcal{P}}(L) = P$ is not double-diamond-like, but neither is P an initial segment of an upper semilattice, and so this L does not occur as an interval of $\mathscr{M}_w$.

Corollary 3.28. *A linear order L is isomorphic to an interval of $\mathscr{M}_w$ if and only if it is complete and the set of successors is dense in L and has the countable predecessor property.*

Proof. The successor elements in a linear order are exactly the completely join-prime elements. The set of successors being dense in L is exactly the third condition in Theorem 3.17, so this just says that L is superalgebraic. $\mathcal{J}_{\mathcal{P}}(L)$, the set of successors, is a linear order and thus automatically an initial segment of an upper semilattice; its having the countable predecessor property is just the last condition in Theorem 3.21. □

Corollary 3.29. *A countable lattice L is isomorphic to an interval of $\mathscr{M}_w$ if and only if it is superalgebraic and $\mathcal{J}_{\mathcal{P}}(L)$ is an initial segment of an upper semilattice.*

Proof. $\mathcal{J}_{\mathcal{P}}(L)$ is countable and thus automatically has the countable predecessor property. □

As a corollary to the *proof* of Theorem 3.21, we also obtain a nice result about initial segments of $\mathscr{M}_w$.

Corollary 3.30. *A lattice L satisfying the conditions in Theorem 3.21 is isomorphic to a closed initial segment of $\mathscr{M}_w$ if and only if the reduced lattice L^- obtained by removing the least element of L is either empty or has a least element.*

Proof. The case when L^- is empty corresponds to the interval $[\emptyset, \emptyset]$ in $\mathscr{M}_w$. Otherwise, the condition that L^- has a least element is necessary, since, because $\mathscr{D}$ has a least element $\mathbf{0}$, every initial segment of $\mathscr{M}_w$ has both a least element ($\emptyset$) and a second-least element ($\{\mathbf{0}\}$). On the other hand, it is sufficient, because applying the proof of the main theorem to L^- we see that L^- is in fact isomorphic to an interval of the form $[\{\mathbf{0}\}, \mathcal{B}]$ for some Muchnik degree $\mathbf{B}$, and hence L is isomorphic to the initial segment $[\emptyset, \mathcal{B}]$. □

3.4 Join-Irreducibles and Intervals with no Uncountable Antichains

The reader will observe that the partial order $\mathcal{J}_{\mathcal{P}}(L)$ of completely join-prime elements plays in our characterization and proof roughly the role that the partial order $\mathcal{J}(L)$ of join-irreducible elements played in Terwijn's original paper. Indeed, as we observed in Corollary 3.27, these notions coincide in the finite case, but not more generally. Nevertheless, the join-irreducible elements of $\mathscr{M}_w$ and its intervals do hold some interest.

Definition 3.31. Given a partial order P, an *ideal* of P is a nonempty downward closed subset $S \subseteq P$ such that for every $x, y \in S$, there is some $z \in S$ such that $x, y \leq z$. A *principal ideal* is the downward closure of a single element of P.

We have already observed (Lemma 3.12) that the completely join-prime elements of $\mathscr{M}_w$ are exactly the principal ideals of $\mathscr{D}$; that is, the principal Turing ideals. Since principal ideals and ideals coincide the the finite case, this may suggest that the join-irreducible elements are the ideals of $\mathscr{D}$. This, and something more general, is true.

Theorem 3.32. *Let L be a superalgebraic lattice and identify L with $\mathcal{O}(\mathcal{J}_{\mathcal{P}}(L))$ via the canonical isomorphism (from Theorem 3.17). Then the join-irreducible elements of L are exactly the ideals of $\mathcal{J}_{\mathcal{P}}(L)$ and the completely join-prime elements of L are exactly the principal ideals of $\mathcal{J}_{\mathcal{P}}(L)$.*

Proof. Suppose that $\mathcal{A} \subseteq \mathcal{J}_{\mathcal{P}}(L)$ is an ideal, and suppose for a contradiction that $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ for some incomparable $\mathcal{B}, \mathcal{C} \in \mathcal{O}(\mathcal{J}_{\mathcal{P}}(L))$. Then there is some $x \in \mathcal{B}$ and $y \in \mathcal{C}$ such that $x \notin \mathcal{C}$ and $y \notin \mathcal{B}$. But since $\mathcal{A}$ is an ideal, there is some $z \in A$ such that $x, y \leq z$; this z must be in either $\mathcal{B}$ or $\mathcal{C}$, which (since $\mathcal{B}$ and $\mathcal{C}$ are downward closed), implies that both x and y are in one of them. This is a contradiction, so $\mathcal{A}$ must be join-irreducible.

Conversely, suppose that $\mathcal{A} \subseteq \mathcal{J}_{\mathcal{P}}(L)$ is join-irreducible. Let $x, y \in \mathcal{A}$, supposing for contradiction that there is no $z \in \mathcal{A}$ such that $x, y \leq z$, and define $\mathcal{B} = \{z \in \mathcal{A} \mid y \not\leq z\}$ and $\mathcal{C} = \{z \in \mathcal{A} \mid x \not\leq z\}$. Then $\mathcal{B}$ and $\mathcal{C}$ are downward closed, $y \notin \mathcal{B}$ and $x \notin \mathcal{C}$, and $\mathcal{B} \cup \mathcal{C} = \mathcal{A}$ since x and y have no mutual upper bound in $\mathcal{A}$. This contradicts $\mathcal{A}$ being join-irreducible, so in fact $\mathcal{A}$ must be an ideal.

Now consider the principal ideals. Let $\mathcal{A} \subseteq \mathcal{J}_{\mathcal{P}}(L)$ be a principal ideal. Then $\mathcal{A} = \mathrm{dcl}(x)$ for some $x \in \mathcal{J}_{\mathcal{P}}(L)$; hence $\mathcal{A} \subseteq \mathcal{B} \in \mathcal{O}(\mathcal{J}_{\mathcal{P}}(L))$ if and only if $x \in \mathcal{B}$. Let $S \subseteq \mathcal{O}(\mathcal{J}_{\mathcal{P}}(L))$ and suppose that $\mathcal{A} \leq \bigcup S$; then $x \in \bigcup S$ and hence there is some $\mathcal{B} \in S$ such that $x \in \mathcal{B}$, whence $\mathcal{A} \subseteq \mathcal{B}$. This shows that $\mathcal{A}$ is completely join-prime.

Finally, suppose that $\mathcal{A} \subseteq \mathcal{J}_{\mathcal{P}}(L)$ is completely join-prime. Writing $\mathcal{A} \subseteq \bigcup_{x \in \mathcal{A}} \mathrm{dcl}(x)$ yields that $\mathcal{A} \subseteq \mathrm{dcl}(x)$ for some $x \in \mathcal{A}$ since $\mathcal{A}$ is completely join-prime, and of course $\mathrm{dcl}(x) \subseteq \mathcal{A}$. Hence $\mathcal{A}$ is a principal ideal. □

As a corollary we have a characterization of the join-irreducible elements of $\mathscr{M}_w$.

Corollary 3.33. *The join-irreducible elements of $\mathscr{M}_w$ are exactly the ideals of $\mathscr{D}$.*

If we restrict our attention to lattices with no uncountable antichains, it turns out that everything can be expressed nicely in terms of join-irreducibles.

