Relations between Set Theory and Computability Theory: the case for localization numbers and sets closed under Turing equivalence

By

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Abstract

This work is divided in two distinct parts. In both of them, we use different techniques of Set Theory in the setting of Computability Theory.

In the first part we work with the effective analogues of the localization and prediction numbers, define by Roslanowski-Newilski and Blass, respectively. We use the effectivization method proposed by Rupprecht in order to work with new computability theoretic notions in the effective Cichon's Diagram. The effective analogue of the localization numbers are the surviving degrees. The effective analogue of the globally adaptive prediction numbers are the globally surviving degrees.

At the end of the first part, we used the methods develop in computability theory to realize a new set theoretic split between these prediction numbers and the localization numbers.

In the second part we explore the different ways in which a set closed under Turing equivalence sits inside the real line. We investigate this from the perpective of algebra, measure theory and orders.

At the end of this part we find an application of a pathological order (achieved by a set closed under Turing equivalence) to the Turing degrees automorphism problem.

Acknowledgements

Cada cuerpo tiene

su armonía y

su desarmonía.

En algunos casos

la suma de armonías

puede ser casi

empalagosa.

En otros

el conjunto

de desarmonías

produce algo mejor

que la belleza.

Teoría de Conjuntos, Mario Benedetti

For my wife and my son. Their light and support showed me the way.

For my family back in Mexico, those who I know since my youth and those who we have happily include through the years.

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Contents

Chapter 1

General motivations and questions

Of all the applications of forcing that affect the real line we want to focus on two. One of them, a "global" one, is the ability to add to ZFC different axioms or sets that affect the real line. For example, demanding that all $\aleph_1$-dense sets of the real line are isomorphic (as Baumgartner did in [3]) eliminates a number of pathological sets and adds multiple reals. This work ignite multiple questions about the possible behavior of $\aleph_1$-dense orders giving birth to a plethora of axioms with different effects over the real numbers (to see two examples, see Avraham, Rudin and Shelah [1] and Todorcevic [37]).

On the other we can study the "local" differences between models of ZFC. We can ask ourselves which properties have the reals added by a single forcing (if any) and find which properties of the forcing makes the special real appear (and which other properties may avoid it). For example, Sacks studied minimal forcing extensions inspired by the possibility of having a real such that it can only code ground model reals or reals equivalent to it (he was inspired by Spector's work in Computability Theory [34]). From his work, Sacks' property was isolated. This property corresponds exactly to the forcings that keep $cof(\mathcal{N})$ small.

While thinking of these ideas, it is easy to wonder, can this be done within ZFC? Furthermore, can this be done within structures that usually are absolute? Like computability?

In this work we make an approach to these questions. In Chapter 2 we deal with the "local" line of thought. Based on work done by Rupprecht [27] and by Brendle, Brooke-Taylor, Nies and Ng [7], we continue to explore relations inside the effective Cichon diagram. Although there are different ways to explain how this diagram came to be, there is one that aligns with our line of study: giving that set theoretic forcing can add a real number with certain properties over the ground model, it is often the case that effective (or computable) versions of the same forcing construct a real number with the same property over the computable objects. When studying these computability theoretic properties, clear analogues to cardinal characteristics are made.

In this sense, in Chapter 2 we study how forcing acts in a computable setting in the case of localization and prediction numbers.

Now, in Chapter 3 we deal with our "global" line of thinking. In that chapter we investigate if sets closed under Turing equivalence can have pathological properties with respect to order or measure; and we also investigate which consequences those pathologies can have within the Turing degrees. A diversity of set theoretic techniques are applied to sets closed under Turing equivalence, sometime leaving more open questions than answers.

In the long run, we hope that all these questions can be answered and that the techniques showed in Chapter 3 find other applications.

About the structure of this dissertation, both chapters have their own introduction and both have sections giving the necessary background, notation and definition. Appendix A has two diagrams and a list of definitions related to cardinal characteristics and the properties in the effective Cichon diagram.

It is important to comment that Chapter 2 is a combination of a solo paper by the author (that is under review) [22] and a joint work with Noah Schweber [23] who we thank for his permission to publish it here. As a trivia fact, those two papers were originally thought of as one. Nevertheless, due to the different amount of work done by each author it was convenient that they were publish separately. We are glad that, in the end, they do appear together somewhere.

At the moment, this text is the first time that Chapter 3 is publish. All the chapter is work done by the author alone, with some input from UW-Madison faculty and students (thank you very much). We have plans, for the near future, to publish this chapter in a journal with, most likely, minor changes.

Finally, we want to thank Paul Tveite for his permission to use the diagrams that appears in Appendix A which originally were done for his doctoral thesis [38] and a joint work with the author (under revision) [24].

Chapter 2

Localization and prediction numbers

2.1 Introduction

In this chapter we will work with the relations between cardinal characteristic and computability-theoretic cardinal characteristics of the continuum. Classically, a cardinal characteristic is a cardinal which measures how large a set of reals with a certain "sufficiency" property must be: for example, the least size of a set of functions $\omega \to \omega$ such that every function $\omega \to \omega$ is dominated by some function in the set. It is possible that all reasonable cardinal characteristics are equal — this would follow from the continuum hypothesis — but a rich structure is revealed when we look at *consistent* separations: the study of provable (weak) cardinal characteristic inequalities, of possible simultaneous separations of more than two characteristics at once, and of the interactions between cardinal characteristic inequalities and properties of forcing notions which preserve or induce them are important aspects of modern set theory.

As is often the case, the theory of cardinal characteristics has an "effective" counterpart. The explicit analogy was first drawn by Rupprecht [27],[1] who analyzed the characteristics occurring in Cichon's diagram, and was further studied by others including Brendle, Brooke-Taylor, Ng, and Nies [7]. Given a relation $R \subseteq (\omega^\omega)^2$, an effective

[1] An early effective analogue of a cardinal characteristic equality was provided by Terwijn and Zambella [36], although they did not draw this connection explicitly.

cardinal characteristic emerges when we ask a real s to satisfy that cRs for every computable real c. Associated to this question is a corresponding "highness" property, and Rupprecht showed that these highness properties are often of independent interest in computability theory. We compare these highness properties by measuring how hard is to achieve one respect the other: for example, it is harder to produce a set of functions which dominates every function $\omega \to \omega$ than it is to produce one which escapes every function $\omega \to \omega$, and the corresponding inequality on the computability-theoretic side is "high implies hyperimmune." Some cardinal characteristics require some appropriate coding to effectivize, such as $cov(\mathcal{N})$ = the smallest number of null sets which cover $\mathbb{R}$, but such coding can be done in a natural way via effective notions of null/meager sets.

Working within the effective Cichon diagram usually allows to study particular forcing notions in a meticulous and thorough way in order to effectivize set theoretic notions. This is really helpful in order to have a better understanding of the subtleties of each forcing and is analogous to the analysis done when wondering which kind of reals a forcing adds (see the contructibility diagram of Switzer [35]). In this chapter we will work with two chardinal characteristics whose translation to the computability theoretic setting inspired us to realize a new set theoretic result.

The first of them are the localization numbers. In 1993, Newelski and Roslanowski defined the k-localization number, $\mathfrak{L}_k$ (see [20]), as the minimal cardinality of a family $\mathcal{T}$ of k-trees such that every element $(k+1)^\omega$ is a branch of a tree in $\mathcal{T}$.[2]

In their paper, they proved that $\mathfrak{L}_{k+1} \leq \mathfrak{L}_k$ and that it is consistent to have $\mathfrak{L}_{k+1} < \mathfrak{L}_k$. In order to do this, they introduce the k-localization property[3] that was later studied

[2]In their work, they originally studied ideals of unsymmetric games. The covering numbers of those ideals are the ones that we called k-localization numbers.

[3]The k-localization property says "all the reals in ω^ω of the generic extension are a branch of a k-tree

by Roslanowski [26] and Zapletal [39]. These properties were also used by Geschke [11] to show that it is consistent to have $\mathfrak{L}_i = f(i)$ for any non-increasing function from a natural number to the cardinals with uncountable cofinality.

These characteristics lend themselves to multiple computability-theoretic interpretations which we explore, especially in connection with computable traceability. We show that the various notions so resulting are reasonably distinct, and exhibit a mixture of strength and weakness: for example, the simplest effective localization notions are "is not a path through a computable k-branching subtree of $(k+1)^{<\omega}$" (*k-surviving*) for $k > 1$. We study these in Section 2.4. Our main result in this section is that these notions form a strict hierarchy, and interact with computable traceability in a nice way:

Theorem 2.1. *For each k there is a k-surviving, not $(k+1)$-surviving degree which is computably traceable.*

A more complicated picture emerges in Section 2.5 when we consider covering ω^ω with closed sets. Here a difficulty arises in the effective setting with no classical analogue: there is no computable way to pass from a tree which branches at most n times at each branching node ("n-tree") to a tree which branches exactly n times at each branching node ("n-branching tree"). This gives us two separate tracks of cardinal characteristics, and leads to a somewhat messy picture. We show, for example:

Theorem 2.2. *There is a computably traceable degree containing a globally branch surviving function that is not 3-globally tree surviving.*

(These notions are defined in the beginning of Section 2.5.)

from the ground model."

Theorem 2.3. *There is a real $A \in \omega^\omega$ which is not a path through any computable k-tree for any $k \in \omega$ but which does not compute any $f \in 3^\omega$ which is not a path through any computable 2-branching tree.*

It is natural to ask if there is a set theoretic chardinal characteristic that is analogous to the globally surviving degrees (as the surviving degrees are analogous to the localization numbers). There is one. In his chapter of the Handbook of Set Theory [6], Andreas Blass talks about cardinal characteristics related to the concepts of evasion and prediction. At the end of that section, he introduces 36 variations of these cardinals and left as an open question to pin down 4 of them whose identity didn't appear to be one of the known cardinal characteristic. It turns out that the same proof of Newelski and Roslanowski shows that one of these variations, specifically the prediction number for global adaptive k predictors, is not one of the known cardinal characteristics and, actually, gives countable many cardinal characteristics which, consistently, can take different values (see Theorem 2.7).

This triggers the following question: are the variation of prediction and the k-localization number equal? No. As mention in Theorem 2.6 of Brendle-Garcia [8], you can split them making the localization number strictly less than $\mathfrak{d}$. This thesis shows a different way to separate them keeping all the values of the Cichon diagram small. It is consistent to have all the prediction numbers mention above at value $\aleph_2 = \mathfrak{c}$ and all localization numbers and all numbers in Cichon at value $\aleph_1$ (see Theorem 2.61).

To do this we use a forcing that appears in the proof of Theorem 2.3 that Noah Schweber and the author called accelerating tree forcing. We will show in section 2.7 that countable support product of the accelerating tree forcing has the 3^ω-localization property (defined in Section 2.6).

It has been pointed out to us that the accelerating tree forcing could be related to bushy tree forcing (as done by Khan and Miller [15]) or other fast-growing tree forcing (as done by Ciesielski-Shelah [9]). Nevertheless, this is a forcing of a different kind. Around the time when [23] was publish, we find out that Miller Lite forcing (which Geschke uses and describes in [10]) is equivalent to the accelerating tree forcing. Altough both forcings are equivalent, we decide to keep using the accelerating tree description since it fits better our needs.

It is also important to remark that countable support iteration and product of forcings with the $(k+1)^{\omega}$-localization property could also have the $(k+1)^{\omega}$-localization property, as the k-localization property (see [39]). Notice that these two properties are in the same line as the Sacks property. It is unknown to the author if there is a bigger theory or theorem that handle all of them at once. This, we believe, is an interesting topic.

About the structure of the chapter, it has a first section with definition and background. In section 2.3 we explore the ZFC relations between $\mathfrak{L}_k$ and $\mathfrak{v}_k^g$. Then, we explore the effective analogues of the cardinal characteristics above describe in Sections 2.4 and 2.5. In Section 2.6, we prove lemmas involving the $(k+1)^{\omega}$-localization property in order to prepare the splitting. Section 2.7 has the set theoretic split (Theorem 2.61). The last section has some conclusions and open problems.

2.2 Definitions and background

These first definitions will be useful during the rest of the chapter:

Definition 2.4. *1. We say that $T \subseteq \omega^{<\omega}$ is a tree if and only if given $\sigma \in T$ we have* *that* $\sigma \restriction j \in T$ *for all* $j < |\sigma|$.

2. *A* k*-branching tree, is a tree such that every node has either* 1 *successor or* k *of them. and every node has an extension with more than* 1 *successor.*

3. *A* k*-tree is a tree such that every node has at least* 1 *successor and no more than* k.

4. *We say that* r *is a branch of* T*, or* $r \in [T]$ *if and only if* $r \restriction n \in T$ *for all* $n \in \omega$.

Now, the following definition is due to [20] (they express it as the covering number of an ideal):

Definition 2.5. *The* k*-localization number,* $\mathfrak{L}_k$*, is the smallest cardinality of a family of* k*-branching trees that cover* $(k+1)^\omega$.

Notice that the definition is not trivial for $k \geq 2$. Furthermore, Newelski and Roslanowski showed in [20] that, for $k \geq 2$, $\mathfrak{L}_k \geq \max\{cov(\mathcal{M}), cov(\mathcal{N})\}$, that $\mathfrak{L}_{k+1} \leq \mathfrak{L}_k$ and that it is consistent that $\mathfrak{L}_{k+1} < \mathfrak{L}_k$.

On the other hand, in Blass's chapter of the Handbook of Set Theory [6] he defines:

Definition 2.6. 1. *A* k *globally adaptive predictor is a sequence of functions* $\pi = \langle \pi_n : n \in \omega \rangle$ *with* $\pi_n : \omega^n \to [\omega]^k$*. We say that a function* $f \in \omega^\omega$ *is predicted by* π *if there is* $m \in \omega$ *such that for all* $n > m$, $f(n) \in \pi_n(f \restriction n)$.

2. *The* k *globally prediction number,* $\mathfrak{v}_k^g$*, is the minimal cardinality of a set of* k *globally adaptive predictors that predict all functions in* ω^ω.

3. *The* k *globally evasion number,* $\mathfrak{e}_k^g$*, is the minimal cardinality of a set of functions in* ω^ω *such that the whole set is not predicted by a single* k *globally adaptive predictor.*

It is important to make some remarks about the last definition:

- The 'adaptive' part refers to the fact that π_n is not constant. Non-adaptive objects are closer to slaloms (or traces).
- The 'globally' part of the definition refers to the fact that we have π_n for all $n \in \omega$. It is possible to define predictors using π_n for $n \in D \subsetneq \omega$.
- Blass does not give a notation for this number, so the notation $\mathfrak{v}_k^g$ and $\mathfrak{e}_k^g$ is introduced here.
- These definitions are not trivial for $k \geq 2$.

The numbers $\mathfrak{v}_k^g$ and $\mathfrak{e}_k^g$ are mutually dual and, by the work done in [6], we know that $\mathfrak{m}_{\sigma-k-\text{linked}} \leq \mathfrak{e}_k^g \leq \text{add}(\mathcal{N})$. So, by duality, we know that $\text{cof}(\mathcal{N}) \leq \mathfrak{v}_k^g \leq \mathfrak{c}$. Also, from the definition, we have that $\mathfrak{v}_{k+1}^g \leq \mathfrak{v}_k^g$.

Reading the definition more carefully we can notice that all the functions that are predicted by a k-globally adaptive predictor are covered by $\aleph_0$ many k-branching trees, so $\mathfrak{v}_k^g$ is also the minimum cardinal of a set of k-branching trees (or k-trees) that cover ω^ω (see the following section for a proof of this).

Furthermore, in [20], we have the following result:

Theorem 2.7. *Given $k \geq 2$, it is consistent to have ZFC+$cof(\mathcal{N}) = \mathfrak{v}_{k+1}^g < \mathfrak{v}_k^g = \mathfrak{c}$.*

This theorem is a corollary of the proof of:

Theorem 2.8 (Newelski, Roslanowski [20])**.** *Given $k \geq 2$, it is consistent to have ZFC+$cof(\mathcal{N}) = \mathfrak{L}_{k+1} < \mathfrak{L}_k = \mathfrak{c}$.*

This result comes from two facts: first, that the forcings that were used have the k-localization property. This is that "every real in ω^ω is a branch of a k-tree from the ground model", this keeps $\mathfrak{L}_{k+1}$ and $\mathfrak{v}^g_{k+1}$ at $\aleph_1$; and the forcing adds a function in $(k+1)^\omega$ that is not the branch of any k-tree from the ground model. Notice that this function is also a function in ω^ω that is not the branch of any k-tree from the ground model. Once you take a countable support product, this makes $\mathfrak{L}_k$ and $\mathfrak{v}^g_k$ of size $\mathfrak{c}$.

The relation between these two cardinal characteristics is more evident once we realize, as we will do in the following section, that $\mathfrak{L}_k \leq \mathfrak{v}^g_k$.

2.3 ZFC relations

In this section, we will exhaust all the relations between $\mathfrak{L}_k$ and $\mathfrak{v}^g_k$ that are true in ZFC.

In order to do this we need to show an equivalent definition for $\mathfrak{v}^g_k$.

Lemma 2.9. *Let $\mathcal{T}$ be a family of k-subtrees of $\omega^{<\omega}$.*

$$\mathfrak{v}^g_k = min\{|\mathcal{T}| : \forall r \in \omega^\omega \exists T \in \mathcal{T}(r \in [T])\}.$$

Proof. First of all, given a k-globally adaptive predictor π and $\sigma \in \omega^\omega$ let

$$T_{\pi,\sigma} = \{\tau \in \omega^{<\omega} : \forall i \in \omega((\tau(i) = \sigma(i)) \vee (i \geq |\sigma| \wedge \tau(i) \in \pi_i(\tau \restriction i)))\}.$$

Notice that, if $f \in \omega^\omega$ is predicted by π then there is m such that for all $n \geq m$ $f(n) \in \pi_n(f \restriction n)$ and this happens if and only if $f \in [T_{\pi,f\restriction m}]$.

This shows that

$$\{f \in \omega^\omega : f \text{ is predicted by } \pi\} = \bigcup_{\sigma \in \omega^{<\omega}} [T_{\pi,\sigma}].$$

Therefore, if we can cover ω^ω with certain family of k globally adaptive predictors, we can change each one of them for countably many k-trees and still cover ω^ω.

On the other hand, given a k-tree T you can define π_T such that given $\sigma \in \omega^{<\omega}$, if $\sigma \in T$, $(\pi_T)_i(\sigma) = \{n \in \omega : \sigma^\frown n \in T\} \cup A$ where A has extra natural numbers to make $(\pi_T)_i(\sigma)$ of size k or $(\pi_T)_i(\sigma) = k \subseteq \omega$ if $\sigma \notin T$.

This shows that

$$[T] \subseteq \{f \in \omega^\omega : f \text{ is predicted by } \pi_T\}.$$

Therefore, if we can cover ω^ω with certain amount of k-trees, we can cover it with the same amount of k globally adaptive predictors.

Joining this two pieces of information we have that

$$\mathfrak{v}_k^g \leq min\{|\mathcal{T}| : \forall r \in \omega^\omega \exists T \in \mathcal{T}(r \in [T])\} \leq \aleph_0 \cdot \mathfrak{v}_k^g = \mathfrak{v}_k^g.$$

□

Corollary 2.10. $\mathfrak{L}_k \leq \mathfrak{v}_k^g$.

Proof. Notice that if we cover ω^ω with k-trees we also cover $(k+1)^\omega$. □

There is one more ZFC relation between these cardinal characteristics. This relation restricts greatly the kind of splits that these numbers can have.

Theorem 2.11. *The following equality is true* $\mathfrak{v}_k^g = \max\{\mathfrak{v}_{k+1}^g, \mathfrak{L}_k\}$. *Furthermore, if* $\mathfrak{v}_{k+1}^g < \mathfrak{v}_k^g$ *then* $\mathfrak{L}_{k+1} < \mathfrak{L}_k$.

Proof. Let $\mathfrak{v}_{k+1}^g = \kappa$. This means that ω^ω can be covered by κ many $k+1$-trees.

Now, if $\mathfrak{L}_k = \lambda$, this means that $(k+1)^\omega$ can be covered by λ many k-trees. Notice that, given T a $k+1$-tree of $\omega^{<\omega}$, there is a onto function from $(k+1)^{<\omega}$ to T. This transforms any cover by k-trees of $(k+1)^\omega$ into a cover of k-trees of $[T]$.

Now, given a cover of ω^ω with $k+1$-trees, we can cover each one of them with λ many k-trees creating a cover of $\kappa \cdot \lambda$ many k-trees of ω^ω. This means that $\mathfrak{v}^g_k \leq \kappa \cdot \lambda = \max\{\mathfrak{v}^g_{k+1}, \mathfrak{L}_k\}$. Since $\mathfrak{v}^g_k \geq \mathfrak{v}^g_{k+1}$ and $\mathfrak{v}^g_k \geq \mathfrak{L}_k$, we have that $\mathfrak{v}^g_k = \max\{\mathfrak{v}^g_{k+1}, \mathfrak{L}_k\}$.

For the furthermore, if $\mathfrak{v}^g_{k+1} < \mathfrak{v}^g_k$ then $\mathfrak{L}_k = \mathfrak{v}^g_k$ and

$$\mathfrak{L}_{k+1} \leq \mathfrak{v}^g_{k+1} < \mathfrak{v}^g_k = \mathfrak{L}_k$$

□

Inspired by the proof above, we can obtain the following result in the flavor of the contructibility degrees (see [35]):

Lemma 2.12. *Given $V \subseteq W$ models of ZFC such that for every $a \in \omega^\omega \cap W$ there is a $k+1$-tree $T \in V$ such that $a \in [T]$ but there is $b \in \omega^\omega \cap W$ such that for all k-trees $U \in V$ we have $b \notin [U]$ then there is $c \in (k+1)^\omega \cap W$ such that for all k-trees $S \in V$ we have $c \notin [S]$.*

Proof. In W let $b \in \omega^\omega$ be such that it is not in any k-tree from the ground model. Now, let T be a $k+1$-branching tree from V such that $b \in [T]$. Notice that, in V, there is a bijection $f : T \to (k+1)^{<\omega}$ so, in W, this induces a function $f^* : [T] \to (k+1)^\omega$.

Notice that $f^*(b)$ is also not in any k-tree from the ground model. If it were, say in S, we will have that $b \in f^{-1}(S)$, but $f^{-1}(S)$ is a k-tree from V. □

Corollary 2.13. *If a forcing doesn't add a function in ω^ω that escapes all $k+1$-trees of the ground model but it adds a function in ω^ω that escapes all ground model k-trees then this forcing adds a function in $(k+1)^\omega$ that escapes all ground model k-trees.*

In order to start thinking about computability theory, let's do a mental exercise. In the lemma above, instead of thinking about V and W as models of Set Theory, we can see them as the computable reals (the analogue for V) and reals computed from a non-computable real (analogue for W).

Lemma 2.14. *If a Turing degree has the property that every total function it computes is a branch of a $k+1$-branching computable tree but it computes a function that escapes every k-branching computable tree then it computes a function in $(k+1)^\omega$ that escapes every k-branching tree.*

Proof. The same proof of the above theorem works, once we define a computable function from $T \subseteq \omega^\omega$, a computable k-branching tree, to $(k+1)^{<\omega}$.

Let's think of $f : T \to (k+1)^{<\omega}$ as a partial function from $\omega^{<\omega}$ to $(k+1)^{<\omega}$. First of all $\sigma \in dom(f)$ if and only if $\sigma \in T$.

Now, we will demand that $|f(\sigma)| = |\sigma|$ and that if τ is a successor of σ then $f(\tau)$ is a successor of $f(\sigma)$. Finally, if τ is the i-th successor of σ to halt (in other words, the i-th successor of σ to enter T), then $f(\tau)(|\sigma|) = i$.

This function f is computable. □

Notice that, in order to make the proof work, f cannot be a bijection anymore. This kind of small changes are common in the translations from set theoretic result to computability results. But deeper changes can be seen, although it does not affect these results. In general, we need to be careful between k-trees and k-branching trees in the effective setting (more on that on section 2.4 and 2.5).

Finally, we want to end this section showing the translation of $\mathfrak{v}^g_k = \max\{\mathfrak{v}^g_{k+1}, \mathfrak{L}_k\}$ to the effective Cichon diagram. Although this is a trivial result, it makes clears that we need a better language to talk about these degrees.

