

# Complexity Classifications in Model Theory and Computable Structures

By

**Tamvana P. Makuluni**

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE  
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY  
(MATHEMATICS)

at the

**UNIVERSITY OF WISCONSIN – MADISON**

2018

Date of final oral examination: 04/24/2018

The dissertation is approved by the following members of the Final Oral Committee:

Uri Andrews, Assistant Professor, Mathematics

Steffen Lempp, Professor, Mathematics

Paul M. Terwilliger, Professor, Mathematics

Noah D. Schweber, Van Vleck Visiting Assistant Professor, Mathematics

© Copyright by Tamvana P. Makuluni 2018

All Rights Reserved

To my parents, and my first home.

To my husband, and our home together.

To my POC and queer communities, and my home in the world.

# Acknowledgements

I would like to thank everybody who has helped to make this thesis a reality. First and foremost is my advisor Uri Andrews, of course, who was an invaluable help throughout all of this work. He has patiently listened to me talk through everything in this thesis, and offered many ideas about directions this work could take.

In particular, he led the work in Chapter 3, which he gave me my first opportunity as a graduate student to truly participate in research. Thanks also goes to Alice Medvedev, whose suggestions directly inspired our proof.

I would also like to thank Turbo Ho, my academic brother, and my friend and companion throughout much of this work. Our conversations not only inspired my work, but provided much a needed break when I was stressed out.

I would like to thank Asher Kach and Antonio Montalbán for inviting me to UChicago to work on the proof in Chapter 2. Our conversations on the topic brought the result from a sketch of a proof to a rigorous result. Thanks also to Joe Miller who was the first to look at my idea for that proof and who encouraged me to pursue it.

The work in Part II owes its inspiration to Vince Guingona, who introduced me to convex orderability, and I thank him for the conversations that we had on the topic.

I would of course like to thank my friends and my family for putting up with my anxiety throughout graduate school, and especially through the writing process. I have often been poor company, and no one has been more patient than my husband, who even allowed me to talk to him about my work, though I suspect very little bores him more.

Finally, I extend my thanks to Amtrak, as much of the work to compile this thesis was done while riding the Southwest Chief, and to Next Door Brewing, where I have done much of my revision.

# Abstract

We explore various questions of complexity in model theory and computable structures. Part I focuses on computable complexity. In Chapter 2, we show that the theory of the collection of isomorphism classes of at-most countable groups pre-ordered by embedability is 1-reducible to the true theory of second-order arithmetic. In Chapter 3, we show that uncountable categoricity is a  $0'$ -d.c.e. property, work previously published with Uri Andrews. Part II focuses on one specific dividing line in model theory called convex orderability. We show that convexly orderable linear orders can be divided into dense and discrete orders. We show that definable sets in a convexly orderable discrete orders are piecewise periodic, and that convexly orderable dense orders have the nowhere dense graph property.

# Contents

<b>Acknowledgements</b>	<b>ii</b>
<b>Abstract</b>	<b>iii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Notation and Conventions . . . . .	2
<b>I Complexity in Computable Structures</b>	<b>3</b>
<b>2 The Preorder of Groups</b>	<b>4</b>
2.1 Introduction . . . . .	4
2.2 The sets $X_a$ and $Y$ . . . . .	5
2.3 Defining the Set of Possible Parameters . . . . .	7
<b>3 The Index Set of Uncountable Categoricity</b>	<b>9</b>
3.1 Introduction . . . . .	9
3.2 Description . . . . .	10
3.3 Hardness . . . . .	12
<b>II Convex Orderability</b>	<b>14</b>
<b>4 Introduction to Convex Orderability</b>	<b>15</b>
4.1 Notation . . . . .	15
4.2 An introduction to Convex Orderability . . . . .	16
4.3 Observations on Convex Orderability . . . . .	20
<b>5 Convex Orderings of Ordered Structures</b>	<b>22</b>
5.1 Decompositions of Orders . . . . .	22
5.2 Discretely Ordered CO Structures . . . . .	30
5.3 Dense Orders . . . . .	35

<b>6 Monotonicity in Ordered Convexly Orderable Structures</b>	<b>43</b>
6.1 Introduction and Notation . . . . .	43
6.2 Density and Grids . . . . .	44
6.3 Monotonicity . . . . .	47
<b>Bibliography</b>	<b>51</b>

# Chapter 1

## Introduction

If you ask two mathematicians what is mathematics is, you're as likely as not to get a different answers from them. We write here from a structural perspective, i.e., from the point of view that mathematics is the study not of any particular objects, real or imagined, but rather the study of the relationships between objects. From an ontological point of view this means that we are not concerned with understanding what things are, but rather with understanding the rules for how they interact with each other. From a practical point of view, this is not only a declaration of philosophy, but also a way of saying that the work in this thesis is in the realm of model theory and computable structure theory.

