RELATIVE RANDOMNESS VIA RK-REDUCIBILITY

by

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Abstract

This is a dissertation in the field of MATHEMATICS: LOGIC: COMPUTABILITY THEORY: ALGORITHMIC RANDOMNESS (Mathematics Subject Classification 03D80, 68Q30). Its focus is relative randomness as measured by rK-reducibility, a refinement of Turing reducibility defined as follows. An infinite binary sequence A is **rK-reducible** to an infinite binary sequence B, written $A \leq_{\mathrm{rK}} B$, if

$$\exists d \, \forall n \, . \, K(A \upharpoonright n | B \upharpoonright n) < d,$$

where $K(\sigma|\tau)$ is the conditional prefix-free descriptional complexity of σ given τ. Herein i study the relationship between relative randomness and (standard) absolute randomness and that between relative randomness and computable analysis.

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There are, it seems, two Muses: the Muse of Inspiration, who gives us inarticulate visions and desires, and the Muse of Realization, who returns again and again to say, "It is yet more difficult than you thought."

Wendell Berry (1934–)

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CHAPTER 1

Introduction

One of the most popular definitions of absolute algorithmic randomness states that an infinite binary sequence R is random if it is incompressible, that is, if

$$\exists d\,\forall n\,.\,K(R\restriction n)\geq n-d,$$

where $K(\sigma)$ is the prefix-free descriptional complexity of the string σ. Under this same paradigm of incompressibility, one can define relative algorithmic randomness as follows. An infinite binary sequence A is less random than an infinite binary sequence B if A is completely compressible given B, that is, if

$$\exists d\,\forall n\,.\,K(A\restriction n|B\restriction n)<d,$$

where $K(\sigma|\tau)$ is the conditional prefix-free descriptional complexity of σ given τ. In this case, we write $A\leq_{\mathrm{rK}} B$ for short and say "A is rK-reducible to B".[1]

This dissertation continues the study of relative randomness via rK-reducibility initiated in [**DHL04**]. Herein i consider the simplest of questions, at times finding answers, at times finding nothing but bafflement.

1.1. Summary of Results

Chapter 2 addresses four basic questions on relative randomness via rK-reducibility, namely, Question ①: Is there a sequence of minimal relative randomness, that is, a sequence with only the computable sequences strictly less random ($<_{\mathrm{rK}}$) than it? Question ②: Is there a sequence of maximal relative randomness, that is, a sequence with no sequences strictly more random than

[1]The 'rK' stands for 'relative Kolmogorov' complexity, another name for conditional prefix-free descriptional complexity.

it? Question ③: Are there two random sequences, one strictly less random than the other? Question ④: Is every sequence less random than some random sequence? The answer to Question ① is YES and the proof, joint with Frank Stephan, is a sneaky variation on the construction of a Turing minimal sequence. Questions ②–④ are left unresolved, though not without a fight. Most notably and surprisingly, Joseph Miller and i show that the rK-upper cones of random sequences are countable.

Chapter 3 leaves behind Questions ①–④ and pursues the study of relative randomness via rK-reducibility in the context of computable analysis. Therein i prove that the class of d.c.e. reals, the class of reals less random than the halting probability Ω, and the class of computably approximable reals form countable real closed fields, each strictly contained in the next, respectively. The real closure proofs all use the same general method of closure under weakly computable locally Lipschitz functions. This method can also be used to show that the class of K-trivial reals form a countable real closed field.

Chapter 4 contains odds and ends that do not fit well with the previous chapters. Among other things, i show there that, in contrast to the case of the c.e. reals, the rK-degrees of the d.c.e. reals have no least upper bound and that there exists a repetition-free effective enumeration of the family of all Σ^0_1 classes.

1.2. Notation and Conventions

The following notation and conventions will apply throughout; additional notation and conventions will be introduced when necessary.

$\mathbb{N}$ will denote the set of natural numbers $\{0, 1, 2, \ldots\}$, ${}^n2$, for a natural n, the set of binary strings of length n (functions from n to 2), ${}^{<\mathbb{N}}2$ the set of all binary strings (functions from initial segments of $\mathbb{N}$ to 2), and ${}^{\mathbb{N}}2$ the set of infinite binary sequences (functions from $\mathbb{N}$ to 2). For the most part, 'string', 'sequence', and 'class' without further qualification will mean 'binary string', 'infinite binary sequence', and 'set of infinite binary sequences', respectively. For strings σ and τ, $|\sigma|$ will denote the length of σ, and $\sigma\tau$ or, when that might cause confusion,

$\sigma^\frown\tau$ the concatenation of σ and τ. Also, σ^n will denote the string $\sigma\sigma\cdots\sigma$ (n times) with the understanding that $\sigma^0 = \emptyset$, the empty string. $\overline{\sigma}$ will denote the string $0^{|\sigma|}1\sigma$. Also, $\sigma \subseteq \tau$ and $\sigma \subset \tau$ will mean σ is a substring of τ and σ is a proper substring of τ, respectively. By substring i mean initial segment. A set of strings S is called **prefix-free** if every string in S has no proper substring in S. For a string or sequence X and a positive natural n, $X \upharpoonright n$ will denote the length n initial segment of X, that is, the string $\langle X(0), X(1), \ldots, X(n-1)\rangle$. $\langle\ \ \rangle$ will delimit ordered tuples and sequences. For a set of strings W, $\mathcal{O}(W)$ will denote the class $\bigcup\{X \in {}^{\mathbb{N}}2 : X \supset \sigma \wedge \sigma \in W\}$, and for a string σ, $\mathcal{O}(\sigma)$ will be shorthand for $\mathcal{O}(\{\sigma\})$. Finally, μ will denote the uniform (fair-coin) probability measure on ${}^{\mathbb{N}}2$.

For computability-theoretic notions i will mostly follow the notation and conventions of [**Soa87**]; i will assume you are very familiar with computability theory. I will also assume you are fairly familiar with the rudiments of absolute algorithmic randomness as described in [**DH**], say. See also Appendix A for a brief review. From here on out a sequence is **random** (more precisely, 1-random) if it is incompressible (via prefix-free descriptional complexity), typical (via Σ^0_1 tests), or unpredictable (via Σ^0_1 supermartingales).

One last remark. C.p.f. will abbreviate 'computable partial function(s)' and p.c.p.f. will abbreviate 'prefix-free computable partial function(s)'; a prefix-free computable partial function is a (oracle) computable partial function with prefix free domain (regardless of the oracle), or prefix-free first-coordinate domain in the case of binary functions.

CHAPTER 2

Basic Questions

As i mentioned in the introduction, the study of relative randomness via rK-reducibility began with [**DHL04**]. In that article Downey, Hirschfeldt, and LaForte proved the following fundamental facts about the $\leq_{\mathrm{rK}}$ relation (which is fairly easily seen to be reflexive and transitive), all of which we will use throughout.

2.0.1 Theorem ([DHL04]). *For $A, B \in {}^{\mathbb{N}}2$, $A \leq_{\mathrm{rK}} B$ is equivalent to both of*

- $\exists d\, \forall n\, .\, C(A \upharpoonright n | B \upharpoonright n) < d$
- $\exists$*c.p.f.* $\varphi\, \exists d\, \forall n\, \exists i < d\, .\, \varphi(i, B \upharpoonright n) = A \upharpoonright n$

and implies all three of

- $\exists d\, \forall n\, .\, K(A \upharpoonright n) \leq K(B \upharpoonright n) + d$
- $\exists d\, \forall n\, .\, C(A \upharpoonright n) \leq C(B \upharpoonright n) + d$
- $A \leq_{\mathrm{T}} B$

Proof. The first two bullets follow easily from the definitions of conditional C and K. Still, let us show that the first bullet implies $A \leq_{\mathrm{rK}} B$ to better familiarize ourselves with these definitions. Suppose there exists a natural number d such that for all natural numbers n, $C(A \upharpoonright n | B \upharpoonright n) < d$. Let $\rho_0, \ldots, \rho_{e-1}$ be a listing of the strings of length $< d$, and let θ be the p.c.p.f. defined by

$$\theta(\rho, \tau) = \begin{cases} \widehat{\varphi}(\rho_i, \tau) & \text{if } \rho = \overline{\rho}_i \text{ for some } i < e \\ \uparrow & \text{else.} \end{cases}$$

Here $\widehat{\varphi}$ is the universal binary c.p.f. defined in Appendix A. Then for any n, $\theta(\overline{\rho}_i, B \upharpoonright n) = \widehat{\varphi}(\rho_i, \tau) = A \upharpoonright n$ for some $i < e$ (by construction and hypothesis). So up to an additive constant independent of n, $K(A \upharpoonright n | B \upharpoonright n) \leq K_\theta(A \upharpoonright n | B \upharpoonright n) \leq e$ (by the optimality of K), as desired.

For the third bullet, suppose there exists a natural number d such that for all natural numbers n, $K(A\upharpoonright n|B\upharpoonright n) < d$. Let θ be the p.c.p.f. defined by

$$\theta(\overline{\rho}\tau) = \widehat{\psi}(\rho, \widehat{\psi}(\tau, \emptyset)).$$

Then given n, there is a ρ of length $< d$ such that

$$\theta(\overline{\rho}^\frown(B\upharpoonright n)^*) = \widehat{\psi}(\rho, \widehat{\psi}((B\upharpoonright n)^*, \emptyset)) = \widehat{\psi}(\rho, B\upharpoonright n) = A\upharpoonright n,$$

where σ^* stands for the length-lexicographic least string such that $\widehat{\psi}(\sigma^*, \emptyset) = \sigma$. So up to a uniform additive constant,

$$K(A\upharpoonright n) \leq |\rho^\frown(B\upharpoonright n)^*| \leq d + K(B\upharpoonright n),$$

as desired. A similar argument proves the fourth bullet.

For the fifth, choose d least such that there exists a c.p.f. φ with $\forall n\ \exists i < d\ .\ \varphi(i, B\upharpoonright n) = A\upharpoonright n$. Working with a witness φ for d, notice that by the minimality of d

$$W := \{n : \forall i < d\ .\ \varphi(i, B\upharpoonright n)\downarrow\}$$

is infinite. Since W is also B-c.e., it contains an infinite B-computable subset C. Thus T, the downward closure of

$$\{\varphi(i, B\upharpoonright n) : n \in C \wedge i < d\},$$

is a B-computable subtree of ${}^{<\mathbb{N}}2$ with $< d$ paths. Since T has only finitely many paths, each path is computable from T and so computable from B.[1] Since A is one of these paths, $A \leq_{\mathrm{T}} B$, as desired. □

Let us reflect on this theorem. Notice that the second bullet says that $\leq_{\mathrm{rK}}$ really is a reducibility in the computability-theoretic sense; from B there is a way to compute A. Moreover, it implies that every computable sequence is rK-reducible to any given sequence. Also, from the

[1] We will use this fact often: that every isolated path in a tree is computable from the tree.

fifth bullet, any sequence rK-reducible to a computable sequence is itself computable. So the computable sequences are those of least relative randomness, as they should be.

> **Question** ①: Is there a sequence of minimal relative randomness, that is, a sequence with only the computable sequences strictly less random ($<_{\mathrm{rK}}$) than it?

Flipping this around,

> **Question** ②: Is there a sequence of maximal relative randomness, that is, a sequence with no sequences strictly more random than it?

Notice that the third bullet implies that rK-reducibility preserves absolute randomness, that is, if A is random and $A \leq_{\mathrm{rK}} B$, then B is random (since $K(B \upharpoonright n) \geq K(A \upharpoonright n) > n$, up to an additive constant independent of n), as should be the case.

> **Question** ③: Are there two random sequences, one strictly less random than the other?

Also,

> **Question** ④: Is every sequence less random than some random sequence?

These questions are the thrust of this chapter.

2.1. Question ①

This section is joint work with Frank Stephan and also appears as [**RS**].

2.1.1 Definition. *The* **rK-degrees** *are the members of the partial order* $\langle {}^{\mathbb{N}}2/\equiv_{\mathrm{rK}}, \leq \rangle$, *where* $\boldsymbol{a} \leq \boldsymbol{b}$ *iff* $A \leq_{\mathrm{rK}} B$ *for some (all)* $A \in \boldsymbol{a}$ *and* $B \in \boldsymbol{b}$

Indeed, there is a sequence of minimal relative randomness. Phrased in terms of the rK-degrees, we have

2.1.2 Theorem. *There is a minimal rK-degree.*

Proof. Tweaking the proof of the existence of a minimal Turing degree, we construct a special binary tree, suitable paths of which will have minimal rK-degree. Roughly speaking, we make the set of splitting nodes of our tree very sparse so that any noncomputable path of hyperimmune-free Turing degree can be recovered in two guesses from its image under an rK-reduction.

More precisely, we build a Π^0_1 tree T (a tree whose complement is computably enumerable) such that

(1) T has no computable paths;
(2) for every computable partial function $\Phi :\subseteq \mathbb{N} \to \mathbb{N}$ (thought of as a functional) and for every path X of T there is a string $\star \subset X$ such that either
   (a) for every path Y of T extending $\star$, $\forall t\, \forall n\, .\, \Phi^Y_t(n)\!\downarrow\ \wedge\, \Phi^X_t(n)\!\downarrow\, \to \Phi^Y_t(n) = \Phi^X_t(n)$, or
   (b) for every path Y, Z of T extending $\star$, Φ^Y and Φ^Z are incompatible;
(3) the set S of splitting nodes of T is very sparse, to wit, for all computable functions $g : \mathbb{N} \to \mathbb{N}$ we have

$$\forall^\infty \sigma \in S\ \forall \tau \in S\ .\ \sigma \subset \tau \to g(|\sigma|) < |\tau|.$$

Constructing T. We build T in stages, beginning with the full binary tree and pruning it computably. To describe this pruning we use moving markers in the style of [**Ste01**]. For notational niceness, stage subscripts are suppressed whenever possible.

Let $\{m_\sigma : \sigma \in {}^{<\mathbb{N}}2\} \subseteq {}^{<\mathbb{N}}2$ denote the set of markers of T. These are/lie on the splitting nodes of T. At stage zero, $T = {}^{<\mathbb{N}}2$ and each $m_\sigma = \sigma$. At later stages when necessary T is pruned via the Cut procedure. For $\sigma \subset \tau$, Cut(m_σ, m_τ) cuts off all paths of T that extend m_σ but not m_τ and then updates the positions of all the markers, preserving their order, as follows: m_σ moves to m_τ, each $m_{\sigma\epsilon}$ moves to $m_{\tau\epsilon}$, and all other markers stay put. Since Cut is the only action ever taken, T will be a perfect tree without leaves at every stage.

At stage $s > 0$ the construction runs as follows, where each check is performed only when the markers involved have indices of length $\leq s$; also, the computations involved are only up to stage s.

(i) If there exist σ, $i < 2$, and $e \leq |m_\sigma|$ such that for all $x \leq |\sigma|$, $\Phi_e(x) = m_{\sigma i}(x)$, then $\textsc{Cut}(m_\sigma, m_{\sigma(1-i)})$.

(ii) If there exist σ, τ, and $e, x \leq |\sigma|$ such that $\sigma \subset \tau$, $\Phi_e^{m_\sigma}(x) \uparrow$, and $\Phi_e^{m_\tau}(x) \downarrow$, then $\textsc{Cut}(m_\sigma, m_\tau)$.

If there exist σ, δ, ϵ, and $e \leq |\sigma|$ such that $\Phi_e^{m_{\sigma 0}}$ and $\Phi_e^{m_{\sigma 1}}$ are compatible for all arguments $\leq |\sigma|$, but $\Phi_e^{m_{\sigma 0 \delta}}$ and $\Phi_e^{m_{\sigma 1 \epsilon}}$ are incompatible at some argument $\leq |\sigma|$, then $\textsc{Cut}(m_{\sigma 0}, m_{\sigma 0 \delta})$ and $\textsc{Cut}(m_{\sigma 1}, m_{\sigma 1 \epsilon})$.

(iii) If there exist σ, τ, ν, and $e \leq |\sigma|$ such that $\sigma \subset \tau \subset \nu$ and $|m_\tau| \leq \Phi_e(|m_\sigma|) < |m_\nu|$, then $\textsc{Cut}(m_\tau, m_\nu)$.

It is not difficult to check that each marker eventually settles and that, in the end/limit, T satisfies properties (1)-(3).

A suitable path of T. Let A be a path of T of hyperimmune-free Turing degree, that is, A has the property that for every total function $f \leq_{\mathrm{T}} A$, there exists a computable function g such that for all x, $g(x) \geq f(x)$. Put more concisely, every total function computable from A has a computable majorant. Such a path exists by the Hyperimmune-free Basis Theorem ([**JS72**]) since $[T]$ is a nonempty Π^0_1 class (the set of paths through a Π^0_1 tree). We show that A has minimal rK-degree. By (1), A is noncomputable. Let $B \leq_{\mathrm{rK}} A$ be a noncomputable set. We need to show that $A \leq_{\mathrm{rK}} B$. To this end, observe that $B \leq_{\mathrm{T}} A$, and, in fact, $B \leq_{tt} A$ since A has hyperimmune-free Turing degree (see [**Odi89**, page 589]). Let Φ be a computable functional (total on all oracles) that witnesses the truth-table reduction.

We come now to the heart of the argument: building an rK-reduction from B to A. Let $\star$ be the magic string of (2) for A. Given $B \upharpoonright n$ for n sufficiently large, run through the computable approximation (that thins) to T until a stage t is reached such that T_t (the stage t approximation of T) has at most two superstrings of $\star$ of length n with extensions in T_t that map to $B \upharpoonright n$ under

Φ. The key here is that such a stage is guaranteed to exist by Lemma 2.1.3 below. To find these superstrings and extensions computably from $B \upharpoonright n$ we use the fact that Φ is total on all oracles and has a computable use function. Output the (at most) two strings of length n found; one will be $A \upharpoonright n$. Except for finitely many short lengths, this procedure describes an rK-reduction from B to A. Extending it to all lengths gives the final reduction. $\square$

2.1.3 Lemma. *Let $\star$ be the magic string of (2) for A. For almost all lengths n and almost all stages t, T_t has at most two superstrings of $\star$ of length n with extensions in T_t that map to $B \upharpoonright n$ under Φ.*

Proof. Let f be the function defined for $m \geq |\star|$ by $f(m)$ equals the first stage s such that for all strings $\nu \supset A \upharpoonright m\,\widehat{}\,(1 - A(m))$ either $\nu \notin T_s$, or for some $x \leq s$, $\Phi^\nu(x)\downarrow \neq \Phi^A(x)$. (Notice that all ν extend $\star$.) For $m < |\star|$, define $f(m)$ to be 0, say. It is unimportant. Since B is noncomputable, (2b) for $X = A$ holds, and since Φ is total on all oracles, f is total and A-computable. Since A has hyperimmune-free Turing degree, f has a computable increasing majorant g.