Lemma 2.15. *Any Turing degree that computes a function $f \in \omega^\omega$ that escapes all k computable trees either computes a function that escapes all $k+1$ computable trees (that same f) or a function $g \in (k+1)^\omega$ that escapes all k-computable trees (the image of f under a certain computable function).*

2.4 Surviving degrees

Following Rupprecht's analogy, the computability-theoretic version of the localization number is the following highness property:

Definition 2.16. *A function $f \in (k+1)^\omega$ is* k-surviving *if it is not a path through any computable k-branching subtree of $(k+1)^{<\omega}$. We say that a Turing degree is k-surviving if it computes a k-surviving function.*

We call these k-surviving degrees since they are the ones that go into the forest of k-branching trees and are able to escape it: they survive the experience. Note that this definition requires $k > 0$ to make sense, and for $k = 1$ trivializes: "1-surviving" is just "non-computable." So we are only interested in $k \geq 2$.

Before we begin analyzing the k-surviving degrees, there is a subtlety here which will matter later.

Classically, the localization numbers can be equivalently defined in terms of k-trees rather than k-branching trees, and on the computability-theoretic side we can effectively pass from a k-tree contained in $n^{<\omega}$ for finite n to a k-branching tree containing it, so this is not an issue at the moment. However, in general a computable k-tree merely contained in $\omega^{<\omega}$ may not be contained in a computable k-branching tree, as we will see below; this will give us two distinct computable analogues of the class globally adaptive prediction numbers in $\omega^{<\omega}$ in section 2.5.

As an initial observation, it is easy to see that the k-surviving degrees form a hierarchy as k varies:

Lemma 2.17. *Given $k \geq s \geq 2$, a k-surviving degree is also an s-surviving degree.*

Proof. Fix a surjection $g : k+1 \to s+1$, and let $g^* : (k+1)^{<\omega} \to (s+1)^{<\omega}$ and $\widehat{g} : (k+1)^{\omega} \to (s+1)^{\omega}$ be the induced computable surjections on the corresponding sets of finite or infinite strings. If $T \subseteq (s+1)^{<\omega}$ is a computable s-tree, then $(g^*)^{-1}[T] \subseteq (k+1)^{<\omega}$ is a computable k-tree. This means that if $A \in (k+1)^{\omega}$ is a k-surviving function then $\widehat{g}(A) \in (s+1)^{\omega}$ is an s-surviving function: if $\widehat{g}(A)$ were in some computable s-tree, then pushing this forward we would have a computable k-tree containing A. □

The above result is an anologue for to $\mathfrak{L}_{k+1} \leq \mathfrak{L}_k$ proved by Newelski and Roslanowski [20]. Furthermore, we can also mimic in the computable sidet that for all $k \geq 2$, $\mathfrak{L}_k \geq \max\{cov(\mathcal{M}), cov(\mathcal{N})\}$, and that it is consistent that $\mathfrak{L}_{k+1} < \mathfrak{L}_k$. Lemma 2.17 is an analogue for $\mathfrak{L}_{k+1} \leq \mathfrak{L}_k$.

Since, all the subsets of n^{ω} that are covered by a k-branching computable tree (with $k < n$) are effectively meager and null, we obtain the following result (using notation from [7] that you can also find in appendix A):

Theorem 2.18. *All the degrees that compute a Schnorr random real are k-surviving for all $k \geq 2$. In particular, there is a k-surviving degree that is DNC and a k-surviving degree that is not computable traceable.*

Theorem 2.19. *All the degrees that compute a weak 1-generic (equivalently, all hyperimmune degrees) are k-surviving for all $k \geq 2$. In particular, there is a k-surviving degree that is weakly Schnorr engulfing.*

Since there is a Schnorr random which is hyperimmune-free (see e.g. [7] §4.2 (2)) and a hyperimmune degree that does not compute a Schnorr random (See e.g. [21] Theorem 1.8.37) we have:

Corollary 2.20. *For all $k \geq 2$ there is a k-surviving degree which does not compute a Schnorr random.*

Corollary 2.21. *For all $k \geq 2$ there is a k-surviving degree which is hyperimmune free.*

Notice that Theorem 2.18 and Theorem 2.19 are the effective analogues of $\mathfrak{L}_k \geq cov(\mathcal{M})$ and $\mathfrak{L}_k \geq cov(\mathcal{N})$ respectively.

Finally, we turn to the converse of Lemma 2.17: we show that the k-surviving degree are truly hierarchical, i.e., that there are k-surviving degrees that are not s-surviving, with $s > k$. This is the computable analogue of the conditional consistency of $\mathfrak{L}_{k+1} < \mathfrak{L}_k$.

Theorem 2.22. *Given $k \geq 2$, there is a k-surviving degree that is not ℓ-surviving for $\ell \geq k+1$. Furthermore, it is possible to make this degree computable traceable.*

Proof. We will force with computable trees, T, in $(k+1)^{<\omega}$ such that for every $s \in T$ there is t extending s such that t has more than one successor and, for all $s \in T$, we have that $|\{t \in T : s < t \,\&|t| = |s| + 1\}|$ is either 1 or $k+1$.

We will construct a function $A : \omega \to k+1$ that satisfy two types of requirements:

- R_e: Given the e-th computable k-tree in $(k+1)^{<\omega}$, A is not one of its branches.

- P_e: φ_e^A is either partial or is a branch of a computable $k+1$-tree of $\omega^{<\omega}$. In particular, if $\varphi_e^A : \omega \to \ell + 1$ with $\ell \geq k+1$ then φ_e^A is not ℓ-surviving.

We begin at stage $s = 0$ by setting $T_0 = (k+1)^{<\omega}$. Now suppose that at stage $s+1$ we have a computable $k+1$-branching tree T_s such that every branch satisfies all requirements R_j and P_j for $j < s$.

To satisfy R_s we just need to extend the current stem (or root), r_s, in such a way that it is not longer in the s-th k-tree in $(k+1)^{<\omega}$. This is possible because T_s has $k+1$ options every time it branches.

Satisfying P_s is more complicated, however, and we have two cases which need to be handled separately. The first case happens if there is $t \in T_s$ and n such that $r_s \subseteq t$ and $\varphi_s^{t'}(n)$ diverges for all $t' \supset t$. Setting T_{s+1} to be the subtree of T_s consisting of nodes comparable with t then trivially satisfies P_s.

Now suppose we are unable to force partiality in this way. For all $t \in T_s$ extending r_s and $n \in \omega$ there is a $t' \in T_s$ extending t such that $\varphi_s^{t'}(n)$ converges. Now we will create T_{s+1} in such a way that for all branches A of T_{s+1} we have that φ_s^A is total. Furthermore, we can find a computable $k+1$-tree, U_s, such that for all branches of T_{s+1}, φ_s^A is a branch of U_s. For convenience we will describe T_{s+1} as a function $(k+1)^{<\omega} \to (k+1)^{<\omega}$, denote by t_σ, and U_s as function $(k+1)^{<\omega} \to \omega^{<\omega}$, denote by u_σ.

Before starting the construction we can assume one more hypothesis: for all $\tau \in T_s$ there exist $\tau_0, ..., \tau_k$ in T_s extending τ and $n \in \omega$ such that $\varphi_s^{\tau_i}(\ell)$ converges for all $i < k+1$ and all $\ell < n$. Also, we need that $\varphi_s^{\tau_i} \restriction n \neq \varphi_s^{\tau_j} \restriction n$ for all $i \neq j < k+1$.[4]

If there is a τ extending r_s such that the above hypothesis is false, that means that φ_s^A can have at most k different values as long as τ is an initial segment of A. Therefore,

[4]The use of τ in this paragraph instead of t will simplify the reading later.

defining T_{s+1} the subtree of T_s extending τ, we can find a computable tree U_s with at most k branches such that φ_s^A, with $A \in [T_{s+1}]$, is always one of those branches.

Now, back to the construction, our strategy will be define for each node: first find an extension that splits; then, look for extensions of each node in the split (there are exactly $k+1$ of them) that makes the function φ_e^t different to each other, with that we keep the $k+1$-branching and we can use the information to define U_s.

Bringing the strategy to work, at the first stage, let $r_s = t_\emptyset$. Then look for for the first split above $t_\emptyset$ and call those nodes $\tau_0, ..., \tau_k$. Next, look for $t_0, ..., t_k$ extending $\tau_0, ..., \tau_k$ respectively and $n_\emptyset \in \omega$ with $0 < n_\emptyset$ such that $\varphi_s^{t_i}\lceil n_\emptyset \neq \varphi_s^{t_j}\lceil n_\emptyset$ for all $i \neq j < k+1$ and $\varphi_s^{t_i}(\ell)$ converges for all $i < k+1$ and all $\ell < n_\emptyset$. Define $u_\emptyset = \emptyset$ and $u_i = \varphi_s^{t_i}\lceil_{n_\emptyset}$.

In general, given t_σ with $\sigma \in (k+1)^{<\omega}$, look for the first split above t_σ and call those nodes $\tau_{\sigma 0}, ..., \tau_{\sigma k}$. Next, look for $t_{\sigma 0}, ..., t_{\sigma k}$ extending $\tau_{\sigma 0}, ..., \tau_{\sigma k}$ respectively and an $n_\sigma \in \omega$ with $|\sigma| < n_\sigma$ such that $\varphi_s^{t_{\sigma i}}\lceil n_\sigma \neq \varphi_s^{t_{\sigma j}}\lceil n_\sigma$ for all $i \neq j < k+1$ and $\varphi_s^{t_{\sigma i}}(\ell)$ converges for all $i < k+1$ and all $\ell < n_\sigma$. Define $u_{\sigma i} = \varphi_s^{t_{\sigma i}}\lceil_{n_\sigma}$.

Since T_s is computable we have that both T_{s+1} and U_s are computable. Furthermore, each split in U_s is at most of size $k+1$ so it is a $k+1$-tree and, by construction, given a branch A of T_{s+1} we have that φ_s^A is a branch of U_s.

Furthermore, if $A \in [T_{s+1}]$ and φ_s^A is total then we can define a computable trace $\phi_s : \omega \to [\omega]^{<\omega}$ such that $\phi_s(n)$ is the n-th level of U_s. Notice that $|\phi_s(n)| \leq (k+1)^n$ and that for all branches A of T_{s+1} we have that φ_s^A goes through ϕ_s.

Finally, $A \in \bigcap_{s\in\omega}[T_s]$ is a k-surviving degree that is a computably traceable degree and not $k+1$-surviving (or ℓ-surviving for $\ell \geq k+1$). $\square$

Corollary 2.23. *Given $k \geq 2$, there is a k-surviving degree that is not DNC and not weakly Schnorr Engulfing.*

Notice that the above construction for $k = 1$ give us a non-computable set that is not a 2-surviving degree.

To further pin-point the location of the surviving degrees in the effective Chichoń diagram it is necessary to compare them to the DNC degrees. This question is still open:

Question 2.24. *Is there a DNC degree that is not k-surviving?*

In the same spirit, we may also ask:

Question 2.25. *Is it possible to make a k-surviving degree that is not $k + 1$ surviving and not computable traceable?*

2.5 Globally surviving degrees

As seen in the proof of theorem 2.22, it is possible to have degrees such that all their functions $f : \omega \to \omega$ go through a k-subtree of $\omega^{<\omega}$. Therefore, we can define degrees that survive k-trees in ω^{ω}. Here, however, we run into the subtlety mentioned earlier: that since we are no longer working with trees over a finite set, "computable k-branching tree" and "computable k-tree" may behave differently. This leads to two separate tracks of highness notions:

Definition 2.26.

1. *A function $g : \omega \to \omega$ is k-*globally branch surviving* if it is not a path through any computable k-branching tree; a Turing degree B is k-globally branch surviving if it computes a k-globally branch surviving function.*

2. *A function $g : \omega \to \omega$ is* globally branch surviving *if it is k-globally branch surviving for every $k \in \omega$; a Turing degree B is globally branch surviving if it computes a globally branch surviving function.*

3. *A function $g : \omega \to \omega$ is k-*globally tree surviving *if it is not a path through any computable tree; a Turing degree B is k-globally tree surviving if it computes a k-globally tree surviving function.*

4. *A function $g : \omega \to \omega$ is* globally tree surviving *if it is not a path through any computable k-tree for any $k \in \omega$; a Turing degree B is globally tree surviving if it computes a globally tree surviving function.*

The distinction between k-branching trees and k-trees is significant. Trivially a k-globally tree surviving degree is also k-globally branch surviving, and similarly a globally tree surviving degree is globally branch surviving. However, no other coarse implication exists.

We begin by showing that global branch and global tree survival differ wildly on the level of individual functions. Although they are the same for $k = 2$ (a 2-tree is the same as a 2-branching tree), there is no other coincidence for larger k:

Proposition 2.27. *There is a globally-branch surviving function that is not 3-globally tree surviving.*

Proof. We will define a computable 3-tree of $\omega^{<\omega}$ which is not covered by any computable k-branching tree. The right most path of this 3-tree will be globally-branch surviving but, since it is a branch of a 3-tree, it is not 3-globally tree surviving.

We will define T by stages, beginning with $T_0 = \emptyset$ and obeying the following rules:

1. If $p \in T_s$ then $p0 \in T_{s+1}$.

2. If $p \in T_s$, then for each $i < s + 1$ we will decide whether $p^\frown i = pi \in T$ at stage $s + 1$.

3. We will fix a computable permutation that maps ω with all the pairs $\langle e,k\rangle$ with $k>2$. If $p\in T$, $|p|=n=\langle e,k\rangle$, the successors of p will deal with φ_e as if it were a k-branching tree. In particular, if φ_e is a k-branching tree then the right most successor of p is not in it.

(Strictly speaking, at a given stage we have a tree together with a finite set of forbidden nodes, but for simplicity we speak of just building a tree and making declarations.)

Assume that at stage s we have $p\in T_s$, $|p|=n=\langle e,k\rangle$. There are now three cases:

Case 1 No successor of p is in T_s other than perhaps $p0$.

If $\varphi_{e,s}$ does not look like a k-branching tree containing p, then we set $pi\notin T$ for all $0<i\le s$. Otherwise, we check whether $p0\in\varphi_{e,s}$; if it is not, we declare that $ps\notin T$.

If $p0\in\varphi_{e,s}$, we further check if $ps\in\varphi_{e,s}$. If it is then we declare $ps\notin T$; if $ps\notin\varphi_{e,s}$ then we declare that $ps\in T_{s+1}$.

Case 2 There are exactly two successors of p in T_s (one of which is $p0$).

Being in this case means that, at some stage, φ_e looked like a k-branching tree and that $p0\in\varphi_{e,s}$. If there are not exactly k-many successors of p in $\varphi_{e,s}$ then we declare that $ps\notin T$; otherwise, we put $ps\in T_{s+1}$. Note that $ps\notin\varphi_{e,s}$ by use constraints, so we are free to make this decision at this time.

Case 3 If there are three successors of p in T_s we declare $ps\notin T$.

The union $\bigcup_{s\in\omega}T_s$ is a computable 3-tree whose right-most path is not in any computable k-branching tree, and this finishes the proof. □

At a first glance, it may appear that the rightmost branch through the tree constructed above has significant computational power. Interestingly, this is not quite true:

Theorem 2.28. *There is computably traceable degree that computes a globally branch surviving function that is not* 3*-globally tree surviving.*

Proof. We will use the same tree as in the proof of Proposition 2.27, with a slight modification. Having, as above, an effective enumeration $\{\langle e_i, k_i\rangle : i > 0\}$ of $\omega \times \omega_{\geq 3}$ — note that we do not include a 0th term, for notational convenience below — we will fix a computable sequence in which every natural number > 0 occurs infinitely often; we use

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, ...$$

The terms in this sequence tell us what levels of our tree will deal with each pair in $\omega \times \omega_{\geq 3}$. For example, $\langle e_1, k_1\rangle$ will be dealt with at levels $0, 1, 3, ...$ and $\langle e_2, k_2\rangle$ at levels $2, 4, 7, ...$

We build a sequence of triples $\langle p, T, g\rangle$ with the following properties:

- $p \in \omega^{<\omega}$.
- T is a computable 3-tree that is not covered by any computable 3-branching tree, with $p \in T$ (indeed we may assume that p is the stem of T).
- $g : T \to \omega$ is such that: for all $\sigma \in T, n \in \omega$, there is an extension $\tau \in T$ of σ with $g(\tau) = n$.
- Finally, if φ_e is a k-branching tree, then there are infinitely many nodes in T with a successor not in φ_e.

Here, p is the initial segment of the function we are constructing, T is the tree of possible future extensions (so the real we produce is a branch of T), and if $g(\sigma) > 0$

then the successors of σ deal with $\varphi_{e_{g(\sigma)}}$ as if it were a $k_{g(\sigma)}$-tree (where $\{\langle e_i, k_i\rangle : i > 0\}$ is as above) — that is, the labelling function g assigns tasks to each node of the tree.

To start, fix (noneffectively) an enumeration of all computable branching trees (i.e., k-branching trees for some $k \in \omega$). During the construction of A we want to satisfy two families of requirements:

- R_e: If φ_e is a k-branching tree for any $k \in \omega$ then A is not a branch of it. This will make A a globally branch surviving degree.

- P_e: If φ_e^A is total then it goes through some computable trace bounded by $f(n) = 3^n$. This will make A computably traceable.

We will start our construction with $\langle p_0, T_0, g_0\rangle = \langle \emptyset, T, g_0\rangle$ with T as described in the first paragraph and g_0 being constant at every level, and mapping nodes on the ith level to the ith term of the sequence

$$\langle 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, ...\rangle.$$

Our construction breaks into even and odd stages, handling the R- and P-requirements respectively. The former are easily satisfied, while the latter require a construction.

Even stages. At stage $s = 2e$ we have $\langle p_s, T_s, g_s\rangle$. Given the e-th computable 3-branching tree, we look for an extension of p_s that avoids it. To do this, we look for r such that the e-th branching tree is a k_r-branching tree and is describe by φ_{e_r} (we are using $\langle e_r, k_r\rangle$). Now, we look for an extension of p_s in T_s, called it τ, such that $g_s(\tau) = r$, we focus on the successor of τ that avoids φ_{e_r}. That successor will be p_{s+1}. We define T_{s+1} to be the subtree of T_s that extends p_{s+1} and we let g_{s+1} to be 0 for all the initial segments of p_{s+1} and be the same as g_s for the other members of T_{s+1}.

Odd stages. At $s = 2e + 1$ we have $\langle p_s, T_s, g_s\rangle$. If there is $\tau \in T_s$ such that φ_e^A is not total for all branches A of T_s extending τ then we define $p_{s+1} = \tau$ and we make T_{s+1} and g_{s+1} as in stage $2e$. If there is not such an extension, we define p_{s+1} to be the first node extending p_s such that $g_s(p_{s+1}) = 1$. To define T_{s+1}, we want to prune T_s in such a way that φ_e^A is in a computable trace bounded by 3^n whenever A is a branch, and g_{s+1} should then be defined accordingly.

Specifically, under the assumption above we define T_{s+1} and g_{s+1} by the following steps:

1. At every stage t there is at most one node entering $T_{s+1,t}$ that is a successor of a node with $g_{s+1} \neq 0$.

2. At every stage t, if we declare that $\sigma \in T_s$ will belong to T_{s+1} and σ is not the successor of a node with $g_{s+1} \neq 0$ then $\sigma \in T_{s+1,t+1}$.

3. We will fix an enumeration of $\omega^{<\omega}$. After adding the nodes of the rule above, we will give an opportunity to the nodes that are successors of a node with $g_{s+1} \neq 0$ by the order of the enumeration.

4. At stage $t = 0$, $p_{s+1} \in T_{s+1,0}$, $g_{s+1,0}(\sigma) = 0$ for all $\sigma \prec p_{s+1}$ and $g_{s+1}(p_{s+1}) = g_s(p_{s+1}) = 1$. Here we start the next stage.

5. If q just entered $T_{s+1,t}$ and it is a successor of a node with $g_{s+1} \neq 0$, we will look for $\tau_\sigma \in T_s$ extending σ, for each leaf σ of $T_{s+1,t}$, and $\tau \in T_s$ extending q such that there is $n, m \in \omega$ with $m > |q|$, $n < m$ and $\varphi_e^\tau(n) \neq \varphi_e^{\tau_\sigma}(n)$: Also, we want φ_e^τ and $\varphi_e^{\tau_\sigma}$ to converge for the whole interval $[0, m]$. Finally, $g_s(\tau_\sigma) = g_{s+1}(\sigma)$.

6. For every leaf σ of $T_{s+1,t}$ we declare that the nodes between σ and τ_σ will be in T_{s+1}, that there is no split between these nodes, and that $g_{s+1}(\rho) = 0$ for such a node. We also change the value of $g_{s+1}(\sigma)$ to 0 and set $g_{s+1}(\tau_\sigma) = g_s(\tau_\sigma)$.

7. Given τ from step 5 (the one extending q), we look at the subsequence $\langle g_{s+1}(q\restriction i_m)\rangle$ made by all the nonzero values of $\langle g_{s+1}(q\restriction i) : i < |q|\rangle$ and we look for an extension of τ, called it q', in T_s such that $\langle g_{s+1}(q\restriction i_m)\rangle \frown g_s(q')$ is an initial segment of $\langle 1, 1, 2, 1, 2, 3, ...\rangle$.

8. Given q and q' as the rules above, we declare that $q' \in T_{s+1,t}$ as well as all its successors in T_s and initial segments; we also declare that $g_{s+1}(q') = g_s(q') \neq 0$; that there are no splits in T_{s+1} between q and q', and that $g_{s+1}(\sigma) = 0$ for all $q \preceq \sigma \prec q'$. Now we start the next stage.

This construction produces a triple $\langle p_{s+1}, T_{s+1}, g_{s+1}\rangle$ with the desired form. Moreover, the function g_{s+1} changes value from g_s at each node at most once and will not change once a successor of the node enters T_{s+1}. Since all nodes of T_{s+1} have a successor, g_{s+1} is computable. Furthermore, by step 5 we know that φ_e^A is total whenever A is a branch of T_{s+1}. So to complete the proof we just need to check that there is a single computable trace capturing all this functions. This is provided by

$$U_e = \{\varphi_e^\sigma\restriction n : n \in \omega, \sigma \in T_{s+1}\}.$$

That is, we claim that the nth level of U_e has size at most 3^n. This is because T_{s+1} is a 3-tree, so the only way this could fail would be if there were splitting nodes in T_{s+1} which saw no new convergence of φ_e (since then by waiting for more splittings further up the tree, we could produce more than 3^n values of the computation and hence more than

3^n-many nodes on U_e of height n). However, by construction we generate new values of φ_e^σ exactly when we split, so this cannot happen.

So the degree of the function $A = \bigcup_{n\in\omega} p_n$ is as we desired. □

Question 2.29. *Is there a globally-branch surviving degree that is not* 3*-globally tree surviving?*

Question 2.30. *Is it true that $k, s \geq 3$ there is a k-globally branch surviving degree that is not s-globally tree surviving?*

Clearly, a globally-branch surviving degree is a k-globally branch surviving degree. Also, a globally-tree surviving degree is a k-globally tree surviving degree. To really show that this degrees make a hierarchy, we need to make the following observations.

Lemma 2.31. *Given $k \geq s \geq 2$, a k-globally tree surviving degree is also an s-globally tree surviving degree.*

Proof. By definition, an s-tree is also a k-tree, so, if you survive all k-trees, in particular, you survive all s-trees. □

Lemma 2.32. *Given $k \geq s \geq 2$, a k-globally branch surviving degree is also an s-globally branch surviving degree.*

Proof. Given $k \geq s$ notice that if A is not an s-globally surviving degree then all the total functions that it computes are the branch of a computable s-branching tree. Now, notice that we can make a computable s-branching tree of ω^ω into a k-branching tree in a uniform way, so all the total functions that A computes are a branch of a k-branching tree.

This shows that A is not k-globally surviving. □

Note that contrary to what the name suggests, being k-globally surviving is weaker than being k-surviving: a function that is k-surviving is also k-globally surviving since $(k+1)^{<\omega} \subseteq (\omega)^{<\omega}$ and the fact that if $T \subseteq (\omega)^{<\omega}$ is a k-branching tree then $T \cap (k+1)^{<\omega}$ is a k-tree. We have:

Lemma 2.33. *If A is a k-surviving degree then it is a k-globally surviving degree.*

The converse fails badly, however:

Theorem 2.34. *There is a Turing degree A that is globally tree surviving but not 2-surviving.*

Proof. As before, we will do forcing with stems and trees, but this time we will use accelerating subtrees of $\omega^{<\omega}$.