The questions that we answer in this thesis are, in various shapes, questions about complexity. Complexity is, like mathematics, a concept with many definitions and in this work we employ many. We can sometimes determine the relative complexity of two structures by showing that one interprets the other, and is therefore at least as complicated (this general idea is in fact common to both model theory and computability theory). This notion is relative, and some structures simply cannot be compared this way, though it can be quite powerful when it shows two structures to be equally complex. We have, in computability theory, the arithmetic hierarchy, which gives a classification of some structures in terms of the number of “infinite steps” necessary to build them. Model theory, for its part, has many different dividing lines which classify structures as more or less complex. Many of these define complexity in terms of the tools which can be applied in attempting to understand a structure.

The thesis is laid out as follows: In Part I, we focus on complexity in computability theory terms. In Chapter 2 we consider the preorder of groups by the subgroup relation. We show that this structure is interpretable with second-order arithmetic. Chapter 3 contains joint work with Uri Andrews. We classify a model-theoretic dividing line, *uncountable categoricity*, in the arithmetic hierarchy by showing that the set of uncountably categorical computable theories is the intersection of a  $\Pi_2$  and a  $\Sigma_2$  set, and that it is *complete* for such sets (i.e., that other such sets can be computed from it).

In Part II we focus entirely on model theory, and in particular on a relatively new notion of complexity called *convex orderability*. Chapter 4 focuses on the relationship between convex orderability and other notions of minimality, in particular dp-smallness, and weak o-minimality. Chapter 5 focuses on ordered



convexly orderable structures as examples of structures which might have some sort of canonical convex ordering, and which might give more insight into the reducts of structures with weakly o-minimal theories. We show that ordered convexly orderable structures have “nice” definable orders. Finally, in Chapter 6 we use this to show that sets in ordered convexly orderable structures have some reasonable topological properties locally.

## 1.1 Notation and Conventions

Throughout this work, unless it is otherwise specified, the letters  $i, j, k, l, m, n, p$  are used to denote integers. We use  $\omega$  to denote the set of natural numbers and  $\mathbb{Z}$  to denote either the set of integers, or the infinite cyclic group. The script capital letters  $\mathcal{M}$  and  $\mathcal{N}$  refer to structures, and the corresponding print capitals  $M$  and  $N$  to their universes. Script capital  $\mathcal{L}$  is used to denote a first-order language. For a structure  $\mathcal{M}$  and  $n \in \omega$ , the subsets of  $M^n$  are generally denoted by capital letters  $X, Y$ , and so on, and the elements of  $M^n$  are generally denoted by lowercase letters  $a, b$ , and so on.

The letters  $\eta, \tau$ , and  $\xi$  refer to finite binary strings. We use  $\varepsilon$  to denote the empty string. The letters  $\varphi, \psi$ , and  $\theta$  refer to formulas in a first-order language. When we wish to emphasize the variables we write, for example,  $\varphi(\bar{x}; \bar{y})$ . The semicolon indicates that the variables are partitioned, **instances** of a formula  $\varphi(\bar{x}; \bar{y})$  in a structure  $\mathcal{M}$  are the parametrized formulas  $\varphi(\bar{x}; \bar{b})$  where  $\bar{b} \in M^{|\bar{y}|}$ . When working in a specific structure, we generally treat a formula and the set that it defines as interchangeable.

We write  $a \in M$  when referring to a single element of  $M$ , and we write  $a \in \mathcal{M}$  to refer to a tuple of elements from  $M$ , i.e., when referring to  $a \in M^n$  for some  $n$ . We use a bar (e.g.,  $\bar{a}$ ) only when we wish to emphasize that  $\bar{a}$  may not be a single element of  $M$ . Given a definable set  $X = \varphi(\bar{x}, y)^{\mathcal{M}}$ , and  $a \in M$ , we write  $X_a$  for the set  $\varphi(\bar{x}, a)^{\mathcal{M}}$ .

A **definable family** is the collection of instances of a formula. We denote definable families with script capital letters, e.g.,  $\mathcal{F}$ . If  $X_1, X_2, X_3, \dots$  are instances of the same formula, we say that they are **uniformly definable**.

## Part I

# Complexity in Computable Structures

## Chapter 2

# The Preorder of Groups

### 2.1 Introduction

Our goal in this chapter is simply to extend a result of Kach and Montalbán. They showed:

**Theorem 2.1.1 (Kach and Montalbán [13]).** *The first-order theory of isomorphism classes of at most countable groups under the direct product relation and the subgroup relation is 1-equivalent to true second-order arithmetic.*

We will extend this result by showing that the direct product is unnecessary:

**Theorem 2.3.6.** *The first-order theory of isomorphism classes of at most countable groups under the subgroup relation is 1-equivalent to true second-order arithmetic.*

We will use the following notation:

**Notation 2.1.2.** We denote by  $\text{Grp}_{<\kappa}$  the set of (isomorphism classes of) groups of cardinality less than  $\kappa$ .