Now, fix n bigger than the length of $\star$, the length that (3) takes effect for g, and the length of the first splitting node of A on T. Let τ be the last splitting node of T on $A \upharpoonright n$, and let $\sigma \subset \tau$ be any other splitting node of T extending $\star$. Then by (3) we have that

$$s := f(|\sigma|) \leq g(|\sigma|) < |\tau| \leq n.$$

So by stage s, every string $\nu \in T_s$ extending $A \upharpoonright |\sigma|\,\widehat{}\,(1 - A(|\sigma|) = \sigma\,\widehat{}\,(1 - A(|\sigma|)$ will have some number $x \leq s < n$ such that $\Phi^\nu(x)\downarrow \neq \Phi^A(x) = B(x)$, so that ν cannot map to $B \upharpoonright n$ under Φ. Since σ was an arbitrary splitting node of T below the last splitting node of $A \upharpoonright n$, we see that only the strings extending the last splitting node of $A \upharpoonright n$ can map to $B \upharpoonright n$ under Φ. So the result holds. $\square$

In fact, by a generalized hyperimmune-free basis theorem below, the tree of the proof of Theorem 2.1.2 has continuum many paths of hyperimmune-free Turing degree. Thus, since every rK-degree is countable, there are continuum many minimal rK-degrees.

2.1.4 Theorem. *Every nonempty computably bounded Π^0_1 class with no computable members has $2^{\aleph_0}$ paths of hyperimmune-free Turing degree.*

Proof. By basic facts from the theory of Π^0_1 classes, we can assume without loss of generality that our Π^0_1 class is the set of paths through a binary tree T_0 that is infinite, computable, and has no computable paths. We modify slightly the proof of the Hyperimmune-free Basis Theorem in [**JS72**] by way of an extra parameter sequence X. For each sequence X we construct (computably in $X \oplus \emptyset''$) computable subtrees $S_1 \supset T_1 \supseteq S_2 \supset T_2 \supseteq \cdots$ of T_0 such that their only common path Y has hyperimmune-free Turing degree. We then show that the map $X \mapsto Y$ is one-to-one.

To this end, fix X and, starting from T_0, let S_e and T_e be defined recursively as follows. Let $U_{e,x}$ be the computable tree $\{\tau : \Phi^{\tau}_{e,|\tau|}(x)\uparrow\}$.

(i) If for all x, $T_e \cap U_{e,x}$ is finite, then $S_e := T_e$. Otherwise, choose x least such that $U_{e,x}$ is infinite and $S_e := T_e \cap U_{e,x}$.

(ii) Since S_e is an infinite tree with no computable paths, it has at least two paths. Let σ be the length-lexicographic least node of S_e such that $\sigma 0$ and $\sigma 1$ have paths in S_e through them.

(iii) $T_{e+1} := \{\tau \in S_e : \tau \subseteq \sigma \frown X(e) \vee \tau \supset \sigma \frown X(e)\}$.

By induction each $[T_e]$ and $[S_e]$ is nonempty, so that $\bigcap_e [T_e] \cap [S_e]$ is nonempty, being the intersection of a decreasing sequence of closed nonempty sets in the compact space ${}^{\mathbb{N}}2$. Choose (the unique) sequence $Y \in \bigcap_e [T_e] \cap [S_e]$. It will have hyperimmune-free Turing degree, for fix a natural number e and consider the function Φ^Y_e. If for every x, $T_e \cap U_{e,x}$ is finite, then the following function is total, computable, and majorizes Φ^Y_e.

$$g(x) = \max\{\Phi^{\tau}_{e,|\tau|}(x) : \tau \in T_e \wedge |\tau| = l_x\},$$

where l_x is least such that $\Phi^{\tau}_{e,|\tau|}(x)$ is defined for each $\tau \in T_e$ of length l_x. If there exists some x such that $T_e \cap U_{e,x}$ is infinite, then $\Phi^{\tau}_{e,|\tau|}(x)$ is undefined for infinitely many $\tau \in T_e$ and S_e is the set of all these τ. Since all prefixes of Y are in S_e, this means $\Phi^Y_e(x)$ is undefined, so that Φ^Y_e is not total.

Also, the map $X \mapsto Y$ is one-to-one, for if two sequences X_1 and X_2 differ, and e is the first place at which this happens, then the corresponding trees $S_e(X_1)$ and $S_e(X_2)$ are the same, but the intersection of $T_{e+1}(X_1)$ and $T_{e+1}(X_2)$ is finite since one is all but finitely contained in the nodes above $\sigma 0$ and the other in the nodes above $\sigma 1$. Thus $Y(X_1)(|\sigma|) \neq Y(X_2)(|\sigma|)$. $\square$

A sequence of minimal relative randomness is also minimal in terms of absolute randomness in the sense of the next proposition. From now on let us call a sequence with minimal rK-degree a 'minimal sequence'. Recall from [**Cha76**] that a set X is computable iff $\exists d\, \forall n \,.\, C(X \upharpoonright n) < C(n) + d$.

2.1.5 Proposition. *If A is a minimal sequence, then for any computable unbounded nondecreasing function $g : \mathbb{N} \to \mathbb{N}$,*

$$\exists d\, \forall n \,.\, C(A \upharpoonright n) < C(n) + g(n) + d \quad \textit{and}$$

$$\exists d\, \forall n \,.\, K(A \upharpoonright n) < K(n) + g(n) + d.$$

In particular, A cannot be random.

We prove this with dilutions.

2.1.6 Definition. *For $X \in {}^{\mathbb{N}}2$ and $f : \mathbb{N} \to \mathbb{N}$ increasing, the* **f-dilution of X** *is the sequence defined by*

$$X_f(n) = \begin{cases} X(m) & \textit{if } n = f(m) \textit{ for some (unique) } m \\ 0 & \textit{else.} \end{cases}$$

Notice that for any sequence X and any increasing computable function f, $X_f \leq_{\mathrm{rK}} X$ and $X_f \equiv_{\mathrm{T}} X$.

Proof of Proposition 2.1.5. Fix A and g as in the hypothesis. The idea is that since A is a minimal sequence, it is rK-reducible to every one of its computable dilutions. Picking a dilution appropriate to g will give the desired complexity bound.

We prove the bound for K. The argument for C is identical. Define the function $f : \mathbb{N} \to \mathbb{N}$ recursively by

$$f(0) = 0;$$

$$f(x) = \text{the least } n \text{ such that } n > f(x-1) \text{ and } g(n) \geq 4x.$$

Since g is unbounded and nondecreasing, f is well-defined. Also, by construction f is computable, increasing, and for any given n, if x is greatest such that $f(x) \leq n$, then $g(n) \geq 4x$.

Since A is minimal, $A \leq_{\mathrm{rK}} A_f$ via some $[\varphi, d]$. Now fix n and choose x greatest such that $f(x) \leq n$. Observe that inserting zeros into $A \upharpoonright x$ in the appropriate computable places produces $A_f \upharpoonright n$. So to describe $A \upharpoonright n$, besides a few computable partial functions given ahead of time, one only needs the correct $i < d$ such that $\varphi(i, A_f \upharpoonright n) = A \upharpoonright n$, n, and $A \upharpoonright x$. This information can be coded, up to a uniform constant, by a string of length $K(n) + 2K(A \upharpoonright x)$. The factor of 2 comes from concatenating strings in a prefix-free way. So, up to a uniform additive constant, for all n

$$K(A \upharpoonright n) \leq K(n) + 2K(A \upharpoonright x) \leq K(n) + 4x \leq K(n) + g(n),$$

as desired.

Letting $g(n) = \lfloor \lg(n+1) \rfloor$, say, we see that A cannot be random. □

Using dilutions again, we get

2.1.7 Proposition. *Every minimal sequence is rK-reducible to a random sequence.*

Proof. Fix a minimal sequence A, and choose a random sequence $R \geq_{\mathrm{wtt}} A$ with use majorized by $f(n) = 2n$. This is possible since every sequence has such a random sequence ([**Kuč85**],[**Gác86**]; see also [**MM04**] for a more recent proof using martingales). Then $R \geq_{\mathrm{rK}} A_f \geq_{\mathrm{rK}} A$, by the minimality of A, as desired. □

Do all sequences have random sequences rK-above them? This is Question ④, to which we return in the next section. We end with one last note, a contrast to Proposition 2.1.7.

2.1.8 Proposition. *There is a random sequence with no minimal sequence rK-reducible to it.*

Proof. Let R be a random sequence of hyperimmune-free Turing degree. Such a sequence exists by the Hyperimmune-free Basis Theorem applied to the complement of any member of a universal Martin-Löf test. Then R has no minimal sequence reducible to it.

To see this, assume (toward a contradiction) there is some minimal sequence A such that $A \leq_{\mathrm{rK}} R$. Since R has hyperimmune-free Turing degree, so does A and $A \leq_{\mathrm{tt}} R$. Since A is noncomputable and truth-table reducible to a random sequence, A is Turing equivalent to some random sequence S (see [**Dem88**]). Since A has hyperimmune-free Turing degree, $S \leq_{\mathrm{tt}} A$ via some computable partial function with computable use function f. Thus, disregarding floor functions and uniform constants for ease of reading, we have that for all n

$$\begin{aligned} n &\leq K(S \upharpoonright n) && \text{(since } S \text{ is random)} \\ &\leq 2K(A \upharpoonright f(n)) && \text{(using the tt-reduction)} \\ &\leq 2K(f(n)) + 2\lg n && \text{(by Proposition 2.1.5)} \\ &\leq 2K(n) + 2\lg n && \text{(since } f \text{ is computable)} \\ &\leq 4\lg n, \end{aligned}$$

a contradiction. □

2.2. Questions ②–④

Whereas Question ① concerned lower rK-cones, building sequences rK-below given sequences, Questions ②–④ concern upper rK-cones, building sequences rK-above given sequences, or so it seems. These questions are more difficult to answer, and i have only partial results here.

The following definition will be useful.

2.2.1 Definition. *For a function* $\varphi :\subseteq d \times {}^{<\mathbb{N}}2 \to {}^{<\mathbb{N}}2$ *(usually an rK-reduction), the* **preimage** *of a string* τ *under* φ *is*

$$\varphi^{-1}(\tau) := \{\sigma \in {}^{|\tau|}2 : \forall m \leq |\tau| \; \exists i < d \; . \; \varphi(i, \sigma \restriction m) = \tau \restriction m\}.$$

With respect to Questions ② and ③, it is tempting to conjecture that all random sequences are maximal. Nothing should be more random than a random sequence, right? To show this we would have to fix a random sequence A, suppose that $A \leq_{\mathrm{rK}} B$, and prove that $B \leq_{\mathrm{rK}} A$. To this end, we might try to reverse the rK-reduction from B to A, call it $[\varphi, d]$. The simplest way to do this is to show that, as a function of n, $|\varphi^{-1}(A \restriction n)|$ is bounded (by a constant). This bound will then be the constant of the reverse reduction. Indeed, this is the case for a norm-1 reduction. By the **norm** of an rK-reduction i mean its constant.

2.2.2 Proposition. *If A is random and $A \leq_{\mathrm{rK}} B$ via a norm-1 reduction, then $A \equiv_{\mathrm{rK}} B$.*

Proof. Fix A and B as in the hypotheses and let φ be a norm-1 rK-reduction witnessing $A \leq_{\mathrm{rK}} B$. We need to show that $B \leq_{\mathrm{rK}} A$.

Consider the function $f : {}^{<\mathbb{N}}2 \to \mathbb{N}$ defined by

$$f(\sigma) = |\varphi^{-1}(\sigma)|.$$

f is computably approximable from below and, since φ has norm 1, for all σ,

$$f(\sigma 0) + f(\sigma 1) = |\varphi^{-1}(\sigma 0)| + |\varphi^{-1}(\sigma 1)| \leq 2|\varphi^{-1}(\sigma)| = 2f(\sigma).$$

So f is a Σ^0_1 supermartingale. Since A is random, there exists a natural number e such that $\forall n \, . \, f(A \restriction n) < e$, that is,

$$\forall n \, . \, |\varphi^{-1}(A \restriction n)| < e.$$

Since $B \upharpoonright n \in \varphi^{-1}(A \upharpoonright n)$ for all n, $B \leq_{\text{rK}} A$ via a norm-e reduction. Specifically, let θ be the c.p.f. defined by

$$\theta(i, \sigma) = \text{the (i+1)st new string}$$
$$\text{that appears in the enumeration of } \varphi^{-1}(\sigma).$$

Then $B = [\theta, e]^A$. □

2.2.3 Remark. *The proof above can easily be modified to show that if A is random and $A \leq_{sw} B$, then $B \leq_{\text{rK}} A$. Here $A \leq_{sw} B$ means that B Turing computes A with use bounded by the identity function plus a constant.*

When we move to norm-2 reductions, however, the reversal strategy of Proposition 2.2.2 fails.

2.2.4 Proposition. *There is an rK-reduction $[\varphi, 2]$ such that for every random sequence A, $|\varphi^{-1}(A \upharpoonright n)|$ is an unbounded function of n.*

Proof. Firstly we set $\varphi(0, \sigma) = \sigma$ for all strings σ. Secondly we recursively define $\varphi(1, \sigma)$ for various strings σ along with prefix-free sets of strings $P_0, P_1, \ldots \subseteq \text{dom } \varphi(1, \cdot)$ as follows. Let $P_0 = \{\emptyset\}$. Fix $k > 0$ and assume P_{k-1} has been defined. For all $\rho \in P_{k-1}$, $\delta \in ({}^k2 \setminus \{0^k\})^*$, and $\epsilon \in {}^k2 \setminus \{0^k\}$, define $\varphi(1, \rho\delta\epsilon) = \rho\delta 0^k$, extending this definition to substrings as well, and put $\rho\delta 0^k$ into P_k. Here S^*, for a set of strings S, denotes the set of all possible concatentations of strings in S including the empty string. Notice that P_k is prefix-free and that for all $\sigma \in P_k$, $|\varphi^{-1}(\sigma)| = 2^k$.

Now fix a random sequence A. It suffices to show that for all k, $A \in \mathcal{O}(P_k)$. For this it suffices to show that each $\mathcal{O}(P_k)$ is a Σ^0_1 class of measure one, since a random sequence is in every Σ^0_1 class of measure one (see [**Kur81**]). Since each P_k is computable, each $\mathcal{O}(P_k)$ is a Σ^0_1 class. To show each of these classes has measure one, we use induction. The measure of $\mathcal{O}(P_0) = \mathcal{O}(\emptyset) = {}^{\mathbb{N}}2$ is

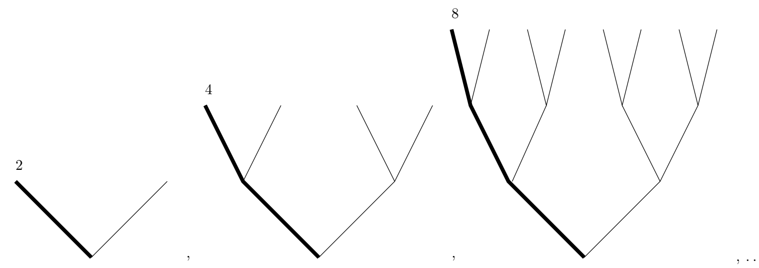

FIGURE 1. The building blocks for stages $1, 2, 3, \ldots$ of the construction along with the number of preimages they contribute.

one. Assume $k > 0$ and $\mu\ \mathcal{O}(P_{k-1}) = 1$. By the definition of P_k, for each $\rho \in P_{k-1}$ we have that

$$\begin{aligned}\mu\left[\mathcal{O}(P_k) \cap \mathcal{O}(\rho)\right] &= 2^{-|\rho|}\left[2^{-k} + (2^k-1)2^{-2k} + (2^k-1)^2 2^{-3k} + \cdots\right] \\ &= 2^{-|\rho|}\frac{2^{-k}}{1-(2^k-1)2^{-k}} \\ &= 2^{-|\rho|}.\end{aligned}$$

Thus, since P_{k-1} is prefix-free,

$$\begin{aligned}\mu\ \mathcal{O}(P_k) &= \sum_{\rho \in P_{k-1}} \mu\left[\mathcal{O}(P_k) \cap \mathcal{O}(\rho)\right] \\ &= \sum_{\rho \in P_{k-1}} 2^{-|\rho|} \\ &= \mu\ \mathcal{O}(P_{k-1}) \\ &= 1,\end{aligned}$$

as desired. □

But, upon further reflection, perhaps the reversal strategy of Proposition 2.2.2 still holds hope. In the proof of Proposition 2.2.4, if $A = [\varphi, 2]^B$ (the specific φ constructed), then $A = [\mathrm{id}, 1]^B$ and $A = B$. If we are going to show maximality of random sequences we need to assume that A is random and $A \leq_{\mathrm{rK}} B$ via a reduction of *least norm*.

At this point i should mention that, in general, the norm of an rK-reduction does matter.

2.2.5 Proposition. *For each natural number d there are Δ^0_2 sequences A and B and a computable function f such that $A = [f,e]^B$, where $e = 2^{\lceil \lg(d+1) \rceil}$, but for all c.p.f. φ, $A \neq [\varphi, d]^B$.*

Proof. Fix a natural number d, let $l = \lceil \lg(d+1) \rceil$, let $e = 2^l$, and let $\varphi_0, \varphi_1, \ldots$ be a computable enumeration of all c.p.f. from $\mathbb{N} \times {}^{<\mathbb{N}}2$ to ${}^{<\mathbb{N}}2$. We construct A and B in stages by a $\emptyset'$-finite extension.

At stage s suppose A and B have been constructed up to length $2sl$. Using oracle $\emptyset'$, choose the lexicographic least string τ of length l such that

$$\forall i < d \,.\, \varphi_n(i, B_s 0^l) \neq A_s \tau.$$

Such a τ exists since there are $e = 2^l \geq 2^{\lg(d+1)} = d+1$ strings of length l and only d possibilities for $\varphi_n(i, B_s 0^l)$ with $i < d$. Let $B_{s+1} = B_s 0^l \tau$ and $A_{s+1} = A_s \tau 0^l$. This ends the construction.

By construction, for any c.p.f. φ_n, $A \neq [\varphi_n, d]^B$. Moreover, it is easy to see from the diagram below how to construct a computable function f such that $A = [f,e]^B$.

$$\begin{array}{lccccc} B: & 0\ldots0 & \#\ldots\# & 0\ldots0 & \star\cdots\star & \ldots \\ A: & \#\ldots\# & 0\ldots0 & \star\cdots\star & 0\ldots0 & \ldots \end{array}$$

In the diagram the length of each block is l. □

More hope, if not evidence, for the maximality of random sequences comes from the following two theorems. The rK-upper cones of random sequences are countable!

2.2.6 Theorem. *If A is 2-random, then for any rK-reduction $[\varphi, d]$ there are only finitely many sequences rK-above A via $[\varphi, d]$. Moreover for each such sequence B we have that $B \leq_{\mathrm{T}} A'$.*

Proof. Let A be 2-random and fix an rK-reduction $[\varphi, d]$. We show that every sequence rK-above A is a path through an A'-computable tree with only finitely many paths. The result then follows.