Theorem 3.34. *Let L be a superalgebraic lattice with no uncountable antichains. Then every element of L is the join of finitely many join-irreducible elements of L.*

Observe that if we remove "finitely many" this is true of all superalgebraic lattices by the third condition in Theorem 3.17. On the other hand, with the "finitely many" condition, it is not: consider, for instance, $L = \mathcal{O}(P)$ where P consists of uncountably many incomparable elements.

Proof. Continue to identify L with $\mathcal{O}(\mathcal{J}_{\mathcal{P}}(L))$. Let $\mathcal{X} \in L$ be arbitrary. Because the interval $M = [\emptyset, \mathcal{X}]$ is again superalgebraic by Lemma 3.19, every ideal of $\mathcal{J}_{\mathcal{P}}(L)$ contained in M is again an ideal of $\mathcal{J}_{\mathcal{P}}(M)$, and every ideal of $\mathcal{J}_{\mathcal{P}}(M)$ is an ideal of $\mathcal{J}_{\mathcal{P}}(L)$, it suffices to consider the *largest* element $\mathcal{X} \in L$.

Since the union of a chain of ideals is again an ideal, it follows that every ideal of $\mathcal{J}_{\mathcal{P}}(L)$ is contained in a maximal ideal. Since $\mathcal{X}$ is the union of (principal) ideals, it is therefore the union of *maximal* ideals. Our goal will be to get a handle on the maximal ideals of L and show that they are finite in number.

Lemma 3.35. *There are at most countably many maximal ideals of $\mathcal{J}_{\mathcal{P}}(L)$.*

Proof. Any pair of maximal ideals is necessarily incomparable, so the collection of all maximal ideals is an antichain of L. By assumption L has no uncountable antichains, so $\mathcal{J}_{\mathcal{P}}(L)$ has at most countably many maximal ideals. □

Lemma 3.36. *Let $\mathcal{A}$ and $\{\mathcal{B}_n\}_{n \leq N}$ be distinct maximal ideals of $\mathcal{J}_{\mathcal{P}}(L)$ for some $N \in \omega$. Then $\mathcal{A} \not\subseteq \bigcup_{n \leq N} \mathcal{B}_n$.*

Proof. For every $n \leq N$, $\mathcal{A} \not\subseteq \mathcal{B}_n$, and so there is some $x_n \in \mathcal{A}$ such that $x_n \notin \mathcal{B}_n$. Since $\mathcal{A}$ is an ideal, there is some $z \in \mathcal{A}$ such that $x_n \leq z$ for each $n \leq N$; this z can therefore not be contained in any $\mathcal{B}_n$, and hence $\mathcal{A} \not\subseteq \bigcup_{n \leq N} \mathcal{B}_n$. □

Next, we make a somewhat topological definition, broadly inspired by an analogy between maximal ideals and paths in an infinite binary tree.

Definition 3.37. A maximal ideal $\mathcal{A}$ of a partial order P is a *limit ideal* if it is contained in the union of all other maximal ideals of P. Otherwise, $\mathcal{A}$ is said to be *isolated*. Equivalently, $\mathcal{A}$ is isolated if and only if there is some $x \in \mathcal{A}$ such that $x \notin \mathcal{B}$ for any other maximal ideal $\mathcal{B}$ of P.

Lemma 3.38. *If $\mathcal{A} \subseteq \bigcup_{n\in\omega} \mathcal{B}_n$ is a limit ideal (the union is necessarily countable by Lemma 3.35), then $\mathcal{A} \subseteq \bigcup_{n\geq N} \mathcal{B}_n$ for any $N \in \omega$.*

Proof. Let $N \in \omega$ be arbitrary. By Lemma 3.36, there is some $x \in \mathcal{A}$ which is not contained in any $\mathcal{B}_n$ for $n < N$. Let $y \in \mathcal{A}$ be arbitrary; then because $\mathcal{A}$ is an ideal there is some $z_y \in \mathcal{A}$ such that $x, y \leq z_y$. Since $x \leq z_y$, $z_y \notin \mathcal{B}_n$ for $n < N$; since $z_y \in \mathcal{A}$, $z_y \in \mathcal{B}_n$ for some n, so this must occur for some $n \geq N$. Since $\mathcal{B}_n$ is downward closed, $y \in \mathcal{B}_n$ for some $n \geq N$, and as this held for every $y \in \mathcal{A}$, $\mathcal{A} \subseteq \bigcup_{n\geq N} \mathcal{B}_n$. $\square$

We are now ready to prove that $\mathcal{J}_{\mathcal{P}}(L)$ has finitely many maximal ideals. A priori, there are three possible cases:

1. There are finitely many maximal ideals.
2. There are infinitely many isolated ideals.
3. There are finitely many isolated ideals and infinitely many limit ideals.

Our goal is to show that only the first case can hold, so we will prove that each of the other cases is impossible by showing that each leads to a contradiction.

Lemma 3.39. *If there are infinitely many isolated ideals, then there is an uncountable antichain in L, a contradiction. So the second case is impossible.*

Proof. Each isolated ideal $\mathcal{A}_n$ contains an element x_n not in any other isolated ideal; thus, any union of isolated ideals contains that element x_n if and only if $\mathcal{A}_n$ is present in the union. Let infinitely many isolated ideals be given by $\{\mathcal{A}_n\}_{n\in\omega}$; it follows that $\bigcup_{n\in S}\mathcal{A}_n$ and $\bigcup_{n\in T}\mathcal{A}_n$ are incomparable in L if and only if the subsets S and T of ω are $\subseteq$-incomparable. Since 2^ω under $\subseteq$ has an uncountable antichain (by Lemma 3.22), it follows that there are uncountably many pairwise incomparable unions of isolated ideals, and hence L has an uncountable antichain, contradicting our hypothesis about L. □

Lemma 3.40. *If there are finitely many isolated ideals and infinitely many limit ideals, then there are uncountably many limit ideals, contradicting Lemma 3.35. So the third case is impossible.*

Proof. Since there are only finitely many isolated ideals, by Lemma 3.38 every limit ideal is contained in the union of other limit ideals. Let $\{\mathcal{A}_n\}_{n\in\omega}$ be all of the (countably many) limit ideals, with $A_i \neq A_j$ for $i \neq j$. We will construct a limit ideal not on the list by a diagonalization argument, giving a contradiction.

We construct $\mathcal{B} = \bigcup_{n\in\omega}\mathcal{B}_n$, with the $\mathcal{B}_n$ constructed as follows. Let $b_0 \in \mathcal{A}_0$ and b_0 is not contained in any of the finitely many isolated ideals (we can do this by Lemma 3.36), and let $\mathcal{B}_0 = \mathrm{dcl}(b_0)$. We will then construct by induction $\mathcal{B}_n$ for $n \geq 0$, satisfying the following conditions:

1. $\mathcal{B}_n$ is a principal ideal.

2. $\mathcal{B}_n \subseteq \mathcal{A}_{f(n)}$ for some $f(n)$.

3. $\mathcal{B}_n \supseteq \mathcal{B}_{n-1}$ if $n > 0$.