We denote by  $\mathcal{G}^{<\kappa}$  the structure  $(\text{Grp}_{<\kappa}, \leq)$  where  $A \leq B$  if  $A$  is isomorphic to a subgroup of  $B$ .

We prove this by finding a 0-interpretation of second-order arithmetic in  $\mathcal{G}^{<\aleph_1}$ . Our basic strategy will be to start by showing there are two countable definable sets,  $X_a$  and  $Y$ , so that:

1.  $X_a$  is definable with parameter  $a$ , and has order-type  $\omega$ ,
2.  $Y$  is 0-definable,
3.  $|X_a \cap Y| = 1$ , and
4. the (true) set  $X^{\subseteq Y}(a)$  of functions from subsets of  $Y \setminus X_a$  to  $X_a$  is a definable family.

This is enough to show that there is an interpretation of second-order arithmetic with a parameter; we can define the successor function on  $X_a$  since it has order type  $\omega$ , which allows us to define addition and multiplication inductively by quantifying over functions from  $X_a$  to  $X_a$  (which are the same as functions  $g^{-1}f$  where  $g, f \in X^{\subseteq Y}(a)$ ). The subsets of  $X_a$  are uniformly definable as images of functions in  $X^{\subseteq Y}(a)$ .

To complete the proof, we then show that there is a 0-definable set  $A$  so that as long as  $a \in A$ , the sets  $X_a$  and  $Y$  are as above. For  $a, b \in A$ , every map from  $X_a$  to  $X_b$  is of the form  $f^{-1}g$  where  $f: X_b \rightarrow Y \setminus (X_a \cup X_b)$  and  $g: X_a \rightarrow Y \setminus (X_a \cup X_b)$ . We can therefore quantify over functions  $X_a \rightarrow X_b$ . It follows that the unique order-preserving bijection  $\mu_{a,b}: X_a \rightarrow X_b$  is uniformly definable in parameters  $a$  and  $b$ .

Our 0-interpretation of second-order arithmetic, then, will take for its copy of  $\omega$  the set of pairs  $(a, x)$  where  $a \in A$  and  $x \in X_a$  modded out by the equivalence relation  $(a, x) \equiv (b, y)$  if and only if  $\mu_{b,a}(x) = y$ , i.e., sets  $[n]$  consisting of  $(a, x) \in A \times X$  so that  $x$  is the interpretation of  $n$  in  $X_a$ .

## 2.2 The sets $X_a$ and $Y$

We use for our prototypical parameter  $a \in A$ , the group  $\mathbb{Z}^\omega$ , the direct sum of countably many copies of  $\mathbb{Z}$ .

**Notation 2.2.1.** For  $G \in \mathcal{G}^{<\aleph_1}$  we write  $G^<$  for the set  $G^< = \{H \mid H < G\}$ . Note that  $G^<$  is uniformly definable from  $G$ .

Our  $\omega$ -chain  $X_a$  will simply be  $a^<$ . For our prototype, note that the proper subgroups of  $\mathbb{Z}^\omega$  are just the groups  $\mathbb{Z}^n$  for  $n \in \omega$ , and  $\mathbb{Z}^n$  is a subgroup of  $\mathbb{Z}^m$  exactly when  $n \leq m$ , so  $(\mathbb{Z}^\omega)^<$  is an  $\omega$ -chain.

**Definition 2.2.2.** We say  $H$  is a **successor** of  $G$  if  $H^< = G^< \cup \{G\}$ . The set of successors of  $G$  is denoted  $\text{Succ}(G)$ . If  $|\text{Succ}(G)| = n$ , we say that  $G$  is  **$n$ -branching**.

Of course, for any  $\kappa$ ,  $\mathcal{G}^{<\kappa}$  has a unique minimum element 1, the trivial group (which is 0-definable). Any other group embeds a cyclic group, so  $\text{Succ}(1)$  consists of  $\mathbb{Z}$  and the prime cyclic groups. We write  $Z_p$  for the cyclic group of order  $p$ .

**Lemma 2.2.3.**  $Z_2$  is 2-branching. For any prime  $p > 10^{75}$ ,  $Z_p$  is not 2-branching.

*Proof.* For any prime  $p$ , if  $G \in \text{Succ}(Z_p)$  is finite, then  $G$  is a  $p$ -group. If  $G$  has an element of order  $p^2$ , then we can conclude that  $G \cong Z_{p^2}$ . Otherwise, since  $G$  is a  $p$ -group it has non-trivial center, so we can pick  $g \neq 1$  from the center of  $G$  and  $h$  which is not in the subgroup of  $G$  generated by  $g$  since  $|\langle g \rangle| = p$  and  $|G| > p$ . Since both  $g$  and  $h$  have order  $p$ , it follows that  $g$  is not in the subgroup of  $G$  generated by  $h$  either. Since  $g, h$  commute, it follows that  $G = \langle g, h \rangle \cong Z_p^2$ . So, the only finite successors of  $Z_p$  are  $Z_p^2$  and  $Z_{p^2}$ .