Definition 2.35. *An* accelerating tree *is a subtree $T \subseteq \omega^{<\omega}$ such that if $\sigma \in T$ is a splitting node with n splitting initial segments, then σ has more than $n+2$ immediate successors.*

We will force with conditions of the form $\langle p, T\rangle$ were $p \in \omega^{<\omega}$ and T is a computable accelerating subtree of $\omega^{<\omega}$ extending p.

We will construct $A \in \omega^{<\omega}$ with the following two requirements:

- $R_{e,k}$: A is not a branch of the e-th computable k-subtree of $\omega^{<\omega}$. This will make A a globally tree surviving degree.

- P_e: φ_e^A is either not total or there is n such that $\varphi_e^A(n) \geq 3$ or there is a computable 2-branching tree of $3^{<\omega}$ such that φ_e^A is a branch of it. This will make A not a 2-surviving degree.

We will set $\langle p_0, T_0\rangle = \langle \emptyset, \omega^{<\omega}\rangle$.

At stage $s = 2\langle e, k\rangle$, if U_e is the e-th computable k-branching subtree of $\omega^{<\omega}$ then we look for an extension of p_s that has at least $k+1$ successors and we define p_{s+1} to be the successor that is not in U_e. We define T_{s+1} to be the subtree of T_s extending p_{s+1}. Since T_s is computable, given p_{s+1}, T_{s+1} is computable.

At stage $s = 2e+1$ we have four cases:

Case 1 If there is $\sigma \in T_s$ extending p_s such that φ_e^σ is not total, then let $p_{s+1} = \sigma$ and define T_{s+1} to be the subtree of T_s extending p_{s+1}. Here we satisfy P_e by avoiding totality.

Case 2 If there is $\sigma \in T_s$ extending p_s and $n \in \omega$ such that $\varphi_e^\tau(n) \geq 3$, then let $p_{s+1} = \sigma$ and define T_{s+1} to be the subtree of T_s extending p_{s+1}. This satisfy P_e.

Case 3 If there is $\sigma \in T_s$ extending p_s such that there a no $\tau_1, \tau_2 \in T_s$ extending σ such that $\varphi_e^{\tau_1} \neq \varphi_e^{\tau_2}$. In this case we will satisfy P_e by the fact that φ_e^A with A extending σ is computable if it is total.

Case 4 For this case, we need that the other three cases are not happening. We will define $p_{s+1} = p_s$ and we will prune T_s.

We will define this prune by levels, here nodes at level n will have exactly n splits before them. Furthermore, during the prune we will define a U_e a 2-tree (remember that a 2-tree and a 2-branching tree are the same) such that for all the branches A of T_{s+1}, φ_e^A is a branch of U_e.

At level 0 we will have a unique node: p_{s+1}. Furthermore, we will add $\emptyset$ to U_e.

Now, assume that $\tau \in T_s$ is a node in level $n-2$ with $n \geq 2$. We know that there are exactly $n-2$ splits before τ and that, in order to make T_{s+1} accelerating, the next split should have at least n nodes.

To do this we will take $\sigma \in T_s$ extending τ that has at least 3^n successors $\tau_0^0, ..., \tau_{3^n-1}^0$ in T_s. We will look for $m \in \omega$, and $\sigma_i \in T_s$ extending τ_i^0 such that $\varphi_e^{\sigma_i}(t) \downarrow$ for $t < m$ and that there are $i_0, j_0 < 3^n$ such that $\varphi_e^{\sigma_{i_0}} \restriction m \neq \varphi_e^{\sigma_{j_0}} \restriction m$.

Let $m_0 < m$ be the minimal number such that there are $i.j < 3^n$ with $\varphi_e^{\sigma_i}(m_0) \neq \varphi_e^{\sigma_j}(m_0)$. Since we know that $\varphi_e^{\sigma_i}(m_0) < 3$, we have that there is $k_0 < 3$ with at least 3^{n-1} σ_i such that $\varphi_e^{\sigma_i}(m_0) = k_0$.

We will define τ_0' to be one of the σ_i such that $\varphi_e^{\tau_0'}(m_0) \neq k_0$ and we will define $\tau_0^1, ..., \tau_{3^{n-1}}^1$ to be 3^{n-1} of the σ_i with $\varphi_e^{\sigma_i}(m_0) = k_0$.

To clarify, at this moment we have $\tau_0' \in T_s$ (that is a candidate to be a member of the n-th level) and 3^{n-1} nodes of T_s, $\tau_0^1, ..., \tau_{3^{n-1}}^1$, such that for $i, j < 3^{n-1}$ $\varphi_e^{\tau_i^1} \restriction (m_0 + 1) = \varphi_e^{\tau_j^1} \restriction (m_0 + 1)$, $\varphi_e^{\tau_i^1}(m_0) \neq \varphi_e^{\tau_0'}(m_0)$ but $\varphi_e^{\tau_i^1} \restriction m_0 = \varphi_e^{\tau_0'} \restriction m_0$.

We can repeat the process to get $m_1 \in \omega$, $m_1 > m_0$, $\tau_1' \in T_s$ and $\tau_0^2, ..., \tau_{3^{n-2}}^2 \in T_s$ all of them extending one of the nodes τ_j^1 and have similar properties as the above paragraph. In other words: for $i, j < 3^{n-2}$ $\varphi_e^{\tau_i^2} \restriction (m_1 + 1) = \varphi_e^{\tau_j^2} \restriction (m_1 + 1)$, $\varphi_e^{\tau_i^2}(m_1) \neq \varphi_e^{\tau_1'}(m_1)$ but $\varphi_e^{\tau_i^2} \restriction m_1 = \varphi_e^{\tau_1'} \restriction m_1$.

Repeating this process n times, we get $m_0 < ... < m_{n-1} \in \omega$, $\tau_0', ..., \tau_{n-1}' \in T_s$ all of them extending τ (even more specifically, they all come from a single split above τ) such that $\varphi_e^{\tau_{i_0}} \restriction m_j = \varphi_e^{\tau_{i_1}} \restriction m_j$ for all $j \leq i_0, i_1 < n$. In particular, if we find τ_i extending τ_i' such that $\varphi_e^{\tau_i}(t) \downarrow$ for $t < m_n + 1$, we have that the tree created by $\varphi_e^{\tau_i} \restriction m_n$ is a 2-tree. Notice that for all $i < n$, τ_i have exactly n splits before it and the n-th split has n successors. At this moment, we include τ_i at level n and we include $\varphi_e^{\tau_i} \restriction m_n$ to U_e. There will be no more extension of τ in level n.

We now define T_{s+1} to be the subtree generated by all the levels describe above. T_{s+1} is an accelerating tree and for all the branches A of T_{s+1}, φ_e^A is a branch of U_e. This satisfy the requirement P_e.

The above argument shows that the Turing degree of $A = \bigcup_{n\in\omega} p_n$ is globally tree surviving but not 2-surviving, so we are done. □

At the same time, the two notions are not too far apart in some sense.

Theorem 2.36. *Given $k \geq 2$, there is a k-globally surviving degree that is not ℓ-globally surviving for $\ell \geq k+1$. Also, this degree is computable traceable.*

So despite the difference between the two, there is a parallel between survival and global survival (the proof is the same as in Theorem 2.22).

Finally, we rewrite the final results of section 2.3 with all of our new terminology:

Lemma 2.37 (2.14)**.** *If a Turing degree is not a $k+1$-globally surviving degree but it is a k-globally surviving degree, then it is a k-surviving degree.*

Lemma 2.38 (2.15)**.** *Any Turing degree that is a k-globally surviving degree either is a $k+1$-globally surviving degree or a k-surviving degree.*

2.6 Combinatorial lemmas and localization properties

In this section we are going to build tools that will help us split $\mathfrak{L}_k$ and $\mathfrak{v}_k^g$ (see section 2.2. For this setting it is better to understand some of the processes as combinatorial principles instead of parts of a forcing argument. Because of that, the following lemmas come in pairs: one is a combinatorial statement and the following one is the forcing result. The first pairs of result has a big resemblance with the proof of Theorem 2.34.

Lemma 2.39. *Given* $\{f_i : i \in I\} \subseteq 3^\omega$ *with* $|I| = 3^n$ *you can find* $S \subseteq I$ *with* $|S| = n$ *such that* $\{f_i \restriction n : i \in S, n \in \omega\}$ *is a 2-tree.*

Proof. We will do the proof by induction.

For $n = 0$ and $n = 1$ it is trivially true.

Now, assume that it is true for n, we will prove it for $n + 1$.

Given $\{f_i : i \in I\} \subseteq 3^\omega$ with $|I| = 3^{n+1}$ if all of them are the same function then take the first $n + 1$ of them, they make trivially a 2-tree. On the other hand, if there are two of them that are different, find the first natural number m such that two of them differ. Notice that, using a pigeon hole principle, there is a value $k \in 3$ such that there is $J \subseteq I$, with $|J| \geq 3^n$ such that for all $i \in J$ we have $f_i(m) = k$.

Now, take $i_0 \in I$ such that $f_{i_0}(m) \neq k$ and let $S' \subseteq J$ be the index set of size n given after using the induction hypothesis over J. Notice that $\{f_{i_0} \restriction j : j \in \omega\} \cup \{f_i \restriction j : i \in S', j \in \omega\}$ forms a 2-tree and that $S = S' \cup \{i_0\}$ has size $n + 1$. □

Lemma 2.40. *There is a forcing notion that adds a function from* ω *to* ω *that is a branch of any* k*-tree from the ground model but such that all reals in* 3^ω *are a branch of a 2-tree in the ground model.*

Proof.

Definition 2.41. *We say that* $T \subseteq \bigcup_{m\in\omega} \prod_{n\in m}^{m} (n+1)$ *is an accelerating tree if and only if it is a subtree of* $\bigcup_{m\in\omega} \prod_{n\in m}^{m} (n+1)$, *if every node has an extension that splits and given* $\sigma \in T$ *such that there are* $k_i \in \omega$, $i < n$, *such that* $\sigma \restriction k_i$ *is a splitting node (i.e.,* σ *has* n *splits before it) then* σ *has either* 1 *successor or at least* $n + 2$.[5]

[5] *During the conference 'Set Theory of the Reals', BIRS-CMO Oaxaca, August 2019, it was brought to our attention that this forcing was originally defined by Geschke in [10] as Miller Lite Forcing.*

Let $\mathbb{P}$ be the forcing notion whose conditions are of the form $\langle \tau, T \rangle$ with $\tau \in \bigcup_{m\in\omega} \prod_{n\in m}^{m} (n+1)$ and T an accelerating subtree of $\bigcup_{m\in\omega} \prod_{n\in m}^{m} (n+1)$ extending τ. We say that $\langle \tau', T' \rangle \leq \langle \tau, T \rangle$ if and only if $\tau \subseteq \tau'$, $T' \subseteq T$ and $\tau' \in T$.[6]

For a node $\rho \in T$, let

$$T_\rho = \{\tau \in T : \tau \subseteq \rho \ \vee \ \rho \subseteq \tau\}.$$

Notice that given any k-tree $U \subseteq \omega^{<\omega}$ and a condition $\langle \tau, T \rangle$, there is $\rho \in T$ that is not a node in U (for example, go to a split with $k+1$ nodes, one of them is not in U). Furthermore, if we take the condition $\langle \rho, T_\rho \rangle$, none of the branches of T_ρ are branches of U. This shows that forcing with accelerating tree forcings adds a function from ω to ω that is not a branch of any k-tree from the ground model.

Now, we need to argue that all reals in 3^ω are a branch of a 2-tree in the ground model. Notice that the forcing is the set theoretical version of the forcing used in Theorem 2.34 (originally appear in [23] joint work with Noah Schweber). From that proof, translating from computability theory to set theory, as follows, give us the desired result: instead of talking about a Turing functional φ_e we will talk about a name of a function $\dot{f}$, then we change all requirements that ask φ_e^τ to do "blah" to finding a condition $\langle \tau, T_\tau \rangle$ that forces $\dot{f}$ to do "blah".

The final part of the theorem is also a corollary of Lemma 2.48 letting $\kappa = 1$. □

Lemma 2.42. *Given $\{f_i^j : i \in I, j \in k\} \subseteq 3^\omega$ with $k \in \omega$, $|I| = N(n,k)$ a big enough number and $m \in \omega$ such that*

(a) $\{f_i^j \upharpoonright l : i \in I, j \in k, l \in m\}$ is a 2-tree and

[6] We decide to define the acceleration tree forcing using pairs to create a stronger resemblance to the effective analogue of accelerating trees of ω^ω . Furthermore, this will allow us to easily define $(T)^0$ in Lemma 2.48.

(b) such that if $f_i^j\restriction m = f_s^t\restriction m$ *with* $t \neq j$ *we have that* $f_i^j = f_s^t$

then you can find $S \subseteq I$ *with* $|S| = n$ *such that* $\{f_i^j\restriction l : i \in S, j \in k, l \in \omega\}$ *is a 2-tree.*

Proof. We will prove this by induction over k.

At $k = 1$, we need $N(n, 1) \geq 3^n$ so that we can use Lemma 2.39 to be done.

Now, assuming we have the case for k we will prove it for $k+1$. We need $N(n, k+1) \geq 3^{N(n,k)}$, with this we can use Lemma 2.39 over $\{f_i^k : i \in I\}$ to get $J \subseteq I$ such that $|J| = N(n, k)$ and $\{f_i^t\restriction l : i \in J, t = k, l \in \omega\}$ is a 2-tree. Now, we can use our induction hypothesis over $\{f_i^t : i \in J, t \in k\}$ to get $S \subseteq J$ of size n such that $\{f_i^j\restriction l : i \in S, j \in k, l \in \omega\}$ is a 2-tree.

We just need to show that

$$\{f_i^j\restriction l : i \in S, j \in k+1, l \in \omega\} = \{f_i^k\restriction l : i \in S, l \in \omega\} \cup \{f_i^j\restriction l : i \in S, j \in k, l \in \omega\}$$

is a 2-tree.

Assume that we have $a \in \omega$ and $\langle i, j\rangle, \langle s, t\rangle, \langle g, h\rangle \in S \times (k+1)$ different between them such that $f_i^j\restriction a = f_s^t\restriction a = f_g^h\restriction a$. We have to show that

$$|\{f_i^j\restriction(a+1), f_s^t\restriction(a+1), f_g^h\restriction(a+1)\}| \leq 2.$$

Taking into account that $\{f_i^j\restriction l : i \in I, j \in k+1, l \in m\}$, $\{f_i^k\restriction l : i \in S, l \in \omega\}$ and $\{f_i^j\restriction l : i \in S, j \in k, l \in \omega\}$ are 2-trees, the only case left to check is when $a \geq m$ and j, t and h are not all the same but at least one of them is equal to k. Without lost of generality, assume that $h = k$ and $j \neq k$.

Since $a \geq m$ we have that $f_g^k\restriction m = f_i^j\restriction m$. Using the fact that $j \neq k$, and our theorem's hypothesis, we have that $f_g^k = f_i^j$, so

$$\{|f_i^j\restriction(a+1), f_s^t\restriction(a+1), f_g^h\restriction(a+1)\}| \leq 2.$$

□

It is important to remark that in these combinatorial lemmas it is never used that the domain of the functions is ω, so these lemmas are also true for 3^n.

The following definitions will facilitate our technical discussion.

Definition 2.43. *A forcing notion has the k-localization property if and only if every function in ω^ω in the generic extension is a branch of a k-tree from the ground model.*

Definition 2.44. *A forcing notion has the $(k+1)^\omega$-localization property if and only if every function in $(k+1)^\omega$ in the generic extension is a branch of a k-tree from the ground model.*

Two examples of simplification done by this definitions can be seen in Corollary 2.13 and Lemma 2.40 since they can be rewritten as

Corollary 2.45 (2.13)**.** *If a forcing has the $k+1$-localization property but it does not have the k-localization then it do not have the $(k+1)^\omega$-localization property.*

Corollary 2.46. *If a forcing notion has the $k+1$-localization property and the $(k+1)^\omega$-localization property then it has the k-localization property.*

Lemma 2.47 (2.40)**.** *There is a forcing notion with the $(2+1)^\omega$-localization property but without the k-localization property, for any $k \in \omega$.*

Newelski and Roslanowski, in [20], define the k-localization property. This property was deeply study later by Roslanowski, in [26], and by Zapletal, in [39]. They found that the k-localization property is preserved under most of the used countable support product and iteration of proper forcings.

Our forcing does not have the 2-localization property, it will have a version of that for 3^ω: the 3^ω-localization property. Our proof will resemble the one did by Newelski and Roslanowski, nevertheless, it is possible that there are results in the lines of the other two papers.

Lemma 2.48. *Countable product of the accelerating tree forcing has the 3^ω-localization property.*

Proof. First, given a tree and $n > 0$, we let $(T)^n$ be the set of all nodes such that they are the successors of the n-th split. As a convention, given $p = \langle s, T\rangle$ a forcing condition, we have that $(T)^0 = \{s\}$. Now, given elements of the accelerating tree forcing we will define for $n \geq 1$, $p = \langle s, T\rangle \leq_n p' = \langle s', T'\rangle$ if and only if $\langle s, T\rangle \leq \langle s', T'\rangle$ and $(T')^k = (T)^k$, for all $1 \leq k \leq n$, and $p \leq_0 p'$ if and only if $p \leq p'$. Notice that, since these are subtrees of $\bigcup_{m\in\omega} \prod_{n\in m}^{m} (n+1)$, these orders have the fusion property and satisfy Axiom A (as in [4]).

Assume that we have a countable support product of the accelerating tree forcing of length κ. Call the final partial order $\mathbb{P}_\kappa$, as notation we will express $q \in \mathbb{P}_\kappa$ as $q = \langle r, T\rangle$ and $q(\alpha) = \langle r(\alpha), T(\alpha)\rangle$.

Given $F \in [\kappa]^{<\omega}$ and $\eta : F \to \omega$, we define $p \leq_{F,\eta} q$ if and only if $p \leq q$ and for all $\alpha \in F$ we have that $p(\alpha) \leq_{\eta(\alpha)} q(\alpha)$. Furthermore, given $\sigma \in \prod_{\alpha\in F}(T(\alpha))^{\eta(\alpha)}$ and $p \in \mathbb{P}_\kappa$ we define $(p * \sigma)(\beta)$ to be $p(\beta)$ if $\beta \notin F$ and $(p * \sigma)(\beta) = \langle \sigma(\beta), T_{\sigma(\beta)}\rangle$ if $\beta \in F$ (following the notation of Lemma 2.40).

The orders $\leq_{F,\eta}$ have the fusion property under the following conditions: given $p_{n+1} \leq_{F_n,\eta_n} p_n$ with $\bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} supp(p_n)$ and $\lim_{n\to\infty} \eta_n(\alpha) = \infty$ for all $\alpha \in \bigcup_{n\in\omega} F_n$ we have that there exist $q \in \mathbb{P}_\kappa$ such that $q \leq_{F_n,\eta_n} p_n$ for all $n \in \omega$.

In order to complete the proof, it is enough to define the following concept and show the following claim:

Definition 2.49. *Given* $\Vdash_{\mathbb{P}}$ *"$\dot{f} \in 3^\omega$". We say that the 5-tuple* $\langle q, F, \eta, m, A\rangle$ *consolidates* $\dot{f}$ *if and only if the following is satisfied:*

1. $q = \langle r, T\rangle \in \mathbb{P}_\kappa$, $F \in [\kappa]^{<\omega}$, $\eta : F \to \omega$, $m \in \omega$.

2. $A \subseteq 3^{<m}$ *is a 2-tree,* $q \Vdash$ "$\dot{f}\restriction m \in A$".

3. *For each* $\sigma \in \prod_{\alpha\in F}(T(\alpha))^{\eta(\alpha)}$ *there is* $g \in A$ *such that* $q * \sigma \Vdash$ "$\dot{f}\restriction m = g$".

4. *If there is a condition* $q^* \leq_{F,\eta} q$, $M \in \omega$, $h \in 3^M$ *and* $\sigma_1 \neq \sigma_2 \in \prod_{\alpha\in F}(T(\alpha))^{\eta(\alpha)}$ *such that* $q^* * \sigma_1 \Vdash$ "$\dot{f}\restriction M = h$" *but* $q^* * \sigma_2 \Vdash$ "$\dot{f}\restriction M \neq h$" *then there is* $g \in A$ *such that* $q * \sigma_1 \Vdash$ "$\dot{f}\restriction m = g$" *and* $q * \sigma_2 \Vdash$ "$\dot{f}\restriction m \neq g$".

Claim 2.50. *Working in* V*, suppose that* $\Vdash_{\mathbb{P}}$ *"$\dot{f} \in 3^\omega$" and that* $\langle q, F, \eta, m, A\rangle$ *consolidates* $\dot{f}$*. Then there are* $M' > m$*,* $A' \subset 3^{<M'+1}$ *a 2-tree with* $A = A' \cap 3^{<m}$ *and* $q' = \langle r', T'\rangle \leq_{F,\eta} q$ *such that* $\langle q', F, \eta + 1, M', A'\rangle$ *also consolidates* f*.*

If we prove this claim, given $p \in \mathbb{P}_\kappa$ such that $p \Vdash$ "$\dot{f} \in 3^\omega$" we can define q_n, F_n, η_n, A_n, m_n as follows:

1. $q_0 = p$, $A_0 = \{\emptyset\}$ and $m_0 = 0$.

2. We write $supp(q_0) = \{\alpha_0^i : i \in \omega\}$ and let $F_0 = \{\alpha_0^0\}$.

3. We let $\eta_0(\alpha_0^0) = 0$. Clearly, $\langle q_0, F_0, \eta_0, m_0, A_0\rangle$ consolidates $\dot{f}$.

4. We define q_{n+1}, A_{n+1} and m_{n+1} as the result of the claim using q_n, A_n, F_n, η_n and m_n.

5. We write $supp(q_{n+1}) = \{\alpha^i_{n+1} : i \in \omega\}$ and let $F_{n+1} = F_n \cup \{\alpha^{j_n}_{i_n}\}$ with $\langle i_n, j_n\rangle$ following the usual enumeration of $\omega \times \omega$.

6. Finally, we let $\eta_{n+1}(\alpha) = \eta_n(\alpha) + 1$ for $\alpha \in F_n$ and $\eta_{n+1}(\alpha^{j_n}_{i_n}) = 0$. Again, notice that $\langle q_{n+1}, F_{n+1}, \eta_{n+1}, m_{n+1}, A_{n+1}\rangle$ consolidates $\dot{f}$.

With this, we can use the fusion property with $q_{n+1} \leq_{F_n,\eta_n} q_n$ and get $q \in \mathbb{P}_\kappa$ such that $q \leq_{F_n,\eta_n} q_n$ for all n so we have that $q \Vdash$ "$\dot{f} \in [\bigcup_{n\in\omega} A_n]$".

This shows that all the functions in 3^ω in the extension are a branch of a ground model 2-tree.

It is important to notice that the properties given to the 2-tree in the above definition and claim align with those in the hypothesis of Lemma 2.42. Specifically, clauses $1-3$ of the definition make A a tree like the one ask in part (a) of the lemmas hypothesis. Furthermore, clause 4 aligns with hypothesis (b). It should be no surprise that we will use Lemma 2.42 in the proof. In order to do that, we need a couple of observations and reductions.

Below we assume that $\langle q, F, \eta, m, A\rangle$ consolidates $\dot{f}$ and we fix $\beta \in F$. Let $\nu : F \to \omega$ such that $\nu(\alpha) = \eta(\alpha)$ if $\alpha \neq \beta$ and $\nu(\beta) = \eta(\beta) + 1$. To show Claim 2.50 we will look for $q' \leq_{F,\eta} q$ such that $\langle q', F, \nu, M', A'\rangle$ consolidates $\dot{f}$ (instead of $\langle q', F, \eta + 1, M', A'\rangle$). This is enough since, changing the β we are using, we can go from η to $\eta + 1$ using $|F|$ intermediate ν functions.

Let $n = \nu(\beta) + 2$ and we let $k = \left|\prod_{\alpha\in F}(T(\alpha))^{\eta(\alpha)}\right|$. We will also use $N(n, k)$ as defined in Lemma 2.42.