4. $\mathcal{B}_n \not\subseteq \mathcal{A}_{n-1}$ if $n > 0$.

Certainly $\mathcal{B}_0$ satisfies these conditions with $f(0) = 0$. Now, given $\mathcal{B}_n$ satisfying these conditions, we construct $\mathcal{B}_{n+1}$. Let b_n be the largest element of $\mathcal{B}_n$. Choose $m > n$ such that $b_n \in \mathcal{A}_m$; such an m exists by Lemma 3.38 applied to $\mathcal{A}_{f(n)}$. Since $m > n$, $\mathcal{A}_m \neq \mathcal{A}_n$, so there is some $a \in \mathcal{A}_m$ such that $a \notin \mathcal{A}_n$. Since $\mathcal{A}_m$ is an ideal, there is some $b_{n+1} \in \mathcal{A}_m$ such that $a, b_n \leq b_{n+1}$. Define $\mathcal{B}_{n+1} = \mathrm{dcl}(b_{n+1})$. Then $\mathcal{B}_{n+1}$ is a principal ideal, $\mathcal{B}_{n+1} \subseteq \mathcal{A}_{f(n+1)}$ where $f(n+1) = m$, $b_n \in \mathcal{B}_{n+1}$ so that $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$, and finally, $\mathcal{B}_{n+1} \not\subseteq \mathcal{A}_n$, since it contains $a \notin \mathcal{A}_n$. Thus by induction we can construct a sequence of $\mathcal{B}_n$ satisfying all four conditions.

Finally, define $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$. Since $\mathcal{B}$ is the union of a chain of ideals, it is an ideal. By the fourth condition, $\mathcal{B} \not\subseteq \mathcal{A}_n$. Let $\mathcal{B}^*$ be any maximal ideal containing $\mathcal{B}$ (this may or may not be $\mathcal{B}$ itself). Since $\mathcal{B} \not\subseteq \mathcal{A}_n$ for any n, $\mathcal{B}^* \neq \mathcal{A}_n$ for any n. On the other hand, $\mathcal{B}^*$ is not isolated, since it contains b_0 which is not an element of any isolated ideal. It must therefore be a new limit ideal which was not on our list, a contradiction. So there are uncountably many limit ideals, which itself contradicts Lemma 3.35. □

It therefore follows that there are only finitely many maximal ideals. Since $\mathcal{X}$ is the union (join) of all the maximal ideals, $\mathcal{X}$ is therefore the join of finitely many ideals, which by Theorem 3.32 is the join of finitely many join-irreducibles in L.

This applies when $\mathcal{X}$ is the largest element of L, but as we observed at the beginning, by passing to the interval $[\emptyset, \mathcal{X}]$ we can obtain the result for an interval in which $\mathcal{X}$ is the largest element and then pull back to the original lattice. □

As a corollary we get the following result about intervals in the Muchnik lattice.

Corollary 3.41. *Let L be an interval of the Muchnik lattice $\mathscr{M}_w$ with no uncountable antichain. Then every element of L is the join of finitely many join-irreducible elements* *of L.*

3.5 Intervals with Large Antichains

The characterization of Theorem 3.21 is not quite a complete characterization of the intervals of $\mathscr{M}_w$. There is one pesky condition: that L not have any antichains of size $2^{\aleph_2}$. What about lattices L with larger antichains?

It is certainly possible for L to have antichains larger than that. For example, by taking a set $\mathcal{X}$ of infinite size κ ($\kappa \leq 2^{\aleph_0}$) of minimal Turing degrees, the interval $L = [\emptyset, \mathcal{X}]$ in $\mathscr{M}_w$ has, by Lemma 3.22, an antichain of size 2^{κ}. What prevents our using the techniques in this paper to characterize all these intervals of $\mathscr{M}_w$ is exactly the fact that it is unknown which posets of size greater than $\aleph_1$ can be initial segments of the Turing degrees.

Indeed, Groszek and Slaman [9] show that it is consistent with ZFC that $2^{\aleph_0} > \aleph_2$ and that there is a locally finite upper semilattice P of cardinality $\aleph_2$ which cannot be embedded into $\mathscr{D}$. In that case, the lattice $\mathcal{O}(P)$ would satisfy all the conditions of Theorem 3.21, except for having an antichain of cardinality $2^{\aleph_2}$, but it would not be isomorphic to an interval of $\mathscr{M}_w$ despite having cardinality smaller than that of $\mathscr{M}_w$. This means that Theorem 3.21 is best possible in the sense that the same characterization does not necessarily apply when we relax the constraint on antichain cardinality.

On the other hand, we can ask how incomplete our characterization is, in the sense of asking which intervals in $\mathscr{M}_w$ it does not catch. This also depends on our model of set theory. For example, if $2^{2^{\aleph_0}} < 2^{\aleph_2}$ (which in particular holds under GCH), our characterization is complete. On the other hand, under Martin's Axiom + $(2^{\aleph_0} > \aleph_2)$, we have $2^{\aleph_2} = 2^{\aleph_0}$, so our characterization fails to tell us about any of the intervals in $\mathscr{M}_w$ with antichains the size of the continuum, which presumably would be nice to know about. We can get around a few (but not all) of these issues if we are willing to restrict the class of lattices L under consideration by the properties of $\mathcal{J}_{\mathcal{P}}(L)$, rather than just by the size of their antichain, thus incorporating some parts of the characterization into the choice of domain. Our definition of $\mathcal{J}_{\mathcal{P}}(L)$ was only for lattices in which every subset of L had a supremum in L, but if we let $\mathcal{J}_{\mathcal{P}}(L)$ be empty for other lattices, the following is true:

Theorem 3.42. *Let L be a complete lattice such that $\mathcal{J}_{\mathcal{P}}(L)$ has no antichains of cardinality greater than $\aleph_1$. Then L is isomorphic to an interval of $\mathscr{M}_w$ if and only if L is superalgebraic and $\mathcal{J}_{\mathcal{P}}(L)$ is an initial segment of an upper semilattice and has the countable predecessor property.*

Proof. Exactly the same as the proof of Theorem 3.21, except that Lemma 3.23 is a hypothesis rather than a lemma. □

Depending on set theory, this version of the theorem may include more intervals of $\mathscr{M}_w$ than the other. Whether that makes it better is largely a matter of aesthetics. The author feels that restricting the domain based on the cardinality of antichains in the lattice L itself is more natural than doing so based on $\mathcal{J}_{\mathcal{P}}(L)$, and the characterizations are identical in a wide range of cases including under GCH and under the Proper Forcing Axiom.

Chapter 4

Small Mass Problems and Alternative Reducibilities

In addition to the full structures, it is worth studying smaller suborders of the Medvedev and Muchnik lattices. In his PhD thesis, Binns [2] looked at the structures of the Medvedev and Muchnik lattices of Π^0_1 classes of 2^ω (called $\mathcal{P}_M$ and $\mathcal{P}_w$ respectively), determining a great deal about the which lattices embed into these structures.

This chapter is an exploration of some other sublattices of the Muchnik and Medvedev lattices. Sections 4.1 and 4.2 deal with the sublattice of *finite* mass problems, while subsequent sections explore other sublattices and some alternative reducibilities for mass problems.