Now, assume that  $G \neq Z_4$  and  $G \in \text{Succ}(Z_2)$ . If  $a, b$  are distinct non-identity elements of  $G$ , then they each have order 2, so  $[a, b] = a^{-1}b^{-1}ab = abab = (ab)^2 = 1$ . It follows that  $\langle a, b \rangle \cong Z_2^2$ . So,  $Z_4$  and  $Z_2^2$  are the only successors of  $Z_2$ .

In fact, an infinite successor of  $Z_p$  must be an infinite group whose non-isomorphic proper subgroups are all copies of  $Z_p$ . Such a group is known as a **Tarski  $p$ -group**. So to show that  $Z_p$  is not 2-branching when  $p > 10^{75}$ , we appeal to the following theorem of Ol'shanskii:

**Theorem 2.2.4** (Ol'shanskiĭ [16]). *There is a Tarski  $p$ -group for every prime  $p > 10^{75}$ .*

Such a group is necessarily countable, since it is generated by any pair of non-commuting elements, so in fact  $Z_p^2$ ,  $Z_{p^2}$ , and a Tarski  $p$ -group are all distinct successors of  $Z_p$ .  $\square$

**Lemma 2.2.5.**  *$\mathbb{Z}$  is not 2-branching.*

*Proof.* For any prime  $p$ , let  $H_p = \{m/p^n \mid n, m \in \mathbb{Z}\} \subset \mathbb{Q}$  (this is a group under addition). Suppose  $G \subset H_p$  is a non-trivial subgroup of  $H_p$ . For any  $m/p^n, r/p^n \in G$ , there are some  $j, k$  so that  $(mj + rk)/p^n = \gcd(m, r)/p^n$ , and both  $m/p^n$  and  $r/p^n$  are in the subgroup generated by  $\gcd(m, r)/p^n$ , it follows that either  $G$  is generated by a single element, in which case  $G \cong \mathbb{Z}$  because that element must have infinite order, or  $G$  is not finitely generated. In particular, if  $G$  is infinitely generated then it cannot be contained in  $H_{p,j} = \{m/p^j \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$  for any  $j \in \omega$ , so we can fix  $m/p^{j+1} \in G \setminus H_{p,j}$ . For any  $r/p^j \in H_{p,j} \cap G$ , then, we will have  $\gcd(m, r/p^{j+1}) = \gcd(m, rp)/p^{j+1} \in G$ , so  $r/p^j$  is divisible by  $p$ . It follows that  $m/p^n \in G$  if and only if  $m/p^k \in G$  for all  $k \in \mathbb{Z}$ . In particular, the set of such  $m$  is an infinite subgroup of  $\mathbb{Z}$ , therefore isomorphic to  $\mathbb{Z}$ , and it follows that  $G \cong H_p$ .

So every proper subgroup of  $H_p$  is isomorphic to  $\mathbb{Z}$ , which proves that each  $H_p$  is a successor of  $\mathbb{Z}$ .  $\square$

Now, we are ready to define  $Y$ :

**Definition 2.2.6.** We define  $Y = \text{Succ}'$  to be the set of  $c \in \text{Succ}(1)$  so that  $c$  is not 2-branching. Note that  $Y$  is a countably infinite, and it consists of  $\mathbb{Z}$  and cyclic groups of odd prime order.

So, now we have defined  $X_a$  and  $Y$ , and we've picked  $a = \mathbb{Z}^\omega$  as our prototype. We next need to show that the functions from subsets of  $Y \setminus X_a$  to  $X_a$  are uniformly definable. We use the set-theoretic definition of a function as a collection of pairs.

**Definition 2.2.7.** We say that  $G \in \mathcal{G}^{<\aleph_1}$  is an  **$a$ -pair** if there are  $U \in Y \setminus X_a$  and  $V \in X_a$  so that:

1.  $G^<$  is a partial order.
2.  $G^< \cap \text{Succ}(1) \subseteq Y$ .
3.  $U$  is the unique element of  $G^< \cap Y \setminus X_a$ .
4.  $V$  is the maximum element of  $G^< \cap X_a$ .

We say that  $G$  is an  **$a$ -code** for  $(U, V)$ . As these are all first-order conditions, the set of  $a$ -pairs is uniformly definable from  $a$ , and the groups  $U$  and  $V$  are each uniformly definable from an  $a$ -code for  $(U, V)$  and the parameter  $a$ .