Notice that, pruning the trees of q if necessary, we can find $p_0 = \langle r_0, T_0\rangle \leq_{F,\eta} q$ such that for each $t \in (T_0(\beta))^{\eta(\beta)} = (T(\beta))^{\eta(\beta)}$ we have that

$$|\{s \in (T_0(\beta))^{\nu(\beta)} : s \text{ extends } t\}| = |\{s \in (T_0(\beta))^{\eta(\beta)+1} : s \text{ extends } t\}| = N(n,k).$$

It is important to remark that, in general, $p_0 \not\leq_{F,\nu} q$.

Observation A *Suppose $M \in \omega$ and $\sigma \in \prod_{\alpha \in F}(T_0(\alpha))^{\nu(\alpha)}$. Then there exists $q^* \in \mathbb{P}_\kappa$ such that $q^* \leq_{F,\nu} p_0$ and $q^* * \sigma$ forces a value to $\dot{f}\restriction M$.*

Proof. First find $p^* \leq p_0 * \sigma$ that forces a value to $\dot{f}\restriction M$. Afterwards, carefully define q^* in such a way so that $q^* * \sigma = p^*$ and whenever $\tau \in \sigma \in \prod_{\alpha \in F}(T_0(\alpha))^{\nu(\alpha)}$ do not share any coordinate with σ then $(q^* * \tau)(\alpha) = p_0(\alpha)$ for $\alpha \in F$. For a full detailed proof, see Lemma 1.7 of [5]. □

Observation B *For $M \in \omega$ there exists there exists $q^* \in \mathbb{P}_\kappa$, $q^* \leq_{F,\nu} p_0$, such that for every $\sigma \in \prod_{\alpha \in F}(T_0(\alpha))^{\nu(\alpha)}$ we have that $q^* * \sigma$ forces a value to $\dot{f}\restriction M$.*

Proof. You can apply observation A finitely many times. For a full detailed proof, see Corollary 1.10 of [5]. □

Now, given $p \leq_{F,\nu} p_0$ and $M > m$ we define

$$z_{p,M} = |\{a \in 3^M : \exists \sigma \in \prod_{\alpha \in F}(T(\alpha))^{\nu(\alpha)}(p * \sigma \Vdash \dot{f}\restriction M = a)\}|.$$

Notice that $z_{p,M} \leq k \cdot N(n,k)$. Therefore, we can find $p^+ = \langle r^+, T^+ \rangle \leq_{F,\nu} p_0$ and $M' > m$ such that $z_{p^+,M'}$ has the maximum value.

Passing to a $\leq_{F,\nu}$ condition we may also demand that for every $\sigma \in \prod_{\alpha \in F}(T^+(\alpha))^{\nu(\alpha)} = \prod_{\alpha \in F}(T_0(\alpha))^{\nu(\alpha)}$, the condition $p^+ * \sigma$ forces a value to $\dot{f}\restriction M'$.

Observation C *Suppose that* $\sigma_0, \sigma_1 \in \prod_{\alpha\in F}(T^+(\alpha))^{\nu(\alpha)}$*,* $M'' \geq M'$*,* $p \leq_{F,\nu} p^+$ *and* $a_0, a_1 \in 3^{M''}$*. If* $a_0 \neq a_1$ *and*

$$p * \sigma_0 \Vdash \dot{f}\restriction M'' = a_0 \textit{ and } p * \sigma_1 \Vdash \dot{f}\restriction M'' = a_1$$

then $a_0\restriction M' \neq a_1\restriction M'$*.*

Proof. Suppose towards a contradiction that $a_0\restriction M' = a_1\restriction M'$. We can find $p'' \leq_{F,\nu} p \leq_{F,\nu} p^+ \leq_{F,\nu} p_0$ such that for every $\sigma \in \prod_{\alpha\in F}(T^+(\alpha))^{\nu(\alpha)}$, the condition $p'' * \sigma$ forces a value to $\dot{f}\restriction M''$. Then, $z_{p'',M''} > z_{p^+,M'}$, a contradiction. □

For every $\sigma \in \prod_{\alpha\in F}(T^+(\alpha))^{\eta(\alpha)} = \prod_{\alpha\in F}(T(\alpha))^{\eta(\alpha)}$ there are $N(n,k)$ many $\rho^\sigma \in \prod_{\alpha\in F}(T^+(\alpha))^{\nu(\alpha)} = \prod_{\alpha\in F}(T_0(\alpha))^{\nu(\alpha)}$ such that for all $\alpha \in F \setminus \{\beta\}$ we have that $\sigma(\alpha) = \rho^\sigma(\alpha)$. Fix an enumeration of these ρ and define $\sigma^\frown i = \rho^\sigma_i$ (if we have $\sigma_1(\beta) = \sigma_2(\beta)$ then $\rho^{\sigma_1}_i = \rho^{\sigma_2}_i$ for all i).

Given $\sigma \in \prod_{\alpha\in F}(T^+(\alpha))^{\eta(\alpha)}$ and $i \in N(n,k)$, define $f^\sigma_i \in 3^{M'}$ to be such that $p^+ * \sigma^\frown i \Vdash$ "$\dot{f}\restriction M' = f^\sigma_i$". Since $\langle q, F, \eta, m, A\rangle$ consolidates $\dot{f}$, we have that $\{f^\sigma_i\restriction l : i \in N(n,k), t \in \prod_{\alpha\in F}(T^+(\alpha))^{\eta(\alpha)}, l \in m\} \subseteq A$ is a 2-tree such that if $f^{\sigma_1}_i\restriction m = f^{\sigma_2}_j\restriction m$ with $\sigma_1 \neq \sigma_2$ we have that $f^{\sigma_1}_i = f^{\sigma_2}_s$.

Furthermore, using observation C, we ensure clause (4) of the definition of consolidation. So any condition below p^+ will still satisfy it.

Now we can use Lemma 2.42 on $\{f^\sigma_i : i \in N(n,k), \sigma \in \prod_{\alpha\in F}(T^+(\alpha))^{\eta(\alpha)}\}$ so we can find $S \subseteq N(n,k)$ of size n such that

$$\{f^\sigma_i : i \in S, \sigma \in \prod_{\alpha\in F}(T^+(\alpha))^{\eta(\alpha)}\}$$

is a 2-tree.

To complete the claim, we use:

- M',
- $A' = \{f_i^t \upharpoonright l : i \in S, t \in k, l \leq M'\}$ and
- $q' \leq_{F,\eta} q$ define as $q'(\alpha) = p^+(\alpha)$ for all $\alpha \neq \beta$ and $q'(\beta) = \langle r'(\beta), T'(\beta)\rangle$ where $r'(\beta) = r^+(\beta)$ and $T'(\beta)$ is an accelerating subtree of $T^+(\beta)$ such that

$$(T'(\beta))^{\nu(\beta)} = \{\rho_i^\sigma(\beta) : \sigma \in \prod_{\alpha \in F} (T^+(\alpha))^{\eta(\alpha)}, i \in S\}.$$

□

Corollary 2.51. *Countable support product of accelerating tree forcing has the $(k+1)^\omega$-localization property for all $k \geq 2$.*

Proof. To prove this, it is enough to show that the $(k+1)^\omega$-localization property is implied by the $(s+1)^\omega$-localization property for $k \geq s \geq 2$, then, the result is a corollary of Lemma 2.48.

Fix a surjective function $f : (k+1) \to (s+1)$. Notice that this function induces a surjective function $f^* : (k+1)^\omega \to (s+1)^\omega$. Now, working in a generic extension given a s-tree T from the ground model, $(f^*)^{-1}[T]$ is a k-tree from the ground model.

Therefore, if in the generic extension $(s+1)^\omega$ is covered by s-trees from the ground model, then $(k+1)^\omega$ is covered by k-trees from the ground model. □

Now, the following definition can let us expand our last result a little more.

Definition 2.52. *Forcing with k-branching trees of $k^{<\omega}$ is the forcing notion that uses subtrees of $k^{<\omega}$ such that every node has either 1 or k successors.*

This forcing is used in [20] where Newelski and Roslanowski showed that this forcing has the k-localization property, i.e., that every function of ω^ω in the generic extension is the branch of a k-tree from the ground model. Notice that this property implies the $(k+1)^\omega$-localization property. A first step in order to investigate if the countable support products of forcings with the $(k+1)^\omega$-localization property still has the $(k+1)^\omega$-localization is true for a bigger spectrum of forcings than the accelerating tree forcing is to show the following lemmas, that are analogues of Lemma 2.42 and 2.48:

Lemma 2.53. *Given $\{f_i^j : i \in I, j \in a\} \subseteq (k+1)^\omega$ with $a \in \omega$, $|I| = N(n,a)$ a big enough number and $m \in \omega$ that makes $\{f_i^j \restriction l : i \in I, j \in a, l \in m\}$ a k-tree such that if $f_i^j \restriction m = f_s^t \restriction m$ with $t \neq j$ we have that $f_i^j = f_s^t$ then you can find $S \subseteq I$ with $|S| = n$ such that $\{f_i^j : i \in S, j \in l\}$ is a k-tree.*

Proof. This follows from the proofs of Lemma 2.39 and Lemma 2.42, in those lemmas we had $k = 2$. The same reasoning will give us this lemma. □

Lemma 2.54. *Countable support product of alternating accelerating tree forcing and forcing with k-branching trees of $k^{<\omega}$ has the $(k+1)^\omega$-localization property.*

Proof. Notice that the orders $\leq_n$ also make sense when forcing with k-branching trees of k^ω.

The proof in full detail will have the same extension as the proof of Lemma 2.48. Nevertheless, here we give a sketch of how to combine the technique used in [20] and the proof of 2.48.

Everything works the same changing 2 for k and 3 for $k+1$. Now, to show the analogue of Claim 2.50 we will have two cases:

1. If you are extending a node that comes from an accelerating tree, then use Lemma 2.54 instead of Lemma 2.48. Everything else works the same.

2. If you are extending a node that comes from a k-branching tree instead of using Lemma 2.54, it is enough to find a condition like p^+. Since the next split only has k successors, they naturally form a k-tree. Everything else works the same as the proof of Claim 2.50 or you can use the technique used in [20].

□

2.7 The split

This first results is the result of a conversation with Corey Switzer during the XIX Graduate Student Conference in Logic in Madison, Wisconsin, April 2018.

Lemma 2.55. *The accelerating tree forcing has the Sacks property.*

Proof.

Definition 2.56. *Given a function $f : \omega \to \omega \setminus \{0\}$ an slalom of growth f is a function $s : \omega \to [\omega]^{<\omega}$ such that $|s(n)| \leq f(n)$ for all n. We say that $g \in \omega^\omega$ goes thorugh s if and only if $g(n) \in s(n)$ for all n.*

Definition 2.57. *[30] We say that a forcing has the Sacks property if and only if there is $g \in V$, $g : \omega \to \omega \setminus \{0\}$ that diverges to infinity, such that for all $f \in \omega^\omega \cap V[G]$ there is a tree $T \in V$ such that f is a branch of T and the n-th level of T has size $g(n)$.*

Notice that, given an slalom of growth f, we can generate a tree T such that its n-th level has size $\prod_{i=0}^{n} f(i)$.

To show that the accelerating tree forcing has the Sacks property we will show that every real in $\omega^\omega \cap V[G]$ goes through an slalom $s \in V$ such that $|s(n)| \leq n!$.

Let $\mathbb{P}$ be the accelerating tree forcing. From Lemma 2.48 we know that given a name $\dot{f}$ such that $\Vdash_{\mathbb{P}} \dot{f} \in \omega^\omega$ then there is a condition $\langle p, T \rangle \in \mathbb{P}$ such that given $\sigma \in T^n$ (notation defined in Lemma 2.48) we have that there is $\tau_\sigma \in \omega^n$ such that $\langle \sigma, T_\sigma \rangle \Vdash \dot{f} \restriction n = \tau_\sigma$.

In V define $s : \omega \to [\omega]^{<\omega}$ such that

$$s(n) = \{\tau_\sigma(n) : \sigma \in T^n\}.$$

Since T is accelerating, we have that $|s(n)| \leq n!$. □

It is important to mention that in August 2019 it was brought to our attention that, in [10], Geschke showed indirectly that the accelerating tree forcing has the Sacks property[7]. We hope that this more direct proof is both more convinient for the reader and, maybe, useful for future works.

Theorem 2.58. *In the forcing extension generated after forcing with a countable support product of accelerating tree forcing* $cof(\mathcal{N}) = \aleph_1$.

Proof.

Theorem 2.59. *(from [6])* $cof(\mathcal{N})$ *is the cardinality of the smallest family* $\mathcal{F}$ *of slaloms of growth* f *(for* $f : \omega \to \omega \setminus \{0\}$ *increasing and diverging to infinity) such that all reals in* ω^ω *go through a slalom in* $\mathcal{F}$.

Theorem 2.60. *(from [30]) The countable support product of forcings that have the Sacks property have the Sacks property.*

Notice that, if a forcing has the Sacks property then $cof(\mathcal{N})^V = cof(\mathcal{N})^{V[G]}$.

[7]He forced the Dual Coloring Axiom and showed that this axiom implies $cov(\mathcal{N}) = \aleph_1$.

Since the accelerating tree forcing has the Sacks property, this shows that the model generated in Theorem 2.61 satisfies $cof(\mathcal{N})^{V[G]} = \aleph_1$.

In Theorem 2.62 we can also get the same using the fact that forcing with k-branching trees also has the Sacks property. □

Theorem 2.61. *It is consistent with ZFC that* $\forall k \geq 2(cof(\mathcal{N}) = \mathfrak{L}_k < \mathfrak{v}_k^g = \mathfrak{c})$.

Proof. Starting with a model of $ZFC+GCH$ we can make a countable support product of $\aleph_2$ accelerating tree forcing described in Lemma 2.40. Using Axiom A, as in [4], we know that the product preserves cardinals and that $\mathfrak{c} = \aleph_2$. Also, by Lemma 2.48, the resulting model will have $\mathfrak{L}_k = \mathfrak{L}_2 = \aleph_1$. Furthermore, by Lemma 2.58, $cof(\mathcal{N}) = \aleph_1$. We just need to show that in the extension $\mathfrak{v}_k^g = \aleph_2 = \mathfrak{c}$. In order to do this, we will use the tree equivalence show in Lemma 2.9 instead of k-globally adaptive predictors.

Let $\mathbb{P}_{\omega_2} = \prod_{\alpha\in\omega_2} \mathbb{Q}_\alpha$ be the countable support product of accelerating tree forcings. Let $G = \{c_\alpha : \alpha \in \omega_2\}$ be generic over $\mathbb{P}_{\omega_2}$. Now, for all $\beta < \omega_1$ let $k(\beta) \in \omega$ and let $T_\beta \subseteq \omega^\omega$ be a $k(\beta)$-tree in $V[G]$.

Now, in V, we can find $\dot{T}(\beta)$ a $\mathbb{P}_{\alpha(\beta)}$-name for some $\alpha(\beta) \in \omega_2$. So, there is $\gamma \in \omega_2$ such that $\alpha(\beta) < \gamma$ for all β. Therefore, we have that $T_\beta \in V[\{c_\alpha : \alpha < \gamma\}]$ for all $\beta \in \omega_1$.

Notice that if T is an accelerating tree in V, then at the split $k(\beta)+1$ it has a node that is not in T_β in $V[\{c_\alpha : \alpha < \gamma\}]$. Then avoiding T_β is a dense condition (in $V[\{c_\alpha : \alpha < \gamma\}]$) for accelerating trees from V.

Since c_γ is a V-accelerating forcing generic over $V[\{c_\alpha : \alpha < \gamma\}]$, then c_γ is not a branch of any k-tree in $V[\{c_\alpha : \alpha < \gamma\}]$, $k \in \omega$. Therefore, c_γ is not a branch of any T_β.

This shows that, in $V[G]$, ω^ω is not cover by $\{T_\beta : \beta \in \omega_1\}$. Since this was an arbitrary collection we have that $\mathfrak{v}_k^g = \aleph_2$ for all $k \in \omega$. □

This theorem proves that it is consistent that $\mathfrak{v}_k^g \neq \mathfrak{L}_k$ and answers the question from Blass about the identity of $\mathfrak{v}_k^g$: they indeed are a different cardinal characteristic from the ones that are known.

Furthermore, we can see that there are more ways to do this split:

Theorem 2.62. *For all $s \geq 2$ it is consistent with ZFC that $\forall k \geq 2(cof(\mathcal{N}) = \mathfrak{L}_{s+1} < \mathfrak{L}_s = \mathfrak{v}_k^g = \mathfrak{c})$.*

Proof. Following the same strategy as above, starting with a model of $ZFC + GCH$ we can make a countable support product of the accelerating tree forcing alternated with forcing with $s+1$-branching trees of $(s+1)^\omega$. Just as before, we know that the product preserves cardinals and that $\mathfrak{c} = \aleph_2$. Also, by Lemma 2.54, the resulting model will have $\mathfrak{L}_{s+1} = \aleph_1$ and by the final comment of Lemma 2.58, $cof(\mathcal{N}) = \aleph_1$. We just need to show that, in the extension, $\mathfrak{L}_s = \mathfrak{v}_k^g = \aleph_2 = \mathfrak{c}$.

Let $\mathbb{P}_{\omega_2} = \prod_{\alpha \in \omega_2} \mathbb{Q}_\alpha$ be the countable support product of accelerating tree forcings, when α is even and forcing with $s+1$ subtrees of $(s+1)^\omega$ when α is odd. Let $G = \{c_\alpha : \alpha \in \omega_2\}$ be generic over $\mathbb{P}_{\omega_2}$.

To see that $\mathfrak{v}_k^g = \aleph_2 = \mathfrak{c}$, we can do the same as Theorem 2.61, above. Now, showing that $\mathfrak{L}_s = \aleph_2 = \mathfrak{c}$ can be found in [20]. Nevertheless, for convenience to the reader, we give an argument here:

For all $\beta < \omega_1$ let $T_\beta \subseteq (s+1)^\omega$ be an s-tree in $V[G]$. In V, we can find $\dot{T}(\beta)$ a $\mathbb{P}_{\alpha(\beta)}$-name for some $\alpha(\beta) \in \omega_2$. So, there is $\gamma \in \omega_2$ such that $\alpha(\beta) < 2 \cdot \gamma + 1$ for all β. Therefore, we have that $T_\beta \in V[\{c_\alpha : \alpha < 2 \cdot \gamma + 1\}]$ for all $\beta \in \omega_1$.

Since $c_{2\cdot\gamma+1}$ is a generic for the forcing using $s+1$-branching trees (from V) of $(s+1)^\omega$ over $V[\{c_\alpha : \alpha < \gamma\}]$, then $c_{2\cdot\gamma+1}$ is not a branch of any s-tree in $V[\{c_\alpha : \alpha < \gamma\}]$ (same reasoning as in Theorem 2.61). Therefore, $c_{2\cdot\gamma+1}$ is not a branch of any T_β.

This shows that, in $V[G]$, $(s+1)^\omega$ is not cover by $\{T_\beta : \beta \in \omega_1\}$. Since this was an arbitrary collection we have that $\mathfrak{L}_k = \aleph_2$ for all $k \in \omega$. □

Theorem 2.62 shows that in order to have different values for $\mathfrak{v}_k^g$ and $\mathfrak{L}_k$ it is not necessary that every $\mathfrak{L}_s$ have the same value.

2.8 Further questions

First, although we have established a number of facts about the effective localization numbers (or rather, the highness properties corresponding to localization numbers), there are still important questions left open regarding to their interactions with better-understood highness properties. For example, we showed that computably traceable degrees could witness separations between levels of the global survival hierarchy, but the general role of computable traceability here is open. We do not know whether there are non-computably traceable degrees which are *not* k-surviving for $k \geq 2$, or whether "k-surviving for every k" implies not computably traceable. Similarly we can ask about the role of DNC in these contexts.

Second, returning to the relationship between local and global survivability notions, it seems likely that the combinatorial arguments of Theorems 2.9 and 3.10 used to establish that the respective degrees are not surviving can be modified to prove more than was necessary for those theorems — namely that for every k, every $(k+1)$-branching subtree of $(k+2)^{<\omega}$ (not just element of $(k+2)^\omega$) computable from the generic added by either of the following two forcings is contained in some computable $(k+1)$-branching subtree of $(k+2)^{<\omega}$:

- Forcing with computable accelerating trees with explicit stems.
- Forcing with k-branching subtrees of $(k+1)^{<\omega}$ which have a splitting node above every node, again with explicit stems.

Appropriately relativized, these preservation results would show that forcing with the product of the forcings above produces a real which is not $(k+1)$-surviving; meanwhile, by the previous results the real produced *is* both globally branch surviving and k-surviving. This would separate the global and local survivability notions in a strong way.

To finish this first set of effective questions, the following where suggested by the referee of [23], where these results appeared first:

Question 2.63. *Is there a degree which is k-globally branch surviving for every $k \in \omega$, but is not globally branch surviving? Similarly, is there a degree which is k-globally tree surviving for every k but not globally tree surviving?*

Question 2.64. *There is also a "dual" notion to survival: say that a k-tree T is* weakly k-engulfing *if for every computable $f \in (k+1)^\omega$ there is some path g through T which is eventually equal to f. What can we say about the degrees of weakly k-engulfing k-trees? (And we can ask the analogous question for k-branching trees.)*

Now, during this chapter, the $(k+1)^\omega$-localization property played a really important role. In order to show that it was preserved the proofs showed above are really case specific. This is useful for our purposes, but a question arises:

Question 2.65. *Can we show that the $(k+1)^\omega$-localization property is preserved under countable support iteration and products?*

This is likely to be possible. In [39], Zapletal showed that the n-localization property is preserved under countable support product and iteration of a broad variety of forcings (some kind of definable proper forcings).

Finally, notice that $\mathfrak{v}_k^g$ is a cardinal characteristic that is usually really closed to $\mathfrak{c}$. This is not true in cardinal arithmetic, but it is true in the Chicon diagram: all of these numbers are above $cof(\mathcal{N})$. So, in order to work with them, it is important to use forcing notions that are tame somehow (they cannot add Cohen or random reals, for example). In this case, we used a forcing notion with the $(k+1)^\omega$-localization property but, in the literature, there are examples of properties like the Sacks property, the n-localization property and, most recently, the shrink wrapping property (see Guzman-Hathaway [12]) that are also tame with reals. It is important to notice that most of these 'tameness' properties relates to the idea of keeping the new reals inside a tree of some sort.

Question 2.66. *Is there an underlying theorem (or meta theorem) that relates all (of some) of this tameness properties?*

One possible result could be that all of them are preserved under countable support product of a variety of forcings, but we do not have any good guess of whether this is possible or not.

Chapter 3

Sets of reals closed under Turing equivalence

3.1 Introduction

Studying sets of reals has been one of the main objectives of set theory since it was conceived by Cantor (while studying derived sets). Impressively, as remarked by Löwe [18], not much attention has been given to the study of sets of reals closed under Turing equivalence (except in a small list of articles mentioned in [18]). Although it looks simple, this question can have many modifications that span a really rich area of study.

For example, focusing on algebra, we can ask which degrees are needed to form a field closed under Turing equivalence (or a vector space over $\mathbb{Q}$). Furthermore, given $A \subseteq \mathbb{R}$ we wonder how does the operator of 'smallest field that contains A' and the operator 'close A under Turing equivalence' relate to each other. Do they conmute?

Focusing on measure we can ask which degrees are needed to form a measurable set that is closed under Turing equivalence. Finally, from the order perspective, we wonder which order types are attainable by sets closed under Turing equivalence.

All these questions are addressed in this work and, surprisingly, we discover that our work involving order types has an application to the automorphism problem of the

Turing degrees. This work, as well as recent work in effective cardinal invariants, adds support to the old standing idea that studying computability related questions with set theoretic tools, and vice-versa, gives birth to new and interesting results in both areas.

The chapter is organized as follows: the rest of this section will introduce some notation and give some observations that help explain why we choose to study certain questions in later sections. Sections 3.3, 3.4, 3.5 study sets of reals closed under Turing equivalence from the perspective of algebra, measure and order (respectively). In Section 3.6 we prove that any Borel function is the countable union of monotone functions (Theorem 3.43). This theorem is a key component of our application to the automorphism problem.

Section 3.7 centers on Theorem 3.51. This theorem is a restriction in the way that nontrivial automorphism of the Turing degrees interact with 1-generic degrees. Finally, Section 3.8 talks of possible ways to modify, or improve, Theorem 3.51 and Section 3.9 has open questions and conclusions.