Consider the class of extensions of φ to total functions

$$\begin{aligned}\mathcal{E} :=\{f : d \times {}^{<\mathbb{N}}2 \to {}^{<\mathbb{N}}2 : \forall i\, \forall \tau\ .\ |f(i,\tau)| = |\tau| \wedge \\ \forall i\, \forall \tau\, \forall s\ .\ [\varphi_s(i,\tau)\!\downarrow \to f(i,\tau) = \varphi_s(i,\tau)]\}.\end{aligned}$$

Since $\mathcal{E}$ is a nonempty computably bounded Π^0_1 class (after identifying $d \times {}^{<\mathbb{N}}2$ and ${}^{<\mathbb{N}}2$ with $\mathbb{N}$), $\mathcal{E}$ has a low member f by the Low Basis Theorem. For each natural number k, let

$$\begin{aligned}S_k &:= \{\sigma \in {}^{<\mathbb{N}}2 : |f^{-1}(\sigma)| \geq d2^k\};\\ \mathcal{U}_k &:= \{X \in {}^{\mathbb{N}}2 : \forall^\infty n\ .\ |f^{-1}(X \restriction n)| \geq d2^k\}\\ &= \{X : \exists m\, \forall n \geq m\ .\ X \restriction n \in S_k\}\\ &= \bigcup_m \bigcap_{n \geq m} \mathcal{O}(S_k \cap {}^n2).\end{aligned}$$

Observe that, since $\langle S_k : k \in \mathbb{N}\rangle$ is a computable sequence of f-computable sets, $\langle \mathcal{U}_k : k \in \mathbb{N}\rangle$ is a computable sequence of Σ^f_2 classes.

By a simple counting argument we have that for all k and n, $|S_k \cap {}^n2| \leq 2^{n-k}$, so that $\mu\ \mathcal{O}(S_k \cap {}^n2) \leq 2^{-k}$. (To see that $|S_k \cap {}^n2| \leq 2^{n-k}$, observe that, assuming $S_k \cap {}^n2$ is nonempty, each string in $S_k \cap {}^n2$ has a persistent preimage 'blob' in ${}^n2$, so that $|S_k \cap {}^n2|$ also counts the total number of these blobs. In terms of a Venn diagram, each blob has area at least $d2^k$, and at most d blobs can overlap on a given region since f (thought of as a multimap from ${}^{<\mathbb{N}}2$ to ${}^{<\mathbb{N}}2$) maps ${}^n2$ to ${}^n2$ at most 1-to-d. Therefore, at most $d(2^n d^{-1} 2^{-k}) = 2^{n-k}$ blobs can occupy the total 2^n area of space available. So the inequality holds.) Consequently,

$$\begin{aligned}
\mu\,\mathcal{U}_k &= \mu \bigcup_m \bigcap_{n\geq m} \mathcal{O}(S_k \cap {}^n2) \\
&= \lim_{m\to\infty} \mu \bigcap_{n\geq m} \mathcal{O}(S_k \cap {}^n2) \qquad \text{(since the union is nondecreasing)} \\
&\leq \lim_m 2^{-k} \\
&= 2^{-k}.
\end{aligned}$$

Thus $\langle \mathcal{U}_k : k \in \mathbb{N}\rangle$ is a Σ_2^f test.

Since A is 2-random, A is $\emptyset'$-1-random, hence f'-1-random (since f is low), hence f-2-random. Thus for some k, $A \notin \mathcal{U}_k$, that is $\exists^\infty n\ .\ |f^{-1}(A \restriction n)| < d2^k$. Since f extends φ, there exist a natural number e such that

$$\exists^\infty n\ .\ |\varphi^{-1}(A \restriction n)| < e.$$

Therefore the A'-computable tree

$$\begin{aligned}
T :=& \bigcup_n \varphi^{-1}(A \restriction n) \\
=& \{\tau : \forall m \leq |\tau|\ \exists i < d\ .\ \varphi(i, \tau \restriction m) = A \restriction n\}
\end{aligned}$$

has $< e$ paths, and so every path is computable from A'. Since B is one of these paths, the result holds. $\square$

Actually, using two powerful lemmas, Theorem 2.2.6 can be improved to

2.2.7 Theorem (with Joseph Miller). *If A is 1-random, then for any rK-reduction $[\varphi, d]$ there are only finitely many sequences rK-above A via $[\varphi, d]$. Moreover for each such sequence B we have that $B' \equiv_{tt} A'$.*

2.2.8 Lemma ([Mil]). *If A is random, then*

$$\exists e\ \forall \sigma\ .\ K^A(\sigma) = \min\{K(A \restriction \backslash\sigma, m\backslash) - \backslash\sigma, m\backslash : m \in \mathbb{N}\} + e.$$

Here $\backslash\cdot,\cdot\backslash$ *is any computable bijection from strings and natural numbers to natural numbers.*

2.2.9 Lemma ([Nie05]). *If there exists a natural number* e *such that for natural numbers* n*,* $K^X(Y\restriction n) \leq K^X(n) + e$*, then* $Y' \leq_{tt} X'$*.*

Proof of 2.2.7. Let A be random and fix an rK-reduction $[\varphi, d]$. We show that every sequence rK-above A is a path through an A'-computable tree with only finitely many paths. The first part of the result then follows. The second part will follow from Nies's Lemma.

Suppose $A = [\varphi, d]^B$. Then B is random, $A \leq_{\mathrm{T}} B$, and $A \leq_{\mathrm{K}} B$, where the last inequality abbreviates

$$\exists d' \, \forall n \, . \, K(A\restriction n) < K(B\restriction n) + d'.$$

(The uniform constant d' depends in part on d, but not on B, of course.)

Using Lemma 2.2.8, up to a uniform additive constant —call it e— we have that for all n,

$$\begin{aligned}
K^A(B\restriction n) =& \min\{K(A\restriction \backslash B\restriction n, m\backslash) - \backslash B\restriction n, m\backslash : m \in \mathbb{N}\} \\
& \quad \text{(since } A \text{ is random)} \\
\leq& \min\{K(B\restriction \backslash B\restriction n, m\backslash) - \backslash B\restriction n, m\backslash : m \in \mathbb{N}\} \\
& \quad \text{(since } A \leq_{\mathrm{K}} B\text{)} \\
=& K^B(B\restriction n) \\
& \quad \text{(since } B \text{ is random)} \\
\leq& K^B(n) \\
\leq& K^A(n) \\
& \quad \text{(since } A \leq_{\mathrm{T}} B\text{)}.
\end{aligned}$$

Call this inequality ☺.

Now, by relativizing [**DHNS03**, Theorem 6.6],

$$\exists e' \, \forall n \, . \, |\{\sigma \in {}^n 2 : K^A(\sigma) < K^A(n) + e\}| < e'.$$

Thus the A'-computable tree

$$T := \{\sigma : \forall m \leq |\sigma| \;.\; K^A(\sigma \restriction m) < K^A(n) + e\}$$

has $< e'$ paths, and so every path is computable from A'. Since by ☺, every sequence rK-above A via $[\varphi, d]$ is a path through this tree, the first part of the theorem holds. Moreover, by ☺ and Lemma 2.2.9, $B' \leq_{\mathrm{tt}} A'$. Also $A \leq_{\mathrm{T}} B$, so $A' \leq_1 B'$, so $A' \leq_{\mathrm{tt}} B'$. □

In contrast to the previous two theorems, highly nonrandom sequences have rK-upper cones of size continuum. Here a sequence A is **highly nonrandom** if $\forall n \;.\; C(A \restriction n) < n - f(n)$ for some unbounded computable function $f : \mathbb{N} \to \mathbb{N}$.

2.2.10 Theorem. *If A is highly nonrandom, then there are continuum many sequences rK-above A.*

Proof. Suppose A is highly nonrandom via an unbounded computable function f. We construct an rK-reduction $[\varphi, 1]$ such that $|\{Y \in {}^{\mathbb{N}}2 : A = [\varphi, 1]^Y\}| = 2^{\aleph_0}$. The idea is that, since f is unbounded, the proportion of strings $\tau \in {}^{n}2$ with $C(\tau) < n - f(n)$ shrinks to zero as n increases, so that we can map a lot of strings $\sigma \in {}^{n}2$ to each τ. Since f is computable, we can carry out this process computably.

First, define the computable function $l : \mathbb{N} \to \mathbb{N}$ recursively by

$$l(0) := \text{the least } x \text{ such that } \frac{2^{x-f(x)}}{2^x} \leq 1/2;$$

$$l(n+1) := \text{the least } x \text{ such that } x > l(n) \text{ and } \frac{2^{x-f(x)}}{2^{x-l(n)}} \leq 1/2.$$

This is possible since f is computable and unbounded. Second, define the Σ^0_1 tree $T := \bigcup_n T_{l(n)}$, where

$$T_L = \{\tau \in {}^{L}2 : \forall m \leq N \;.\; C(\tau \restriction m) < m - f(m)\}.$$

Note that $|T_L| \leq 2^{L-f(L)}$. Now, construct the c.p.f. $\varphi :\subseteq 1 \times {}^{\mathbb{N}}2 \to {}^{\mathbb{N}}2$ as follows. Enumerate T. Whenever a new τ appears in $T_{l(0)}$, choose in some computable way two unused strings of length $l(0)$ and map them both to τ under φ, that is, for each such σ set $\varphi(0, \sigma \restriction m) = \tau \restriction m$ for all $m \leq |\sigma|$. Label τ as 'covered'. Whenever a new τ appears in $T_{l(n+1)}$, for each preimage ρ of $\tau \restriction l(n)$ (of length $l(n)$) (which inductively exists), choose in some computable way two unused strings of length $l(n+1)$ extending ρ and map them both to τ under φ. After this is done, label τ as covered. Continue.

Observe that this definition of φ is consistent, that is, we never have $\varphi(0, \sigma_1) = \tau$ and $\varphi(0, \sigma_2) = \tau$ with σ_1 and σ_2 incomparable. This is because for all L, $T_L \supseteq T_{L+1} \restriction L$. Also, observe that at each stage of the construction we do, in fact, have enough strings at our disposal, that is, for each $\tau \in T_{l(n+1)}$ with $\tau \restriction l(n)$ covered we can map two length $l(n+1)$ strings to τ under φ, because

$$\begin{aligned}
&\frac{\#\text{ extensions of } \tau \restriction l(n) \text{ in } T_{l(n+1)}}{\#\text{ extensions of some preimage of } \tau \restriction l(n)} \\
&\leq \frac{|T_{l(n+1)}|}{\#\text{ extensions of some preimage of } \tau \restriction l(n)} \\
&\leq \frac{2^{l(n+1)-f(l(n+1))}}{2^{l(n+1)-l(n)}} \\
&\leq 1/2 \quad \text{by definition of } l.
\end{aligned}$$

Lastly, observe that for each A with $\forall n \,.\, C(A \restriction n) < n - f(n)$ (each path in $[T]$) we built a perfect preimage tree T_A such that for all $Y \in [T_A]$, $A = [\varphi, 1]^Y$. Since T_A is perfect, $|[T_A]| = 2^{\aleph_0}$. $\square$

The name 'highly nonrandom' really is appropriate here.

2.2.11 Proposition. *The highly nonrandom sequences form a proper subclass of the nonrandom sequences.*

Proof. First, if $A \in {}^{\mathbb{N}}2$ is highly nonrandom, then A is nonrandom. To see this (through the contrapositive), fix any unbounded computable function f and consider the Σ^0_1 sets

$$S_n := \{\sigma : \forall m \leq n \,.\, C(\sigma \restriction m) < m - f(m)\}, \qquad (n \in \mathbb{N}).$$

As mentioned in the proof of Theorem 2.2.10, for each n, $|S_n| \leq 2^{n-f(n)}$, so that $\mu\mathcal{O}(S_n) \leq 2^{-f(n)}$. Since f is unbounded and computable, $\langle \mathcal{O}(S_n) : n \in \mathbb{N} \rangle$ can be refined to a Σ^0_1-test. Thus if A

is random, $A \notin \mathcal{O}(S_n)$ for some n, so that A is not highly nonrandom via f. Since f was an arbitrary unbounded computable function, the result holds.

To show proper containment, we build, via a finite extension argument, a nonrandom sequence A that is not highly nonrandom. Suppose $A \restriction l$ has been constructed already and consider the next unbounded computable function in our (noncomputable) list of all such functions. We extend $A \restriction l$ in such a way that it cannot be highly nonrandom via f. This is pretty easy, because there are 2^{n-l} possible length n extensions of $A \restriction l$, but only $\leq 2^{n-f(n)}$ of them can have complexity $< n - f(n)$. So for n large enough $f(n) > l$ since f is unbounded, so that some extension of $A \restriction l$ will not have low complexity. Let $A \restriction n$ be such an extension. Then extend $A \restriction n$ again by adding on $2n$ zeros. This ends the construction.

Clearly A is not highly nonrandom. Also, A has at least twice as many zeros as ones in the limit, so it is nonrandom; random sequences must satisfy the Weak Law of Large Numbers (see [**Cal94**]). □

The sizes of rK-upper cones of nonrandom sequences are still unknown. (Always the continuum? Or are some countable?)

But what of Questions ② and ③? Despite the hard work and partial results of this section, i can find no proof or disproof of the maximality of random sequences. Question ④ stands in a similar state. While minimal sequences always have random sequences rK-above them (Theorem 2.1.7), the general case of arbitrary sequences is still open.

CHAPTER 3

Real closed fields

In this chapter we depart from Questions ①–④ and explore instead rK-reducibility in the context of computable analysis. We group reals into relative randomness classes and study these classes in relation to well-known computational classes of reals. This chapter also appears as [**Rai05**].

Before beginning, let us set some relevant notation and conventions. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $x \upharpoonright n$ denotes the truncation of the binary expansion of x (both the integer and fractional part) up to and including the first n bits past the binary point. Again $\langle \quad \rangle$ delimits ordered tuples and sequences, and for each $s \in \mathbb{N}^+$, let $\backslash \quad \backslash : \mathbb{N}^s \to \mathbb{N}$ be a lexicographically increasing computable bijection (coding function).

Let us also recall the following standard computational classes of reals. A real number x is **computable** iff there exists a computable sequence of rationals $\langle q_s : s \in \mathbb{N} \rangle$ converging effectively to x, that is, there is a computable function $e : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$s \geq e(n) \to |q_s - x| \leq 2^{-n}.$$

Equivalently, x is computable iff the binary sequence of the binary expansion of the fractional part of x is a computable function. A real number x is **computably enumerable (c.e.)** iff there is a computable nondecreasing sequence of rationals converging to x. A real number x is **a difference of c.e. reals (d.c.e.)** iff there exist c.e. reals y, z such that $x = y - z$. A real number x is **computably approximable (c.a.)** iff there is a computable sequence of rationals converging to x (with no further restrictions on the sequence). Let $\mathbb{R}_{\mathrm{c}}$ denote the class of computable reals, $\mathbb{R}_{\mathrm{c.e.}}$ the class of c.e. reals, $\mathbb{R}_{\mathrm{d.c.e.}}$ the class of d.c.e. reals, and $\mathbb{R}_{\mathrm{c.a.}}$ the class of c.a. reals. These classes are properly nested: $\mathbb{R}_{\mathrm{c}} \subset \mathbb{R}_{\mathrm{c.e.}} \subset \mathbb{R}_{\mathrm{d.c.e.}} \subset \mathbb{R}_{\mathrm{c.a.}}$ (see [**ASWZ00**] for instance).

3.1. Real Closed Fields

Since we will be dealing with real numbers, let us rephrase rK-reducibility in terms of them. For $x, y \in \mathbb{R}$, $x \leq_{\mathrm{rK}} y$ iff

$$\exists \text{c.p.f. } \varphi :\subseteq \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \; \exists d \, \forall n \, \exists i < d \, . \, \varphi(y \upharpoonright n, i) \downarrow = x \upharpoonright n.$$

In this case we write $x = [\varphi, d]^y$. Now, given $y \in \mathbb{R}$, let

$$\mathbb{R}_y = \{x \in \mathbb{R} : x \leq_{\mathrm{rK}} y\},$$

the class of reals less random than y.

Perhaps surprisingly, each $\mathbb{R}_y$ has tame algebraic and analytic structure: each is a real closed field. This generalizes the well-known fact (see [**PER89**] for instance) that $\mathbb{R}_{\mathrm{c}}$, the class of computable reals, forms a real closed field in the following sense. $x \in \mathbb{R}$ is computable iff $x \leq_{\mathrm{T}} \emptyset$ (identifying x with the binary sequence of the binary expansion of its fractional part) iff $x \leq_{\mathrm{rK}} 0$ (remember that rK-reducibility is a refinement of T-reducibility). Thus $\mathbb{R}_{\mathrm{c}} = \mathbb{R}_0$, that is, the class of computable reals is the randomness class $\mathbb{R}_0$ (or $\mathbb{R}_a$, for any computable real a). Notice also that $\mathbb{R}_0 \subseteq \mathbb{R}_y$ for all y.

For the rest of this section, fix a randomness class $\mathbb{R}_y$. As a first step to showing $\mathbb{R}_y$ is a real closed field, we introduce a large class of functions under which $\mathbb{R}_y$ is closed, the weakly computable locally Lipschitz functions.

3.1.1 Definition. *Let $s \in \mathbb{N}^+$, $E \subseteq \mathbb{R}^s$ be open, and $f : E \to \mathbb{R}$.*

- *f is* **locally Lipschitz** *iff for each $x \in E$ there is an open set $E_0 \subseteq E$ containing x on which f is Lipschitz, that is*

$$\exists M \in \mathbb{R}^+ \, \forall \vec{x}, \vec{y} \in E_0 \, . \, |f(\vec{x}) - f(\vec{y})| \leq M |\vec{x} - \vec{y}|,$$

where $| \quad |$ is the Euclidean norm.

- *f is* **weakly computable** *iff $f \restriction E \cap \mathbb{Q}^s$ uniformly outputs computable reals in the following sense:*

$$\exists c.p.f.\ \hat{f} :\subseteq \mathbb{Q}^s \times \mathbb{N} \to \mathbb{Q}\ \forall \vec{q}\, \forall n\ .\ \vec{q} \in E \cap \mathbb{Q}^s \to \hat{f}(\vec{q}, n){\downarrow} = f(\vec{q}) \restriction n$$

- *f is* **weakly computable locally Lipschitz (w.c.l.L.)** *iff f is weakly computable and locally Lipschitz.*

3.1.2 Remark. *It is easy to see that weakly computable Lipschitz functions are computable, and computable functions are weakly computable. For a definition of 'computable' in this sense see* [**PER89**] *again. Also, as a fact from elementary real analysis, locally Lipschitz functions on compact domains are Lipschitz. Thus w.c.l.L. functions on compact domains are computable functions. We could use the stronger notion of 'computable function' instead of 'weakly computable function' throughout, but weak computability suffices, and its criterion is slightly easier to check.*

The following two lemmas and short comment thereafter explain why w.c.l.L. functions interact so well with rK-reducibility.

3.1.3 Lemma. *If $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is locally Lipschitz, then for all $\vec{x} \in E$*

$$\exists C\ \forall n{>}C\ .\ |f(\vec{x}) - f(\vec{x} \restriction n)| < 2^{C-n},$$

where $\vec{x} \restriction n = \langle x_0 \restriction n, \ldots, x_{s-1} \restriction n \rangle$.