4.1 Finite Mass Problems

There is a natural way of identifying elements of ω^ω with certain mass problems, and thus Turing degrees with certain Medvedev and Muchnik degrees: for $X \in \omega^\omega$, let $i(X) = \{X\} \subseteq \omega^\omega$. This identification is well-behaved: in particular, the embedding that it induces of the Turing degrees into the Medvedev and Muchnik lattices is in fact an embedding of upper semilattices. (There is another natural embedding of the Turing degrees into the Muchnik lattice: in the language of Chapter 3 where Muchnik degrees are identified with downward closed sets of Turing degrees, for a Turing degree $\mathbf{a}$ we have $f(\mathbf{a}) = \mathrm{dcl}(\mathbf{a})$. This, however, is only an embedding of partial orders, not of upper semilattices.) We may wish to consider extending these embedded copies of $\mathscr{D}$ into lattices by closing under meets and joins. This turns out to yield the notion of finite mass problems.

Definition 4.1. A *finite mass problem* is a mass problem $\mathcal{A} \subseteq \omega^\omega$ with finite cardinality.

For finite mass problems, the notions of Medvedev and Muchnik reducibility coincide.

Proposition 4.2. *If $\mathcal{A}$ and $\mathcal{B}$ are finite mass problems, then $\mathcal{A} \leq_s \mathcal{B}$ if and only if $\mathcal{A} \leq_w \mathcal{B}$.*

Proof. Certainly Medvedev reducibility implies Muchnik reducibility, since that is true in the general case. So suppose that $\mathcal{A} \leq_w \mathcal{B}$. Then, for every $X \in \mathcal{A}$, there is some Φ and some $Y \in \mathcal{A}$ such that $\Phi(X) = Y$. Since $\mathcal{B}$ is finite, say of cardinality n, we can label the elements of $\mathcal{B}$ as X_1 through X_n, the functionals as Φ_1 through Φ_n, and the (not necessarily distinct) elements of $\mathcal{A}$ which are the images of X_1 through X_n as Y_1 through Y_n; thus, $\Phi_i(X_i) = Y_i$ for each $1 \leq i \leq n$. Now we create a new functional Ψ: first, Ψ reads enough of its oracle to distinguish between all of the X's (if its oracle is none of them, it does not matter what Ψ does). Then, having determined that its oracle is X_i (under the hypothesis that the oracle is one of the X's), it runs Φ_i. We therefore have that $\Psi(X_i) = \Phi_i(X_i) = Y_i \in \mathcal{A}$, which means that $\mathcal{A} \leq_s \mathcal{B}$. □

In light of this, we will write $\mathcal{A} \leq_f \mathcal{B}$ in case $\mathcal{A}$ and $\mathcal{B}$ are finite mass problems and one (and therefore both) of the above reducibilities holds. We also write F for the set of all finite mass problems, and $\mathscr{F}$ for the partial order $F/\equiv_f$ of *f-degrees* of finite mass problems. If $\mathcal{A}$ is a finite mass problem, we write $[\mathcal{A}]$ for its f-degree.

It turns out that meets and joins exist in $\mathscr{F}$ are are well-behaved.

Lemma 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite mass problems. Then $[\mathcal{A}] \wedge [\mathcal{B}] = [\mathcal{A} \cup \mathcal{B}]$. Moreover, this meet coincides with the meet of $\mathcal{A}$ and $\mathcal{B}$ in the Medvedev and Muchnik lattices.*

Proof. Certainly the identity functional shows that $\mathcal{A} \cup \mathcal{B} \leq_s \mathcal{A}, \mathcal{B}$ and hence $\mathcal{A} \cup \mathcal{B} \leq_w \mathcal{A}, \mathcal{B}$. On the other hand, suppose that $\mathcal{C} \leq_s \mathcal{A}, \mathcal{B}$ (where $\mathcal{C}$ is any mass problem, not just a finite one). Then there are Φ_1 and Φ_2 such that, for every $X \in \mathcal{A}$ and every $Y \in \mathcal{B}$, $\Phi_1(X) \in \mathcal{C}$ and $\Phi_2(Y) \in \mathcal{C}$. Since $\mathcal{A}$ and $\mathcal{B}$ are finite, we can computably determine, when our oracle is in $\mathcal{A} \cup \mathcal{B}$, whether it is in $\mathcal{A}$ or not. Let Ψ be the functional that first determines, assuming its oracle is in $\mathcal{A} \cup \mathcal{B}$, whether it is in $\mathcal{A}$, and then executes Φ_1 if so and Φ_2 if not. Then, for every $X \in \mathcal{A} \cup \mathcal{B}$, $\Psi(X) \in \mathcal{C}$. It follows that $\mathcal{C} \leq_s \mathcal{A} \cup \mathcal{B}$. The analogous fact for Muchnik reducibility follows from the fact that meets in the full Muchnik lattice are given by unions. □

Lemma 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite mass problems. Then $[\mathcal{A}] \vee [\mathcal{B}] = [\{X \oplus Y \mid X \in \mathcal{A},\ Y \in \mathcal{B}\}]$. Moreover, this join conincides with the join of $\mathcal{A}$ and $\mathcal{B}$ in the Medvedev and Muchnik lattices.*

Proof. Once again, it is not hard to see that $\mathcal{A}, \mathcal{B} \leq_s \{X \oplus Y \mid X \in \mathcal{A},\ Y \in \mathcal{B}\}$, and thus similarly for Muchnik reducibility: just use the identity functional for the even- (for $\mathcal{A}$) or odd- (for $\mathcal{B}$) indexed entries of the oracle. On the other hand, suppose that $\mathcal{A}, \mathcal{B} \leq_s \mathcal{C}$ (again, $\mathcal{C}$ is any mass problem). Then there are Turing functionals Φ and Ψ such that, for each $X \in \mathcal{C}$, $\Phi(X) \in \mathcal{A}$ and $\Psi(X) \in \mathcal{B}$. Then the functional $\Phi \oplus \Psi$ given by $(\Phi \oplus \Psi)(X) = \Phi(X) \oplus \Psi(X)$ has, for each $X \in \mathcal{C}$, $(\Phi \oplus \Psi)(X) \in \{X \oplus Y \mid X \in \mathcal{A},\ Y \in \mathcal{B}\}$.

Hence $\{X \oplus Y \mid X \in \mathcal{A},\ Y \in \mathcal{B}\} \leq_s \mathcal{C}$. The analogous result for Muchnik reducibility follows in the same way. $\square$

From these two lemmas we can conclude that

Proposition 4.5. *The partial order $\mathscr{F}$ of finite mass problems is a lattice.*

Proof. By Lemmas 4.3 and 4.4 both meets and joins exist in $\mathscr{F}$. $\square$

Additionally, $\mathscr{F}$ has a smallest element (the degree of the zero function) and largest element (the degree of the empty set) coinciding with the smallest and largest elements of $\mathscr{M}_w$ and $\mathscr{M}$.