3.2 Background and firsts definitions

Given a real number r, the set $\mathbb{R}_r = \{s \in \mathbb{R} : deg(r) = deg(s)\}$ is countable and dense. It is countable since we only have countably many Turing operators, it is dense since changing finite information does not change the Turing degree of a number. These two properties imply that $\mathbb{R}_r$, as an order, is isomorphic to $\mathbb{Q}$.

On the other hand, given a noncomputable r, $\mathbb{R}_r$ is not $\mathbb{R}_0 + r$ (since $2r \in \mathbb{R}_r$ but $2r - r = r \notin \mathbb{R}_0$) nor $r \cdot \mathbb{R}_0$ (since $0 \notin \mathbb{R}_r$ and, less triavially, $r + \frac{1}{2} \in \mathbb{R}_r$ but $1 + \frac{1}{2r} \notin \mathbb{R}_0$). So $\mathbb{R}_r$ is not a coset of $\mathbb{R}_0$ as a group (either with respect to addition or multiplication), it is not a subfield, ideal or a subgroup of $\mathbb{R}$.

Given $\mathcal{S}$ a subset of the Turing degrees, our objective is to understand how does $\mathbb{R}_{\mathcal{S}} = \bigcup_{deg(r)\in S} \mathbb{R}_r$ sit inside $\mathbb{R}$.

3.2.1 Notation

- We will denote real numbers by lower case letters, x, y, z, a, b, c.
- We will denote Turing degrees by bold lower case letters, $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a}, \mathbf{b}, \mathbf{c}$.
- Upper case letters, A, B, E, L, will denote subsets of real numbers.
- φ_e will denote the e-th Turing functional. If no oracle is expressed, we will assume that we are using a computable oracle.
- $\mathcal{D}$ is the set of all Turing degrees.
- Upper case calligraphic letters, $\mathcal{S}, \mathcal{A}, \mathcal{B}$, will denote subsets of the Turing degrees.
- $\mathbb{R}_x$ ($\mathbb{R}_{\mathbf{x}}$) are all the members of the real line, $\mathbb{R}$, that are Turing equivalent to x (resp. that have degree $\mathbf{x}$).
- Given a subset of reals A, $\mathbb{R}_A = \bigcup_{x\in A} \mathbb{R}_x$.
- Given a subset of Turing degrees $\mathcal{S}$, $\mathbb{R}_{\mathcal{S}} = \bigcup_{\mathbf{x}\in\mathcal{S}} \mathbb{R}_{\mathbf{x}}$.

3.3 Algebra

The first question that we will work on is which subsets of the Turing degrees $\mathcal{S}$, make $\mathbb{R}_{\mathcal{S}}$ a subfield of the real numbers.

From Theorem 6 in Rice [25], we know that $\mathbb{R}_0$ is a real closed field. In other words, $\mathbb{R}_0$ is an elementary subfield of the real numbers or, equivalently, a field that is linearly ordered and that has at least one root for all odd degree polynomials. Following the same proof as in [25], which boils down to showing that finding roots is a computable process, we can also show that $\mathbb{R}_{\bigwedge x}$ is a real closed field, where $\bigwedge x$ is the lower cone of x, i.e., all the reals that are computable from x.

Corollary 3.1. *For any $x \in \mathbb{R}$, the set $\mathbb{R}_{\bigwedge x}$ is a real closed subfield of $\mathbb{R}$.*

Furthermore, we can be sure that all of these fields are non-isomorphic.

Lemma 3.2. *Given $S, K \subseteq \mathbb{R}$ real closed subfields of $\mathbb{R}$ we have that S can be embedded into K if and only if $S \subseteq K$.*

Proof. The "if" side of the theorem is clearly true.

For the other implication, assume that S can be embedded into K. Since both are subfields of the real numbers, we have that $\mathbb{Q} \subseteq S \cap K$. Furthermore, any embedding between S and K will fix $\mathbb{Q}$.

Now, since these are ordered fields, given $x \in \mathbb{R}$, the real number is completely defined by it's order relation with respect to elements of $\mathbb{Q}$. Therefore, the only embedding from S to K is inclusion. □

Corollary 3.3. *The Turing degrees, with their order, can be embedded into the countable real subfields of $\mathbb{R}$.*

Notice that $\mathbb{R}_{\bigwedge x}$ is a subfield that contains $\mathbb{R}_x$. Is it the smallest one? It turns out that it is.

Lemma 3.4. *For any $x \in \mathbb{R}$, and $y \in \mathbb{R}_{\bigwedge x}$ there exists $z, w \in \mathbb{R}_x$ such that $z + w \in \mathbb{R}_y$.*

Proof. Without loss of generality, we will assume that $x, y \in [0,1]$

Assume that we have sequences of digits $\langle a_i : i \in \omega\rangle$, $\langle b_i : i \in \omega\rangle$ that represents the decimal expansion of x and y, respectively, and that are computable from x.

Define $s = \sum_{i=0}^{\infty} \frac{a_i}{10^{4i+1}}$ and $t = \sum_{i=0}^{\infty} \frac{b_i}{10^{4i+3}}$. Let $w = s + t$ and $z = -s$. Notice that you can easily distingish the decimal digits of x and y in w.

Clearly, s and t can be computed from x and y can compute t. Furthermore, we have that t can compute y and that s and w can compute x.

With all of this, we have that $w, z \in \mathbb{R}_x$ and that $w + z = t \in \mathbb{R}_y$. □

Theorem 3.5. *Given $y \in \mathbb{R}_{\bigwedge x}$ we can express y as a finite sum of elements of $\mathbb{R}_x$.*

Proof. Notice that, given $y \in [0,1]$ and $\langle b_i : i \in \omega\rangle$ that represents the decimal expansion of y computable from x, we can write $y = a_1 + a_2 + a_3 + a_4$ such that $a_k = \sum_{i=0}^{\infty} \frac{b_{4i+k}}{10^{4i+k}}$.

So, as we did in the proof of the above lemma, we can find $z_k, w_k \in \mathbb{R}_x$ such that $a_k = z_k + w_k$. So, we have that $y = \sum_{k=1}^{4} z_k + w_k$. □

Corollary 3.6. *$\mathbb{R}_{\bigwedge x}$ is the smallest field that contains $\mathbb{R}_x$.*

Corollary 3.7. *If $\mathcal{C}$ is a set of Turing degrees that is linearly ordered under $\leq_T$ then $\mathbb{R}_{\wedge\mathcal{C}}$ is the smallest field that contains $\mathbb{R}_{\mathcal{C}}$.*

Proof. If $\mathcal{C}$ has a maximum element, then this follows from the result above. Otherwise, take $\langle a_i : i \in \alpha\rangle \subseteq \mathcal{C}$ a cofinal sequence, with $\alpha \in \{\omega, \omega_1\}$ and notice that $\bigcup_{i\in\alpha} \mathbb{R}_{\wedge a_i}$ is a field. □

Now we can answer our first question:

Theorem 3.8. *Given $\mathcal{S}$ a set of Turing degrees, we have that $\mathbb{R}_{\mathcal{S}}$ is a field if an only if $\mathcal{S}$ is and ideal, i.e., $\mathcal{S}$ is closed under finite Turing joins (denoted by $\oplus$) and closed downward under Turing reduction.*

Proof. In one direction, if $\mathbb{R}_{\mathcal{S}}$ is a field, then given $x \in \mathbb{R}_{\mathcal{S}}$ we have that $\mathbb{R}_x \subseteq \mathbb{R}_{\mathcal{S}}$. Therefore, by Corollary 3.6, we have that $\mathbb{R}_{\wedge x} \subseteq \mathbb{R}_{\mathcal{S}}$, so $\mathcal{S}$ is closed downwards. Now, if $x, y \in \mathbb{R}_{\mathcal{S}}$ then, using an argument like the one in Lemma 3.4, there are $s \in \mathbb{R}_x$ and $t \in \mathbb{R}_y$ such that $s + t = x \oplus y$.

For the other direction, assume that $\mathcal{S}$ is downward closed and closed under joins. First, notice that $0, 1 \in \mathbb{R}_{\mathcal{S}}$ and that given $x \in \mathbb{R}_{\mathcal{S}}$ we have that $\frac{1}{x} \in \mathbb{R}_{\mathcal{S}}$. Finally, if $x, y \in \mathbb{R}$ we have that $x + y, x \cdot y \leq_T x \oplus y$. So, given $x, y \in \mathbb{R}_{\mathcal{S}}$, since $x \oplus y \in \mathcal{S}$, we can conclude that $x + y, x \cdot y \in \mathbb{R}_{\wedge x \oplus y} \subseteq \mathbb{R}_{\mathcal{S}}$. □

The following result shows that, when studying countable subfields of $\mathbb{R}$ closed under Turing equivalence, it is enough to look at chains of Turing degrees.

Corollary 3.9. *Given $F \subseteq \mathbb{R}$ a countable subfield closed under Turing equivalence there is a set $\mathcal{C}$ of Turing degrees that is linearly order under $\leq_T$, such that $F = \mathbb{R}_{\wedge \mathcal{C}}$.*

Proof. Let $F = \{a_i : i \in \omega\}$ be a countable subfield closed under Turing equivalence and denote by $\mathbf{a}_i$ the Turing degree corresponding to each element. Let $\mathcal{C} = \{\bigoplus_{i \in n} \mathbf{a}_i : n \in \omega\}$. From the theorem above, we know that there is a set $\mathcal{S}$ of Turing degrees that is and ideal such that $F = \mathbb{R}_{\mathcal{S}}$.

Notice that $\mathcal{C} \subseteq \mathcal{S}$ and that $\mathcal{S} \subseteq \wedge \mathcal{C}$. Therefore, $F = \mathbb{R}_{\mathcal{S}} \subseteq \mathbb{R}_{\wedge \mathcal{C}}$, $\mathbb{R}_{\mathcal{C}} \subseteq \mathbb{R}_{\mathcal{S}} = F$ and $\mathbb{R}_{\wedge \mathcal{C}} \subseteq F$ (since $\mathbb{R}_{\wedge \mathcal{C}}$ is the smallest field containing $\mathbb{R}_{\mathcal{C}}$). □

In view of these result, we can conjecture that the smallest subfield containing $\mathbb{R}_x \cup \mathbb{R}_y$ is $\mathbb{R}_{\wedge x \oplus y}$. Nevertheless, the following result hints to the contrary.

Definition 3.10. *Given $A \subseteq \mathbb{R}$ let $\langle A \rangle_{\mathbb{Q}}$ be the smallest $\mathbb{Q}$-vector space containing A.*

Corollary 3.11. *Given $x \in \mathbb{R}$, $\langle \mathbb{R}_x \rangle_{\mathbb{Q}} = \mathbb{R}_{\wedge x}$.*

Proof. This follows from Theorem 3.5. □

Theorem 3.12. *There are $x, y \in \mathbb{R}$ such that $\langle \mathbb{R}_x \cup \mathbb{R}_y \rangle_{\mathbb{Q}} \neq \mathbb{R}_{\wedge x \oplus y}$.*

Proof. First of all notice that, if $w \in \langle \mathbb{R}_x \cup \mathbb{R}_y \rangle_{\mathbb{Q}}$ then there are $s \in \mathbb{R}_{\wedge x}$ and $t \in \mathbb{R}_{\wedge y}$ such that $w = s + t$. Then, if w computes s it follows that w also computes t.

Using the classic result of embedability of upper semilattices as initial segments of the Turing degrees, by Lachlan and Lebeuf [17], there are degrees $\mathbf{x}, \mathbf{y}, \mathbf{w}$ such that $\mathbf{x}$ and $\mathbf{y}$ are distinct minimal degrees (the only degree below them is $\mathbf{0}$) and such that $\mathbf{x} <_T \mathbf{w} <_T \mathbf{x} \oplus \mathbf{y}$.

Notice that, given $w \in \mathbf{w}$, $w \neq s + t$ for any $s \in \mathbb{R}_{\wedge x}$ and $t \in \mathbb{R}_{\wedge y}$. To show this, suppose that $w = s + t$. Since w computes s we have that w computes t. We know that $\mathbf{y} \not\leq_T \mathbf{w}$, therefore t is computable. This means that $w = s + t$ is such that $w \in \mathbb{R}_x$ which is impossible since $\mathbf{x} <_T \mathbf{w}$. This shows that $w \notin \langle \mathbb{R}_x \cup \mathbb{R}_y \rangle_{\mathbb{Q}}$.

Furthermore, it also shows that $\mathbb{R}_{\mathbf{w}} \cap \langle \mathbb{R}_x \cup \mathbb{R}_y \rangle_{\mathbb{Q}} = \emptyset$ and that $\mathbb{R}_{x \oplus y} \not\subseteq \langle \mathbb{R}_x \cup \mathbb{R}_y \rangle_{\mathbb{Q}}$ (although, $x \oplus y \in \langle \mathbb{R}_x \cup \mathbb{R}_y \rangle_{\mathbb{Q}}$). In particular, $\mathbb{R}_{\langle \mathbb{R}_x \cup \mathbb{R}_y \rangle_{\mathbb{Q}}} = \mathbb{R}_{x \oplus y} \cup \mathbb{R}_x \cup \mathbb{R}_y \cup \mathbb{R}_0 \neq \mathbb{R}_{\wedge x \oplus y}$, since $\mathbb{R}_{\wedge x \oplus y} = \mathbb{R}_{x \oplus y} \cup \mathbb{R}_w \cup \mathbb{R}_x \cup \mathbb{R}_y \cup \mathbb{R}_0$. □

Corollary 3.13. *There is a collection of Turing degrees $\mathcal{S}$ such that $\langle \mathbb{R}_{\mathcal{S}} \rangle_{\mathbb{Q}}$ is not closed under Turing equivalence.*

Corollary 3.14. *There is a Q-vector space, V, inside $\mathbb{R}$ such that $\mathbb{R}_V$ is not a vector space.*

Thanks to a suggestion from Josiah Jacobsen-Grocott, we can translate these results to the field case

Theorem 3.15. *There are real numbers $x, y \in \mathbb{R}$, $i \in \omega$ such that the minimal field containing $\mathbb{R}_x \cup \mathbb{R}_y$ is not $\mathbb{R}_{\wedge x \oplus y}$.*

Proof. Let $x, y \in \mathbb{R}$ be such that $\mathbf{0}' = deg(x)' = deg(y)' = deg(x \oplus y)$. A construction of this pair of low degrees can be found making a small modification of the classic Sacks Splitting Theorem [29].

Notice that the elements of the minimal field containing $\mathbb{R}_x \cup \mathbb{R}_y$ are of the form $\frac{\sum_{i=0}^{n} s_i t_i}{\sum_{j=0}^{m} a_j b_j}$ with $s_i, a_j \in \mathbb{R}_{\wedge x}$ and $t_i, b_j \in \mathbb{R}_{\wedge y}$.

Since $\mathbf{0}' = deg(x)' = deg(y)'$, we know that $\mathbf{0}'$ can compute a list $\{f_e\}_{e \in \omega}$ with all the x computable reals, i.e., members of $\mathbb{R}_{\wedge x}$ and a list $\{g_e\}_{e \in \omega}$ with all the y computable reals. Therefore we can make a list of all $h_e = \frac{\sum_{i=0}^{n} f_{e_i} g_{h_i}}{\sum_{j=0}^{m} f_{k_j} g_{\ell_j}}$.

Using a diagonalization, there is a real w computable from $\mathbf{0}'$ such that for all $e \in \omega$ $w \neq h_e$.

This means that $w \in \mathbb{R}_{\wedge \mathbf{0}'} = \mathbb{R}_{\wedge x \oplus y}$, but w is not in the minimal field generated by $\mathbb{R}_x \cup \mathbb{R}_y$. □

These few examples do not exhaust the questions related to $\mathbb{R}_{\mathcal{S}}$ from an algebraic perspective. Nonetheless, we will leave some open questions and list them in the last section. Meanwhile, we will study these sets from a different perspective.

3.4 Measure

One of the firsts examples of pathological subsets of the real line appeared in the early 1900's when Guiseppe Vitali consruct a set that was not Lebesgue measurable. From that moment onward, finding strange subsets of the real line has been a usual part of the mathematical world. And, as we will see multiple times in these sections, there are examples of sets of reals A that have pathological properties but such that $\mathbb{R}_A$ does not have the same properties. Because of this, we believe that 'exploding' a set into a set that is closed under Turing equivalence is an operation that 'tames' certain sets.

Definition 3.16. *1. A* Vitali set*, $V \subseteq \mathbb{R}$, is a set such that for any $r \in \mathbb{R}$ there is $x \in V$ such that $x - r \in \mathbb{Q}$ and given different $x, y \in V$, $x - y \notin \mathbb{Q}$.*

2. A multiplicative Vitali set*, $M \subseteq \mathbb{R}$, is a set such that for any $r \in \mathbb{R} \setminus \{0\}$ there is $x \in M$ such that $\frac{x}{r} \in \mathbb{Q}$ and given different $x, y \in M$, $\frac{x}{y} \notin \mathbb{Q}$.*

Theorem 3.17 (Folklore)**.** *Vitali sets and multiplicative Vitali sets are not measurable.*

Theorem 3.18. *Given V a Vitali set, $\mathbb{R}_V = \mathbb{R}$. Analogously, given M a multiplicative Vitali set, $\mathbb{R}_M = \mathbb{R}$.*

Proof. We will use that addition and multiplication by a computable number is a computable operation.

In the additive case, we know that $\mathbb{R} = V + \mathbb{Q} = \{r + q : q \in \mathbb{Q}, r \in V\}$ and that $r + q$ is Turinq equivalent to r. Therefore, $\mathbb{R}_V = \mathbb{R}$.

We can do an analogue argument for the multiplicative case. □

This shows that there are non measurable sets that become measurable once they are closed under Turing equivalence.

In the literature we can find a direct generalization of Vitali sets focussing on groups different than $\mathbb{Q}$. These examples will behave differently.

Definition 3.19. *1. Given a group G and an action a of G over $\mathbb{R}$, we say that $S \subseteq \mathbb{R}$ is an a-*selector *(or G-*selector *when the action is understood) if and only if for every $x \in \mathbb{R}$ there is $v \in S$ and $g \in G$ such that $g \cdot x = v$ and for all $x, y \in S$, $x \neq y$, there is no $g \in G$ such that $g \cdot x = y$.*

2. *We say that an action a of G over $\mathbb{R}$ is* paradoxical *if and only if every selector of a is non measurable.*

3. *If a, an action of G over $\mathbb{R}$, is paradoxical we call the selectors a-*Vitali sets *and, in case that the action is clear, we call them G-*Vitali sets*.*

Lemma 3.20 (Folklore)**.** *Given a measurable set A, $\lim_{r \to 0} \lambda(A \cap (A + r)) = \lambda(A)$ where λ is the Lebesgue measure (in any dimension) and $A + r = \{a + r : a \in A\}$.*

Lemma 3.21. *For any computable group G, i.e. a countable group with a computable bijection with ω such that the group operation is computable, and a computable action a of G over $\mathbb{R}$ we have that for every a-selector set A, $\mathbb{R}_A = \mathbb{R}$.*

Theorem 3.22. *There is a group G such that for all its G-Vitali sets, A, $\mathbb{R}_A$ is non-measurable.*

Proof. Let $r \in \mathbb{R}$ be noncomputable and let $G = r \cdot \mathbb{Q} = \{rq : q \in \mathbb{Q}\}$. This set is a group under addition and it has a cannonical action over $\mathbb{R}$ (again, addition). Since G is isomorphic to $\mathbb{Q}$ as a group, the same proof that shows that any $\mathbb{Q}$-selector is nonmeasurable shows that any G-selector is nonmeasurable.

Now, let A be a G-Vitali set. We want to show that $\mathbb{R}_A$ is nonmeasurable. In order to do this, we will show that $\mathbb{R}_A \times \mathbb{R}_A \subseteq \mathbb{R}^2$ is nonmeasurable.

Notice that, since $A+G=\mathbb{R}$ we also have that $\mathbb{R}_A+G=\mathbb{R}$, then $\mathbb{R}_A \times \mathbb{R}_A + G \times G = \mathbb{R}^2$. This means that, $\mathbb{R}_A \times \mathbb{R}_A$ is not of measure zero (since G is countable).

To finish the proof, we would like to show that $\mathbb{R}_A \cap (\mathbb{R}_A + qr) = \emptyset$. Nevertheless, this is not true. Given $x \in \mathbb{R}_A$ and $q \in \mathbb{Q}$ if $x + qr = y \in \mathbb{R}_A$ then $x \oplus y$ computes r. We will have to modify $\mathbb{R}$ by a null set to finish our proof.

Let $B = \mathbb{R}_A \times \mathbb{R}_A \setminus N$ where

$$N = \{(x, y) : r \leq_T x \oplus y\}.$$

The classical proof of Sacks that the upper cone of non-computable reals have measure zero in $\mathbb{R}$, in [28], also shows that N is a null set of $\mathbb{R}^2$ (for non computable r). It is enough to show that B is nonmeasurable to finish the proof.

Notice that, given $h \in G \times G \setminus \{(0,0)\}$, $(B+h) \cap B = \emptyset$. Therefore, for any sequence from $G \times G$, h_i, that converges to $(0,0)$ we have that $\lim_{i \to \infty} \lambda((B+h_i) \cap B) = 0$. If B was a measurable set, it would have to have measure 0, but we know that B is not a null set. Therefore, B is not measurable. □

Corollary 3.23. *There is a measurable set L such that $\mathbb{R}_L$ is nonmeasurable.*

Proof. Let C be the ternary Cantor set. Take a function, f, that bijects 2^ω with the usual Cantor set inside the reals, C, and maintains degree, in other words, given $a \in 2^\omega$ then $deg(a) = deg(f(a))$.

This functions shows that C has a representative of every degree in $\mathbb{R}$. Using the set A from the theorem above, let $L = \mathbb{R}_A \cap C$.

Since C has measure zero we know that L is a measurable set of measure zero. Now, since for all real number there is a Turing equivalent real in C we have that $\mathbb{R}_L = \mathbb{R}_A$. □

Although it would have been interesting to find a positive measure measurable set whose closure under Turing equivalence is not measurable, we found out that this is not possible thanks to a suggestion from James Hanson.

Theorem 3.24. *Given $A \subseteq \mathbb{R}$ such that $\lambda(A) > 0$, we have that $\mathbb{R}_A$ is measurable.*

Proof. Remember that the Lebesgue Density Theorem states that given a measurable set A, for almost all points of A, $\lim_{\epsilon \to 0} \frac{\lambda(A \cap (x-\epsilon, x+\epsilon))}{\lambda((x-\epsilon, x+\epsilon))}$ is either 0 or 1. Furthermore, if it it the case that for almost all points the Lebesgue density is 0, then A is a measure zero set.

We will show that if $\lambda(A) > 0$ then $\mathbb{R} \setminus (\mathbb{Q} + A)$ has measure 0, i.e., $\mathbb{Q} + A$ has full measure. Since $\mathbb{Q} + A \subseteq \mathbb{R}_A$, we have that $\mathbb{R}_A$ has also full measure, hence, it is measurable.

Now, to show that $\mathbb{Q} + A$ has full measure we will show that $\lambda((\mathbb{Q} + A) \cap B) > 0$ for all B such that $\lambda(B) > 0$.

Let B be such that $\lambda(B) > 0$. Using the Lebesgue Density Theorem, find $a \in A$, $b \in B$ and $\epsilon > 0$ such that $\lambda(A \cap (a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2})) > \frac{3\epsilon}{4}$ and $\lambda(B \cap (b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})) > \frac{3\epsilon}{4}$.

Now, let $r \in \mathbb{Q}$ such that $\lambda((a + r - \frac{\epsilon}{2}, a + r + \frac{\epsilon}{2}) \cap (b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})) > \frac{3\epsilon}{4}$. Since each of the intervals lost at most $\frac{\epsilon}{4}$ lenght when intersected, we have that

$$\lambda((A + r) \cap (a + r - \frac{\epsilon}{2}, a + r + \frac{\epsilon}{2}) \cap (b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})) > \frac{2\epsilon}{4}$$

and

$$\lambda(B \cap (a + r - \frac{\epsilon}{2}, a + r + \frac{\epsilon}{2}) \cap (b - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})) > \frac{2\epsilon}{4}.$$

Therefore,

$$\lambda((A+r)\cap B\cap(a+r-\frac{\epsilon}{2},a+r+\frac{\epsilon}{2})\cap(b-\frac{\epsilon}{2},b+\frac{\epsilon}{2}))>\frac{\epsilon}{4}.$$

This shows that $\lambda((A+r)\cap B)>0$ which implies that $\lambda((A+\mathbb{Q})\cap B)>0$. □

We think that it is important to investigate which non-measurable sets become measurable once they are closed under Turing equivalence.