Proof. Suppose $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is locally Lipschitz and $\vec{x} \in E$. Then there is an open $E_0 \subseteq E$ containing $\vec{x}$ such that f is Lipschitz on E_0. Thus

$$\begin{aligned} & \exists M \in \mathbb{Q}^+\ \forall \vec{y} \in E_0\ .\ |f(\vec{x}) - f(\vec{y})| \le M|\vec{x} - \vec{y}| \\ \Rightarrow\ & \exists M \in \mathbb{Q}^+\ \forall^\infty n\ .\ |f(\vec{x}) - f(\vec{x} \restriction n)| \le M\sqrt{s}2^{-n} \\ & (\text{since } \forall^\infty n\ .\ \vec{x} \restriction n \in E_0 \text{ and } \forall^\infty n\ .\ |\vec{x} - \vec{x} \restriction n| \le \sqrt{s2^{-2n}} = \sqrt{s}2^{-n}) \\ \Rightarrow\ & \exists C\ \forall n{>}C\ .\ |f(\vec{x}) - f(\vec{x} \restriction n)| < 2^{C-n}. \end{aligned}$$

□

3.1.4 Lemma. *Let $x, y \in \mathbb{R}$ and $C, n \in \mathbb{N}$ with $n > C$. If $|x - y| < 2^{C-n}$, then there exist $j < 2$ and $\rho \in {}^{C+1}2$ such that $[y + (-1)^j 0.0^{n-C-1}{}^\frown\rho] \restriction n = x \restriction n$.*

Proof. An easy exercise in binary addition. □

Using Lemma 3.1.3 and Lemma 3.1.4 we can now show that $\mathbb{R}_y$ is closed under w.c.l.L. functions. The basic idea is this. Suppose $\vec{x} \in (\mathbb{R}_y)^s$ and f is a weakly computable locally Lipschitz function. Since f is locally Lipschitz, the first n bits of $f(\vec{x})$, which we want via an rK-computation from y, are just the first n bits of $[f(\vec{x} \restriction n) +$ fuzz$]$, which we can get via an rK-computation from y since the fuzz is of bounded variability. The hypothesis of weak computability on f ensures that the partial function we build witnessing rK-reducibility is computable.

3.1.5 Lemma. *Let $s \in \mathbb{N}^+$. If $\vec{x} \in (\mathbb{R}_y)^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\vec{x} \in E$, then $f(\vec{x}) \in \mathbb{R}_y$.*

Proof. For notational niceness let us prove the special case $s = 2$. The general proof is no more difficult. Suppose $\vec{x} = \langle x_0, x_1 \rangle \in (\mathbb{R}_y)^2$, say $x_0 = [\varphi_0, d_0]^y$ and $x_1 = [\varphi_1, d_1]^y$. Since f is locally Lipschitz, there exists, by Lemma 3.1.3, $C \in \mathbb{N}$ such that for all $n > C$,

$$|f(\vec{x}) - f(\vec{x} \restriction n)| < 2^{C-n}.$$

So by Lemma 3.1.4, $\forall n{>}C \; \exists j{<}2 \; \exists \rho{\in}{}^{C+1}2$

$$\left[f(\vec{x} \restriction n) + (-1)^j 0.0^{n-C-1}{}^\frown\rho\right] \restriction n = f(\vec{x}) \restriction n.$$

Now, list ${}^{C+1}2$ as $\rho_0, \ldots, \rho_{2^{C+1}-1}$, and let $\theta :\subseteq \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$ be defined by

$$\theta(\tau, \backslash i_0, i_1, j, k\backslash) = \left[f(\varphi_0(\tau, i_0), \varphi_1(\tau, i_1)) + (-1)^j 0.0^{|\tau|-C-1}{}^\frown\rho_k\right] \restriction |\tau|$$

if $i_0 < d_0, i_1 < d_1, j < 2, k < 2^{C+1}, |\tau| > C$ and undefined otherwise. θ is a c.p.f. since f is weakly computable, and for all $n > C$ there is $\backslash i_0, i_1, j, k\backslash < \backslash d_0, d_1, 2, 2^{C+1}\backslash$ such that

$$\begin{aligned} \theta(y \upharpoonright n, \backslash i_0, i_1, j, k\backslash) &= [f(\vec{x} \upharpoonright n) + \text{ correct fuzz}] \upharpoonright n \\ &\quad (\text{since } \vec{x} \leq_{\mathrm{rK}} y) \\ &= f(\vec{x}) \upharpoonright n \end{aligned}$$

So $f(\vec{x}) \leq_{\mathrm{rK}} y$ via a slightly altered constant that only depends on d_0, d_1, and C and a slightly altered c.p.f. θ' that deal with the (finitely many) exceptional $n \leq C$. □

Of course, this result is vacuous unless w.c.l.L. functions actually exist. They certainly do. To see this, let us dig up a helpful fact from real analysis: if f is differentiable on E (with E open), then f is locally Lipschitz on E. Since $+$, $-$, $\cdot$, $/$, and $\sqrt{}$ are differentiable and certainly weakly computable, they are examples of w.c.l.L. functions (restricting domains where necessary). Key examples, in fact, because with these and just a little more real analysis we can reach our goal.

3.1.6 Theorem. *$\langle \mathbb{R}_y, +, \cdot, < \rangle$ is a countable real closed field.*

Proof. First we show that $\mathbb{R}_y$ forms a countable ordered field. $\mathbb{R}_y$ is nonempty since it contains the computable reals. It is countable since rK-reducibility implies Turing reducibility and the Turing cone below a function is countable. $\mathbb{R}_y$ is certainly ordered by $<$, since it is a set of real numbers. Also, given $a, b \in \mathbb{R}_y$, $a - b$ and a/b (for $b \neq 0$) are both in $\mathbb{R}_y$ by Lemma 3.1.5, since, as mentioned previously, subtraction and division are w.c.l.L. functions.

Lastly, we show that the field is real closed, that is, every positive real number in $\mathbb{R}_y$ has a square root in $\mathbb{R}_y$, and every odd degree polynomial with coefficients in $\mathbb{R}_y$ has a root in $\mathbb{R}_y$.

A positive real less random then y has a square root less random than y by Lemma 3.1.5 since $\sqrt{}$ (away from 0) is w.c.l.L.

Odd-degree polynomial roots present a little more difficulty. Let $f(x) = c_0 + c_1 x + \cdots + c_m x^m \in \mathbb{R}_y[x]$ be of odd degree. Then f has a root $r \in \mathbb{R}$ and there exists an open interval with rational endpoints (a, b) on which f changes sign and has no other roots. We show $r \in \mathbb{R}_y$.

First off, we may assume without loss of generality that r is a root of multiplicity 1. To see this, note that if r has multiplicity $k > 1$, then k must be odd, because $f(x) = (x-r)^k g(x)$ for some polynomial $g(x)$ which does not change sign on (a, b). (If $g(x)$ changed sign on (a, b), then, by the Intermediate Value Theorem, g and hence f would have a root different from r on (a, b), a contradiction). Thus $f^{(k-1)}$, the $(k-1)$st derivative of f, is an odd-degree polynomial with coefficients in $\mathbb{R}_y$ having r as a root of multiplicity 1, and so we can work with $f^{(k-1)}$ instead of f.

Now, to do the heavy lifting we bring in some more real analysis. Let $O \subseteq \mathbb{R}^{m+1}$ be an open ball containing $\vec{c} = \langle c_0, \ldots, c_m \rangle$ and let $F : (a, b) \times O \to \mathbb{R}$ be the polynomial defined by $F(x, \vec{v}) = w_0 + w_1 x + \cdots + w_m x^m$. Then F is continuously differentiable on $(a, b) \times O$, $F(r, \vec{c}) = 0$, and $\partial F / \partial x(r, \vec{c}) = f'(r) \neq 0$ (since r has multiplicity 1). Thus, by the Implicit Function Theorem and its proof (see [**Rud76**] by Rudin for instance), there are open balls U and V such that

(i) $r \in U \subseteq (a, b)$, $\vec{c} \in V \subseteq O$ (and U has rational endpoints)

(ii) for all $\vec{v} \in V$, $F(x, \vec{v})$ is 1-1 on U

(iii) there is a unique continuously differentiable $G : V \to U$ such that
$\forall \vec{v} \in V \,.\, F(G(\vec{v}), \vec{v}) = 0.$

With this we show that G is w.c.l.L. and conclude that $r = G(\vec{c}) \in \mathbb{R}_y$ (by Lemma 3.1.5 since $\vec{c} \in (\mathbb{R}_y)^{m+1}$). Since G is differentiable (by iii), G is locally Lipschitz. So we just need to show that G is weakly computable. For any $\vec{q} \in V \cap \mathbb{Q}^{m+1}$, $F(x, \vec{q})$ is 1-1 on U (by ii) and has exactly one root in U (by iii). So, by applying the standard binary search algorithm on $U \subseteq (a, b)$ (which is independent of $\vec{q}$; see [**PER89**] by Pour-El and Richards), $F(x, \vec{q})$ has a computable real root. (Note that for any rational d, $F(d, \vec{q})$ is rational, hence it can be decided whether $F(d, \vec{q}) = 0$, $F(d, \vec{q}) < 0$, or $F(d, \vec{q}) > 0$.) Since $G(\vec{q})$ is that real (by iii), it follows that G is weakly computable. □

3.2. The Reals Less Random Than Ω

We now narrow our view and look more closely at one particular randomness class, the class of reals less random than the halting probability Ω. In [**DHL04**] Downey, Hirschfeldt, and LaForte showed that, in analogy to every c.e. set being T-reducible to the halting set, every c.e. real is rK-reducible to Ω; in symbols, $\mathbb{R}_{\text{c.e.}} \subseteq \mathbb{R}_\Omega$. In fact, even more is true.

3.2.1 Proposition. $\mathbb{R}_{\text{d.c.e.}} \subseteq \mathbb{R}_\Omega \subseteq \mathbb{R}_{\text{c.a.}}$.

Proof. Since $\mathbb{R}_{\text{c.e.}} \subseteq \mathbb{R}_\Omega$ and $\mathbb{R}_\Omega$ is closed under subtraction (by Lemma 3.1.5), $\mathbb{R}_{\text{d.c.e.}} \subseteq \mathbb{R}_\Omega$. Also, if $x \in \mathbb{R}_\Omega$, then $x \leq_{\text{rK}} \Omega$, implying that $x \leq_{\text{T}} \Omega \equiv_{\text{T}} K$. Therefore the fractional part of x is the characteristic function of a Δ^0_2 set, so that $x \in \mathbb{R}_{\text{c.a.}}$. Thus $\mathbb{R}_\Omega \subseteq \mathbb{R}_{\text{c.a.}}$. □

The last implication in the proof above follows from a result essentially due to Ho [**Ho99**]:

3.2.2 Lemma. *$x \in \mathbb{R}_{\text{c.a.}}$ iff x is $\emptyset'$-computable (there is a $\emptyset'$-computable sequence of rationals converging effectively to x) iff the fractional part of x is the characteristic function of a Δ^0_2 set.*

Moreover, using the same technique from the previous section, we get the following two theorems.

3.2.3 Theorem. *$\langle \mathbb{R}_{\text{d.c.e.}}, +, \cdot, < \rangle$ is a countable real closed field.*

3.2.4 Theorem. *$\langle \mathbb{R}_{\text{c.a.}}, +, \cdot, < \rangle$ is a countable real closed field.*

$\mathbb{R}_{\text{d.c.e.}}$ and $\mathbb{R}_{\text{c.a.}}$ are clearly countable since there are only countably many computable sequences of rationals. They are also real closed fields via the same proof used in Theorem 3.1.6, because they are closed under w.c.l.L. functions. This closure follows from the lemmas below.

3.2.5 Lemma ([ASWZ00]). *$x \in \mathbb{R}_{\text{d.c.e.}}$ iff there is a computable sequence of rationals $\langle q_i : i \in \mathbb{N} \rangle$ converging to x such that $\sum_{i \in \mathbb{N}} |q_{i+1} - q_i| < \infty$.*

Recall that a sequence of reals $\langle x_i : i \in \mathbb{N}\rangle$ is **computable** iff there is a double computable sequence of rationals $\langle q_{ij}\rangle_{i,j\in\mathbb{N}}$ and a computable function $e : \mathbb{N}^2 \to \mathbb{N}$ such that for all i, n

$$j \geq e(i,n) \to |q_{ij} - x_i| \leq 2^{-n}.$$

3.2.6 Lemma ([ASWZ00]). *If a computable sequence of reals $\langle x_i : i \in \mathbb{N}\rangle$ converges to x such that $\sum_{i\in\mathbb{N}} |x_{i+1} - x_i| < \infty$, then $x \in \mathbb{R}_{\text{d.c.e.}}$.*

3.2.7 Lemma. *Let $s \in \mathbb{N}^+$. If $\vec{x} \in (\mathbb{R}_{\text{d.c.e.}})^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\vec{x} \in E$, then $f(\vec{x}) \in \mathbb{R}_{\text{d.c.e.}}$.*

Proof. Let $\vec{x}$ and f be as above. By Lemma 3.2.5 there is a computable sequence of vectors $\langle \vec{q}_i : i \in \mathbb{N}\rangle$ from $\mathbb{Q}^s$ that converges to $\vec{x}$ such that $\sum_{i\in\mathbb{N}} |\vec{q}_{i+1} - \vec{q}_i| < \infty$. Since f is locally Lipschitz, there is an open neighborhood E_0 of $\vec{x}$ on which f is Lipschitz, that is

$$\exists M \in \mathbb{R}^+ \, \forall \vec{u}, \vec{v} \in E_0 \, . \, |f(\vec{u}) - f(\vec{v})| \leq M|\vec{u} - \vec{v}| \qquad (\star).$$

Without loss of generality, assume that $\langle \vec{q}_i : i \in \mathbb{N}\rangle \subseteq E_0$. Since f is weakly computable, $\forall i \; \forall n \; . \; |\hat{f}(\vec{q}_i, n) - f(\vec{q}_i)| \leq 2^{-n}$, so that $\langle f(\vec{q}_i)\rangle_{i\in\mathbb{N}}$ is a computable sequence of reals. Also, $\lim_{i\to\infty} f(\vec{q}_i) = f(\lim_{i\to\infty} \vec{q}_i) = f(\vec{x})$ (since locally Lipschitz functions are continuous). Lastly, by $(\star)$,

$$\sum |f(\vec{q}_{i+1}) - f(\vec{q}_i)| \leq M \sum |\vec{q}_{i+1} - \vec{q}_i| < \infty.$$

So $f(\vec{x}) \in \mathbb{R}_{\text{d.c.e.}}$ by Lemma 3.2.6. □

That $\mathbb{R}_{\text{d.c.e.}}$ forms a real closed field was also proved nearly simultaneously and independently by Ng [**Ng**].

3.2.8 Lemma ([ZW01]). *If a computable sequence of reals $\langle x_i : i \in \mathbb{N}\rangle$ converges to x, then $x \in \mathbb{R}_{\text{c.a.}}$.*

3.2.9 Lemma. *Let $s \in \mathbb{N}^+$. If $\vec{x} \in (\mathbb{R}_{\text{c.a.}})^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\vec{x} \in E$, then $f(\vec{x}) \in \mathbb{R}_{\text{c.a.}}$.*

Proof. This follows from a simplified version of the proof of Lemma 3.2.7 and from Lemma 3.2.8. Of course, this also follows by relativizing the argument for the real closure of $\mathbb{R}_c$ using Lemma 3.2.2, but the first approach illustrates the power of w.c.l.L. functions. □

3.3. Proper Containment

So $\mathbb{R}_{\text{d.c.e.}} \subseteq \mathbb{R}_\Omega \subseteq \mathbb{R}_{\text{c.a.}}$, and all three classes form countable real closed fields. Is $\mathbb{R}_\Omega$ equal to either $\mathbb{R}_{\text{d.c.e.}}$ or $\mathbb{R}_{\text{c.a.}}$? Notice that both can not be true since $\mathbb{R}_{\text{d.c.e.}} \subset \mathbb{R}_{\text{c.a.}}$. An affirmative answer would yield intriguing alternate characterizations of both classes involved. However, this is not the case.

3.3.1 Theorem. $\mathbb{R}_{\text{d.c.e.}} \neq \mathbb{R}_\Omega$.

3.3.2 Theorem. $\mathbb{R}_\Omega \neq \mathbb{R}_{\text{c.a.}}$.

Proof of 3.3.1. We need to construct $\alpha \in {}^{\mathbb{N}}2$ such that $\alpha \leq_{\text{rK}} \Omega$ and $0.\alpha$ is not a d.c.e. real. Instead of making $\alpha \leq_{\text{rK}} \Omega$ directly, we construct a c.e. real $0.\beta$ such that $\alpha \leq_{\text{rK}} \beta$; here we use the fact that all c.e. reals are rK-reducible to Ω. The construction is a $\emptyset'$-priority argument, where we meet, for all c.p.f. $x :\subseteq \mathbb{N} \to \mathbb{Q}$ (possible computable sequences of rationals), the following requirements.

Requirements.

$$\mathcal{R}_x : \qquad (\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq 1 \to 0.\alpha \neq \lim_{s\to\infty} x_s) \wedge \exists\theta \,.\, \alpha = [\theta, 2]^\beta.$$

These requirements are sufficient since, by a slight modification of Lemma 3.2.5, every d.c.e. real x has a computable sequence of rationals $\langle x_s : s \in \mathbb{N}\rangle$ converging to it such that $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq 1$.

Plan for $\mathcal{R}_x$. To ensure $0.\alpha \neq \lim_{s\to\infty} x_s$, we flip a big bit of α exponentially often so that $0.\alpha$ becomes a super jumping bean. Eventually x_s will tire and fail to keep up, for x_s, being restricted by the condition $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq 1$, can make at most 2^k jumps of distance at least 2^{-k} (for any k).

More formally, a worker for this requirement proceeds as follows.

(1) Pick a big number ('bigbit') n. In particular, n should be bigger than $n_r + \sum_{i \leq n} 2^{2n_i}$, where $n_0, \ldots, n_r$ are all the bigbits mentioned so far in the construction. Extend α and β (which were formerly of length n_r) to length n by padding them with zeros. We call $(n_r, n]$ 'n's gap'. Also,

$$\begin{aligned}\theta(\beta \upharpoonright (n+1), 0) &:= \alpha \upharpoonright n ^\frown \langle 0 \rangle \\ \theta(\beta \upharpoonright (n+1), 1) &:= \alpha \upharpoonright n ^\frown \langle 1 \rangle \\ \theta(\beta \upharpoonright w, 0) &:= \alpha \upharpoonright w \qquad \text{for all } w \in (n_r, n).\end{aligned}$$

(2) Wait for

$$\sum_{s=0}^{t} |x_s - x_{s+1}| \leq 1 \quad \text{and} \quad |0.\alpha - x| < \epsilon := 2^{-n-3},$$

(with the convention that all the terms of the sum must be defined) where t is the current stage of the construction. While waiting, each time $\alpha \upharpoonright (n+1)$ changes below n_r (because of higher priority workers), add 2^{-n-1} to $0.\beta$, that is, increment n's gap in β by the minimum amount for each change, and redefine θ for bits $[n_r, n+1)$ just as in (1). Notice that changes in $\alpha \upharpoonright (n+1)$ above n_r require no redefining of θ, because the $\langle 0 \rangle$ and $\langle 1 \rangle$ cover these.

(3) $\alpha(n) := 1 - \alpha(n)$.

(4) Go back to (2).