As mentioned above, the upper semilattice of Turing degrees $\mathscr{D}$ embeds into the Muchnik and Medvedev lattices $\mathscr{M}_w$ and $\mathscr{M}$ as an upper semilattice in the natural way. Moreover, it is also known [19] that the Muchnik lattice embeds into the Medvedev degrees, but only as an upper semilattice and not as a lattice. On the other hand, $\mathscr{F}$ is well-behaved with respect to these embeddings:

Theorem 4.6. *As an upper semilattice, $\mathscr{D}$ embeds into $\mathscr{F}$, and $\mathscr{F}$ embeds as a lattice into both $\mathscr{M}_w$ and $\mathscr{M}$.*

Proof. The map i given by $i(X) = \{X\}$ described above induces an embedding of $\mathscr{D}$ as an upper semilattice into $\mathscr{M}_w$ (and into $\mathscr{M}$); since its image consists only of finite mass problems, this gives an embedding $\mathscr{D} \hookrightarrow \mathscr{F}$ of upper semilattices. Since the f-degrees of finite mass problems correspond uniquely to degrees in $\mathscr{M}_w$ and $\mathscr{M}$ by Proposition 4.2, and by Lemmas 4.3 and 4.4, the joins and meets in $\mathscr{F}$ coincide with the joins and meets of the corresponding degrees in $\mathscr{M}_w$ and $\mathscr{M}$, these correspondences yield lattice embeddings $\mathscr{F} \hookrightarrow \mathscr{M}_w$ and $\mathscr{F} \hookrightarrow \mathscr{M}$. $\square$

Lastly, we observe that, as remarked at the beginning of the section, $\mathscr{F}$ is the closure of $\mathscr{D}$, in either $\mathscr{M}_w$ or $\mathscr{M}$, under iterated meets and joins. Indeed, since $\mathscr{D}$ embeds into $\mathscr{F}$, a lattice, and the meets and joins of $\mathscr{F}$ correspond to meets and joins in $\mathscr{M}_w$ and $\mathscr{M}$, $\mathscr{F}$ must contain this closure. On the other hand, every element of $\mathscr{F}$ is the f-degree of a finite set of Turing degrees, which is the meet of those Turing degrees in $\mathscr{F}$. Hence $\mathscr{F}$ is indeed the closure we want.

4.2 The Intervals of $\mathscr{F}$

Now that we have some preliminary information about the lattice $\mathscr{F}$, a natural question is to determine some properties of its intervals. We know that $\mathscr{M}_w$ admits all manner of finite and countable intervals (as does $\mathscr{M}$, though there are fewer); on the other hand, Binns [2] showed that $\mathcal{P}_M$ is dense on every (nontrivial) interval and that in $\mathcal{P}_w$ every nonzero element has infinitely many predecessors. The main result of this section is the following theorem:

Theorem 4.7. *Let* $\mathbf{B} <_f \mathbf{A}$ *be* f*-degrees. Then there is a set of cardinality* $2^{\aleph_0}$ *of incomparable* f*-degrees* $\{\mathbf{X}_\alpha\}_{\alpha<\in 2^\omega}$ *such that* $\mathbf{B} < \mathbf{X}_\alpha < \mathbf{A}$ *for all* $\alpha \in 2^\omega$*.*

In order to prove this, we will need the following result. Rather than using elements of ω^ω, for simplicity we will use the equivalent formulation of problems as elements of 2^ω.

Theorem 4.8. *Suppose that* $n \in \omega$ *and that* $A_1, A_2, \ldots, A_n, B \in 2^\omega$ *satisfy* $A_k \not\leq_T B$ *for all* $1 \leq k \leq n$*. Then there are* $\{X_\alpha\}_{\alpha\in 2^\omega}$ *with* $X_\alpha \in 2^\omega$ *such that*

1. *For all* $1 \leq k \leq n$ *and all* $\alpha \in 2^\omega$*,* $A_k \not\leq_T B \oplus X_\alpha$*.*

2. *For all* $\alpha \in 2^\omega$*,* $B <_T B \oplus X_\alpha$*.*

3. *For all* $\alpha, \beta \in 2^\omega$, *if* $\alpha \neq \beta$ *then* $B \oplus X_\alpha \not\leq_T B \oplus X_\beta$.

Proof. For each $\sigma \in 2^{<\omega}$, we will construct $X_\sigma \in 2^{<\omega}$ inductively and in stages, satisfying the requirements

1. If $\sigma \prec \tau$, then $X_\sigma \prec X_\tau$.

2. $P(k, \sigma, e)$: For any $Y \in 2^\omega$ with $X_\sigma \prec Y$, we have $\Phi_e(B \oplus Y) \neq A_k$.

3. $R(\sigma, \tau, e)$: For any $Y, Z \in 2^\omega$ with $X_\sigma \prec Y$ and $X_\tau \prec Z$, we have $\Phi_e(B \oplus Y) \neq B \oplus Z$.

where there is a requirement $P(k, \sigma, e)$ for each $1 \leq k \leq n$, each $\sigma \in 2^{<\omega}$, and each $0 < e \leq |\sigma|$, and there is a requirement $R(\sigma, \tau, e)$ for each $\sigma, \tau \in 2^{<\omega}$ such that $|\sigma| = |\tau|$ and each $0 < e \leq |\sigma|$.

We begin with $X_\varepsilon = \varepsilon$ as the base. Suppose that at the beginning of stage $m \geq 1$ we have constructed all X_σ for $|\sigma| < m$, and all requirements involving such X_σ are satisfied.

First, let $X_{\sigma 0,0} = X_\sigma 0$ and $X_{\sigma 1,0} = X_\sigma 1$ for each σ of length $m - 1$.

Next, we will ensure that the requirements $P(k, \sigma 0, m)$ (and $P(k, \sigma 1, m)$) are satisfied at substage k for each $1 \leq k \leq n$. At the beginning of substage k we have defined $X_{\sigma 0,k-1}$.

Claim 4.9. *One of the following holds:*

1. *There is some* $Y \succ X_{\sigma 0,k-1}$ *and* $r \in \omega$ *such that* $\Phi_m^{B \oplus Y} \downarrow \neq A_k(r)$.

2. *There is some* $r \in \omega$ *such that for every* $Y \succ X_{\sigma 0,k-1}$, $\Phi_m^{B \oplus Y}(r) \uparrow$.

Proof. Suppose for contradiction that both cases fail. In that case, we can compute $A_k(r)$ from B: search all finite $Y \succ X_{\sigma 0,k-1}$ in order of increasing length (and up to computation length of the length of Y) until $\Phi_m^{B\oplus Y}(r)\downarrow$. This eventually happens for some Y because the second case fails. Because the first case fails, it follows that $\Phi_m^{B\oplus Y}(r)\downarrow= A_k(r)$. Since this procedure is computable in B (and the finitely much information in $X_{\sigma 0,k-1}$, which does not matter), it follows that $A_k \leq_T B$, contradicting the hypothesis of the theorem. □

In first case, let $Y \succ X_{\sigma 0,k-1}$ satisfying $\Phi_m^{B\oplus Y}(r)\downarrow\neq A_k(r)$ for some $r\in\omega$, and then define $X_{\sigma 0,k}$ to be the longer of $X_{\sigma 0,k-1}$ and the initial segment of Y with length equal to the use of $\Phi_m^{B\oplus Y}(r)$. In the second case, let $X_{\sigma 0,k} = X_{\sigma 0,k-1}$. We will ensure that $X_{\sigma 0} \succ X_{\sigma 0,k}$, so it follows that $P(k,\sigma 0,m)$ will be satisfied.