**Lemma 2.2.8.** *When  $a = \mathbb{Z}^\omega$ , there is a code for each pair  $(U, V)$  where  $U \in Y \setminus X_a$  and  $V \in X_a$*

*Proof.* In particular, there are  $p, n \in \omega$  so that  $U = Z_p$  and  $V = \mathbb{Z}^n$ . Let  $G = Z_p \times \mathbb{Z}^n$ . The subgroups of  $G$  are isomorphic to either  $Z_p \times \mathbb{Z}^m$  or  $\mathbb{Z}^m$  where  $m \leq n$ . They are partially ordered by embeddings, so  $G$  satisfies the first requirement, and  $G^< \cap \text{Succ}(1)$  contains only  $Z_p$  and  $\mathbb{Z}$ , which are both in  $Y$  so  $G$  satisfies the second requirement. Since  $\mathbb{Z} \in X_a$ , we have  $G^< \cap Y \setminus X_a = \{Z_p\}$ , so  $G$  satisfies the third requirement. Finally,  $G^< \cap X_a = \{\mathbb{Z}^m \mid m \leq n\}$  has maximum element  $\mathbb{Z}^n$  so  $G$  satisfies the fourth requirement.  $\square$

**Definition 2.2.9.** An  $a$ -code for the partial function  $f: Y \setminus X_a \rightarrow X_a$  is a group  $G \in \mathcal{G}^{<\aleph_1}$  so that for every  $U \in Y \setminus X_a$  either:

1.  $f(U)$  is undefined and there are no  $V \in X_a$  and  $a$ -code  $H$  for  $(U, V)$  so that  $H \leq G$ .
2. if  $G_X$  is the set of  $V$  so that some  $a$ -code for  $(U, V) < G$ , then  $G_X$  has maximum element  $f(U)$ .

We make the following important observations about this definition:

1. The set of groups which code functions from  $Y \setminus X_a \rightarrow X_a$  is uniformly definable in the parameter  $a$ .
2. The function coded by a group is uniformly definable with  $a$  and the group as parameters.
3. The equivalence relation “ $G$  and  $H$  code the same function  $Y \setminus X_a \rightarrow X_a$ ” is uniformly definable in the parameter  $a$ .

We delay the verification that for any partial function  $f: Y \setminus X_a \rightarrow X_a$  there is a code for  $f$  until we have determined our conditions on  $a$ , but the code for  $f$  will be given by the free product of the groups  $(B, f(B))$  where  $B$  ranges over the domain of  $f$ .

## 2.3 Defining the Set of Possible Parameters

We now want to pick out which properties of  $\mathbb{Z}^\omega$  make the construction work. We want to make sure that we always choose a parameter  $a$  so that:

1.  $X_a$  has order type  $\omega$  (at least in the structure  $\mathcal{G}^{<\aleph_1}$ )
2. There are codes for any functions  $f: Y \setminus X_a \rightarrow X_a$ .

To this end, we define:

**Definition 2.3.1.**  $a$  is a **sufficient parameter** if:

1.  $a$  bounds a unique element of  $Y$ ,

2.  $X_a$  is linearly ordered and each element of  $X_a$  has a successor,
3. for every  $U \in Y \setminus X_a$  and every  $V \in X_a$  there is some  $a$ -pair  $Z$  which codes  $(U, V)$ , and
4. for partial  $f, g: Y \setminus X_a \rightarrow X_a$  coded by  $G, H$ , if  $\text{Dom}(f) \subseteq \text{Dom}(g)$  and both  $f$  and  $g$  are injections into proper initial segments of  $Y$ , then  $\text{Img}(f) \subseteq \text{Img}(g)$ .

**Example 2.3.2.**  $\mathbb{Z}^\omega$  is a sufficient parameter. We have already validated requirements one through three, and the fourth requirement is a direct result of the fact that  $(\mathbb{Z}^\omega)^\prec$  is an  $\omega$ -chain.

We are now prepared to provide the main technical lemma which makes our interpretation work:

**Lemma 2.3.3.** *If  $b$  is a sufficient parameter in  $\mathcal{G}^{<\aleph_1}$ , then any partial function from  $Y \setminus X_a$  to  $X_a$  is coded by some element of  $\mathcal{G}^{<\aleph_1}$ .*

*Proof.* For each  $U \in \text{Dom } f$ , let  $F_U$  be a code for  $(U, f(U))$ . Let  $F$  be the free product of the  $F_U$  where  $U$  ranges over  $\text{Dom } f$ . Assume that  $G < F$  is a maximal  $a$ -pair. There are two possibilities:

1.  $G$  is not itself a free product, so by the Kurosh subgroup theorem,  $G \leq F_U$  for some  $U \in \text{Dom } f$ . Since  $G$  is maximal,  $G = F_U$ .
2. Suppose  $G = C * D$ . Neither  $C$  nor  $D$  has an element of order 2 because any element of  $F$  with finite order is conjugate to an element of one of the groups  $F_U$ , and none of the groups  $F_U$  contains an element of order 2 by the second requirement in the definition of an  $a$ -pair. So, for any  $c \in C$  and  $d \in D$ , the group  $\langle cdc, dc^{-1}d \rangle$  is isomorphic to the free group of rank 2, and since  $F_2 < F_3 < F_2$ , we have a contradiction to the first requirement in the definition of an  $a$ -pair.