Outcomes for $\mathcal{R}_x$. As we show in the verification, there is only one final outcome, namely waiting at (2) forever; there is no infinite cycling through the plan's loop. In this case, $0.\alpha \neq \lim_{s \to \infty} x_s$.

Construction. We do a simple (relatively speaking) tree construction (see [**Lem**] for instance). The requirements are ordered effectively with order type ω, and requirement R is assigned to a worker sitting on level/node R of a unary branching tree (which grows down, say).

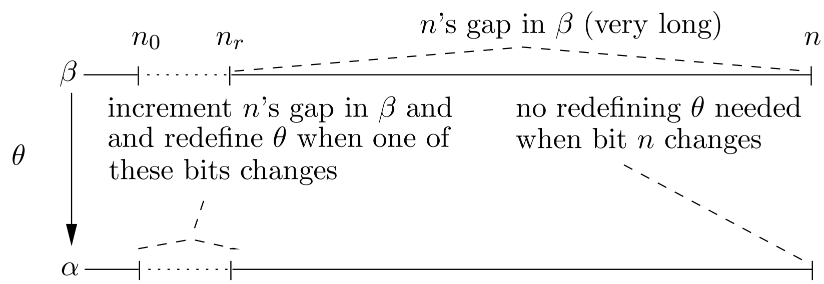

FIGURE 1. How bits change.

At each stage $t \in \mathbb{N}^+$ of the construction, workers $s < t$ act in order down the tree carrying out their plans from where they last left off at the previous stage up until time t, with the last/new worker at level $t - 1$ beginning at step (1). Here time is measured by the number of stages in the simulation of the Turing machines involved. At each stage each worker has only one current outcome, namely waiting at step (2) (steps (3) and (4) do not count as using up time).

In this construction the workers do not really interfere with each other; there is no firing or rehiring of workers depending on different current outcomes. When higher priority workers (closer to the top/root of the tree) change α or β, lower priority workers (farther from the top/root of the tree) deal with the behavior easily by incrementing their gaps in β according to step (2).

Verification. Each worker on the final (only) path satisfies its $\mathcal{R}_x$ requirement.

To see this, fix a worker with plan $\mathcal{R}_x$ and bigbit n. When the worker goes from (2) to (3), bit n of α flips so that (the current approximation of) $0.\alpha$ changes/jumps by 2^{-n-1}. For the worker to reach (3) again (the current approximation of) x must jump by more than $2^{-n-2} = 2^{-n-1} - 2\epsilon$ to get back inside $0.\alpha$'s ϵ-ball. Actually, x might not jump by that much, because once x is outside of $0.\alpha$'s ϵ-ball, $0.\alpha$ might move toward x due to α's other bigbit flips. However, considering these flips, we get that x jumps by more than 2^{-n-3} (bigbit flips by bits smaller than n, move $0.\alpha$ tremendously so that x will certainly have to jump by more than 2^{-n-2}; bigbit flips by bits bigger than n move $0.\alpha$ by less than 2^{-n-3} (a bound obtained from a simple geometric series calculation; remember that bigbits are chosen extremely far apart from each other) so that x jumps by more than $2^{-n-3} = 2^{-n-2} - 2^{-n-3}$).

Now, to maintain the condition $\sum |x_s - x_{s+1}| \leq 1$, x can jump by $\geq 2^{-n-3}$ only $\leq 2^{n+3}$ times. Thus after 2^{n+3} passes from step (2) to (3) in the plan's loop, the worker must wait forever at (2).

Also, $\alpha = [\theta, 2]^{\beta}$ by construction.

Lastly, $0.\beta$ is a well-defined c.e. real. Each n_i flips $\leq 2^{n_i+3}$ times and each flip increments n's gap in β by 2^{-n}, but the gap is at least $\sum_{i \leq n} 2^{2n_i}$ long and is therefore big enough to absorb these additions without spilling carry bits into other gaps. □

Proof of 3.3.2. We construct $\alpha \in {}^{\mathbb{N}}2$ as the characteristic function of a Δ^0_2 set such that $\alpha \not\leq_{\text{rK}} \Omega$ via simple diagonalization. By Lemma 3.2.2, $0.\alpha$ will be a c.a. real.

Let $\#$ be a computable bijection from the set of all triples $\langle \varphi, i, c \rangle$, where φ is a c.p.f. from ${}^{<\mathbb{N}}2 \times \mathbb{N}$ to ${}^{<\mathbb{N}}2$ and $i < c$ are natural numbers. Also, let l be the function defined by $l(\varphi, c) = \max\{\#(\varphi, i, c) : i < c\} + 1$.

Now, using oracle $\emptyset'$ we define α as follows. If $\varphi(\Omega \upharpoonright l(\varphi, c), i)\downarrow$ and is of length $l(\varphi, c)$, then let

$$\alpha(\#(\varphi, i, c)) = 1 - \varphi(\Omega \upharpoonright l(\varphi, c), i)(\#(\varphi, i, c)).$$

Otherwise, let $\alpha(\#(\varphi, i, c)) = 0$.

For all pairs $\langle \varphi, c \rangle$, $\alpha \neq [\varphi, c]^{\Omega}$, because for all $i < c$, $\alpha \upharpoonright l(\varphi, c) \neq \varphi(\Omega \upharpoonright l(\varphi, c), i)$, as witnessed by bit $\#(\varphi, i, c)$. Thus $\alpha \not\leq_{\text{rK}} \Omega$. □

Let us end with one last question. We now know that $\mathbb{R}_{\text{d.c.e.}} \subset \mathbb{R}_{\Omega} \subset \mathbb{R}_{\text{c.a.}}$. Is $\mathbb{R}_{\text{d.c.e.}}$ or $\mathbb{R}_{\text{c.a.}}$ a randomness class? That is, does $\mathbb{R}_{\text{d.c.e.}}$ or $\mathbb{R}_{\text{c.a.}}$ equal $\mathbb{R}_y$ for *any* real number y?

By the proper inclusion of Theorem 3.3.1 and the technique in the proof of Theorem 3.3.2, it follows that, here again, the answer is negative.

3.3.3 Theorem. *For all $y \in \mathbb{R}$, $\mathbb{R}_{\text{d.c.e.}} \neq \mathbb{R}_y$.*

3.3.4 Theorem. *For all $y \in \mathbb{R}$, $\mathbb{R}_{\text{c.a.}} \neq \mathbb{R}_y$.*

Proof of 3.3.3. Assume (toward a contradiction) that for some $y \in \mathbb{R}$, $\mathbb{R}_{\text{d.c.e.}} = \mathbb{R}_y$. Since $\Omega \in \mathbb{R}_{\text{d.c.e.}} = \mathbb{R}_y \subseteq \mathbb{R}_\Omega$, $\Omega \leq_{\text{rK}} y \leq_{\text{rK}} \Omega$, so that $y \equiv_{\text{rK}} \Omega$. Thus $\mathbb{R}_{\text{d.c.e.}} = \mathbb{R}_y = \mathbb{R}_\Omega$, a contradiction. □

Proof of 3.3.4. Assume (toward a contradiction) that for some $y \in \mathbb{R}$, $\mathbb{R}_{\text{c.a.}} = \mathbb{R}_y$. Thus every c.a. real is $\leq_{\text{rK}} y$. But carrying out the same construction as in the proof of Theorem 3.3.2 with y in place of Ω —note that in the proof no special properties of Ω, besides it being $\leq_{\text{T}} \emptyset'$, were used— yields a c.a. real $\not\leq_{\text{rK}} y$, a contradiction. □

3.4. Alternative Proofs

When i submitted this chapter's work for publication the anonymous referee suggested some interesting alternative proofs of the results of section 3.3. Based directly on prefix-free conditional complexity, they offer a different and valuable perspective.

Alternative proof of Theorem 3.3.1. Let Ω^{shift} be the shift of powers-of-two-position bits of Ω (as a binary sequence), that is, for $n \in \mathbb{N}$

$$\Omega^{\text{shift}}(n) := \begin{cases} \Omega(2n) & \text{if } n = 2^m \text{ for some } m \in \mathbb{N} \\ \Omega(n) & \text{otherwise.} \end{cases}$$

Notice that $\Omega^{\text{shift}} \in \mathbb{R}_\Omega$ since at each length n, a program using $\Omega \restriction n$ needs to guess only one bit to compute $\Omega^{\text{shift}} \restriction n$. However, $\Omega^{\text{shift}} \notin \mathbb{R}_{\text{d.c.e.}}$, so that $\mathbb{R}_{\text{d.c.e.}} \neq \mathbb{R}_\Omega$.

To see this, assume (toward a contradiction) that $\Omega^{\text{shift}} \in \mathbb{R}_{\text{d.c.e.}}$. From this assumption we will build a computable set C and a c.p.f. φ such that for all $m \in C$

$$\varphi(\Omega \restriction m) = \Omega(m).$$

This contradicts the fact that Ω is random (in the sense of c.e. martingales).

Since $\Omega^{\text{shift}} \in \mathbb{R}_{\text{d.c.e.}}$, there is a computable sequence of rationals $\langle \Omega^{\text{shift}}_s : s \in \mathbb{N} \rangle$ converging to Ω^{shift} such that $J := \sum_{s \in \mathbb{N}} |\Omega^{\text{shift}}_s - \Omega^{\text{shift}}_{s+1}| < \infty$. Let $\langle \Omega_s : s \in \mathbb{N} \rangle$ be an nondecreasing sequence of rationals converging to Ω and for each $t \in \mathbb{N}$, let $J_t = \sum_{s=0}^{t-1} |\Omega_s^{\text{shift}} - \Omega_{s+1}^{\text{shift}}|$ with $J_0 = 0$. Notice that $\langle J_t : t \in \mathbb{N}\rangle$ is a computable sequence of rationals converging nondecreasingly to J. Now, define two sequences of natural numbers $\langle s_n : n \in \mathbb{N}\rangle$ and $\langle t_n : n \in \mathbb{N}\rangle$ as follows.

$$s_n := \text{the least } s \text{ such that } \Omega_s \restriction 2^{n+1} = \Omega \restriction 2^{n+1} \text{ and}$$

$$\Omega_s^{\text{shift}} \restriction I = \Omega^{\text{shift}} \restriction I, \text{ where } I = [0, 2^{n+1}) \setminus \{2^n\};$$

$$t_n := \text{the least } t \text{ such that } J_t \restriction (2^n + 2) = J \restriction (2^n + 2).$$

Lastly, let $A = \{n \in \mathbb{N} : s_n \leq t_n\}$ and $B = \{n \in \mathbb{N} : n \geq 1 \wedge s_n > t_n\}$.

Notice that if $s_n \leq t_n$, then $\Omega \restriction 2^{n+1}$ can be computed from $J \restriction (2^n+2)$, for given $J \restriction (2^n+2)$ we can compute the least t $(= t_n)$ such that $J_t \restriction (2^n + 2) = J \restriction (2^n + 2)$. Then, since $s_n \leq t_n$, $\Omega_t \restriction 2^{n+1} = \Omega \restriction 2^{n+1}$. Thus, there is a constant c_0 such that for all $n \in A$ $K(\Omega \restriction 2^{n+1} | J \restriction (2^n + 2)) \leq c_0$, implying that there are constants c_1, c_2, c_3 such that for all $n \in A$

$$\begin{aligned} K(\Omega \restriction 2^{n+1}) &\leq K(J \restriction (2^n + 2)) + c_1 \\ &\leq 2^n + 2 + 2\lg(2^n + 2) + c_2 \\ &\leq 2^n + 2n + c_3. \end{aligned}$$

Since Ω is random, this can happen for only finitely many n. So A is finite. Thus B is cofinite, hence computable.

Let C be the computable set $\{2^{n+1} : n \in B\}$ and let φ be the c.p.f. defined by

$$\varphi(\sigma) := \Omega_s^{\text{shift}}(2^n)$$

if s is the least number such that $\Omega_s \restriction 2^{n+1} = \sigma$ and $\Omega_s^{\text{shift}} \restriction I = \sigma^{\text{shift}} \restriction I$ (where the shift operation and I are as above), and let φ be undefined if there is no such s. Then for $2^{n+1} \in C$

$$\varphi(\Omega \restriction 2^{n+1}) = \Omega_s^{\text{shift}}(2^n) = \Omega_{s_n}^{\text{shift}}(2^n).$$

by definition of φ and s_n.

If $\Omega^{\text{shift}}_{s_n}(2^n) \neq \Omega^{\text{shift}}(2^n)$, then there is an $m > s_n$ such that $\Omega^{\text{shift}}_m \restriction 2^{n+1} = \Omega^{\text{shift}} \restriction 2^{n+1}$. Since $\Omega^{\text{shift}}_{s_n} \restriction 2^{n+1}$ and $\Omega^{\text{shift}}_m \restriction 2^{n+1} = \Omega^{\text{shift}} \restriction 2^{n+1}$ differ only on bit 2^n and $n \geq 1$, we have

$$J_m - J_{s_n} \geq |\Omega^{\text{shift}}_m - \Omega^{\text{shift}}_{s_n}| > 2^{-2^n-2}$$

so that $J_m \restriction (2^n+2) \neq J_{s_n} \restriction (2^n+2)$. This is a contradiction, since $J_{s_n} \restriction (2^n+2) = J_m \restriction (2^n+2) = J \restriction (2^n+2)$ since $t_n < s_n$ for $n \in B$.

Thus for $2^{n+1} \in C$, $\varphi(\Omega \restriction 2^{n+1}) = \Omega^{\text{shift}}_{s_n}(2^n) = \Omega^{\text{shift}}(2^n) = \Omega(2^{n+1})$, that is, for all $m \in C$

$$\varphi(\Omega \restriction m) = \Omega(m),$$

a contradiction. □

Alternative proof of Theorem 3.3.2. Let Ω^{even} and Ω^{odd} be the the even-position bits of and the odd-position bits of Ω (as a binary sequence), respectively, that is, for $n \in \mathbb{N}$

$$\Omega^{\text{even}}(n) := \Omega(2n) \qquad \text{and}$$

$$\Omega^{\text{odd}}(n) := \Omega(2n+1).$$

Notice that $\Omega^{\text{even}}, \Omega^{\text{odd}} \in \mathbb{R}_{\text{c.a.}}$ since $\Omega \in \mathbb{R}_{\text{c.a.}}$. However, both Ω^{even} and Ω^{odd} can not be in $\mathbb{R}_\Omega$, so that $\mathbb{R}_\Omega \neq \mathbb{R}_{\text{c.a.}}$.

To see this, assume (toward a contradiction) that $\Omega^{\text{even}}, \Omega^{\text{odd}} \in \mathbb{R}_\Omega$. Then, there are constants c_0 and c_1 such that for all n, $K(\Omega^{\text{even}} \restriction n | \Omega \restriction n) \leq c_0$ and $K(\Omega^{\text{odd}} \restriction n | \Omega \restriction n) \leq c_1$. Thus there is a c_2 such that for all n, $K(\Omega \restriction 2n | \Omega \restriction n) \leq c_2$ (by combining the two underlying algorithms), implying that there are c_3, c_4 such that for all n

$$K(\Omega \restriction 2n) \leq K(\Omega \restriction n) + c_3 \leq n + 2 \lg n + c_4.$$

This is a contradiction, since Ω is random. □

Alternative proof of Theorem 3.3.4. Actually, we show that for all $y \in \mathbb{R}$, $\mathbb{R}_{\text{c.a.}} \not\subseteq \mathbb{R}_y$. Assume (toward a contradiction) that for some $y \in \mathbb{R}$, $\mathbb{R}_{\text{c.a.}} \subseteq \mathbb{R}_y$. Then $\Omega^{\text{even}}, \Omega^{\text{odd}} \in \mathbb{R}_{\text{c.a.}} \subseteq \mathbb{R}_y$. However, this implies (just like in the previous proof) that up to a uniform additive constant

$$K(\Omega \restriction 2n) \leq K(y \restriction n) \leq n + 2\lg n,$$

a contradiction. □

3.5. A Remark on the K-trivial Reals

The methods of the previous sections can also be used to show that the class of K-trivial reals $\mathbb{R}_{\text{Kt}} = \{x \in \mathbb{R} : \exists d\, \forall n\, .\, K(x \restriction n) < K(n) + d\}$, a class of considerable recent interest, forms a real closed field. Again, it suffices to show closure under w.c.l.L. functions.

3.5.1 Lemma. *Let $s \in \mathbb{N}^+$. If $\vec{x} \in (\mathbb{R}_{\text{Kt}})^s$, $f : E \subseteq \mathbb{R}^s \to \mathbb{R}$ is w.c.l.L, and $\vec{x} \in E$, then $f(\vec{x}) \in \mathbb{R}_{\text{Kt}}$.*

Proof. Let $\vec{x}$ and f be as above. By Lemma 3.1.5, $f(\vec{x}) \leq_{\text{rK}} \vec{x}$, so that up to a uniform additive constant,

$$K(f(x) \restriction n) \leq K(x \restriction n) \leq K(n).$$

So $f(\vec{x}) \in \mathbb{R}_{\text{Kt}}$. □

Moreover, since there are only countably many K-trivial reals (an unpublished result of Zambella; see [**DHNS03**]) we have

3.5.2 Theorem. *$\langle \mathbb{R}_{\text{Kt}}, +, \cdot, < \rangle$ is a countable real closed field.*

CHAPTER 4

Odds and Ends

4.1. Other Strong Reducibilities

While rK-reducibility is a strong reducibility, that is, a refinement of Turing reducibility, it is incomparable with the standard strong reducibilities: 1-reducibility, m-reducibility, tt-reducibility, wtt-reducibility (in decreasing order of strength).

4.1.1 Proposition. *1-reducibility does not imply rK-reducibility, and rK-reducibility does not imply wtt-reducibility.*

Proof. For the first non-implication fix a random sequence R. Then $R \leq_1 R \oplus R$ (and this is true for any sequence), but since $R \oplus R$ is not random, $R \not\leq_{\mathrm{rK}} R \oplus R$.

For the second non-implication we build Δ^0_2 sequences A and B and a computable function f such that $A = [f, 2]^B$, but for all wtt-reductions Φ, $A \neq \Phi^B$. To this end, let $\langle \Phi_0, \varphi_1 \rangle, \langle \Phi_0, \varphi_2 \rangle, \ldots$ be a computable enumeration of all pairs of c.p.f. (wtt-reductions with their uses). We build A and B by a $\emptyset'$-finite extension argument.

Fix a noncomputable sequence U. At stage $s+1$, suppose A and B have been defined up to some length $l \leq s$. Take the sth pair $\langle \Phi, \varphi \rangle$ and, using $\emptyset'$, choose $x \geq l$ least such that

$$\Phi^{B_s 0^{\varphi(x)}}(l) \neq [A_s \,\widehat{}\, U(x)](l) = U(x).$$

Such an x exists, for otherwise $\forall x \geq l \,.\, \Phi^{B_s 0^{\varphi(x)}}(l) = U(x)$, so that U is computable using $\langle \Phi, \varphi \rangle$ and the current finite string B_s, a contradiction. Let $m = \max\{1, \varphi(x)\}(= 1 \text{ if } \varphi(x)\!\uparrow)$ and let $B_{s+1} = B_s 0^m 1 \,\widehat{}\, U(x)$ and $A_{s+1} = A_s \,\widehat{}\, U(x) \,\widehat{}\, 0^m 1$. End of construction.