Having gone through all substages $1\leq k\leq n$ for each string σ of length $m-1$, we will ensure that the requirements $R(\sigma a,\tau b,e)$ are satisfied for each σ,τ of length $m-1$, each $a,b\in\{0,1\}$, and every $e\leq m$.

For each ordered triple $(\sigma a,\tau b,e)$ we have a separate substage s, for s beginning with $n+1$. The following is what we do at each substage s.

If ρc is neither the string σa or τb associated with substage s, then let $X_{\rho c,s} = X_{\rho c,s-1}$.

Otherwise, let $r > |X_{\tau b,s-1}|$. If there is no $Y \succ X_{\sigma a,s-1}$ such that $\Phi_e^{B\oplus Y}(2r+1)\downarrow$, then $R(\sigma a,\tau b,e)$ is guaranteed to be satisfied (since we will ensure that $X_{\sigma a}\succ X_{\sigma a,s-1}$ and $X_{\tau b}\succ X_{\tau b,s-1}$). If there is such a Y, then take $X_{\sigma a,s}\succ X_{\sigma a,s-1}$ finite such that $\Phi_e^{B\oplus X_{\sigma a,s}}(2r+1)\downarrow$ and choose $X_{\tau b,s}\succ X_{\tau b,s-1}$ so that $X_{\tau b,s}(r)\neq \Phi_e^{B\oplus X_{\sigma a,s}}(2r+1)$. This will ensure that $R(\sigma a,\tau b,e)$ is satisfied.

Finally, at the end of the last substage, s, we set $X_{\sigma a} = X_{\sigma a,s}$ for each σ of length $m-1$ and each $a\in\{0,1\}$. We observe that all requirements P and R are satisfied, and that $X_\sigma \prec X_{\sigma a}$ (and hence by induction the first, unnamed, requirement is also satisfied).

This is the end of the construction.

It remains to find the required $X_\alpha \in 2^\omega$ for each $\alpha \in 2^\omega$. This we do in the natural way: Since $X_\sigma \prec X_\tau$ whenever $\sigma \prec \tau$, for each $\alpha \in 2^\omega$ it makes sense to define $X_\alpha = \bigcup_{\sigma \prec \alpha} X_\sigma$.

Then, by requirements $P(k, \sigma, e)$ for $\sigma \prec \alpha$, we have, for all e, $\Phi_e(B \oplus X_\alpha) \neq A_k$, and so $A_k \not\leq_T B \oplus X_\alpha$, satisfying the first condition of the theorem. By requirements $R(\sigma, \tau, e)$ for $\sigma \prec \alpha$ and $\tau \prec \beta$, we have, for all e, $\Phi_e(B \oplus X_\alpha) \neq B \oplus X_\beta$, and so $B \oplus X_\beta \not\leq_T B \oplus X_\alpha$. This satisfies the third condition of the theorem. Finally, since $B \leq_T B \oplus X_\alpha$, it follows that $B \oplus X_\beta \not\leq_T B$, and hence $B <_T B \oplus X_\beta$, satisfying the second condition of the theorem. □

It remains to prove Theorem 4.7.

Proof of Theorem 4.7. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B}$ be finite mass problems with f-degrees $\mathbf{A}$ and $\mathbf{B}$ respectively. Since $\mathcal{B} <_f \mathcal{A}$, it follows that there is some $B \in \mathcal{B}$ such that for all $1 \leq k \leq n$, $A_k \not\leq_T B$. Let $\{X_\alpha\}_{\alpha \in 2^\omega}$, $X_\alpha \in 2^\omega$ be the sets guaranteed to exist by Theorem 4.8. Consider $\mathcal{X}_\alpha := \mathcal{A} \cup \{B \oplus X_\alpha\}$ for each $\alpha \in 2^\omega$. Because $A_k \not\leq_T B \oplus X_\alpha$ for all $1 \leq k \leq n$, it follows that $\mathcal{A} \not\leq_f \mathcal{X}_\alpha$ and hence $\mathcal{X}_\alpha <_f \mathcal{A}$. Similarly, because $B <_T B \oplus X_\alpha$, it follows that $\mathcal{X}_\alpha \not\leq_f \mathcal{B}$ and hence $\mathcal{B} <_f \mathcal{X}_\alpha$. Finally, because $B \oplus X_\alpha \not\leq_T B \oplus X_\beta$ for $\alpha \neq \beta$ (and because $A_k \not\leq_T B \oplus X_\beta$) it follows that $\mathcal{X}_\alpha \not\leq_T \mathcal{X}_\beta$ for $\alpha \neq \beta$, and hence that the $\mathcal{X}_\alpha$ are all incomparable.

If we let $\mathbf{X}_\alpha = [\mathcal{X}_\alpha]$ be the f-degrees of these finite mass problems, we get a cardinality-$2^{\aleph_0}$ antichain of degrees in the interval $[\mathbf{B}, \mathbf{A}]$ as required. □

Because the cardinality of $\mathscr{F}$ is itself $2^{\aleph_0}$, this means that every interval of $\mathscr{F}$ has an antichain that is as large as possible. This is in stark contrast to its superstructures $\mathscr{M}_w$ and $\mathscr{M}$ and to its substructure $\mathscr{D}$.

Because Theorem 4.7 applies to any interval, we have an immediate corollary.

Corollary 4.10. *The lattice $\mathscr{F}$ is dense.*

4.3 Countable and Cocountable Mass Problems

Having studied finite mass problems, it is natural to define countable mass problems in the same way:

Definition 4.11. A *countable mass problem* is an at most countable subset of ω^ω.

In contrast to the finite case, Muchnik and Medvedev reducibility do not coincide for countable mass problems. We will focus on the countable mass problems under Muchnik reducibility.

It is clear that the Muchnik degrees of countable mass problems form a lattice, where meets and joins correspond to meets and joins in the Muchnik lattice, for the same reasons that the degrees of finite mass problems do. What can we say about the intervals in this lattice? It turns out that a direct analogue of Theorem 4.7 is true.

Theorem 4.12. *Suppose that $A_1, A_2, \ldots, B \in 2^\omega$ (with countably many A_k, possibly not all different) satisfy $A_k \not\leq_T B$ for all $k \geq 1$. Then there are $\{X_\alpha\}_{\alpha \in 2^\omega}$ with $X_\alpha \in 2^\omega$ such that*

1. *For all $k \geq 1$ and all $\alpha \in 2^\omega$, $A_k \not\leq_T B \oplus X_\alpha$.*

2. *For all $\alpha \in 2^\omega$, $B <_T B \oplus X_\alpha$.*

3. For all $\alpha, \beta \in 2^\omega$, if $\alpha \neq \beta$ then $B \oplus X_\alpha \not\leq_T B \oplus X_\beta$.

Proof. The same as the proof of Theorem 4.8, with the change that in the first set of substages of stage m, instead of satisfying all requirements $P(k, \sigma, m)$ with $1 \leq k \leq n$, we instead satisfy (in the same way) all requirements $P(k, \sigma, e)$ with $k, e \leq m$. This ensures that by the end of the construction we will have satisfied all such requirements, and the conclusion will hold as before. □

Theorem 4.13. *Let $\mathbf{B} <_w \mathbf{A}$ be Muchnik degrees of countable mass problems. Then there is an set of cardinality $2^{\aleph_0}$ of incomparable Muchnik degrees of countable mass problems $\{\mathbf{X}_\alpha\}_{\alpha<\in 2^\omega}$ such that $\mathbf{B} < \mathbf{X}_\alpha < \mathbf{A}$ for all $\alpha \in 2^\omega$.*

This implies that, in the sublattice of countable mass problems, every interval has an antichain of cardinality $2^{\aleph_0}$; this is, once again, the size of the entire sublattice.