Because it is well known that countable sets have measure zero and that, depending on set theory axioms, uncountable sets of size strictly less than the continuum are either measure zero or non measurable, while working on this topic we should focus our attention exclusively on sets of size $\mathfrak{c}$ (continuum). Nevertheless, in this work we will not investigate other examples related to measure.

3.5 Order types

A different question about $\mathbb{R}_{\mathcal{S}}$ is whether it can be isomorphic to any order type that sits inside the reals.

A really quick answer for this is "no". To see this, we need a definition:

Definition 3.25. *Given $A\subseteq\mathbb{R}$ and a cardinal $\kappa\leq 2^{\aleph_0}=\mathfrak{c}$ we say that A is κ-*dense *if and only if for any nonempty interval I we have that $|I\cap A|=\kappa$.*

Necessarily, $\mathbb{R}_{\mathcal{S}}$ is going to be a $|\mathcal{S}|\cdot\aleph_0$-dense subset of reals (because each $\mathbb{R}_x$ is $\aleph_0$-dense). Therefore, any order that is not κ-dense is not going to have an isomorphic copy as $\mathbb{R}_{\mathcal{S}}$.

By a classical result of Cantor, we know that every $\aleph_0$-dense subset of reals is order isomorphic to $\mathbb{Q}$. Now, if there is $\aleph_0<\kappa<\mathfrak{c}$, then the study of κ-dense subsets of reals is completely independent of ZFC (see Baumgartner [3]). So we will only focus on $\mathfrak{c}$-dense subsets of the reals (or subsets such that $|\mathcal{S}| = \mathfrak{c}$).

Even in this situation, we know that we cannot get all possible order types, for example $\mathbb{R} \setminus \{0\}$ will not be isomorphic to any $\mathbb{R}_{\mathcal{S}}$ (any $\mathbb{R}_{\mathcal{S}}$ that contains an interval will be $\mathbb{R}$).

Trying to characterize which order types can be attained is still an open question. Towards giving more insight into the problem, we will focus our attention to specific $\mathfrak{c}$-dense subsets of reals.

3.5.1 Luzin and Sierpinski sets

Definition 3.26. *We say that $A \subseteq \mathbb{R}$ is a* Luzin set *if and only if A is uncountable and the intersection of A with any meager set (equivalently, no-where-dense set) is countable.*

A is a Sierpinski set *if, in the above definition, we replace meager with null (or measure zero) set.*

The following lemma also appears in Löwe [18] with a more complicated proof, since they work in 2^{ω} instead of the real line. Although we are sure that the results are the same, this proof shows how different it is to work with sets of real numbers depending on if you see them as contained in 2^{ω}, ω^{ω} or the real line.

Lemma 3.27. *There is no $\mathcal{S}$ such that $\mathbb{R}_{\mathcal{S}}$ is a Luzin or Sierpinski set.*

Proof. Let C be the ternary Cantor set. As we remark in Corollary 3.23, C has a representative of all Turing degrees in $\mathbb{R}$.

Notice that, $|\mathbb{R}_{\mathcal{S}} \cap C| = |\mathcal{S}|$. Then, if $\mathcal{S}$ is countable, $\mathbb{R}_{\mathcal{S}}$ is also countable, so it is not Luzin; and if $\mathcal{S}$ is uncountable, $\mathbb{R}_{\mathcal{S}}$ intersects a meager set, C, in uncountably many points, so it is not Luzin.

An analogous reasoning shows that it cannot be Sierpinski using that C is a null set. □

We believe that it is important to remark that the above result is not as trivial as it seems; notice that:

Lemma 3.28. *Given a Luzin set L (resp. a Sierpinski set), the set $L + \mathbb{Q}$ is also a Luzin set (resp. a Sierpinski set).*

Proof. Let L be an uncountable set and assume that $L + \mathbb{Q} = \{l + r : l \in L, r \in \mathbb{Q}\}$ is not Luzin. Therefore, there is a meager set M such that $M \cap (L + \mathbb{Q})$ is uncountable.

Define $L + r$ as $\{l + r : l \in L\}$. Since $\mathbb{Q}$ is countable, there should be $r \in \mathbb{Q}$ such that $(L + r) \cap M$ is uncountable.

Nevertheless, $(M - r) \cap L$ is uncountable since $|(M - r) \cap L| = |(L + r) \cap M|$ and $M - r$ is also meager. This means that L is not Luzin.

In the case for Sierpinski, the argument is the same but replacing M with a null set. □

Taking into account that every uncountable subset of a Luzin (resp. Sierpinski) set is also Luzin (resp. Sierpinski) set. The above result can be improved to:

Corollary 3.29. *Given a set $L \subseteq \mathbb{R}$, we have that L is a Luzin set (resp. a Sierpinski set) if and only if the $L + \mathbb{Q}$ is a Luzin set (resp. a Sierpinski set).*

Analyzing the proof of the lemma above we can notice that the key point is that given M meager, $M - r$ is also meager. If we think of translation by r as a function, say f, we have that for all meager sets M, $f^{-1}(M)$ is also meager.

This observation lets us expand our lemma:

Corollary 3.30. *Given a countable set $\mathcal{F}$ of functions whose inverses preserve category (resp. measure) closed under composition and a Luzin set (resp. Sierpinski set) L, we have that the closure of L under $\mathcal{F}$ is also a Luzin set (resp. Sierpinski set).*

Proof. We can follow the same proof as in Lemma 3.28 using the comments made in the observation above. □

Corollary 3.31. *It is consistent with ZFC that there is no countable set $\mathcal{F}$ of functions whose inverses preserve category (resp. measure) closed under composition such that for all reals, r, we have that $\mathbb{R}_r$ is the closure under $\mathcal{F}$ of the set $\{r\}$.*

Proof. Given a countable collection of functions, $\mathcal{F}$, call $\mathcal{F}_r$ the closure of $\{r\}$ under $\mathcal{F}$. Notice that, since $\mathcal{F}$ is countable, $\mathcal{F}_r$ is also countable. Given $S \subseteq \mathbb{R}$, we define analogously $\mathcal{F}_S$.

It is consistent with ZFC, regardless of the cardinal arithmetic, that there exists a Luzin set L (of any desired size).

If $\mathcal{F}$ is a countable set of functions whose inverses preserve category and that is closed under composition. Then, by the above corollary, we have that $\mathcal{F}_L$ is Luzin.

We can show that it is not the case that for all $r \in \mathbb{R}$ we have that $\mathbb{R}_r \subseteq \mathcal{F}_r$: assume otherwise, then we will have that $\mathbb{R}_L \subseteq \mathcal{F}_L$. Nevertheless, $\mathcal{F}_L$ is Luzin and $\mathbb{R}_L$ is not, but this is a contradiction. Therefore, there exist uncountably many $r \in L$ such that $\mathcal{F}_r \neq \mathbb{R}_r$.

The proof for measure preserving functions is analogous, but it assumes the existence of a Sierpinski set instead of a Luzin set. □

Corollary 3.32. *It is consistent with ZFC that there is no countable set $\mathcal{F}$ of functions whose inverses preserve category (resp. measure) closed under composition such that for all reals, r, we have that $\mathbb{R}_{\wedge r}$ is the closure under $\mathcal{F}$ of the set $\{r\}$.*

This last corollary can also be shown directly (in ZFC) by constructing a computable function that does not preserve category or measure. For the measure case, a candidate is a computable isomorphism between a measure zero Cantor set and a positive measure Cantor set.

3.5.2 Entangled sets

Definition 3.33. *Given a collection of 2-tuples $\{(x_\alpha, y_\alpha) : \alpha \in I\}$ we say that these tuples are* disjoint *if and only if $|\{x_\alpha, y_\alpha\}| = 2$ for all α and $\{x_\alpha, y_\alpha\} \cap \{x_\beta, y_\beta\} = \emptyset$ for all $\alpha \neq \beta$.*

Given $\aleph_0 < \kappa \leq \mathfrak{c}$, we say that $A \subseteq \mathbb{R}$ is κ-2-entangled *if and only if $|A| \geq \kappa$ and for every collection of size κ of disjoint 2-tuples of A there are (x_1, y_1), (x_2, y_2), (w_1, z_1), (w_2, z_2) such that $x_1 < x_2$ and $y_1 < y_2$, $w_1 < w_2$ but $z_1 > z_2$.*

If a set is $\aleph_1$-2-entangled we just say that it is 2-entangled.

Notice that, any one-to-one function between two κ size disjoint subsets of a κ-2-entangled set can be seen as a collection of size κ of disjoint 2-tuples. Then, by the defining property of κ-2-entangled sets, this one-to-one function cannot be monotone increasing nor monotone decreasing. This implies that, given any two disjoint subsets of size κ inside of a κ-2-entangled, the subsets are not order isomorphic.

Lemma 3.34. *There is no uncountable $\mathcal{S}$ such that $\mathbb{R}_{\mathcal{S}}$ is $|\mathcal{S}|$-2-entangled.*

Proof. Notice that, for any $\mathcal{S}$, $\mathbb{R}_{\mathcal{S}} + 2 = \mathbb{R}_{\mathcal{S}}$, therefore, $([0,1] \cap \mathbb{R}_{\mathcal{S}}) + 2 \subseteq \mathbb{R}_{\mathcal{S}}$ and $([0,1] \cap \mathbb{R}_{\mathcal{S}}) + 2 \subseteq [2,3]$. Which means that $(([0,1] \cap \mathbb{R}_{\mathcal{S}}) + 2) \cap ([0,1] \cap \mathbb{R}_{\mathcal{S}}) = \emptyset$.

If $\mathcal{S}$ is uncountable, we have that $[0,1] \cap \mathbb{R}_{\mathcal{S}}$ is uncountable. So the function $x \mapsto x+2$ is a one-to-one order preserving function[1] that shows that $([0,1] \cap \mathbb{R}_{\mathcal{S}})+2$ and $[0,1] \cap \mathbb{R}_{\mathcal{S}}$ are order isomorphic.

This shows that $\mathbb{R}_{\mathcal{S}}$ is not $|\mathcal{S}|$-2-entangled. □

Upon closer inspection of the proof above, we can see that what makes $\mathbb{R}_{\mathcal{S}}$ not $|\mathcal{S}|$-2-entangled is that there are order preserving functions that preserve the degree. Making a slight modification to the definition of an entangled set, we can create an $\mathcal{S}$ such that $\mathbb{R}_{\mathcal{S}}$ is almost $|\mathcal{S}|$-2-entangled.

Definition 3.35. *Given $\aleph_0 < \kappa \leq \mathfrak{c}$, we say that $A \subseteq \mathbb{R}$ is* layered κ-2-entangled *(or* σ-2-entangled *if the κ is understood) if and only if $|A| = \kappa$ and there is a function $ht : A \to \kappa$ countable-to-one such that for every collection of size κ of disjoint 2-tuples of A such that the entries of a given pair have different values of ht then there are (x_1, y_1), (x_2, y_2), (w_1, z_1), (w_2, z_2) such that $x_1 < x_2$ and $y_1 < y_2$, $w_1 < w_2$ but $z_1 > z_2$.*

Notice that any layered κ-2-entangled set contains a κ-2-entangled: every collection of κ many elements that have different values of ht is κ-2-entangled.

Lemma 3.36. *If A is κ-2-entangled and $|A| = \kappa$ then $A + \mathbb{Q}$ is layered κ-2-entangled.*

Proof. Let $A = \{x_\alpha : \alpha < \kappa\}$, given $s \in A + \mathbb{Q}$ let $ht(s) = \min\{\alpha : \exists r \in \mathbb{Q}\ (s = x_\alpha + r)\}$. Since $\mathbb{Q}$ is countable, this function is countable-to-one. This function makes $A + \mathbb{Q}$ layered κ-2-entangled.

[1] An order preserving function is a function such that $x < y$ then $f(x) < f(y)$. On the other hand, an order reversing function is such that if $x < y$ then $f(y) < f(x)$.

To show this, assume that $A+\mathbb{Q}$ is not layered κ-2-entangled. Then, there is a collection of size κ of disjoint 2-tuples of $A+\mathbb{Q}$ such that the entries of a given pair have different values of ht such that, as a function, call it f, from a subset of $A+\mathbb{Q}$ to $A+\mathbb{Q}$ it is order preserving or order reversing.

Without loss of generality, let's say that it is order preserving. Now, given any element $a \in dom(f)$ there is $r_a \in \mathbb{Q}$ such that $a = x_{ht(a)} + r_a$. We know that $|dom(f)| = \kappa$, so, shrinking f, we can find a single r such that for all $a \in dom(f)$, $a = x_{ht(a)} + r$. Since f is one-to-one, we can do the same with $range(f)$, i.e., we can shrink f in such a way that there is $q \in \mathbb{Q}$ such that for all $b \in range(f)$ we have $b = x_{ht(b)} + q$.

Finally, notice that the function $f(x+r)-q$ is an order preserving one-to-one function from A to itself with domain of size κ. This contradicts the fact that A is κ-2-entangled.

□

Corollary 3.37. *Given a countable set $\mathcal{F}$ of strictly monotone functions closed under composition and a κ-2-entangled set A of size κ we have that the closure of A under $\mathcal{F}$ is layered κ-2-entangled.*

Proof. The proof is the same as above as long as we replace addition by a rational with other one-to-one monotone function. □

From now on, we will fix a bijection between the Turing degrees and $\mathfrak{c}$. While working with collections of Turing degrees our function ht will be the composition of the function that maps a point to its degree followed by the bijection fixed above. In case that the size of our set is not $\mathfrak{c}$, we create a bijection with the corresponding cardinal. Nevertheless, to simplify notation, we will refer to ht as the degree of a point.

Given an entangled set A, is it true that $\bigcup_{a\in A} \mathbb{R}_a = \mathbb{R}_A$ is layered entangled? At this moment we can show this to be true using the, really strong, Proper Forcing Axiom (PFA, see Chapter 5 section 7 of [16]). Later, using Theorem 3.43, we will see that PFA was not necessary for this result. Nevertheless, this first approach helps to show which steps need to be followed.

Corollary 3.38. *PFA implies that for every κ-2-entangled set, A, we have that $\mathbb{R}_A$ is layered κ-2-entangled.*

Proof. PFA implies that any countable-to-one function between subsets of reals is the union of countably many monotone functions. Now, given a computable function $\varphi_e : \omega \to \omega$ we can associate it to $f_e : \mathbb{R} \to \mathbb{R}$ where $f_e(x) = \varphi_e^x$ (whenever φ_e^x is total). Now, let $D_e = \{x : x \equiv_T f_e(x)\}$. We have that $f_e\restriction_{D_e}$ is countable-to-one.

Notice that, using the notation of 3.31, $\mathcal{F} = \{f_e\restriction D_e : e \in \omega\}$ is such that for all $x \in \mathbb{R}$, $\mathcal{F}_x = \mathbb{R}_x$. Using PFA, we can change $\mathcal{F}$ for a countable collection of monotone functions. Combining this fact with Corollary 3.37 we are done. □

Later we will show in Theorem 3.45 that ZFC implies that given a 2-entangled set E, $\mathbb{R}_E$ is layered-2-entangled set. Nevertheless, it is important to remark that we can get certain control over a layered entangled set with the following techniques:

Recall that, given $\mathcal{S}$ a subset of the Turing degrees we define $\mathbb{R}_\mathcal{S} = \bigcup_{deg(a)\in\mathcal{S}} \mathbb{R}_a$. We will write Turing degrees with bold case to differentiate them from real numbers.

Theorem 3.39. *Given continuum many Turing degrees, $\mathcal{A}$, there is a continuum size collection of degrees $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathbb{R}_\mathcal{S}$ is layered $\mathfrak{c}$-2-entangled.*

Proof. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions from G_δ subsets of $\mathbb{R}$ to $\mathbb{R}$ and let $\mathcal{A} = \{\mathbf{a}_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of continuum many Turing degrees. Given a Turing degree $\mathbf{a}$, we will denote by $\mathbb{R}_{\mathbf{a}}$ all reals that have Turing degree $\mathbf{a}$.

We will construct the set $\mathcal{S} \subseteq \mathcal{A}$ by recursion.

Assume that we already have $\mathcal{S}_\alpha = \{\mathbf{a}_{\gamma_\xi} : \xi < \alpha\}$. Let

$$\gamma_\alpha = \min\{\beta : \mathbb{R}_{\mathbf{a}_\beta} \cap (\bigcup_{\xi<\alpha} \mathbb{R}_{\mathbf{a}_{\gamma_\xi}} \cup B_\alpha) = \emptyset\}$$

where

$$B_\alpha = \{f_\xi(\overline{v}) : \xi < \alpha, \overline{v} \in \bigcup_{\xi<\alpha} \mathbb{R}_{\mathbf{a}_{\gamma_\xi}}\}.$$

We know that $\mathbb{R} \setminus (\bigcup_{\xi<\alpha} \mathbb{R}_{\mathbf{a}_{\gamma_\xi}} \cup B_\alpha)$ is of size continuum because

$$\left|\bigcup_{\xi<\alpha} \mathbb{R}_{\mathbf{a}_{\gamma_\xi}} \cup B_\alpha\right| = |\alpha| < \mathfrak{c}.$$

This means that there are at most $|\alpha|$ ordinals β such that $\mathbb{R}_{\mathbf{a}_\beta} \cap (\bigcup_{\xi<\alpha} \mathbb{R}_{\mathbf{a}_{\gamma_\xi}} \cup B_\alpha) \neq \emptyset$ (this is because the Turing equivalent classes are disjoint between them).

The set $\mathcal{S} = \mathcal{S}_\mathfrak{c} = \{\mathbf{a}_{\beta_\alpha} : \alpha < \mathfrak{c}\}$ is the one that we are looking for. To show that $\mathbb{R}_\mathcal{S}$ has the property that for every collection of size continuum of disjoint 2-tuples of $\mathbb{R}_\mathcal{S}$ such that each entry of each pair has different degree then there are (x_1, y_1), (x_2, y_2), (w_1, z_1), (w_2, z_2) such that $x_1 < x_2$ and $y_1 < y_2$, $w_1 < w_2$ but $z_1 > z_2$, we will follow a technique used by Todorcevic in [37]. We reproduce the argument here for the convenience of the reader.

Let $\{(x_{\alpha_\xi}, x_{\beta_\xi}) : \xi < \mathfrak{c}\} \subseteq \mathbb{R}^2_\mathcal{S}$ be a collection of size continuum of disjoint 2-tuples of $\mathbb{R}_\mathcal{S}$ such that each entry of each pair has different degree.

Let

$$K = \{x_{\alpha_\xi} : \alpha_\xi < \beta_\xi < \mathfrak{c}\}.$$

We can assume that this set is of size continuum. If not, we can run the argument interchanging the roles of α_ξ and β_ξ.

Now, we can define the function $g : K \to \mathbb{R}$ such that

$$x_{\alpha_\xi} \mapsto x_{\beta_\xi}.$$

Furthermore, define the set

$$K_0 = \{s \in K : |\omega_g(s)| \geq 2\},$$

where[2] $\omega_g(s) = \bigcap_{n\in\omega} \overline{g[B^K_{\frac{1}{n}}(s)]}$ is the oscillation of g at s, i.e., all the accumulation points of the images (under g) of sequences that converges to s. Notice that $|\omega_g(s)| = 1$ if and only if g is continuous at s.

Recall that any partial continouos function from $\mathbb{R}^n$ to $\mathbb{R}$ can be extended with a partial function whose domain is a G_δ set. With this, our construction of $\mathbb{R}_{\mathcal{S}}$, and the fact that $\mathbb{R}_{x_{\alpha_\xi}} \neq \mathbb{R}_{x_{\beta_\xi}}$, we have that the set K_0 is of size continuum.

Given $s \in K_0$ call a_s, b_s two elements in $\omega_g(s)$. Without loss of generality, we can assume that $a_s < b_s$. Let $r \in \mathbb{Q}$ such that $a_s < r < b_s$. Since we only have countably many options, we may shrink K_0 in such a way that for all $s \in K_0$ the rational number r is the same. Notice that K_0 still has size continuum.

Take $t, s \in K_0$ such that $t < s$ and take disjoint intervals I_t, I_s such that $t \in I_t$ and $s \in I_s$. By the definition of a_t, a_s, b_t and b_s there are $t_0, t_1 \in K \cap I_t$ and $s_0, s_1 \in I_s$ such that $g(t_0), g(s_0) < r < g(t_1), g(s_1)$. Then for the pairs $(t_0, g(t_0))$, $(s_1, g(s_1))$ we have $t_0 < s_1$ and $g(t_0) < g(s_1)$; and for the pair $(t_1, g(t_1))$, $(s_0, g(s_0))$ we have $t_1 < s_0$ but $g(t_1) > g(s_0)$. □

[2]Here $B^K_{\frac{1}{n}}(s)$ is the ball of radius $\frac{1}{n}$ with center s in K, a subset of $\mathbb{R}$.

The above proof works as an example of how to use the technique and paves the way for the following theorem.

Theorem 3.40. *There are* $2^{\mathfrak{c}}$ *many* $\mathfrak{c}$*-dense order types inside* $\mathbb{R}$ *that are closed under Turing equivalence.*

Proof. Let $\mathcal{S}$ be a collection of Turing degrees such that $\mathbb{R}_{\mathcal{S}}$ is layered $\mathfrak{c}$-2-entangled.

First, fix two disjoint subsets of size continuum of $\mathcal{S}$, call them $\mathcal{A}$ and $\mathcal{B}$. We have that any one-to-one function between $\mathbb{R}_{\mathcal{A}}$ and $\mathbb{R}_{\mathcal{B}}$ is going to be a collection of size continuum of disjoint 2-tuples of $\mathbb{R}_{\mathcal{S}}$ such that each entry of the pair has a different Turing degree ($\mathcal{A}$ and $\mathcal{B}$ are disjoint), therefore, there are (x_1, y_1), (x_2, y_2), (w_1, z_1), (w_2, z_2) such that $x_1 < x_2$ and $y_1 < y_2$, $w_1 < w_2$ but $z_1 > z_2$. In other words, the function cannot be order preserving or order reversing. This shows that $\mathbb{R}_{\mathcal{A}}$ and $\mathbb{R}_{\mathcal{B}}$ are not order isomorphic.

Now, assume that $\mathcal{A}$ and $\mathcal{B}$ are different subsets of $\mathcal{S}$ with $|\mathcal{A}\Delta\mathcal{B}| = \mathfrak{c}$, where $a\Delta b = (a \setminus b) \cup (b \setminus a)$. Without lost of generality, assume that $\mathbb{R}_{\mathcal{A}\setminus\mathcal{B}}$ is of size continuum. Since $\mathbb{R}_{\mathcal{A}\setminus\mathcal{B}}$ is disjoint from $\mathbb{R}_{\mathcal{B}}$, and $\mathbb{R}_{\mathcal{S}}$ is layered $\mathfrak{c}$-2-entangled, we have that $\mathbb{R}_{\mathcal{A}\setminus\mathcal{B}}$ is not order isomorphic to any subset of $\mathbb{R}_{\mathcal{B}}$. Then, there cannot be any order isomorphism between $\mathbb{R}_{\mathcal{A}} = \mathbb{R}_{\mathcal{A}\cap\mathcal{B}} \cup \mathbb{R}_{\mathcal{A}\setminus\mathcal{B}}$ and $\mathbb{R}_{\mathcal{B}}$. So, again, this shows that $\mathbb{R}_{\mathcal{A}}$ and $\mathbb{R}_{\mathcal{B}}$ are not order isomorphic.

Since there are $2^{\mathfrak{c}}$ subsets of size continuum such that $|\mathcal{A}\Delta\mathcal{B}| = \mathfrak{c}$ between any two of them, we have $2^{\mathfrak{c}}$ non-isomorphic order types. □

It is interesting to wonder which kind of relation have the Turing degrees of the reals in the set created in Theorem 3.39. There is the possibility that all of them are inside a cone or that all of them have some intricate relationship. Because any Turing degree has, at most, countably many degrees below it, as long as $\neg CH$ holds, there is no way to make these degrees a tower. At the end, the relation depends on how $\mathcal{S}$ was originally.

Now, it is possible to combine topological or measure arguments with the proof of Theorem 3.39. For example, assuming that $\max\{cov(\mathcal{M}), cov(\mathcal{N})\} = \mathfrak{c}$, it is possible to produce an antichain directly. Although it is possible for $\mathcal{S}$ to be an antichain to begin with in (in Theorem 3.39), the following theorem is an interesting application.