So, the maximal  $a$ -pairs below  $F$  are exactly  $F_U$  for  $U \in \text{Dom}(f)$ . It follows that  $F$  is a code for  $f$ . □

**Corollary 2.3.4.** *If  $a$  is a sufficient parameter in  $\mathcal{G}^{<\aleph_1}$ , then  $X_a^{\mathcal{G}^{<\aleph_1}}$  has order type  $\omega$ .*

*Proof.* Since  $X_a$  has a least element, and each element of  $X_a$  has a successor, there is at least an infinite initial segment  $I$  of  $X_a$  with order type  $\omega$ . Assume toward a contradiction that there is some  $G \in X_a \setminus I$ , then there is a partial function  $f$  from  $Y \setminus X_a$  with image  $G^\prec \cup \{G\} \supseteq I$ , and there is another function  $g$  from  $Y \setminus X_a$  so that  $\text{Dom}(f) = \text{Dom}(g)$  and  $\text{Img}(g) = I$ . This contradicts the fourth requirement on a sufficient parameter, so in fact  $I$  must not be a proper initial segment, i.e.,  $X_a^{\mathcal{G}^{<\aleph_1}}$  has order type  $\omega$ . □

**Corollary 2.3.5.**  $\mathcal{G}^{<\aleph_1}$  interprets second-order arithmetic. □

**Theorem 2.3.6.** *The first-order theory of isomorphism classes of at most countable groups under the subgroup relation is 1-equivalent to true second-order arithmetic.*

*Proof.* This now follows directly from Corollary 2.3.5 □

## Chapter 3

# The Index Set of Uncountable Categoricity

### 3.1 Introduction

The work in this chapter is joint with Uri Andrews, previously published in [3]. One important dividing line for complexity in model theory is uncountable categoricity. Our goal here is to address the question of how complicated the dividing line itself is; to wit: how difficult is it to determine whether a theory is uncountably categorical? This follows work of Lempp and Slaman [14] who characterized the complexity of countable categoricity ( $\Pi_3^0$ -complete) and Ehrenfeuchtness ( $\Pi_1^1$ -complete), two other important model theoretic dividing lines.

We will categorize uncountable categoricity in the arithmetic hierarchy by looking at its index set:

**Notation 3.1.1.** Let  $\mathbb{T}^A$  denote the set of indices for  $A$ -computable sets of formulae which are uncountably categorical theories, and let  $\mathbb{T}_c^A$  denote the set of indices for complete theories in  $\mathbb{T}^A$

Notice that this means that we are restricting our attention to computable theories, and therefore to theories with countable languages. Our characterization is, however, relativized, so while it applies only to computable theories, the same statement can be made for  $A$ -computable theories where  $A$  is any real. We show:

**Theorem 3.3.1.**  $\mathbb{T}^A$  and  $\mathbb{T}_c^A$  are complete for intersections of  $\Sigma_2^0(A)$  and  $\Pi_2^0(A)$  sets (i.e., for  $A'$ -d.c.e. sets).

We split this into two parts. In Section 3.2 we will show that  $\mathbb{T}^A$  and  $\mathbb{T}_c^A$  are in fact  $A'$ -d.c.e., and in Section 3.3 we will show that  $\mathbb{T}^A$  and  $\mathbb{T}_c^A$  are  $A'$ -d.c.e.-hard, that is, that any other  $A'$ -d.c.e. set can be reduced to them.



## 3.2 Description

We fix a theory  $T$  in a (computable) language  $\mathcal{L}$ . Our goal is to give a criterion which determines whether  $T$  is uncountably categorical. Since we are specifically looking at a computable theory, we can apply the Henkin construction (which is effective) to get a decidable countable model  $\mathcal{M}$  of  $T$ . It follows that we can pass effectively from the theory  $T$  to the theory  $T_{\mathcal{M}} = \text{ElDiag}(\mathcal{M})$  in the language  $\mathcal{L}_{\mathcal{M}}$ .

If  $T$  is an incomplete theory, then the theory  $T_{\mathcal{M}}$  produced by the Henkin construction will be incomplete as well. In fact, the Henkin construction produces an enumeration of  $T_{\mathcal{M}}$  which is total exactly when  $T$  and thus  $T_{\mathcal{M}}$  are complete.