Having considered finite and countable mass problems, it is natural to ask about cofinite and cocountable mass problems. Alas, cofinite mass problems are exceedingly boring: there are infinitely many computable functions, so every cofinite set of functions contains a computable function and thus has both Muchnik and Medvedev degree 0. The cocountable case is a different story, however.

Definition 4.14. A *cocountable mass problem* is a cocountable subset of ω^ω.

We will restrict our attention to the Muchnik degrees of cocountable mass problems (as in the countable case, Muchnik and Medvedev reducibility very much do not coincide). Recalling from Chapter 3 our correspondence between Muchnik degrees and downward closed subsets of Turing degrees, we observe that any cocountable mass problem must correspond to an at most countable downward closed subset of $\mathscr{D}$, and conversely, since there are only countably many functions of each degree, any at most countable downward closed subset of $\mathscr{D}$ corresponds to the Muchnik degree of some cocountable mass problem; namely, the problem consisting of all functions not of one of those countably many degrees.

From this it follows that the Muchnik degrees of cocountable mass problems are closed downward in $\mathscr{M}$, and since the union and intersection of at most countable sets are at most countable, it immediately follows that suborder of Muchnik degrees of cocountable mass problems, which we will call $\mathscr{M}_{ccw}$, has meets and joins coinciding with meets and joins in $\mathscr{M}_w$, and indeed form a lattice initial segment of $\mathscr{M}_w$. On the other hand, $\mathscr{M}_{ccw}$ is itself uncountable, having cardinality $2^{\aleph_0}$ (for instance, $\{0, \mathbf{m}\}$ corresponds to a cocountable mass problem for each minimal Turing degree $\mathbf{m}$), and does not have a largest element.

Because $\mathscr{M}_{ccw}$ is an initial segment of $\mathscr{M}_w$, we can use the results of Chapter 3 to obtain a characterization of the intervals of $\mathscr{M}_{ccw}$. In particular, if L is such an interval, its largest element corresponds to a countable downward closed set of Turing degrees, so $\mathcal{J}_{\mathcal{P}}(L)$ is in fact countable. It follows by the proof of Theorem 3.21 that

Theorem 4.15. *A lattice L is isomorphic to an interval in $\mathscr{M}_{ccw}$ if and only if L is superalgebraic and $\mathcal{J}_{\mathcal{P}}(L)$ is a countable initial segment of an upper semilattice.*

4.4 Some Alternative Reducibilities for Mass Problems

An informal way of expressing both Muchnik and Medvedev reducibilities is the following: a mass problem $\mathcal{A}$ is reducible to another mass problem $\mathcal{B}$ if *any* element of $\mathcal{A}$ can compute *some* element of $\mathcal{B}$. This is the natural way to think if we are thinking about mass problems in terms of the task of finding *one* element: that is, if "solving" $\mathcal{A}$ corresponds to finding some element of $\mathcal{A}$, then $\mathcal{A}$ is easier than $\mathcal{B}$ just in case finding any element of $\mathcal{B}$ allows us to find some element of $\mathcal{B}$. It is reasonable to consider a different possible interpretation. Suppose that we think of a mass problem in terms of the task of finding *all* elements. Then we would like to say that $\mathcal{A}$ is easier than $\mathcal{B}$ just in case knowing *every* element of $\mathcal{B}$ allows us to find *every* element of $\mathcal{B}$.

Just as there are two ways (Medvedev and Muchnik reducibility) to formalize our original intuition, there are several ways to approach this dual kind of reducibility. In the remainder of this chapter we introduce several such reducibilities and make a small study of some of their properties.

Definition 4.16. Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. Then $\mathcal{A}$ is *dual-weak reducible* to $\mathcal{B}$, written $\mathcal{A} \leq_{dw} \mathcal{B}$, if for every $X \in \mathcal{A}$, there is $Y \in \mathcal{B}$ and a Turing functional Φ such that $\Phi(Y) = X$. Dual-weak reducibility is transitive, and the dual-weak degrees are defined in the natural way, forming the partially ordered set D_{dw}.

In fact, D_{dw} is isomorphic to a structure we know and love.

Theorem 4.17. $D_{dw} \cong \mathscr{M}_w$.

Proof. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are mass problems. First observe that if $\mathcal{A} \subseteq \mathcal{B}$, then

$\mathcal{A} \leq_{dw} \mathcal{B}$. Hence, $\mathcal{A} \equiv_{dw} \mathrm{dcl}(\mathcal{A})$, where $\mathrm{dcl}(\mathcal{A})$ is the set of all functions Turing computable from some element of $\mathcal{A}$. It follows that $\mathcal{A} \leq_{dw} \mathcal{B}$ if and only if $\mathrm{dcl}(\mathcal{A}) \leq_{dw} \mathrm{dcl}(\mathcal{B})$. Additionally, $\mathrm{dcl}(\mathcal{A}) \leq_{dw} \mathrm{dcl}(\mathcal{B})$ implies that $\mathrm{dcl}(\mathcal{A}) \subseteq \mathrm{dcl}(\mathcal{B})$, as otherwise $\mathcal{A}$ would contain a set not computable from any element of $\mathcal{B}$. Moreover, the downward closure $\mathrm{dcl}(\mathcal{A})$ of a mass problem $\mathcal{A}$ depends only on the Turing degrees represented in $\mathcal{A}$, and corresponds to the downward closure in $\mathscr{D}$ of the Turing degrees represented in $\mathcal{A}$. Hence the map from $\mathcal{A}$ to the set of Turing degrees $\mathrm{dcl}(\mathcal{A})$ is an isomorphism from D_{dw} to $\mathcal{O}(\mathscr{D})$, which from Lemma 3.5 we know is isomorphic to $\mathscr{M}_w$. □

Despite the fact that this structure is isomorphic to the Muchnik lattice, the isomorphism is not given by the identity map on mass problems, and thus certain natural substructures are not the same. For instance, the suborder of finite mass problems under dual-weak reducibility does *not* form a lattice. If one takes two Turing degrees such that the intersection of their downward cones is an infinite ascending chain of order type ω, any mutual lower bound corresponding to a finite mass problem is the downard closure of some single Turing degree, and is bounded above by another such mutual lower bound. On the other hand, the suborder of finite mass problems is an upper semilattice, since the join of the degrees of two finite mass problems corresponds to the union of two downward closures of finite sets, which is a downward closure of a finite set and thus the degree of a finite mass problem.