Definition 3.41. *Denote by $cov(\mathcal{M})$ the least amount of meager sets that are needed to cover $\mathbb{R}$. Analogously, $cov(\mathcal{N})$ denotes the least amount of null sets that cover $\mathbb{R}$.*

Corollary 3.42. *($\max\{cov(\mathcal{M}), cov(\mathcal{N})\} = \mathfrak{c}$) There is a continuum size antichain of Turing degrees $\mathcal{S}$ such that $\mathbb{R}_{\mathcal{S}}$ is layered $\mathfrak{c}$-2-entangled.*

Proof. The proof is analogous to the one above.

Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions from a G_δ subset of $\mathbb{R}$ to $\mathbb{R}$ and let $\{\mathbf{a}_\alpha : \alpha < \mathfrak{c}\}$ an enumeration of all Turing degrees. Given a Turing degree $\mathbf{a}$, we will denote by $\bigvee \mathbf{a}$ all reals that have Turing degree that computes $\mathbf{a}$ or that is computed by $\mathbf{a}$.

Notice that $\bigvee \mathbf{a}$ is both meager and null: it is a well known result that the upper cones of Turing degrees are meager and null (see [14] and [28], respectively), furthermore, the downward cone is countable, so, it is also meager and null.

We will construct the set by recursion.

Assume that we already have $\mathcal{S}_\alpha = \{\mathbf{a}_{\beta_\xi} : \xi < \alpha\}$. Let

$$\beta_\alpha = \min\{\beta : \mathbb{R}_{\mathbf{a}_\beta} \cap (\bigcup_{\xi<\alpha} \bigvee \mathbf{a}_{\beta_\xi} \cup B_\alpha) = \emptyset\}$$

where

$$B_\alpha = \{f_\xi(\overline{v}) : \xi < \alpha, \overline{v} \in \bigcup_{\xi<\alpha} \mathbb{R}_{\mathbf{a}_{\beta_\xi}}\}.$$

Using $\max\{cov(\mathcal{M}), cov(\mathcal{N})\} = \mathfrak{c}$, we know that $\mathbb{R}\setminus(\bigcup_{\xi<\alpha} \bigvee \mathbf{a}_{\beta_\xi})$ is of size continuum. On the other hand, $|B_\alpha| = |\alpha| < \mathfrak{c}$. This means that $\mathbb{R} \setminus (\bigcup_{\xi<\alpha} \bigvee \mathbf{a}_{\beta_\xi} \cup B_\alpha)$ is of size continuum, so there are at most $|\alpha|$ ordinal β such that $\mathbb{R}_{\mathbf{a}_\beta} \cap (\cup_{\xi<\alpha}\mathbb{R}_{x_\xi} \cup B_\alpha) \neq \emptyset$.

The set $\mathcal{S} = \mathcal{S}_\mathfrak{c} = \{\mathbf{a}_{\beta_\alpha} : \alpha < \mathfrak{c}\}$ is the one that we are looking for. By construction, it is clearly an antichain and, using the same proof as above, we have that $\mathbb{R}_\mathcal{S}$ is layered $\mathfrak{c}$-2-entangled, i.e., has the property that for every collection of size continuum of disjoint 2-tuples of $\mathbb{R}_\mathcal{S}$ such that each entry of the pair has different degree then there are (x_1, y_1), (x_2, y_2), (w_1, z_1), (w_2, z_2) such that $x_1 < x_2$ and $y_1 < y_2$, $w_1 < w_2$ but $z_1 > z_2$. □

Another known construction for an entangled set, due to Avraham, is given by taking a set of $\aleph_1$ many reals that are Cohen generic with respect a countable model of $H(\theta)$ with $\theta > \mathfrak{c}$, this can be found in [2]. A result of Slaman and Woodin [32], proves that the degrees of 5-generic reals form an automorphism base for the Turing degrees. In Section 3.7 we will use an analogous construction to the one done by Avraham to show results related to the existence of an automorphism of the Turing degrees.

3.6 Absolute decomposition of functions

In this section the letters r, x, y will represent real numbers (either elements of $\mathbb{R}$, 2^ω or ω^ω) and the letters s, l, m, n will represent natural numbers (unless otherwise stated).

Fix a countable-to-one function $\varphi : D \subseteq \mathbb{R} \to \mathbb{R}$ such that $\langle x, y\rangle \in \varphi$ is arithmetical (or hyperarithmetical) with respect to x and y. In this section we will analyze the complexity of the sentence:

The function φ is the union of countably many monotone functions.

First, we will analyze how to code a monotone non-decreasing function in a real number. To code a monotone non-increasing function is analogous.

Notice that, given $A \subseteq \mathbb{R}$ and a non-decreasing function $g : A \to \mathbb{R}$ we can define a non-decreasing function $f : \mathbb{R} \to [-\infty, \infty]$ as $f(x) = \sup_{y \leq x} \hat{g}(y)$ where $\hat{g}(y) = -\infty$ whenever $x \notin A$ and $\hat{g}(y) = g(y)$ otherwise. Since $f \restriction A = g$, we can assume that all non-decreasing functions are from the real numbers to $[-\infty, \infty]$.

Now, given a non-decreasing function f, we know that it can only have countably many discontinuities. If we call the discontinuities d_n and we enumerate the rational numbers as q_n then we have that:

$$f(x) = \sup\left(\{f(d_n) : d_n \leq x\} \cup \{f(q_n) : q_n \leq x\}\right).$$

To show that the above equality is true, given x there are only two options: either f is continuous at x in which case $f(x) = \sup\{f(q_n) : q_n \leq x\}$ (and, even if there is a sequence of discontinuities that converges at x, by the continuity at x the value should be the same) or f is not continuous at x so there is some n such that $x = d_n$ and, because f is non-decreasing the above equality is realized.

This means that, given the collection $\{\langle d_n, f(d_n)\rangle : n \in \omega\rangle\} \cup \{\langle q_n, f(q_n)\rangle : n \in \omega\}$ we can decode f. The numbers d_n, $f(d_n)$ and $f(q_n)$ are reals but there are only countably many of them, so we can code them up in a single real, say, r_f.

Notice that, in this situation, $y = f(x)$ if and only if the formula $\langle x, y\rangle \in f$ is the same as the following arithmetic (in x, y and r_f) formula:

$$\forall n, m \left(\ (q_n \leq x \leftrightarrow f(q_n) \leq y) \vee (d_m \geq x \leftrightarrow f(d_m) \geq y) \ \right).$$

Furthermore, we know that it is possible for a single real number to code countably many real numbers and, as a result of the above analysis, countably many monotone functions. Given a real number r, we will call f_n^r the n-th monotone function that r is coding. In the case that r do not satisfy the required conditions to code countably many monotone functions, we let f_n^r be the constant 0 for all $n \in \omega$.

This means that the sentence *the function* φ *is the union of countably many monotone functions* can be represented by the following Σ_2^1 formula:

$$\exists r \forall x, y(\langle x, y\rangle \notin \varphi \vee \exists n \langle x, y\rangle \in f_n^r)$$

With this, we can show the following theorem:

Theorem 3.43. *(ZFC)For every* $w \in \mathbb{R}$ *and every countable-to-one function* $\varphi : D \subseteq \mathbb{R} \to \mathbb{R}$ *that is* $\Sigma_1^1[w]$*, i.e, such that* $\langle x, y\rangle \notin \varphi$ *is a* $\Pi_1^1[w]$ *formula,* φ *is contained in a countable union of monotone functions. In particular, arithmetic and computable functions are the countable union of monotone functions.*

Proof. Fix $w \in \mathbb{R}$ and a countable-to-one function $\varphi : D \subseteq \mathbb{R} \to \mathbb{R}$ that is $\Sigma_1^1[w]$.

We will show that all models of ZFC that have φ satisfy the sentence

The function φ *is the union of countably many monotone functions.*

Given a model M of ZFC, we can take $L^M[\varphi, w]$ which is a model of ZFC+CH and has φ. By a result of Avraham-Rudin-Shelah [1], starting from a model of CH, there is a ccc forcing such that the resulting model, say N, satisfies that given $A, B \subseteq \mathbb{R}$ and $f : A \to B$ countable-to-one function, we have that f is contained in a countable union of monotone functions.

In particular, since $\varphi \in L^M[\varphi, w] \subseteq N$ we have that

$$N \models \exists r \forall x, y(\langle x, y\rangle \notin \varphi \vee \exists n \langle x, y\rangle \in f_n^r)$$

Since the above sentence is $\Sigma^1_2[w]$ (i.e, Σ^1_2 with parameters in $L[\varphi, w]$) , using Shoenfield absolutness Theorem, we know that it is absolute. Therefore we have that

$$M \models \exists r \forall x, y(\langle x, y\rangle \notin \varphi \vee \exists n \langle x, y\rangle \in f_n^r).$$

□

Corollary 3.44. *Given $f : D \subseteq \mathbb{R} \to \mathbb{R}$ a Borel function, f is contained in a countable union of monotone functions.*

Proof. Fix f a Borel function. Using borel codes, we know that there is $x \in \mathbb{R}$ such that f is $\Sigma^1_1[x]$ (actually, $\Delta^1_1[x]$). Now we just need to use Theorem 3.43. □

Theorem 3.45. *Let E be a 2-$\mathfrak{c}$-entangled set. Then $\mathbb{R}_E$ is 2-$\mathfrak{c}$-layered entangled.*

Proof. Towards a contradiction, assume that we have a 2-$\mathfrak{c}$-entangled set E and a one-to-one monotone function $f : D \subseteq \mathbb{R}_E \to \mathbb{R}_E$ such that for continuum many $x \in \mathbb{R}_E$, $f(x)$ is not Turning equivalent to x.

The idea of the proof is to construct a one-to-one (partial) monotone function g from E to E such that $g(r)$ is not equal to r for continuum many $r \in E$.

To do this, we want to first take $r \in E$ with $\mathbb{R}_r \cap D \neq \emptyset$ to a Turing equivalent $x \in D$ with a monotone function and then send $f(x)$ to a Turing equivalent element of it in E using a (most likely different) monotone function.

Given $r \in E$ and $x \in \mathbb{R}_r \subseteq \mathbb{R}_E$ we can use a Turing functional to send r to x. Since there are only countably many of them, we know that continuum many $r \in E$ will use the same Turing functional. We would like to use Lemma 3.43 to have a monotone function. Nevertheless, the problem is that Turing functionals may not be one-to-one, so we cannot use the Lemma. Fortunately, we can fix this.

Let $r \in E$ such that $\mathbb{R}_r \cap D \neq \emptyset$. We know that there exist $e, d \in \omega$ such that $\varphi_e^r \in D$ and $\varphi_d^{\varphi_e^r} = r$. Since there are only countably many $\langle e, d \rangle$ we know that continuum many $r \in E$ use the same pair. Without loss of generality, we can assume that there are $e, d \in \omega$ such that for all $r \in E$, $\varphi_e^r \in D$ and $\varphi_d^{\varphi_e^r} = r$.

Let $B_0 = \{r \in \mathbb{R} : \varphi_d^{\varphi_e^r} = r\}$. This set is arithmetic, so, the function $\varphi_e^{\text{-}} : B_0 \to \varphi_e^{\text{-}}[B_0]$ is one-to-one (with inverse φ_d) arithmetic function and, therefore, a Σ_1^1 function. Notice that, since B_0 is not computable, it may be the case that $\varphi_e^{\text{-}}\restriction_{B_0}$ is not computable. Nevertheless, it will be Σ_1^1.

By Lemma 3.43, $\varphi\restriction_{B_0}$ is contained in the union of countably many monotone functions. One of these functions, call it $h_0 : C_0 \to h_0[C_0]$, will satisfy that $A_0 = C_0 \cap B_0 \cap E$ is of size continuum and $h_0\restriction_{A_0} = \varphi_e\restriction_{A_0}$. Notice that $h_0\restriction_{A_0}$ is an strictly (one-to-one) monotone function such that $h_0[A_0] \subseteq D$ and $A_0 \subseteq E$ is of size continuum.

Doing an analogous argument, we can find an strictly (one-to-one) monotone function such that $h_1[A_1] \subseteq f[h_0[A_0]]$ and $A_1 \subseteq E$ is of size continuum.

Then, the function $g = h_0 \circ f \circ h_1^{-1} : h_0^{-1}[f^{-1}[h_1[A_1]]] \to A_0$ is a one-to-one monotone function from a subset of E to E. Furthermore, we have that, by construction, $deg(g(r)) \neq deg(r)$ which means that $g(r) \neq r$.

There is $g' \subseteq g$ such that g' is a one-to-one function between two disjoint size continuum sets of a $\mathfrak{c}$-entangled set. Notice that the set of g' is a continuum size set of disjoint 2-tuples that contradict the property of E being entangled. □

Corollary 3.46. *There is $\mathcal{F}$, a countable collection of monotone functions, such that*

$\mathcal{F}_x = \mathbb{R}_x$ *for all* $x \in \mathbb{R}$.

It is important to remark that Theorem 3.43 is a little unexpected since you can construct computable functions that are not the countable union of measurable monotone functions. The following example is due to Joe Miller and is publish here with his permission.

Lemma 3.47. *(ZFC) There is a computable function that is not the countable union of countably many measurable monotone functions.*

Proof. Let $f : D \subseteq (0,1) \to \mathbb{R}$ be such that if $x = (x_0, x_1, ...)$ has a unique binary expansion given by $(x_0, x_1, ...)$ then $f(x) = (y_0, y_1, ...)$ where $y_{2n+1} = x_{2n}$ and $y_{2n} = x_{2n+1}$. This is a computable function.

Assume that $f \subseteq \bigcup_{n\in\omega} g_n$ where g_n is a monotone function for all n. Since $f = \bigcup_{n\in\omega} g_n \cap f$, we will show that there is n such that $g_n \cap f$ is nonmeasurable.

Fix n and let $x \in dom(g_n \cap f)$, $m \in \omega$ and $\sigma \in 2^{<\omega}$ such that $m+1$ is even, $|\sigma| = m+1$ and σ is an initial segment of the binary representation of x. Notice that, for any y such that σ is an initial segment of the binary representation of y we have that $|x - y| < 2^{-m}$.

For any $y_{00}, y_{01}, y_{10}, y_{11}$ such that $\sigma 00, \sigma 01, \sigma 10, \sigma 11$ are initial segments of their binary representation we have that $y_{00} < y_{01} < y_{10} < y_{11}$ but $f(y_{00}) < f(y_{10}) < f(y_{01}) < f(y_{11})$. This means that for τ equal to $\sigma 00, \sigma 01, \sigma 10$ or $\sigma 11$ all the extensions of τ are not in $dom(f \cap g_n)$. We can exemplify this as follows: assume that there is $x_{01} \in dom(f \cap g_n)$ such that $\sigma 01$ is an initial segments of it and that g_n is monotone nondecreasing. Then, there is no $y_{10} \in dom(f \cap g)$ such that its binary representation extends $\sigma 10$ since $x_{01} < y_{10}$ but $f(y_{10}) < f(x_{01})$. Notice that the size of the interval of all real numbers such that their binary expansion extends $\sigma 10$ has length, at least, $2^{-|\sigma 10|-2} = 2^{-m-5}$.

Now, in any of the cases, if we use λ to denote the Lebesgue measure and we assume that $dom(f \cap g_n)$ is measurable, we have that

$$\frac{\lambda((x-2^{-m}, x+2^{-m}) \cap dom(f \cap g_n))}{\lambda((x-2^{-m}, x+2^{-m}))} \leq \frac{2^{-m+1}-2^{-m-5}}{2^{-m+1}} = 1-2^{-6},$$

as long as m is big enough for $(x-2^{-m}, x+2^{-m}) \subseteq (0,1)$.

Since the Lebesgue density of any point in $dom(f \cap g_n)$ cannot be 1 (it is less than $1-2^{-6}$), we have that $dom(f \cap g_n)$ is a null set if it is measurable (see Theorem 3.24 to read the statement of the Lebesgue density Theorem).

Since $\lambda(dom(f)) = 1$, it is not a countable union of null sets but, since $dom(f) = \bigcup_{n \in \omega} dom(f \cap g_n)$ then there is n such that $dom(f \cap g_n)$ is not measurable. □

Recall that there is a model of ZF with a certain amount of choice where all subsets of reals are measurable. This fact along with Lemma 3.47 seems to contradict Theorem 3.43. Nevertheless, we want to point at the fact that the proof of the Lebesgue Density Theorem seems to use enough choice to create nonmeasurable sets.

3.7 Towers of Models

Following a result in [2], we can construct an entangled set using a tower of models.

For the convenience of the reader, we reproduce the proof here with more details in some of the steps that are going to be key for us.

Theorem 3.48 (Avraham-Shelah)**.** *Assuming CH there is an uncountable 2-entangled set of reals.*

Proof. We will work with reals as elements of 2^ω.

Let $\langle r_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all real numbers.

Let M_0 be a countable elementary submodel of $H(\aleph_2)$ (the sets whose transitive closure has size less than $\aleph_2$).

Assuming that we have define $\langle e_\xi : \xi < \alpha \rangle$ and $\langle M_\xi : \xi \leq \alpha \rangle$, we define e_α to be a Cohen generic with respect to M_α (which, by the definition of M_ξ, will be also an elementary submodel of $H(\aleph_2)$). Now, if α is a limit ordinal, let $M_\alpha = \bigcup_{\xi < \alpha} M_\xi$.

If $\alpha = \beta + 1$, let M_α be an elementary extension of M_β such that M_α is also an elementary submodel of $H(\aleph_2)$, $M_\beta \in M_\alpha$ looks countable for M_α and, if $e_\beta = r_\gamma$, then $r_\delta \in M_\alpha$ for all $\delta \leq \gamma$.

The set $E = \{e_\alpha : \alpha < \omega_1\}$ is the set that we are looking for. We will see this by contradiction.

Assume that there is an uncountable collection of disjoint pairs

$$A = \{(e_{\alpha_i}, e_{\beta_i}) : i < \omega_1\} \subseteq E^2$$

such that whenever $e_{\alpha_i} < e_{\alpha_j}$ then $e_{\beta_i} < e_{\beta_j}$.

Without loss of generality, we can assume that given $i < j < \omega_1$ we have $\alpha_i < \beta_i < \alpha_j < \beta_j$.

Let N be a countable elementary submodel of $H(\aleph_2)$ such that $A \in N$. Now, let $\gamma_0 = \omega_1 \cap N$ and let ξ_1 be the first ordinal such that $\langle (e_{\alpha_i}, e_{\beta_i}) : i < \gamma_0 \rangle \in M_{\xi_1}$. This ordinal exists because every countable sequence can be coded by a single real number, therefore, the real and the sequence are added at some stage. By our construction of E, for any $\alpha_i > \xi_1$ the pair $(e_{\alpha_i}, e_{\beta_i})$ is M_{ξ_1} generic for the product of two Cohen forcing.

At this moment we need some notation. The expression $(f, g) \leq (r, s)$ for $f, g \in 2^{<\omega}$ and $r, s \in 2^\omega$ means that f is an initial segment of r and g is an initial segment of s.

Also, in the same context, the expression $T((f,g),(r,s))$ means that for any real number $\overline{f}$ and $\overline{g}$ that extend f and g, respectively, we have that $\overline{f} < r$ if and only if $\overline{g} > s$. With this, notice that the set

$$\{(f,g) : f,g \in 2^{<\omega}\, (\forall i < \gamma_0 (f,g) \not\leq (e_{\alpha_i}, e_{\beta_i})) \vee (\exists i < \gamma_0 (T((f,g),(e_{\alpha_i}, e_{\beta_i}))))\},$$

which we will call D, is dense in $2^{<\omega} \times 2^{<\omega}$ and belongs to M_{ξ_1}. By genericity of $(e_{\alpha_j}, e_{\beta_j})$ over M_{ξ_1}, there is $(f,g) \leq (e_{\alpha_j}, e_{\beta_j})$ such that $(f,g) \in D$.

Notice that, since N is an elementary submodel of $H(\aleph_2)$ and $N \cap A = \langle (e_{\alpha_i}, e_{\beta_i}) : i < \gamma_0 \rangle$ we have that for any $j < \omega_1$ and for any finite initial segment $(f,g) \leq (e_{\alpha_j}, e_{\beta_j})$ there is $i < \gamma_0$ such that $(f,g) \leq (e_{\alpha_i}, e_{\beta_i})$. This implies that there is $\alpha_j > \xi_1$ and $i < \gamma_0$ such that $e_{\alpha_i} < e_{\alpha_j}$ but $e_{\beta_i} > e_{\beta_j}$. □

There is a couple of things that are important to notice.

First of all, it is not necessary to add every single real number to the tower of elementary models. As long as we add all countable subsequences of the generic sequence, we can run the argument.

Also, the dense set D can be described in a Σ^0_2 way (actually, Δ^0_2) given that you have the sequence $\langle (e_{\alpha_i}, e_{\beta_i}) : i < \gamma_0 \rangle$. Therefore, in the face of it, the full genericity of the Cohen generic is not necessary, we can use a 2-generic real (or a weak 2-generic, since the set is open dense). Nevertheless, we can do a little better.

Definition 3.49. *Given $g \in 2^\omega$ we say that g is* (weak) 1-generic *if and only if for every Σ^0_1 (dense) $S \subseteq 2^{<\omega}$ there exists $\sigma \in 2^{<\omega}$, $\sigma \leq g$, such that either:*

1. *$\sigma \in S$*

2. *For all τ extending σ, $\tau \notin S$.*

Lemma 3.50. *Given the construction in Theorem 3.48, it is only necessary that* $(e_{\alpha_j}, e_{\beta_j})$ *is* 1*-generic over* $\langle (e_{\alpha_i}, e_{\beta_i}) : i < \alpha_0 \rangle$ *for the proof to work.*

Proof. Notice that the following subset of D is Σ^0_1 over $\langle (e_{\alpha_i}, e_{\beta_i}) : i < \gamma_0 \rangle$:

$$S = \{(f, g) : f, g \in 2^{<\omega} \exists i < \gamma_0 (T((f, g), (e_{\alpha_i}, e_{\beta_i})))\}.$$

Since for every $\alpha_j > \gamma_0$ we have that $(e_{\alpha_j}, e_{\beta_j})$ is 1-generic over $\langle (e_{\alpha_i}, e_{\beta_i}) : i < \alpha_0 \rangle$. We know that there is $(f, g) \leq (e_{\alpha_j}, e_{\beta_j})$ such that either $(f, g) \in S$ or for all $(f', g') \geq (f, g)$, $(f', g') \notin S$.

Nevertheless, notice that, by the definition of γ_0, for all $(f, g) \leq (e_{\alpha_j}, e_{\beta_j})$ there is $i_{(f,g)} < \gamma_0$ such that $(f, g) \leq (e_{\alpha_{i_{(f,g)}}}, e_{\beta_{i_{(f,g)}}})$. Since $e_{\alpha_{i_{(f,g)}}}, e_{\beta_{i_{(f,g)}}}$ are not eventually 0 or 1 (they are 1-generic), for each $(f, g) \leq (e_{\alpha_j}, e_{\beta_j})$ there is $(f', g') \geq (f, g)$ such that $e_{\alpha_{i_{(f,g)}}} \upharpoonright |f'| + 1 = f'^\frown 1$ and $e_{\beta_{i_{(f,g)}}} \upharpoonright |g'| + 1 = g'^\frown 0$, then $T((f'^\frown 0, g'^\frown 1), (e_{\alpha_{i_{(f,g)}}}, e_{\beta_{i_{(f,g)}}}))$ and $(f'^\frown 0, g'^\frown 1) \in S$. This shows that option two of the definition of 1-generic cannot be satisfied, so there must be $(f, g) \leq (e_{\alpha_j}, e_{\beta_j})$ such that $(f, g) \in S$.