From the diagram below it is easy to see how to define a computable function f such that $A = [f, 2]^B$.

					$U(x_1)$						$U(x_2)$		
$B:$	0	...	0	1	#	0	...	0	1		$\star$		...
$A:$	#	0	....		0	$\star$	0	....			0		...

In the diagram $x_1, x_2, \ldots$ are x's from the construction, and the blocks of zeros are of length $\max\{1, \varphi(x_1)\}, \max\{1, \varphi(x_2)\}, \ldots$.

Also, by construction $A \not\leq_{\mathrm{wtt}} B$, for assume (toward a contradiction) that $A = \Phi^B$ with use φ. Then $\langle \Phi, \varphi \rangle$ is the sth pair in our enumeration for some s, and for l, x associated to s, we have that

$$\begin{aligned}
\Phi^{B\restriction\varphi(l)}(l) =& \Phi^{B\restriction\varphi(x)}(l) \\
& \text{(since } x \geq l \text{ and use functions are increasing)} \\
=& \Phi^{B\restriction[\varphi(x)+l]}(l) \\
=& \Phi^{B\restriction l^\frown 0^{\varphi(x)}}(l) \\
\neq& [A \restriction l^\frown U(x)](l) \\
& \text{(by construction)} \\
=& A(l) \\
& \text{(by construction),}
\end{aligned}$$

a contradiction. □

4.2. n-randomness

rK-reducibility preserves not only 1-randomness but randomness of all levels. The following theorem is mostly a rephrasing in the language of rK-reducibility of several results of Miller and Yu from [**MY**].

4.2.1 Theorem. *For $A, B, C \in {}^{\mathbb{N}}2$,* *(1) if $A \oplus C$ is random and $A \leq_{\text{rK}} B$, then $B \oplus C$ is random;*

(2) if A is n-random and $A \leq_{\text{rK}} B$, then B is n-random;

(3) if A is random, B is n-random, and $A \leq_{\text{rK}} B$, then A is n-random;

(4) if $A \oplus B$ is random, then A and B have no rK-upper bound;

(5) if $A \oplus B$ is random, then A, B, and $A \oplus B$ are pairwise rK-incomparable.

PROOF. (1) Suppose $A \oplus C$ is random and $A \leq_{\text{rK}} B$. Since $A \leq_{\text{rK}} B$, $A \oplus C \leq_{\text{rK}} B \oplus C$, so that (up to a uniform additive constant) for all n

$$K(B \oplus C \upharpoonright n) \geq K(A \oplus C \upharpoonright n) \geq n,$$

since $A \oplus C$ is random. Thus $B \oplus C$ is random.

(2) For this we need to recall several important theorems: Kučera's ([**Kuč85**]), which says that every Turing degree $\geq_{\text{T}} \emptyset'$ contains a random sequence; Kautz's ([**Kau91**]), which says that a sequence is $C^{(k)}$-n-random iff it is C-$(n+k)$-random; van Lambalgen's ([**vL90**]), which says that $C \oplus D$ is n-random iff C is n-random and D is C-n-random.

Suppose A is n-random and $A \leq_{\text{rK}} B$. In case $n = 1$, the result holds since $\leq_{\text{rK}}$ preserves randomness. In case $n > 1$, choose a random sequence C Turing equivalent to $\emptyset^{(n-1)}$ (by Kučera's Theorem). Since A is n-random, A is C-random (by Kautz's Theorem), so $A \oplus C$ is random (by van Lambalgen's Theorem), so $B \oplus C$ is random (by (1) since $A \leq_{\text{rK}} B$), so B is C-random (by van Lambalgen's Theorem), so B is n-random (by Kautz's Theorem).

(3) Suppose A is random, B is n-random, and $A \leq_{\text{rK}} B$. In case $n = 1$, the result is true by hypothesis. In case $n > 1$, choose a random sequence C Turing equivalent to $\emptyset^{(n-1)}$ (by Kučera's Theorem). Since B is n-random, B is C-random (by Kautz's Theorem), so $B \oplus C$ is random (by van Lambalgen's Theorem), so C is B-random (by van Lambalgen's Theorem), so C is A-random (since $A \leq_{\text{rK}} B$ and so $A \leq_{\text{T}} B$), so A is C-random (by van Lambalgen's Theorem), so A is n-random (by Kautz's Theorem).

(4) Suppose $A \oplus B$ is random. Assume (towards a contradiction) that $A, B \leq_{\mathrm{rK}} C$ for some sequence C. Since $A \oplus B$ is random and $A \leq_{\mathrm{rK}} C$, $C \oplus B$ is random (by (1)). Since $C \oplus B$ is random and $B \leq_{\mathrm{rK}} C$, $C \oplus C$ is random (by (1)). This, of course, is a contradiction.

(5) Suppose $A \oplus B$ is random. Then A and B are rK-incomparable by (4). Assume (towards a contradiction) that $A \leq_{\mathrm{rK}} A \oplus B$. Since $A \oplus B$ is random and $A \leq_{\mathrm{rK}} A \oplus B$, $A \oplus B \oplus B$ is random (by (1)), a contradiction. Assume (towards a contradiction) that $A \oplus B \leq_{\mathrm{rK}} A$. Since $A \oplus B$ is random, B is A-random (by van Lambalgen's Theorem), so $B \not\leq_{\mathrm{T}} A$. But $A \oplus B \leq_{\mathrm{rK}} A$, so $B \leq_{\mathrm{T}} A \oplus B \leq_{\mathrm{T}} A$, a contradiction. Thus A and $A \oplus B$ are rK-incomparable. By symmetry, B and $A \oplus B$ are also rK-incomparable. □

4.3. Weaker Notions of Randomness

If we restrict ourselves to weaker notions of absolute randomness, we can answer Question ② in the affirmative.

4.3.1 Definition. *A sequence R is* **computably random** *if there is no computable martingale that wins on R, that is, if there is no function $f : {}^{<\mathbb{N}}2 \to \mathbb{Q}^+$ with $f(\sigma 0) + f(\sigma 1) = 2f(\sigma)$ (for all σ) such that $\limsup_n f(R \upharpoonright n) = \infty$. A sequence R is* **Schnorr random** *if there is no computable martingale $f : {}^{<\mathbb{N}}2 \to \mathbb{Q}^+$ and no increasing unbounded computable function $w : \mathbb{N} \to \mathbb{N}$ such that $\exists^\infty n \,.\, f(R \upharpoonright n) \geq w(n)$.*

Random strictly implies (no reverse implication) computably random (see [**Sch71b**]) strictly implies Schnorr random (see [**Wan96**]).

4.3.2 Proposition. *There are computably random (hence Schnorr random) sequences R and S such that $R <_{\mathrm{rK}} S$.*

Proof. By [**NST05**, Theorem 4.2], there is a left-c.e. sequence R (that is, $0.R$ is a c.e. real) that is computably random (but not random!) and not Turing equivalent to $\emptyset' \equiv_{\mathrm{T}} \Omega$. Since R is left-c.e., $R \leq_{\mathrm{rK}} \Omega$, and since R and Ω have different Turing degree, $R \not\equiv_{\mathrm{rK}} \Omega$. Since Ω is random, it is also computably random. Also, since computably random implies Schnorr random, the second part of out theorem also holds. □

The previous proposition is not very exciting, however, because rK-reducibility preserves neither computable randomness nor Schnorr randomness.

4.3.3 Proposition. *There are sequences R and A such that $R \leq_{\mathrm{rK}} A$, R is computably random (hence Schnorr random), and A is not Schnorr random (hence not computably random).*

Proof. Let R be a computably random (hence Schnorr random) sequence and S an infinite computable set such that for all $s \in S$, $R(s)$ can be computed from $R \restriction s$. Such a sequence and set are constructed in [**MM04**, Theorem 4.1] and [**NST05**, Theorem 4.2] for instance. Let A be the sequence that equals R off S and the zero sequence on S. Then $R \leq_{\mathrm{rK}} A$ via a norm-1 reduction, since on bits s of S, $R(s)$ can be computed from $R \restriction s$. In fact, $R \equiv_{\mathrm{rK}} A$. Also, since A contains an infinite computable sequence of zeros, A is not Schnorr random (hence not computably random). □

4.4. The D.C.E. Reals

As we saw in Theorem 4.2.1 there is no least upper bound on the rK-degrees (because there is no upper bound). But, as Downey, Hirschfeldt, and LaForte showed in [**DHL04**], there is one on the rK-degrees of the c.e. reals. (In fact, the rK-degree of the c.e. reals form an upper semi-lattice with least upper bound induced by addition.) Does a least upper bound exist if we widen the degree structure to the next natural class, the rK-degrees of the d.c.e. reals? The answer is no.

4.4.1 Definition. *The* **mind change functions** *for a sequence of binary strings $\langle \sigma_s : s \in \mathbb{N} \rangle$ are defined by*

$$m(\sigma, n, s, t) := |\{u \in [s, t) : \sigma_u(n) \neq \sigma_{u+1}(n)\}|$$

$$m(\sigma, n) := m(\sigma, n, 0, \infty),$$

where $n, s, t \in \mathbb{N}$ and with the convention that before comparing entries of strings of different lengths, we pad the shorter one with enough zeros to bring it up to the length of the longer one.

We will need an easy corollary of Lemma 3.2.5.

4.4.2 Lemma. *Let $x \in \mathbb{R}$. If there is a computable sequence of strings $\langle \alpha_s : s \in \mathbb{N} \rangle$ converging to $\widetilde{x}$ such that $\sum_{n\in\mathbb{N}} 2^{-n} m(\alpha, n) < \infty$, then $x \in \mathbb{R}_{\text{d.c.e.}}$.*

Proof. Fix $x \in \mathbb{R}$, and suppose there is a computable sequence of strings $\langle \alpha_s : s \in \mathbb{N} \rangle$ converging to $\widetilde{x}$ such that $\sum_{n\in\mathbb{N}} 2^{-n} m(\alpha, n) < \infty$. Since $\lim_{s\to\infty} \sigma.\alpha_s = x$, where σ is the binary string such that $\sigma.\widetilde{x} = x$, and for all $s \in \mathbb{N}$

$$|\sigma.\alpha_s - \sigma.\alpha_{s+1}| \leq 2^{-n+1},$$

where n is the first bit on which α_s and α_{s+1} disagree, it follows that

$$\sum_{s\in\mathbb{N}} |\sigma.\alpha_s - \sigma.\alpha_{s+1}| \leq \sum_{n\in\mathbb{N}} 2^{-n+1} m(\alpha, n) < \infty.$$

Thus, by Lemma 3.2.5, x is a d.c.e. real. □

4.4.3 Theorem. *There is no least upper bound on the rK-degrees of $\mathbb{R}_{\text{d.c.e.}}$.*

Proof. We construct d.c.e. reals x_0 and x_1 and, for each d.c.e. real y, a d.c.e. real z such that

$$x_0, x_1 \leq_{\text{rK}} y \rightarrow (x_0, x_1 \leq_{\text{rK}} z \wedge y \not\leq_{\text{rK}} z).$$

(Actually, x_1 and z will be c.e. reals.) To do this, we meet, for all triples of a d.c.e. real, c.p.f. (lower-case Greek letters) and natural (upper-case Roman letters) $\langle y, \nu, J \rangle$, the following

Requirements.

$$\mathcal{R}_{y,\nu,J} : x_0, x_1 = [\nu, J]^y \rightarrow \exists z, \theta \; [x_0, x_1 = [\theta, 2]^z] \, .$$

For each $\mathcal{R}_{y,\nu,J}$ we also meet the following list of subrequirements ranging over pairs of a c.p.f. and natural number $\langle \xi, K \rangle$.

$$\mathcal{S}_{y,\nu,J,\xi,K} : x_0, x_1 = [\nu, J]^y \to (x_0, x_1 = [\theta, 2]^z \wedge y \neq [\xi, K]^z).$$

We use subrequirements here to emphasize that z must be built to thwart *all* possible rK-reductions to y, that is, all ξ and K.

Plan for $\mathcal{S}_{y,\nu,J,\xi,K}$. The basic idea is to pick a big number n and if it looks like $y = [\xi, K]^z$ up to n, which is bad, then we force $y \upharpoonright n$ to change to something completely new while keeping $z \upharpoonright n$ fixed. Repeating this more than K times will meet the requirement. Of course, since $y \upharpoonright n$ is not under our control, we force its changes indirectly by changing x_0 and x_1 past bit n. If $x_0, x_1 = [\nu, J]^y$ is to happen, then y must respond. Lastly, by waiting long enough between changes in x_0 and x_1, we make $x_0, x_1 = [\theta, 2]^z$.

More precisely, a worker for this requirement follows the algorithm below. The algorithm uses the sequence Pairs(b), defined as follows. For each $b \in \mathbb{N}$, let Pairs(b) be any sequence of all pairs of binary strings $\langle \sigma, \tau \rangle$ of length b (that is, $\sigma, \tau \in {}^b 2$) ordered lexicographically on the first coordinate. Note that Pairs(0) is the one-element sequence consisting of just $\langle \emptyset, \emptyset \rangle$.

Outcomes for $\mathcal{S}_{y,\nu,J,\xi,K}$. As we show in the verification, there are only two possible final outcomes, either of which will meet the requirements.

- w1: Wait at line 6 forever. In this case, $x_0 \neq [\nu, J]^y$ or $x_1 \neq [\nu, J]^y$, and requirement $\mathcal{R}_{y,\nu,J}$ is met.
- w2: Wait at line 9 forever. In this case, $y \neq [\xi, K]^z$ and $x_0, x_1 = [\theta, 2]^z$, and subrequirement $\mathcal{S}_{y,\nu,J,\xi,K}$ is met.

Construction. Do the standard tree construction (see [**Lem**]).

Verification. Each worker on the final path satisfies its $\mathcal{S}_{y,\nu,J,\xi,K}$ requirement.

To see this, fix a worker for a requirement $\mathcal{S}_{y,\nu,J,\xi,K}$. At the start, the worker picks a bigbit n bigger than $c + 4c \sum_{i=1}^{c} \left\lceil \lg i \binom{J}{2} \right\rceil$, where $c = n_0 \cdots n_r(K+1)$ and $n_0, \ldots, n_r$ are all the bigbits mentioned in the construction so far. If w1 is the final outcome for the worker, then clearly

1: Pick a big number n, called a 'bigbit', bigger than $c + 4c\sum_{i=1}^{c} \left\lceil \lg i\binom{J}{2} \right\rceil$, where $c = n_0 \cdots n_r(K+1)$ and $n_0, \ldots, n_r$ are all the bigbits mentioned in the construction so far. Extend z, x_0, and x_1 by zeros to length n and let $b = 0$.

2: **for** $i = 1$ to $\mathbb{N}$ **do**

3: Extend z, x_0, and x_1 by b more zeros and define θ for bits $w \in [|x_0| - b, |x_0|)$ as follows.

$$\theta(z \restriction w, 0) := x_0 \restriction w;$$

$$\theta(z \restriction w, 1) := x_1 \restriction w.$$

4: **while** $\textsc{Pairs}(b) \neq \emptyset$ **do**

5: Remove the next (first) pair from $\textsc{Pairs}(b)$ and set the last b bits of $\langle x_0, x_1 \rangle$ equal to it.

6: Wait for

$$\exists j_0 < J \quad \nu(y \restriction (n+b), j_0)\downarrow = x_0 \restriction |x_0| \quad \text{and}$$

$$\exists j_1 < J \quad \nu(y \restriction (n+b), j_1)\downarrow = x_1 \restriction |x_0|.$$

7: Break if $y \restriction n$ changes to something completely new (never seen in the for-loop until now).

8: By now $y \restriction n$ has changed to something new, and the tail b-block of x_0, x_1 will never be changed again. So update θ by setting $z \restriction [|x_0| - b, |x_0|) = 1^b$ and redefining θ for bits $w \in [|x_0| - b, |x_0|)$ as in line 3.

9: Wait for

$$\exists j < K \quad \xi(z \restriction n, j)\downarrow = y \restriction n.$$

10: $b := \left\lceil \lg i\binom{J}{2} \right\rceil + |x_0| - n.$

$x_0 \neq [\nu, J]^y$ or $x_1 \neq [\nu, J]^y$, so that requirement $\mathcal{R}_{y,\nu,J}$ is met. So assume that w1 is not the final outcome. In this case, we show that w2 is the final outcome.

First, every time the while-loop completely finishes, $y \upharpoonright n$ changes to something completely new (never seen in the for-loop until now). This is because, with every pass through the while-loop $x_0, x_1 = [\nu, J]^y$ up to $|x_0|$ (remember that w1 is not the final outcome) and we have seen (inductively) i different $y \upharpoonright n$ since the for-loop began, so that ν with $y \upharpoonright |x_0|$ and J can handle at most $i\binom{J}{2}2^{|x_0|-n} = i\binom{J}{2}2^{|\text{old\_}x_0|+b-n}$ different pairs of strings without changing $y \upharpoonright n$ to something new. But, exhausting $\textsc{Pairs}(b)$ to complete the while-loop produces 2^{2b} different pairs of incarnations of $\langle x_0 \upharpoonright |x_0|, x_1 \upharpoonright |x_0|\rangle$ and

$$\begin{aligned} 2^{2b} &> i\binom{J}{2}2^{|\text{old\_}x_0|+b-n} \quad \text{since} \\ 2^{b} &> i\binom{J}{2}2^{|\text{old\_}x_0|-n} \quad \text{since} \\ b &> \lg i\binom{J}{2} + |\text{old\_}x_0| - n \quad \text{(by design).} \end{aligned}$$

Second, (inductively) $x_0 \upharpoonright n$ and $x_1 \upharpoonright n$ eventually settle, so that $z \upharpoonright n$ eventually settles. Thus, beyond some i, in each full pass through the for-loop $y \upharpoonright n$ changes to something new (as argued in the previous paragraph), $z \upharpoonright n$ does not change, and $\exists j < K \;\; \xi(z \upharpoonright n, j)\downarrow = y \upharpoonright n$. So, by pass $i + K + 1$ at the latest, ξ will have run out of j's and the worker will wait at line 9 and remain waiting forever. So w2 is the final outcome.

Now, if w2 is the final outcome, then clearly $y \neq [\xi, K]^z$. Also $x_0, x_1 = [\theta, 2]^z$ by the time all subrequirements of $\mathcal{R}_{y,\nu,J}$ have been met. To see this, note that x_0 and x_1 change in blocks, and θ becomes defined for those blocks only twice: once when the blocks are all zeros (line 3), and once again after the blocks have settled (line 8; time is on our side). To make the second definition, z is changed to all ones in its corresponding block. So $x_0, x_1 = [\theta, 2]^z$.