Passing to the countable case, however, one does obtain a lattice, and one we already know: since the Turing degrees have the countable predecessor property, the dual-weak degrees of countable mass problems correspond to the at most countable downward closed sets of Turing degrees. The suborder of these degrees is thus isomorphic (by the same isomorphism giving $D_{dw} \cong \mathscr{M}_w$) to the lattice $\mathscr{M}_{ccw}$ described in Section 4.3.

Another natural idea is to require that the reductions be uniform. Unfortunately, this leads to some odd results.

Definition 4.18. *Version 1:* Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. Then $\mathcal{A}$ is *dual-strong reducible* to $\mathcal{B}$, written $\mathcal{A} \leq_{ds} \mathcal{B}$, if there is a Turing functional Φ such that, for every $X \in \mathcal{A}$, there is $Y \in \mathcal{B}$ such that $\Phi(Y) = X$. Dual-strong reducibility is transitive, and the dual-strong degrees are defined in the natural way, forming the partially ordered set D_{ds}.

This leads to some very strange consequences. For instance, the set containing the all-0 and all-1 functions is not equivalent to the set containing just the all-0 function, which clashes with what we are trying to capture.

For finite mass problems, this reducibility is different than dual-weak reducibility, but it can still be characterized. If $\mathcal{A}$ and $\mathcal{B}$ are finite mass problems, then $\mathcal{A} \leq_{ds} \mathcal{B}$ if and only if, for every Turing degree $\mathbf{d}$, the number of elements of $\mathcal{B}$ in the upper cone of $\mathbf{d}$ is at least the number of elements of $\mathcal{A}$ in the upper cone of $\mathbf{d}$.

Unfortunately even this characterization does not work in general. It would be interesting to know whether D_{ds} has nice properties—for instance, whether it is a lattice or even an upper semilattice—but it is unclear whether the structure is worth further attention given that it does not seem to correctly capture a reasonable intuitive notion.

We might try allowing an integer parameter. This fixes the strange problem mentioned above.

Definition 4.19. Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. Then $\mathcal{A}$ is *parametrized-dual-strong reducible* to $\mathcal{B}$, written $\mathcal{A} \leq_{pds} \mathcal{B}$, if there is a Turing functional Φ such that, for every

$X \in \mathcal{A}$, there is $Y \in \mathcal{B}$ and $n \in \omega$ such that $\Phi(n^\frown Y) = X$.

The astute reader will notice that this gives too much away. By taking Φ to be a universal Turing functional, it follows that parametrized-dual-strong reducibility is exactly the same as dual-weak reducibility! Perhaps there is no good analogue of Medvedev reducibility in this case.

These are not the only ways of trying to formalize our intuitive notion. If we want to capture the idea of "all the information in $\mathcal{B}$ is enough to find all the information in $\mathcal{A}$," it may seem odd to require that each element of $\mathcal{A}$ be computable from a *single* element of $\mathcal{B}$. Should we not be able to use several elements of $\mathcal{B}$ at once? This leads to the following definition:

Definition 4.20. Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. Then $\mathcal{A}$ is *n-dual-weak reducible* to $\mathcal{B}$, written $\mathcal{A} \leq_{ndw} \mathcal{B}$, if for every $X \in \mathcal{A}$, there is an integer n, $Y_1, \ldots, Y_n \in \mathcal{B}$, and a Turing functional Φ such that $\Phi(Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n) = X$. N-dual-weak reducibility is transitive, and the n-dual-weak degrees are defined in the natural way, forming the partially ordered set D_{ndw}.

It turns out that we have already met this structure as well:

Theorem 4.21. *The partial order D_{ndw} of n-dual-weak degrees is isomorphic to the partial order of Turing ideals under $\subseteq$ and thus is isomorphic to the suborder of join-irreducible elements of $\mathscr{M}_w$.*

Proof. For a mass problem $\mathcal{A}$, let $T(\mathcal{A})$ be the Turing ideal generated by the degrees of elements of $\mathcal{A}$. Suppose that $\mathcal{A} \leq_{ndw} \mathcal{B}$, and consider $\mathbf{a} \in T(\mathcal{A})$. There are $X_1, \ldots, X_m \in \mathcal{A}$ such that $\mathbf{a} \leq_T d(X_1) \vee \cdots \vee d(X_n)$, where $d(X)$ is the Turing degree of X. Since

$\mathcal{A} \leq_{ndw} \mathcal{B}$, there are, for each $1 \leq k \leq m$, $Y_{k1}, \ldots, Y_{kn} \in \mathcal{B}$ such that $X_k \leq_T Y_{k1} \oplus \cdots \oplus Y_{kn}$, and thus $X_1 \oplus \cdots X_n \leq_T Y_{11} \oplus \cdots Y_{mn}$. Thus $\mathbf{a} \leq_T d(Y_{11} \oplus \cdots Y_{mn}) \in T(\mathcal{B})$, and hence $\mathbf{a} \in \mathcal{B}$ as well. It follows that $T(\mathcal{A}) \subseteq T(\mathcal{B})$.

Conversely, suppose that $T(\mathcal{A}) \subseteq T(\mathcal{B})$. Let $X \in \mathcal{A}$; then $d(X) \in T(\mathcal{A})$ so that $d(X) \in T(\mathcal{B})$, and hence $d(X) \leq_T d(Y_1) \vee \cdots \vee d(Y_n)$ for some integer n and $Y_1, \ldots, Y_n \in \mathcal{B}$. Hence there is some Φ such that $\Phi(Y_1 \oplus \cdots \oplus Y_n) = X$. It follows that $\mathcal{A} \leq_{ndw} \mathcal{B}$.

Thus, $\mathcal{A} \leq_{ndw} \mathcal{B}$ if and only if $T(\mathcal{A}) \leq T(\mathcal{B})$, showing that D_{ndw} is isomorphic to the partial order of Turing ideals under $\subseteq$. By Corollary 3.33, this is isomorphic to the suborder of join-irreducible elements of $\mathscr{M}_w$. □

One immediate consequence of this is that D_{ndw} is a lattice. The join of two Turing ideals is the Turing ideal they generate together, and their meet is their intersection. (Note that the join-irreducible elements of $\mathscr{M}_w$ are not embedded as a sublattice in $\mathscr{M}_w$ despite forming a lattice because their joins do not coincide with joins in $\mathscr{M}_w$.) Additionally, the suborder of D_{ndw} of degrees of finite mass problems is particularly nice. Because every finitely generated Turing ideal is in fact principal, this suborder is isomorphic to the upper semilattice $\mathscr{D}$ of Turing degrees.

Last, one may naturally ask whether there is a notion of *n-dual-strong* reduction as well. Unfortunately, the problems that surface in the attempt to define dual-strong reduction come up here as well. Omitting the parameter n causes the same troubles as before, with the set containing the all-0 function and the all-1 function not being equivalent to the set containing the all-0 function. On the other hand, giving it explicitly leads to the n-dual-strong reduction being equivalent to n-dual-weak reduction.

Whether these or other reducibilities will prove fruitful in the long run is an open question, but the fact that D_{ndw} is isomorphic to the lattice of Turing ideals gives another good reason to study the properties of that lattice.

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