Baldwin and Lachlan showed [5] that strongly minimal sets are key to understanding the definable sets in an uncountably categorical structure. They showed that if  $T$  is uncountably categorical then there is a strongly minimal set definable with parameters from the prime model of  $T$  (hence 0-definable in  $T_{\mathcal{M}}$ ) over which any model of  $T$  is prime.

We base our description here on the notion of a 2-cardinal formula (originally studied by Vaught):

**Definition 3.2.1.** A formula  $\varphi$  is **2-cardinal** if  $|\mathcal{N}| > |\varphi^{\mathcal{N}}|$  for some  $\mathcal{N} \models T$ . We use the non-standard convention that this definition can apply to formulas defining finite sets. A formula is **1-cardinal** if it is not 2-cardinal.

Erimbetov gave the following characterization of uncountable categoricity:

**Theorem 3.2.2 (Erimbetov [7]).** *A complete theory  $T$  is uncountably categorical if and only if  $T$  has a 1-cardinal strongly minimal formula with parameters from any model of  $T$ .*

**Corollary 3.2.3.** *A complete theory  $T$  is uncountably categorical if and only if  $T_{\mathcal{M}}$  has a 1-cardinal strongly minimal formula without parameters.*

*Proof.* If  $T$  is uncountably categorical, then from Baldwin and Lachlan [5], we know that the prime model of  $T$  has a strongly minimal 1-cardinal formula. This formula has parameters in  $\mathcal{M}$ , so in particular it is definable without parameters in  $T_{\mathcal{M}}$ .

Conversely, any formula 0-definable in  $T_{\mathcal{M}}$  is definable in  $T$  with parameters from  $\mathcal{M}$ , so in particular if there is a 1-cardinal strongly minimal formula definable without parameters in  $T_{\mathcal{M}}$ , then there is a 1-cardinal strongly minimal formula in  $T$  with parameters from some model. We apply Erimbetov's theorem.  $\square$

Our description of uncountable categoricity is therefore the following

**Theorem 3.2.4.** *A complete first-order theory  $T$  is uncountably categorical if and only if:*

1.  $T$  defines no infinite 2-cardinal formula (even with parameters), and
2. There is some  $\varphi \in \mathcal{L}_{\mathcal{M}}$  so that  $\varphi$  is 1-cardinal and for all  $\psi(x) \in \mathcal{L}_{\mathcal{M}}$  with no parameters either  $\psi(x) \wedge \varphi(x)$  or  $\neg\psi(x) \wedge \varphi(x)$  is 2-cardinal.

*Proof.* If  $T$  is uncountably categorical, the  $T_{\mathcal{M}}$  contains a 0-definable strongly minimal formula  $\varphi$ . Since  $T_{\mathcal{M}}$  is uncountably categorical, it cannot define any infinite 2-cardinal formulae. Thus we can read condition 2 as saying that for every 0-definable formula  $\psi$ , the formula  $\varphi \wedge \psi$  is either finite or co-finite in  $\varphi$ , which is true because  $\varphi$  is strongly minimal.

For the converse, assume that the two conditions hold, and take  $\varphi$  as given by condition 2.  $\varphi$  is minimal in  $\mathcal{M}$ , and by a theorem of Baldwin and Lachlan [5], this means either it is strongly minimal or there is some infinite 2-cardinal formula (in fact, we will see later that 2-cardinality is elementary, and a minimal but not strongly minimal set has a family of finite sets of unbounded size, i.e., of 2-cardinal formulas of unbounded size). So,  $\varphi$  is a 1-cardinal strongly minimal formula.  $\square$

We now show that  $\mathbb{T}_c^A$  is the intersection of a  $\Pi_2^0(A)$  set and a  $\Sigma_2^0(A)$  set by showing that condition 1 above is  $\Pi_2^0(T)$  and condition 2 above is  $\Sigma_2^0(T)$ . By Vaught's 2-cardinal theorem [11, Thm. 12.1.1],  $\varphi$  is 2-cardinal if and only if there are  $\mathcal{U} \preceq \mathcal{M} \models T$  with  $\varphi^{\mathcal{U}} = \varphi^{\mathcal{M}}$ . Thus, condition 1 is equivalent to stating:

$$\forall \varphi(x, \bar{y}) [T \cup \text{"}\mathcal{U} \prec \mathcal{M}\text{"} \cup \{\exists^{\geq n} x \varphi(x, \bar{c})\} \cup \{\bar{c} \in \mathcal{U}\} \vdash \exists x (x \notin \mathcal{U} \wedge \varphi(x, \bar{c}))]$$

This can all be stated in the language  $\mathcal{L} \cup \{U\}$  where  $U$  is a unary predicate naming  $\mathcal{U}$  as a substructure of  $\mathcal{M}$ . Then condition 1 is the same as checking for the existence of a proof of  $\exists x (x \notin \mathcal{U} \wedge \varphi(x, \bar{c}))$  for each of these computable theories. Checking for a single proof is  $\Sigma_1^0(T)$ , and checking that these proofs all exist is therefore  $\Pi_2^0(T)$ .