Therefore, there is $i < \gamma_0$ such that $T((e_{\alpha_j}, e_{\beta_j}), (e_{\alpha_i}, e_{\beta_i}))$. □

Now, we would like to make an analogous construction of the above proof to show the following theorem:

Theorem 3.51. *Assuming CH, given any non-trivial automorphism of the Turing degrees, call it* a*, there is no family* G *of* 1*-generics degrees such that:*

1. *For every degree* $\mathbf{y}$ *there is a* 1*-generic degree over* $\mathbf{y}$ *in* G*.*

2. *For all* $\mathbf{x} \in G$*, if* $\mathbf{x}$ *is* 1*-generic over* $\mathbf{y}$*, then there are* $x, z \in \mathbb{R}$ *with* $deg(x) = \mathbf{x}$ *and* $deg(z) = a(\mathbf{x})$ *with* (x, z) 1*-generic over* $\mathbf{y}$*.*

By a result of Slaman and Woodin [31, 32], all automorphism of the Turing degrees can be expressed in an arithmetical way. Therefore, assuming that the theorem is false, there is an automorphism of the Turing degrees that will send a family of 1-generic to 1-generics over them. Our strategy for the proof will be to construct a 2-entangled set of 1-generics such that the image of the entangled set under the automorphism is inside the entangled set. Then, since all automorphism can be described in an arithmetical way, using the Decomposition Lemma 3.43, we would generate a monotone function from an entangled set to itself, which is a contradiction.

This approach will run into one problem if we use the construction in [2]: we could have the image of a 1-generic be inside the first model where the real appear (since, elementarity implies arithmetic elementarity). We will be able to solve this by being more careful with the amount of ZFC that our models satisfy.

Also, it is important to notice that, if we change point (2) of the theorem to demand full genericity (or n-generic for all $n \in \omega$) then the theorem is trivially true. This is, again, because any automorphism can be expressed in an arithmetical way. If you have an automorphism a then there is $k \in \omega$ such that, for all $\mathbf{x}$, $a(\mathbf{x}) \leq_T \mathbf{x}^{(k)}$. Then, $a(\mathbf{x})$ cannot be $k+1$-generic over $\mathbf{x}$. Using stronger results, we believe that changing point (2) for 3-generic is trivially true. Either way, this highlights the importance of Lemma 3.50 for Theorem 3.51.

We will break this proof into a definition and couple of lemmas.

Definition 3.52. *A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is* powerful *if and only if*

1. *$x \equiv_T y$ implies $f(x) \equiv_T f(y)$.*

2. *There is a family G of 1-generics degrees such that*

(a) *For all* $x \in \mathbb{R}$, *there is* $\mathbf{y} \in G$ *and* $y \in \mathbf{y}$ *such that* y *is* 1*-generic over* x.

(b) *For all* $\mathbf{x} \in G$ *there is* $x \in \mathbf{x}$ *such that* $x \in A$.

(c) *For all* $\mathbf{x} \in G$, *if* $\mathbf{x}$ *is a* 1*-generic degree over* y, *then there exist* $x_1, x_2 \in \mathbf{x}$ *and* $z \equiv_T f(x_1)$ *such that* (x_2, z) *is* 1*-generic over* y.

Notice that (c), specifically the fact that (x_2, z) is 1-generic, implies that if $deg(x) \in G$ then $x \not\equiv_T f(x)$.

Lemma 3.53. *(CH) Given a powerful function* g, *there is a* 2*-entangled set* E, *made of* 1*-generics, such that* $g(\mathbb{R}_E) \cap \mathbb{R}_E$ *is uncountable.*

Proof. Let $\langle r_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all real numbers. We will start with some notation: given $A \subset \mathbb{R}$ a countable set, let O_A be one of the followings:

1. The minimal real (in the enumeration) that belongs to the minimal Turing degree (respect to Turing reduction) such that it can compute all elements of A.

2. If the above one is not possible, then O_A will be the minimal real (in the enumeration) such that it can compute all elements of A but does not compute $\mathbf{0}'$ (zero jump).

3. If neither of the above ones are possible, O_A will be the minimal real (in the enumeration) such that it can compute all elements of A.

Now, let $M_0 = H(\aleph_0) \cup C$, where C is the set containing all computable subsets of $H(\aleph_0)$. Here, $H(\aleph_0)$ is the set of the hereditary finite sets.

Assuming that we have define $\langle e_\xi : \xi < \alpha \rangle$ and $\langle M_\xi : \xi \leq \alpha \rangle$. If $\alpha = \gamma + 2n$ for $\gamma = 0$ or γ a limit ordinal, we define e_α to be the minimal real (with respect the enumeration) such that it is 1-generic with respect to $O_{\mathbb{R}\cap M_\alpha}$ and such that $deg(e_\alpha) \in G$ (the family associated with the powerful function). Now, let $e_{\alpha+1}$ be such that $(e_\alpha, e_{\alpha+1})$ is 1-generic over $O_{\mathbb{R}\cap M_\alpha}$ and $e_{\alpha+1} \equiv_T g(x_\alpha)$, where $x_\alpha \equiv_T e_\alpha$. We can do this because g is powerful and $deg(e_\alpha) \in G$.

Throughout the construction we will do a bookkeeping (as in [33]) of the countable subsets of $\langle e_\xi : \xi < \alpha\rangle$. The objective is that, once we have the sequence $\langle e_\xi : \xi < \omega_1\rangle$, we can also enumerate all the countable subsets of it as $\langle B_\xi : \xi < \omega_1\rangle$. Notice that this is possible since every countable subset of $\langle e_\xi : \xi < \omega_1\rangle$ is a countable subset of $\langle e_\xi : \xi < \alpha\rangle$, for some $\alpha < \omega_1$ (and by CH).

To finish the construction, if α is a δ limit ordinal, let $M_\alpha = \bigcup_{\xi<\alpha} M_\xi \cup C[O_{B_\delta}]$ where $C[O_{B_\delta}]$ is the set of all the computable objects from O_{B_δ}. Finally, if $\alpha = \beta + 1$, let $M_\alpha = M_\beta \cup C[e_\beta]$.

The set $E = \{e_\alpha : \alpha < \omega_1\}$ is 2-entangled. We will see this by contradiction.

Assume that there is an uncountable collection of disjoint pairs

$$A = \{(e_{\alpha_i}, e_{\beta_i}) : i < \omega_1\} \subseteq E^2$$

such that whenever $e_{\alpha_i} < e_{\alpha_j}$ then $e_{\beta_i} < e_{\beta_j}$.

As in Theorem 3.48, we can ask that given $i < j < \omega_1$ we have $\alpha_i < \beta_i < \alpha_j < \beta_j$.

Let N be a countable elementary submodel of $H(\aleph_2)$ such that $A \in N$. Now, define $\gamma_0 = \omega_1 \cap N$ and let ξ_1 be the first ordinal such that $O_\Gamma \in M_{\xi_1}$, where $\Gamma = \{(e_{\alpha_i}, e_{\beta_i}) : i < \gamma_0\}$. By our construction of E, for any $\alpha_i > \xi_1$ the real $(e_{\alpha_i}, e_{\beta_i})$ is 1-generic over O_Γ.

Using the same notiation as in Theorem 3.48, we know that the set

$$\{(f, g) : f, g \in 2^{<\omega}\, (\forall i < \gamma_0 (f, g) \not\leq (e_{\alpha_i}, e_{\beta_i})) \vee (\exists i < \gamma_0 (T((f, g), (e_{\alpha_i}, e_{\beta_i}))))\},$$

which we will call D, is open dense in $2^{<\omega} \times 2^{<\omega}$.

By Lemma 3.50, we know that there is $\alpha_i > \xi_1$ and $j < \alpha_0$ such that $e_{\alpha_i} < e_{\alpha_j}$ but $e_{\beta_i} > e_{\beta_j}$.

This shows that E is 2-entangled and, by construction, $\mathbb{R}_E \cap g(\mathbb{R}_E)$ is uncountable.

□

Lemma 3.54. *(CH) There is no countable-to-one Σ^1_1 powerful function.*

Proof. Suppose that there exist such a function. Then, by Lemma 3.53, there is an entangled set E, made of 1-generics, such that $g(\mathbb{R}_E) \cap \mathbb{R}_E$ is uncountable. By Theorem 3.45, $\mathbb{R}_E$ is σ-2-entagled.

Since g is a countable-to-one Σ^1_1 function, by lemma 3.43, it contains an uncountable monotone function, g_1, such that $g_1(\mathbb{R}_E) \cap \mathbb{R}_E$ is uncountable.

Nevertheless, by the obsevation made after the definition of powerful, there are uncountably many 1-generics in E such that $g(x) \not\equiv_T x$.

Then we have a strictly monotone function changing uncountably many degrees from an uncountable subset of a σ-2-entangled set to itself. This cannot happen. □

With the above lemmas, we are ready to proof Theorem 3.51.

Proof. By a result in [32], every automorphism has an arithmetic representation, this is, a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ such that f is arithmetic, A contains an element from each Turing degree, $x \equiv_T y$ if and only if $f(x) \equiv_T f(y)$ and the change of degree is done exactly as the automorphism.

Working towards a contradiction, assume that there is an automorphism a and a family G of 1-generics with the properties enlist in the theorem. The representation f of a is an arithmetic powerful function. By lemma 3.53, this function does not exist. □

3.8 Modifications and other possible results

It is important to make a couple of remarks about the construction of Lemma 3.53. First of all, the construction can be stated purely using Turing degrees. The usage of models was to make the analogy with Theorem 3.48 more direct. Also, we just need the degree of (e_α, e_β) to be 1-generic over the degree of each element of $\mathbb{R} \cap M_\alpha$. This means that the usage of $O_{\mathbb{R}\cap M_\alpha}$ was not necessary, nevertheless, it reduces writing. Specifically, because having a degree being 1-generic over countably many degrees does not mean that you have a single element that is 1-generic over all of them at the same time (or not in a trivial way), see question 3.64.

Either way, it is possible to show that the set that you get is 2-entangled only assuming that the degree of (e_α, e_β) to be 1-generic over the degree of each element of $\mathbb{R} \cap M_\alpha$. We could define $e_\alpha = x_\alpha$ and $e_{\alpha+1} = f(x_{\alpha+1})$. Then, using the (e, d) trick in Theorem 3.45, we get a monotone function from pairs of 1-generic degrees over O_Γ (that is in some $\mathbb{R} \cap M_\alpha$) to $(e_{\alpha_i}, e_{\beta_i})$. The last thing to do will be to change the definition of D depending on whether the function is decreasing or increasing. This description is missing details but, we hope that the reader understands why we choose to make the proof differently.

In a different (but related) topic, do not believe that the strange definition of O_A was completely for free. It is important to remark that adding O_{B_α}, contrary to using $O_{\mathbb{R}\cap M_\alpha}$, is a key point in the proof since we need the subset S of D of Lemma 3.50 to be an open Σ^0_1 set of a single degree.

Finally, if it is the case that $g(x)$ is 1-generic over x whenever x is 1-generic, we can show that $0' \notin M_\alpha$ for any $\alpha < \omega_1$. Making $0' \notin M_\alpha$ ensures that for any element in M_α

all the 1-generics, respect to that degree, are not in M_α. Therefore, if we pick 2-generics instead of 1-generics, for any e_α, $g(e_\alpha) \notin M_{\alpha+1}$. We didn't use this in our proof since $g(e_\alpha)$ already had all the wanted properties because g is powerful. Nevertheless, this could be handy if the definition of powerful is weakened or for other applications, as the following.

Corollary 3.55. *Assuming CH, one of the following is true:*

1. *The relation "$\boldsymbol{x}$ is 1-generic over $\boldsymbol{y}$" is not definable in the Turing degrees.*

2. *For any automorphism of the Turing degrees, call it a, there is no family G of 1-generics degrees such that:*

   (a) *For every degree $\mathbf{y}$ there is a 1-generic over $\mathbf{y}$ in G.*

   (b) *For all $\mathbf{x} \in G$, if $\mathbf{x}$ and $a(\mathbf{x})$ are 1-generic degrees over y, then there are $x, z \in \mathbb{R}$ with $deg(x) = \mathbf{x}$ and $deg(z) = a(\mathbf{x})$ with (x, z) 1-generic over y.*

Proof. We can modify the definition of O_A so that O_A computes $0''$. This way, we have that, for any automorphism a, $a(deg(O_A)) = deg(O_A)$ (see [32]). Let f be a representation of a.

Therefore, if the relation "$\mathbf{x}$ is 1-generic over $\mathbf{y}$" is definable, we have that if x is 1-generic over O_A then $f(x)$ is generic over $f(O_A) \equiv_T O_A$. So, if 1 and 2 fails, there is an automorphism a such that $(x, f(x))$ is 1-generic over O_A if $x \in G$.

The rest of the proof follows. □

Corollary 3.56. *Assuming CH, one of the following is true:*

1. *The relation "x is 1-generic over y" is not definable in the Turing degrees.*

2. *For any automorphism of the Turing degrees, call it a, it is not the case that for all* 1*-generic degrees* $\mathbf{x}$*, if* $\mathbf{x}$ *and* $a(\mathbf{x})$ *are* 1*-generic over* y*, then there is* $z \in a(\mathbf{x})$ *such that* z *is* 1*-generic over* y *and over* $\mathbf{x}$*.*

Proof. This follows from the previous lemma using the family G to be all 1-generic degrees. □

Notice that, in both proofs, we are not using that "y is 1-generic over x" is definable but rather that there is a definable relation R such that xRy implies "y is 1-generic over x" and such that for all x there is a y with xRy. This gives us the following corollary:

Corollary 3.57. *Assuming CH, at least, one of the following is true:*

1. *There is no relation R definable over the Turing degrees such that xRy implies "y is* 1*-generic over x" and such that for all x there is a y with xRy.*

2. *For any automorphism of the Turing degrees, call it a, there is no family G of* 1*-generics degrees such that:*

   (a) *For every degree* $\mathbf{y}$ *there is a* 1*-generic over* $\mathbf{y}$ *in G, say* $\mathbf{g}$*, such that* $\mathbf{y}R\mathbf{g}$*.*

   (b) *For all* $\mathbf{x} \in G$*, if* $yR\mathbf{x}$ *and* $yRa(\mathbf{x})$*, for some real* $y \in \mathbb{R}$*, then there are* $x, z \in \mathbb{R}$ *with* $deg(x) = \mathbf{x}$ *and* $deg(z) = a(\mathbf{x})$ *with* (x, z) 1*-generic over* y*.*

Now, a couple of words dedicated to absoluteness are in place. From the proofs it is clear that the use of CH, or of the existance of a tower of models with that property, is essential. Nevertheless, in [32], it was shown that the statement "There is a non-trivial automorphism of the Turing degrees" was absolute. This does not imply that the above Theorem, or Corollaries, are absolute since there is the possibility that the definition of the 1-generic relation is not absolute. Indeed, that will depend on the proof (or disproof) of the existence of such a relation.

To see other situations where towers of models can be generated, see the COMA, defined by Hart and Kunen in [13].

3.9 Questions and conclusions

In this chapter, we studied how do sets closed under Turing equivalance look inside $\mathbb{R}$. Definitively, not all the questions were solved. Questioning how these sets behave from an algebraic perspective left some open questions:

Question 3.58. *In Theorem 3.15, is the use of* $\mathbf{0}'$ *necessary? In other words, are there reals* x, y *such that* $\mathbf{0}' \not\leq_T deg(x \oplus y)$ *and the minimal subfield of* $\mathbb{R}$ *that contains* $\mathbb{R}_x \cup \mathbb{R}_y$ *is not* $\mathbb{R}_{\wedge x \oplus y}$*?*

Question 3.59. *Given a field* F*, is* $\mathbb{R}_F$ *also a field?*

And, of course, we have questions related to measure like:

Question 3.60. *For which subsets* $S \subseteq \mathbb{R}$ *is* $\mathbb{R}_S$ *measurable?*

Even in the case of order type, which was studied in a much more lengthy way, the advance was not fundamental. So we can ask:

Question 3.61. *Which other* $\mathfrak{c}$*-dense order can be obtained (or cannot be obtained) by a set of the form* $\mathbb{R}_S$*?*

Question 3.62. *Is there a topological or model theoretic characterization of all the order types of the form* $\mathbb{R}_S$*?*

Question 3.63. *Is there a collection of countably many monotone functions, $\mathcal{F}$, such that $\mathcal{F}_x = \mathbf{R}_{\wedge deg(x)}$ for all x?*

But this approach to subsets of reals showed that it can interact with other important questions in the area of Computability Theory or, as it has been commented to the author, in Descriptive Set Theory.

Our approach to the automorphism problem gave some restrictions to the way that automorphisms interacts with 1-generics, under the Continuum Hypothesis. The result, as showed in Section 3.8, can be written in multiple ways, but the interactions between the degrees and the reals inside of them are more subtle than they appear.

For example, one way to improve our result, specifically Corollary 3.56, will be to show that there is no automorphism such that $a(\mathbf{x})$ is 1-generic over $\mathbf{x}$ for a big family of 1-generics. Nevertheless, if we attempt to prove it, the following question arises:

Question 3.64. *Given a Turing degree $\mathbf{g}$ that is 1-generic over $\mathbf{a}$ and $\mathbf{b}$, under what conditions is there a real number z, with Turing degree $\mathbf{g}$, such that z is 1-generic over $\mathbf{a}$ and $\mathbf{b}$?*

Question 3.65. *Given a sufficiently generic degree $\mathbf{g}$ over y, is it true that the image of $\mathbf{g}$ under any nontrivial automorphism of the Turing degrees can compute a sufficiently generic degree over y? Can this degree be 1-generic over $\mathbf{g}$?*

Finally, Turing reduction is not the only way to classify the real numbers in degrees that show you how much information they bear. For example, we could use the enumeration reduction (here $A \leq_e B$ if and only if A is c.e. over B) or the constructible reduction (here $x \leq_c y$ if and only if $x \in L[y]$, these also have countable degrees if you assume large cardinals).

Question 3.66. *Which of the results in this chapter are still true, or false, if we change Turing equivalence to enumeration, arithmetic, hyperearithmetic or constructibility equivalence?*

In this same line of thought, we can also wonder about the properties of these sets in different spaces.

Question 3.67. *What can we say about subsets (or subspaces) of the Hilbert cube that are closed under the continuous degree equivalence (see Miller [19])?*

Appendix A

Cichon and Effective Cichon Diagrams

The figures in this Appendix originally appeared in [38] and [24]. We want to thank Paul Tveite for doing them and sharing them with us.

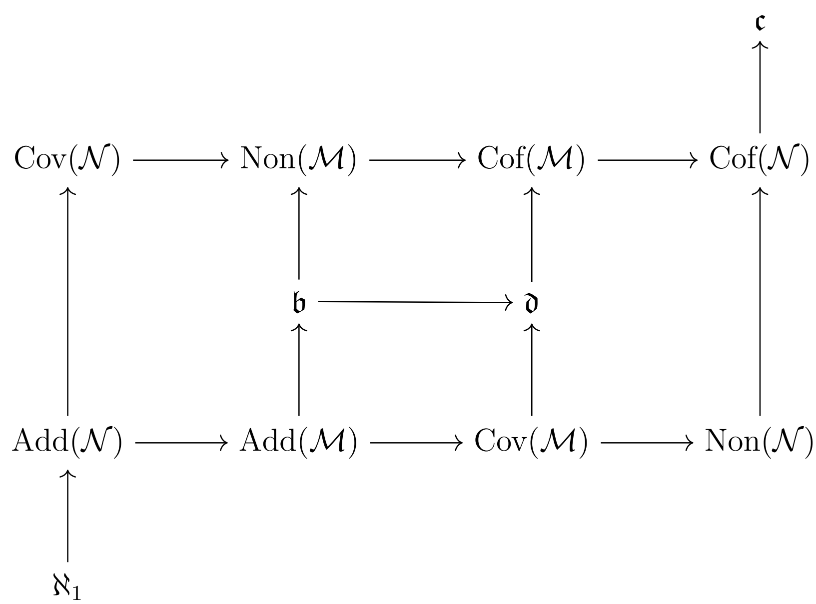
We will first present the diagrams and then list the corresponding definitions or theorems. All of them can be found either in [16], [27] or in [7].

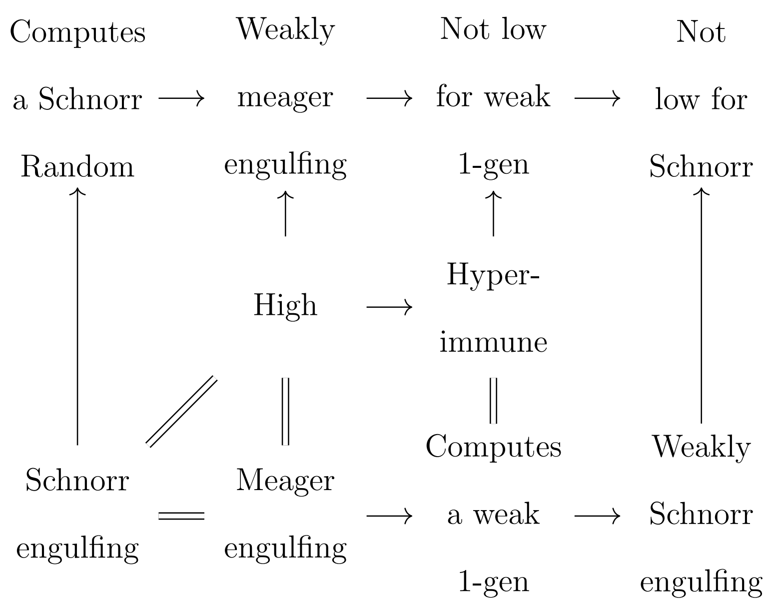
A.1 Diagrams

In the first diagram, elements to the left or bottom are smaller than the ones right or above them. In other words, arrows go from small to big.

In the second diagram, elements to the left or bottom are stronger than the ones right or above them. In other words, arrows represent the usual implication.

In both diagrams, if there is no equal sign between elements it means that there is a way to split the notions/numbers.





A.2 Cichon's diagram elements

Definition A.1. *Given an ideal $\mathcal{I}$ of subsets of X define*

- $Add(\mathcal{I}) = \min\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{I} \vee \bigcup \mathcal{Y} \notin \mathcal{I}\}$

- $Cov(\mathcal{I}) = \min\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{I} \vee X \subseteq \bigcup \mathcal{Y} \notin \mathcal{I}\}$

- $Non(\mathcal{I}) = \min\{|Y| : Y \subseteq X \vee \bigcup Y \notin \mathcal{I}\}$

- $Add(\mathcal{I}) = \min\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{I} \vee \forall I \in \mathcal{I} \exists Y \in \mathcal{Y}(I \subseteq Y)\}$

Definition A.2. *We call $\mathcal{N}$ the ideal of all measure zero sets (also known as null sets) of reals.*

We call $\mathcal{M}$ the ideal of all meager sets (also known as first category sets) of reals.

Definition A.3. *1. Given $f, g \in \omega^\omega$ we say that g eventually dominates f, write as $f \leq^* g$, if and only if there is $n \in \omega$ such that for all $m \geq n$, $f(m) \leq g(m)$.*

2. $\mathfrak{b} = \min\{|Y| : Y \subseteq \omega^\omega \vee \forall f \in \omega^\omega \exists g \in Y(g \not\leq^* f)\}$

3. $\mathfrak{d} = \min\{|Y| : Y \subseteq \omega^\omega \vee \forall f \in \omega^\omega \exists g \in Y(f \leq^* g)\}$

A.3 Effective Cichon's diagram elements

Definition A.4. *1. We say that a degree A is high if and only if $0'' \leq_T A'$.*

2. We say that a degree is hyper-immune if and only if it computes a function $f \in \omega^\omega$ such that $f \not\leq^* g$ for all computable functions g.

Theorem A.5. *A is a high degree if and only if A computes a function $f \in \omega^\omega$ such that $g \not\leq^* f$ for all computable functions g.*

Definition A.6. *1. We say that $\langle U_e : e \in \omega \rangle$ is a Schnorr test if and only if U_e is a c.e. open set such that $\lambda(U_e) = 2^{-e}$.*

2. *We say that r is an Schnorr random reals if and only if $r \notin \bigcap_{e\in\omega} U_e$ for all Schnorr test $\langle U_e : e \in \omega\rangle$.*

Definition A.7. *A degree D is DNC if and only if it computes a function $f \in \omega^\omega$ such that for all $e \in \omega$, $f(e) \neq \varphi_e(e)$.*

Definition A.8. *A degree S is weakly meager engulfing if and only if it computes a meager set containing all computable reals.*

Theorem A.9. *A degree is weakly meager engulfing if and only if it is High or DNC.*

Theorem A.10. *A degree is not low for weak 1-generics if and only if it is Hyperimmune or DNC.*

Theorem A.11. *A degree is not low for Schnorr if and only if it is non-computably treaceable.*

Definition A.12. *A degree S is weakly Schnorr engulfing if and only if it computes a Schnorr test containing all computable reals.*

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