Lastly, we show that x_0 and x_1 and all the z's are d.c.e. reals. It is clear that x_0 and all the z's are c.e. reals since both were constructed via increasing (in value) approximations; ordering $\textsc{Pairs}(b)$ lexicographically on the first coordinate took care of that for x_0. Also, x_1 is a d.c.e. real by Lemma 4.4.2 since a bit w of x_1 flips no more than $2^{w/2}$ times. To see this, let n be the biggest bigbit less than or equal to w and $w = n + m$. If w changes at all, then it must be in a flipping region of x_1. From a slightly tedious calculation assuming the worst case scenario that bigbits $n_0, \ldots, n_r, n$ were chosen in connection with the same $\mathcal{R}_{y,\nu,J}$ requirement, it follows that this region spans no more than $L := c\sum_{i=1}^{c} \left\lceil \lg i\binom{J}{2} \right\rceil$ bits, where $c = n_0 \cdots n_r(K+1)$ and K comes from the plan for $\mathcal{S}_{y,\nu,J,\xi,K}$ in which n was chosen. So, $m \leq L$, and, by construction (see the while-loop), the number of times bit w flips is at most

$$2^L 2^{m+1} < 2^{w/2} \quad \text{since}$$

$$L + m + 1 < w/2 \quad \text{since}$$

$$2L + 2m + 2 < m + n \quad \text{since}$$

$$4L < n \quad \text{(by design).}$$

This ends the proof.

Almost. Let us not forget the tedious calculation mentioned above. Fix a point in the construction, suppose that $n_0 < \cdots < n_r$ are all the bigbits mentioned so far, and assume the worst case scenario that each bigbit n_j $(j < r)$ was chosen in connection with requirement $\mathcal{S}_{\mu,I,\nu,J,\xi_j,K_j}$. In the construction, directly after each n_j a block of bits in x_1 are created, the bits of which flip in a lexicographic pattern. Call these 'n_j-blocks' and the total space they (the finitely many) span, 'n_j's flip region'. We show by induction that, with $n_j > c_j + 4c_j \sum_{i=1}^{c_j} \left\lceil \lg i\binom{J}{2} \right\rceil$, where $c_j = n_0 \cdots n_{j-1}(K_j + 1)$, the number of n_j-blocks is less than c_j and n_j's flip region is shorter than $c_j \sum_{i=1}^{c_j} \left\lceil \lg i\binom{J}{2} \right\rceil$.

First suppose $n_0 > c_0 + 4c_0 \sum_{i=1}^{c_0} \left\lceil \lg i\binom{J}{2} \right\rceil$, where $c_0 = K_0 + 1$. Since n_0 is the first bigbit, x_0 and x_1 do not change below n_0, so that at most $K_0 + 1 < n_0$ n_0-blocks are needed to meet requirement $\mathcal{S}_{\mu,I,\nu,J,\xi_0,K_0}$. Thus n_0's flip region is no longer than

$$\begin{aligned} b_1 + \cdots + b_{K_0+1} &\leq (K_0+1)b_{K_0+1} \\ &= (K_0+1)\sum_{i=1}^{K_0+1}\left\lceil \lg i \binom{J}{2} \right\rceil \\ &= c_0 \sum_{i=1}^{c_0} \left\lceil \lg i \binom{J}{2} \right\rceil \end{aligned}$$

bits, where $b_1, \ldots, b_{K_0+1}$ are the different values variable b assumes in the for-loop of the plan for requirement $\mathcal{S}_{\mu,I,\nu,J,\xi_0,K_0}$.

Now suppose $n_j > c_j + 4c_j \sum_{i=1}^{c_j} \left\lceil \lg i \binom{J}{2} \right\rceil$, where $c_j = n_0 \cdots n_{j-1}(K_j+1)$, and the induction hypothesis holds. In the worst case, higher priority arguments keep changing x_0 and x_1 below n_j, so that each time an n_0-block up to n_{j-1}-block is built, $K_j + 1$ n_j-blocks are built. By the induction hypothesis, at most n_0 n_0-blocks, n_1 n_1-blocks, n_2 n_2-blocks, etc. are built. So, in the worst case, $n_0 \cdots n_{j-1}(K_j+1) = c_j < n_j$ n_j-blocks are built, and n_j's flip region is no longer than $c_j \sum_{i=1}^{c_j} \left\lceil \lg i \binom{J}{2} \right\rceil$ bits, as desired. □

4.5. Enumerating Σ^0_1 Classes

My first computability theory project, given to me by Reed Solomon, was in the study of numberings. A **numbering** of a family of sets S is a surjective map $\nu : \mathbb{N} \to S$, and a numbering is $\mathbf{\Sigma^0_n}$ if the relation '$x \in \nu(e)$' is Σ^0_n.[1] Reed suggested i try to construct an injective Σ^0_1 numbering of the family of all Σ^0_1 classes, more precisely, an injective Σ^0_1 numbering of a family S of Σ^0_1 sets such that $\langle \mathcal{O}(W) : W \in S \rangle$ includes all Σ^0_1 classes without repetition.

Now, Friedberg's Theorem (below) says there is an injective Σ^0_1 numbering of the familiy of all Σ^0_1 sets, and Σ^0_1 classes are generated by Σ^0_1 sets. Moreover, the question of containment of two Σ^0_1 classes given their generating sets is of the same complexity as the question of containment for Σ^0_1 sets, namely Π^0_2. So the task seems possible. Indeed it is. The proof is modeled on a modern

[1]The Russian school calls these 'computable' numberings

proof of Friedberg's Theorem in [**GLS02**] (which is based on the presentation in [**Odi89**]), which i include here for comparison.

4.5.1 Theorem ([Fri58]). *There is an injective Σ^0_1 numbering of the family of all Σ^0_1 sets.*

Proof. Let $\langle \alpha_n : n \in \mathbb{N} \rangle$ be a Σ^0_1 numbering of the family S of all Σ^0_1 sets. Without loss of generality, assume $\alpha_0 = \mathbb{N}$. We construct an injective Σ^0_1 numbering $\langle \beta_n : n \in \mathbb{N} \rangle$ of S and a $\emptyset'$-c.p.f. f approximated by a sequence $\langle f_s : s \in \mathbb{N} \rangle$ of c.p.f. in the sense that $f(n)\downarrow = m$ if $\forall^\infty s \,.\, f_s(n)\downarrow = m$, and $f(n)\uparrow$ otherwise, that meet the following

Requirements.

(1) If $\alpha_n = \alpha_{n'}$ for some $n' < n$ then $f(n)\uparrow$.

(2) If $\alpha_n \neq \alpha_{n'}$ for all $n' < n$ then either $f(n)\downarrow$ and $\alpha_n = \beta_{f(n)}$, or $f(n)\uparrow$ and $\alpha_n = \beta_m = [0, x]$ for some x and some $m \notin \operatorname{ran}(f)$.

(3) For any $m \notin \operatorname{ran}(f)$ there is an x such that $\beta_m = [0, x]$.

(4) For any set $[0, x]$ there is a unique m with $\beta_m = [0, x]$.

Construction. Let $\theta(n, s)$ be the formula

$$\exists n' {<} n \,.\, \alpha_{n',s} \upharpoonright f_s(n) = \alpha_{n,s} \upharpoonright f_s(n),$$

that is, the formula that says 'at stage s, n appears not to be the least index for α_n'. Let $\psi(n, s)$ be the formula

$$\exists s' {<} s \; \exists m {\in} \operatorname{ran}(f_{s'}) \backslash \operatorname{ran}(f_s) \,.\, \beta_{m,s} \upharpoonright f_s(n) = \beta_{f_s(n),s} \upharpoonright f_s(n),$$

that is, the formula that says 'at stage s, β_m appears to occur twice in the β-sequence of sets'.

The construction runs as described in the pseudocode below.

Verification. We check that the requirements are met.

(1) Fix $n \in \mathbb{N}$ and suppose $\alpha_n = \alpha_{n'}$ for some $n' < n$. Then $\theta(n, s)$ will hold for infinitely many s and so $\exists^\infty s \,.\, f_{s+1}(n)\uparrow$ in view of line 8. Thus $f(n)\uparrow$.

1: $f_0(0) := 0$
2: $f(0) := f_0(0)$
3: $\beta_0 := \alpha_0$
4: **for** $s := 0$ to $\mathbb{N}$ **do**
5:     **for** $n := 1$ to s **do**
6:         **if** $f_s(n)\downarrow$ **then**
7:             **if** $\theta(n,s)$ or $\psi(n,s)$ **then**
8:                 $f_{s+1}(n) :=\uparrow$ (let $f_{s+1}(n)$ remain undefined)
9:                 $\beta_{f_s(n),s+1} := [0,x]$ for some $x \in \mathbb{N}$ larger than any number mentioned so far
10:                 $\beta_{f_s(n)} := \beta_{f_s(n),s+1}$
11:             **else**
12:                 $f_{s+1}(n) := f_s(n)$
13:         **else**
14:             $f_{s+1}(n) :=$ least $m \notin \{\mathrm{ran}(f_{s'}) : s' \leq s\} \cup \{f_{s+1}(n') : n' < n\}$
15:         **if** $f_{s+1}(n)\downarrow$ **then**
16:             $\beta_{f_{s+1}(n),s+1} := \alpha_{n,s+1}$

(2) Fix $n \in \mathbb{N}$ and suppose $\alpha_n \neq \alpha_{n'}$ for all $n' < n$. If $f(n)\downarrow$, then $\alpha_n = \beta_{f(n)}$ in view of lines 12 and 16. If $f(n)\uparrow$, then line 8 is reached for infinitely many s, and, by hypothesis, this must be because $\psi(n,s)$ holds for infinitely many s. There are two cases.

<u>case 1</u>: $\psi(n,s)$ holds for infinitely many s with the same m.

Then $\alpha_n = \beta_m = [0,x]$ for some x and some $m \notin \mathrm{ran}(f)$ in view of lines 10 and 16.

<u>case 2</u>: $\psi(n,s)$ holds for infinitely many s with different m's.

Then $\exists^\infty s\ \exists^\infty x\ .\ \alpha_{n,s} = \beta_{f(n),s} = [0,x]$, so that $\alpha_n = \mathbb{N}$. Thus, $n = 0$ by hypothesis. Then $\psi(n,s) = \psi(0,s)$ can't hold for infinitely many s since the inner **for** loop starts with $n = 1$, a contradiction.

(3) Follows from lines 8, 10, and 14.

(4) Fix $x \in \mathbb{N}$. Choose the least n for which $\alpha_n = [0, x]$. By lines 7-10 and 14 there is at most one m such that $\beta_m = [0, x]$. By (2), either $f(n)\downarrow$ and $\beta_{f(n)} = \alpha_n$, or $f(n)\uparrow$ and $\beta_m = \alpha_n$, as desired. □

We now modify this proof slightly for the case of Σ^0_1 classes.

4.5.2 Definition. *Let*

$$F_x = \{0, 10, 110, \ldots, 1^x 0\} \qquad (x \in \mathbb{N});$$

$$F_{\mathbb{N}} = \bigcup_{x \in \mathbb{N}} F_x.$$

Also, for ease of reading, let $\mathcal{O}(W) \Uparrow n$ *stand for* $\mathcal{O}(W) \cap {}^n 2$.

4.5.3 Theorem. *There is an injective* Σ^0_1 *numbering of a family* S *of* Σ^0_1 *sets such that* $\langle \mathcal{O}(W) : W \in S \rangle$ *includes all* Σ^0_1 *classes without repetition.*

Proof. Let $\langle \alpha_n : n \in \mathbb{N} \rangle$ be a Σ^0_1 numbering of the family of all Σ^0_1 sets. Without loss of generality, assume $\alpha_0 = F_{\mathbb{N}}$. We construct a Σ^0_1 numbering $\langle \beta_n : n \in \mathbb{N} \rangle$ of a family of Σ^0_1 sets and a $\emptyset'$-c.p.f. f approximated by a sequence $\langle f_s : s \in \mathbb{N} \rangle$ of c.p.f. in the sense that $f(n) \downarrow = m$ if $\forall^\infty s \,.\, f_s(n) \downarrow = m$, and $f(n) \uparrow$ otherwise, that meet the following

Requirements.

(1) If $\mathcal{O}(\alpha_n) = \mathcal{O}(\alpha_{n'})$ for some $n' < n$ then $f(n) \uparrow$.

(2) If $\mathcal{O}(\alpha_n) \neq \mathcal{O}(\alpha_{n'})$ for all $n' < n$ then either $f(n) \downarrow$ and $\alpha_n = \beta_{f(n)}$, or $f(n) \uparrow$ and $\alpha_n = \beta_m = F_x$ for some x and some $m \notin \operatorname{ran}(f)$.

(3) For any $m \notin \operatorname{ran}(f)$ there is an x such that $\beta_m = F_x$.

(4) For any set F_x there is a unique m with $\beta_m = F_x$.

Construction. Let $\theta(n, s)$ be the formula

$$\exists n' < n \,.\, \mathcal{O}(\alpha_{n',s}) \Uparrow f_s(n) = \mathcal{O}(\alpha_{n,s}) \Uparrow f_s(n),$$

that is, the formula that says 'at stage s, n appears not to be the least index for $\mathcal{O}(\alpha_n)$'. Let $\psi(n,s)$ be the formula

$$\exists s' < s\ \exists m \in \mathrm{ran}(f_{s'}) \backslash \mathrm{ran}(f_s)\ .\ \mathcal{O}(\beta_{m,s}) \upharpoonright f_s(n) = \mathcal{O}(\beta_{f_s(n),s}) \upharpoonright f_s(n),$$

that is, the formula that says 'at stage s, $\mathcal{O}(\beta_m)$ appears to occur twice in the $\mathcal{O}(\beta)$-sequence of classes'.

The construction runs as described in the pseudocode below.

1: $f_0(0) := 0$

2: $f(0) := f_0(0)$

3: $\beta_0 := \alpha_0$

4: **for** $s := 0$ to $\mathbb{N}$ **do**

5:     **for** $n := 1$ to s **do**

6:         **if** $f_s(n)\downarrow$ **then**

7:             **if** $\theta(n,s)$ or $\psi(n,s)$ **then**

8:                 $f_{s+1}(n) := \uparrow$ (let $f_{s+1}(n)$ remain undefined)

9:                 $\beta_{f_s(n),s+1} := F_x$ for some $x \in \mathbb{N}$ larger than any number mentioned so far

10:                 $\beta_{f_s(n)} := \beta_{f_s(n),s+1}$

11:             **else**

12:                 $f_{s+1}(n) := f_s(n)$

13:         **else**

14:             $f_{s+1}(n) :=$ least $m \notin \{\mathrm{ran}(f_{s'}) : s' \leq s\} \cup \{f_{s+1}(n') : n' < n\}$

15:         **if** $f_{s+1}(n)\downarrow$ **then**

16:             $\beta_{f_{s+1}(n),s+1} := \alpha_{n,s+1}$

Verification. We check that the requirements are met.

(1) Fix $n \in \mathbb{N}$ and suppose $\mathcal{O}(\alpha_n) = \mathcal{O}(\alpha_{n'})$ for some $n' < n$. Then $\theta(n,s)$ will hold for infinitely many s and so $\exists^\infty s\ .\ f_{s+1}(n)\uparrow$ in view of line 8. Thus $f(n)\uparrow$.

(2) Fix $n \in \mathbb{N}$ and suppose $\mathcal{O}(\alpha_n) \neq \mathcal{O}(\alpha_{n'})$ for all $n' < n$. If $f(n)\downarrow$, then $\alpha_n = \beta_{f(n)}$ in view of lines 12 and 16. If $f(n)\uparrow$, then line 8 is reached for infinitely many s, and, by hypothesis, this must be because $\psi(n,s)$ holds for infinitely many s. There are two cases.

case 1: $\psi(n,s)$ holds for infinitely many s with the same m.

Then $\alpha_n = \beta_m = F_x$ for some x and some $m \notin \text{ran}(f)$ in view of lines 10 and 16.

case 2: $\psi(n,s)$ holds for infinitely many s with different m's.

Then $\exists^\infty s\ \exists^\infty x\ .\ \alpha_{n,s} = \beta_{f(n),s} = F_x$, so that $\alpha_n = F_{\mathbb{N}}$. Thus, $n = 0$ by hypothesis. Then $\psi(n,s) = \psi(0,s)$ can't hold for infinitely many s since the inner **for** loop starts with $n = 1$, a contradiction.

(3) Follows from lines 8, 10, and 14.

(4) Fix $x \in \mathbb{N}$. Choose the least n for which $\alpha_n = F_x$. By lines 7-10 and 14 there is at most one m such that $\beta_m = F_x$. By (2), either $f(n)\downarrow$ and $\beta_{f(n)} = \alpha_n$, or $f(n)\uparrow$ and $\beta_m = \alpha_n$, as desired. □

4.5.4 Remark. *The proof of Theorem 4.5.3 can be modified slightly to produce a repetition-free effective enumeration of the family of all c.e. reals. For this we use the fact that any c.e. real in the interval* $(0,1]$ *can be written as* $2^{-W} := \sum\{2^{-|\sigma|} : \sigma \in W\}$ *for some prefix-free* Σ^0_1 *set of strings* W *(see* [**CHKW01**]*). So for the modification, we simply let* $\langle \alpha_n : n \in \mathbb{N}\rangle$ *be a* Σ^0_1 *numbering of the family of all prefix-free* Σ^0_1 *sets (of strings) and construct a* Σ^0_1 *numbering* $\langle \beta_n : n \in \mathbb{N}\rangle$ *of a family of prefix-free* Σ^0_1 *sets such that* $\langle 2^{-\beta_n} : n \in \mathbb{N}\rangle$ *includes all c.e. reals in the interval* $(0,1]$ *without repetition by using* 2^{-W}*'s instead of* $\mathcal{O}(W)$*'s. We then take this numbering and adjust it by rationals to get a numbering of the family of all c.e. reals without repetition.*

APPENDIX A

A Brief Review of Absolute Randomness

While this dissertation focuses on relative algorithmic randomness, absolute algorithmic randomness is mentioned throughout. Also relative randomness is based on absolute randomness. So let us review the latter, more primitive concept. Again, see [**DH**] for full details and a brief history.

Any reasonable notion of randomness for sequences makes sense only with respect to a given probability measure on ${}^{\mathbb{N}}2$. For example, in a probability space where zero and one are assigned probabilities $p = q = 1/2$ (a fair coin), we would not consider a sequence with twice as many zeroes as ones random, whereas it could be were we working in a space where $p = 2/3$ and $q = 1/3$ (a biased coin). Herein we deal exclusively with the fair coin probability space explained in more detail below.

The three most popular definitions of absolute algorithmic randomness, which turn out to be *equivalent* in our probability space, come from three different intuitions, namely, a sequence should be called random if

- it is patternless, irregular, or *incompressible*;
- it is unexceptional, ordinary, or *typical*;
- its successive digits are *unpredictable*.

A.1. Random Means Incompressible

The idea here is that the initial segments of a random sequence should have no shorter descriptions than themselves. But what is a description? Here we bring in algorithms and computability theory. Descriptions will be inputs of Turing machines and incompressiblity will be formalized in terms of descriptional complexity.

Descriptional complexity comes in two flavors, plain and prefix-free, both of which are useful. Let $s : \mathbb{N} \rightarrowtail\!\!\!\!\twoheadrightarrow {}^{<\mathbb{N}}2$ be the computable bijection given by

$$s(n) = n + 1 \text{ written in binary without the leading 1,}$$

that is, s numbers ${}^{<\mathbb{N}}2$ length-lexicographically. Note that $|s(n)| = \lfloor \lg(n+1) \rfloor$. For ease of reading and when the context is clear, i will often just write n for $s(n)$.

Let $\langle \varphi_e : e \in \mathbb{N} \rangle$ be a computable enumeration of all (oracle) binary c.p.f. and $\langle \psi_e : e \in \mathbb{N} \rangle$ a computable enumeration of all (oracle) binary p.c.p.f. Here an oracle binary p.c.p.f. is a oracle binary c.p.f. that has prefix-free first-coordinate domain no matter what oracle is used. Let $\widehat{\varphi}$ be the universal c.p.f. defined by

$$\widehat{\varphi}(\sigma, \tau) = \begin{cases} \varphi_e(\rho, \tau) & \text{if } \sigma = \overline{e}\rho \text{ for some } e \text{ and } \rho \\ \uparrow & \text{else,} \end{cases}$$

and let $\widehat{\psi}$ be the universal binary p.c.p.f. defined analogously. Remember that $\overline{e}$ here means $\overline{s(e)}$ which means $0^{|s(e)|}1s(e)$. Of course, we could use unary c.p.f. and p.c.p.f. instead by just adding one more layer of coding: $\widehat{\varphi}(\overline{\tau}\,\overline{e}\rho) = \varphi_e(\overline{\tau}\rho)$. Now the descriptional complexity functions are defined as follows.

$$C_\varphi(\sigma|\tau) := \begin{cases} \min\{|\rho| : \varphi(\rho,\tau)\downarrow\} & \text{if such a } \rho \text{ exists} \\ \uparrow & \text{else} \end{cases}$$

(for each c.p.f. φ);

the conditional complexity function with respect to φ;

$$C(\sigma|\tau) := C_{\widehat{\varphi}}(\sigma|\tau)$$

the conditional complexity function;

$$C(\sigma) := C(\sigma|\emptyset)$$

the complexity function;

$$K_\psi(\sigma|\tau) := C_\psi(\sigma|\tau)$$

(for each p.c.p.f. ψ);

the conditional prefix-free complexity function with respect to ψ;

$$K(\sigma|\tau) := K_{\widehat{\psi}}(\sigma|\tau)$$

the conditional prefix-free complexity function;

$$K(\sigma) := K(\sigma|\emptyset)$$

the prefix-free complexity function.

Note the optimality of C and K (up to an additive constant): for each e, σ, and τ

$$C(\sigma|\tau) \leq C_{\varphi_e}(\sigma|\tau) + 2|e| + 1;$$

$$K(\sigma|\tau) \leq K_{\psi_e}(\sigma|\tau) + 2|e| + 1.$$

With these complexity functions we can now define a sequence R to be incompressible iff for every c.p.f. φ, $\exists d\ \forall n\ .\ C_\varphi(R \upharpoonright n) \geq n - d$, that is, by the universality and optimality of C, iff $\exists d\ \forall n\ .\ C(R \upharpoonright n) \geq n - d$. Unfortunately, this definition is vacuous, as no sequence has this property (see the section on C-oscillations in [**DH**]). We fix this defect by using prefix-free complexity instead. A sequence R is **incompressible** iff for every p.c.p.f. ψ

$$\exists d\, \forall n\ .\ K_\psi(R \upharpoonright n) \geq n - d,$$

that is, by the universality and optimality of K, iff

$$\exists d\, \forall n\ .\ K(R \upharpoonright n) \geq n - d,$$

For any oracle A we can relativize this definition by saying that a sequence R is **A-incompressible** if

$$\exists d\, \forall n\ .\ K^A(R \upharpoonright n) \geq n - d,$$

where K^A uses $\widehat{\psi}^A$.

A.1.1 Theorem ([Cha76]). *The binary sequence of the binary expansion of the real*

$$\Omega := \sum \{2^{-|\sigma|} : \widehat{\psi}(\sigma, \emptyset)\downarrow\}$$

is incompressible.

Finally, i should mention an invaluable tool for dealing with prefix-free complexity (even though it is not used explicitly in this dissertation): the Kraft-Chaitin Theorem.

A.1.2 Theorem ([Cha75]). *Let* $W \subseteq {}^{<\mathbb{N}}2 \times \mathbb{N}$ *be a c.e. set. If* $\sum\{2^{-l} : \langle \sigma, l\rangle \in W\} < \infty$, *then*

$$\exists d\, \forall \langle \sigma, l\rangle \in W\ .\ K(\sigma) < l + d.$$

A.2. Random Means Typical

The idea here is that a random sequence should have no exceptional properties. But what is an exceptional property? Again, we bring in algorithms and computability theory. An exceptional property will be a class of algorithmic measure zero and we will formalize typicality in terms of algorithmic measure theory.

We work in the fair-coin probability space, which can be described as follows (see [**Kec95**, pages 103–107] for more details). Start with Cantor space, the topological space $\langle {}^{\mathbb{N}}2, \mathcal{T} \rangle$, where $\mathcal{T}$ is the topology generated by the basis

$$\mathcal{O}(\rho) = \{X \in {}^{\mathbb{N}}2 : X \supset \rho\}, \qquad (\rho \in {}^{<\mathbb{N}}2).$$

Letting $\overline{\mu}\ \mathcal{O}(\rho) = 2^{-|\rho|}$ uniquely defines a probability measure $\overline{\mu}$ on $\mathbb{B}$, the family of Borel sets of Cantor space (the σ-algebra generated by $\mathcal{T}$), that is, $\overline{\mu} : \mathbb{B} \to [0,1]$, $\overline{\mu}\ {}^{\mathbb{N}}2 = 1$, and $\overline{\mu}\ \bigcup_n \mathcal{B}_n = \sum_n \overline{\mu}\ \mathcal{B}_n$ for every disjoint countable family $\{\mathcal{B}_n\} \subseteq \mathbb{B}$. Let NULL denote the family of subsets of Borel sets of measure zero. Letting $\mu\ (\mathcal{B} \cup \mathcal{N}) = \overline{\mu}\ \mathcal{B}$, for $\mathcal{B} \in \mathbb{B}$ and $\mathcal{N} \in$ NULL, uniquely defines a probability measure μ on the family of measurable sets of Cantor space (the σ-algebra generated by $\mathbb{B} \cup$ NULL, which is easily seen to consist of the sets of the form $\mathcal{B} \cup \mathcal{N}$ with $\mathcal{B} \in \mathbb{B}$ and $\mathcal{N} \in$ NULL) called the **uniform probability measure** on ${}^{\mathbb{N}}2$. This measure is regular, that is, for any measurable set $\mathcal{X}$ of Cantor space

$$\begin{aligned} \mu\mathcal{X} &= \sup\{\mu\mathcal{C} : \mathcal{C} \subseteq \mathcal{X} \wedge \mathcal{C} \text{ is closed}\} \\ &= \inf\{\mu\mathcal{O} : \mathcal{O} \supseteq \mathcal{X} \wedge \mathcal{O} \text{ is open}\}. \end{aligned}$$

Now for the algorithmic measure theory. We deal with arithmetical classes, that is, members of the arithmetical Borel hierarchy, possibly relative to some oracle. An open set generated by a Σ^0_1 set of strings W, that is, a class of the form

$$\mathcal{O}(W) = \bigcup_{\sigma \in W} \mathcal{O}(\sigma),$$

is called a $\mathbf{\Sigma^0_1}$ **class**. A $\mathbf{\Pi^0_1}$ **class** is the complement of a Σ^0_1 class. In general, a Π^0_n class is the complement of a Σ^0_n class, a Σ^0_{n+1} class is a class of the form $\bigcup_i \mathcal{P}_i$, where $\langle \mathcal{P}_i : i \in \mathbb{N}\rangle$ is a computable sequence of Π^0_n classes, and a Π^0_{n+1} class is a class of the form $\bigcap_i \mathcal{S}_i$, where $\langle \mathcal{S}_i : i \in \mathbb{N}\rangle$ is a computable sequence of Σ^0_n classes. A **computable sequence** of Σ^0_n/Π^0_n classes $\langle \mathcal{Q}_i : i \in \mathbb{N}\rangle$ is a sequence for which there is a computable function f such that $f(i)$ is an index for $\mathcal{Q}_i$. For each n we assume there is a canonical assignment of indices to Σ^0_n classes (Π^0_n classes, respectively) with the property that given an index for a Σ^0_{n+1} class (Π^0_{n+1} class) $\mathcal{R}$, we have a uniformly computable way to obtain an index for the sequence $\langle \mathcal{Q}_i : i \in \mathbb{N}\rangle$ of Π^0_n classes (Σ^0_n classes) such that $\mathcal{R} = \bigcup_i \mathcal{Q}_i$ ($\mathcal{R} = \bigcap_i \mathcal{Q}_i$). A Σ^0_0 or Π^0_0 class is one of the form $\mathcal{O}(F)$ for some finite set of strings F. A class is called **arithmetical** if it is Σ^0_n for some n.

Arithmetical classes can also be defined in terms of quantifier complexity (see [**Rog67**, Chapter 15] for more details). A relation $R \subseteq ({}^{\mathbb{N}}2)^k \times \mathbb{N}^l$ is **computable** if

$$\begin{aligned}
&\exists \Phi\ \forall X_1, \dots, X_k\ \forall y_1, \dots, y_l\ . \\
&\quad R(X_1, \dots, X_k, y_1, \dots, y_l) \to \Phi^{X_1 \oplus \cdots \oplus X_k}(y_1, \dots, y_l) = 1\ \wedge \\
&\quad \neg R(X_1, \dots, X_k, y_1, \dots, y_l) \to \Phi^{X_1 \oplus \cdots \oplus X_k}(y_1, \dots, y_l) = 0.
\end{aligned}$$

A Σ^0_n class is then a class of the form

$$\{X \in {}^{\mathbb{N}}2 : \exists y_1\ \forall y_2\ \dots Q y_l.\ R(X, y_1, \dots, y_l)\},$$

for some computable relation R and where Q is $\forall$ if l is even and $\exists$ otherwise. Again, a Π^0_n class is the complement of Σ^0_n class, that is, a class of the form

$$\{X \in {}^{\mathbb{N}}2 : \forall y_1\ \forall y_2\ \dots Q y_l.\ R(X, y_1, \dots, y_l)\},$$

for some computable relation R and where Q is $\exists$ if l is even and $\forall$ otherwise.

The definitions of arithmetical classes can all be relativized. For example, a Σ^A_1 class is one of the form $\mathcal{O}(W)$ for a Σ^A_1 set W. Note also that a $\Sigma^{\emptyset^{(n-1)}}_1$ class is an *open* Σ^0_n class, and a $\Pi^{\emptyset^{(n-1)}}_1$ class is a *closed* Π^0_n class. However —and here we need to be careful when dealing with classes— a Σ^0_n class is not necessarily a Σ^0_1 class relative to $\emptyset^{(n-1)}$. For example, the class of sequences with cofinitely many zeros is a Σ^0_2 class that can *not* be expressed as $\mathcal{O}(W)$ for some Σ^0_2 set W since the former class is not open. Fortunately, this does not present a problem for our notion of randomness (see Theorem A.2.2 below).[1]

For an oracle A, a $\mathbf{\Sigma^A_n}$ **test** is a computable sequence of Σ^A_n classes $\langle \mathcal{S}_i : i \in \mathbb{N} \rangle$ with $\mu\mathcal{S}_i \leq 2^{-i}$. Think of $\bigcap_i \mathcal{S}_i$ as a class of *algorithmic* measure zero. A sequence R is **A-n-typical** ($n \geq 1$) iff for every Σ^A_n test $\langle \mathcal{S}_i : i \in \mathbb{N} \rangle$

$$R \notin \bigcap_i \mathcal{S}_i.$$

In this case we say R ‘passes the test $\langle \mathcal{S}_i : i \in \mathbb{N} \rangle$’. A **generalized** $\mathbf{\Sigma^A_n}$ **test** is a computable sequence of Σ^A_n classes $\langle \mathcal{S}_i : i \in \mathbb{N} \rangle$ with $\sum_i \mu\mathcal{S}_i < \infty$. A sequence R is **generalized A-n-typical** ($n \geq 1$) iff for every generalized Σ^A_n test $\langle \mathcal{S}_i : i \in \mathbb{N} \rangle$

$$\forall^\infty i \ . \ R \notin \mathcal{S}_i.$$

If $A \equiv_{\mathrm{T}} \emptyset$, we simply say R is n-typical or R is generalized n-typical.

A.2.1 Theorem ([Sol75]). *A sequence is A-n-typical iff it is generalized A-n-typical.*

A.2.2 Theorem ([Kau91]). *A sequence R is* $A^{(k)}$*-n-typical iff it is* A*-*$(n+k)$*-typical.*

A.2.3 Theorem ([ML66]). *For any oracle A and any* $n \geq 1$ *there exists a universal* Σ^A_n *test, that is, a* Σ^A_n *test* $\langle \mathcal{U}_i : i \in \mathbb{N} \rangle$ *such that for any sequence R, R is A-n typical iff it passes* $\langle \mathcal{U}_i : i \in \mathbb{N} \rangle$ *only.*

A.2.4 Theorem ([Sch71a]). *For any oracle A and any* $n \geq 1$*, a sequence is A-n-typical iff it is A-n-incompressible.*

[1] Much of the last three paragraphs are quoted from [**Kau91**].

A.3. Random Means Unpredictable

The idea here is that a random sequence should permit no betting strategy to make unboundedly much money wagering on its successive bits. But what is a betting strategy? Again, we bring in algorithms and computability theory. A betting strategy will be an algorithm and unpredictability will be formalized in terms of algorithmic martingales.

A **supermartingale** is a function $f : {}^{<\mathbb{N}}2 \to \mathbb{R}^{\geq 0}$ such that for all strings σ

$$f(\sigma 0) + f(\sigma 1) \leq 2f(\sigma).$$

Think of a supermartingale as the capital of a gambler/betting strategy that plays the following game. At the start, a sequence is chosen by the house and kept hidden. The house then reveals the bits of the sequence, one bit per round. At the beginning of a round the gambler bets some of her money on the next bit of the sequence coming up zero and some, but no more than the rest of her money, on the next bit coming up one. The next bit is revealed, and if it is a zero, the gambler wins her zero wager and loses her one wager and vice versa. The gambler can then tip the house or give to the poor before the next round begins. $f(\sigma)$ is the gambler's money after having bet on all the initial segments of sigma, and the inequality on f arises from tipping. (A capital function with equality, that is no tipping, is called a **martingale**.)

A $\mathbf{\Sigma^0_1}$ **function** is a function $f : {}^{<\mathbb{N}}2 \to \mathbb{R}$ computably approximable from below, that is, one for which there exists a computable function $\hat{f} : \mathbb{N} \times {}^{<\mathbb{N}}2 \to \mathbb{Q}$ such that for all σ

$$f(\sigma) = \sup_i \hat{f}(i, \sigma).$$

Similarly, a $\mathbf{\Pi^0_1}$ **function** is a function $g : {}^{<\mathbb{N}}2 \to \mathbb{R}$ computably approximable from above, that is, one for which there exists a computable function $\hat{g} : \mathbb{N} \times {}^{<\mathbb{N}}2 \to \mathbb{Q}$ such that for all σ

$$g(\sigma) = \inf_i \hat{g}(i, \sigma).$$

In general, a Σ_n^0 function is a function $f : {}^{<\mathbb{N}}2 \to \mathbb{R}$ for which there exists a computable function $\hat{f} : \mathbb{N}^n \times {}^{<\mathbb{N}}2 \to \mathbb{Q}$ such that for all σ

$$f(\sigma) = \sup_{i_1} \inf_{i_2} \sup_{i_3} \cdots Q_{i_n} \hat{f}(i, \sigma),$$

where Q_{i_n} is $\sup_{i_n}$ if n is odd and $\inf_{i_n}$ if n is even; Π_n^0 functions are defined similarly, but starting with an infimum. A Σ_0^0 function is a Π_0^0 function is a computable function. A function is called **arithmetical** if it is Σ_n^0 for some n.

The definitions for arithmetical functions can all be relativized. For example, a Σ_1^A function is one that is approximable from below by an A-computable function.

A.3.1 Theorem ([ZW01]). *A function $f : {}^{<\mathbb{N}}2 \to \mathbb{R}$ is $\Sigma_n^{A \oplus \emptyset^{(k)}}$ ($\Pi_n^{A \oplus (k)}$, respectively) iff it is Σ_{n+k}^A (Π_{n+k}^A).*

For an oracle A, a sequence R is **A-n-unpredictable** ($n \geq 1$) iff for every Σ_n^A supermartingale $f : {}^{<\mathbb{N}}2 \to \mathbb{R}^{\geq 0}$

$$\exists q \, \forall n \, . \, f(R \upharpoonright n) < q,$$

that is, every Σ_n^A supermartingale fails to win on R. If $A \equiv_{\mathrm{T}} \emptyset$, we simply say R is n-unpredictable.

A.3.2 Theorem ([Sch71a]). *For any oracle A and any $n \geq 1$ there exists a universal Σ_n^A supermartingale, that is, a Σ_n^A martingale $u : {}^{<\mathbb{N}}2 \to \mathbb{R}^{\geq 0}$ such that for any sequence R, R is A-n-unpredictable iff u fails to win on R.*

A.3.3 Theorem ([Sch71a]). *For any oracle A and any $n \geq 1$, a sequence is A-n-unpredictable iff it is A-n-typical.*

APPENDIX B

Notation Used

Symbol	Meaning
C	descriptional complexity
K	prefix-free descriptional complexity
$\mathbb{N}$	$\{0, 1, 2, \ldots\}$
${}^{n}2$	the set of binary strings of length n
${}^{<\mathbb{N}}2$	the set of binary strings
${}^{\mathbb{N}}2$	the set of infinite binary sequences
$\sigma \subseteq \tau$	σ is a substring of τ
$\sigma \subset \tau$	σ is a proper substring of τ
$\sigma\tau$ or $\sigma ^\frown \tau$	the concatenation of σ and τ
σ^n	the string $\sigma\sigma\cdots\sigma$ (n times)
$\overline{\sigma}$	$0^{\lvert\sigma\rvert}1\sigma$
$\langle \quad \rangle$	delimits ordered tuples and sequences
$X \restriction n$	the string $\langle X(0), X(1), \ldots, X(n-1)\rangle$
$\mathcal{O}(W)$	$\bigcup\{X \in {}^{\mathbb{N}}2 : X \supset \sigma \wedge \sigma \in W\}$
$\mathcal{O}(\sigma)$	$\mathcal{O}(\{\sigma\})$
μ	the uniform probability measure on ${}^{\mathbb{N}}2$
$x \restriction n$	the truncation of the binary expansion of x (both the integer and fractional part) up to and including the first n bits past the binary point

Symbol	Meaning
$\backslash \quad \backslash : \mathbb{N}^s \to \mathbb{N}$	a lexicographically increasing computable bijection
$\mathbb{R}_{\mathrm{c}}$	the class of computable reals
$\mathbb{R}_{\mathrm{c.e.}}$	the class of computably enumerable reals
$\mathbb{R}_{\mathrm{d.c.e.}}$	the class of differences of computably enumerable reals
$\mathbb{R}_{\mathrm{c.a.}}$	the class of computably approximable reals
$\mathbb{R}_{\mathrm{Kt}}$	the class of K-trivial reals

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