Now, for condition 2, we use the formulation of 1-cardinality in terms of layerings. A layering in terms of  $\varphi$  is a formula  $\theta$  with specific form. It is computable to check whether  $\theta$  is a layering in terms of  $\varphi$ . We use the following unpublished theorem of Gaifman:

**Theorem 3.2.5 (Gaifman [11, Thm. 12.1.5]).** *A 0-definable set  $\varphi$  is 1-cardinal if and only if there exists a layering  $\theta$  (also a 0-definable formula) in terms of  $\varphi$  so that  $T \vdash \forall x \theta(x)$ .*

So, it is  $\Sigma_1^0(T)$  to check whether a formula  $\varphi$  is 1-cardinal. Reading condition 2 with this in mind, we see that condition 2 is  $\Sigma_2^0(T)$ . Finally, note that the condition of being a complete theory is itself  $\Pi_2^0(T)$ , so we have shown the following:

**Theorem 3.2.6.**  $\mathbb{T}_c^A$  is  $A'$ -d.c.e.

The result for  $\mathbb{T}^A$  is immediate:

**Corollary 3.2.7.**  $\mathbb{T}^A$  is  $A'$ -d.c.e.

*Proof.* This many-one reduces to the index set of computable complete uncountably categorical theories because  $T$  is uncountably categorical exactly if the theory of infinite models of  $T$  is uncountably categorical and complete, i.e., if  $T \cup \{\exists^{\geq n} x(x = x) \mid n \in \omega\}$  is both uncountably categorical and complete.  $\square$

### 3.3 Hardness

It remains to show that  $\mathbb{T}^A$  is complete for  $A'$ -d.c.e. theories.

**Theorem 3.3.1.**  $\mathbb{T}^A$  and  $\mathbb{T}_c^A$  are complete for intersections of  $\Sigma_2^0(A)$  and  $\Pi_2^0(A)$  sets (i.e., for  $A'$ -d.c.e. sets).

*Proof.* We have already shown that  $\mathbb{T}^A$  and  $\mathbb{T}_c^A$  are  $A'$ -d.c.e., so we need only show that they are hard for this class. We use a reduction which always produces a complete theory. Since  $\mathbb{T}^A$  and  $\mathbb{T}_c^A$  coincide on complete theories, this will suffice for both.

Since  $\text{FIN}^A$  is  $\Sigma_2^0(A)$ -complete and  $\text{INF}^A$  is  $\Pi_2^0(A)$ -complete, we want to produce a complete theory  $T_{I,F}$  given enumerations  $W_I^A$  and  $W_F^A$  so that  $T_{I,F}$  is uncountably categorical exactly if  $W_I^A$  is infinite and  $W_F^A$  is finite. We fix a language with infinitely many unary relation symbols  $U_j$  and  $V_k$  and infinitely many binary relation symbols  $R_j$ .

For each  $j$ , either  $U_j$  and  $U_{j-1}$  are the same (and  $R_j$  is empty), or  $U_j$  is a subset of  $U_{j-1}$  and  $R_j$  is a bijection between  $U_j$  and  $U_{j-1} \setminus U_j$ . Note then, that if there are infinitely many  $j$  so that  $U_j$  splits  $U_{j-1}$  in half, then there are continuum many 1-types, so  $T_{I,F}$  is not uncountably categorical.

The  $V_k$  will be disjoint subsets of  $\bigcap_{j \in \omega} U_j$ . Either each is finite, or infinitely many of them are infinite. In the latter case, each infinite  $V_k$  is 2-cardinal, so  $T_{I,F}$  is not uncountably categorical.

If there are only finitely many  $j$  so that  $U_j$  splits  $U_{j-1}$  and all the  $V_k$  are finite,  $\bigcap_{j \in \omega} U_j = \bigcap_{j=0}^N U_j$  is a strongly minimal 1-cardinal formula, so  $T_{I,F}$  is uncountably categorical.

For the construction, we enumerate  $W_I^A$  and  $W_F^A$  in stages:

**Stage 0:** At the start of the construction, declare that  $U_0 \supseteq U_1 \supseteq U_2 \supseteq \dots$ , that the  $V_k$  are disjoint subsets of  $\bigcap_j U_j$ , and that  $U_0$  is the entire structure.

**Stage  $s > 0$ :** If  $W_F^A$  enumerates a number at stage  $s$ , then we declare that  $U_s$  splits  $U_{s-1}$ . Otherwise, we declare that  $U_s$  does not split  $U_{s-1}$ . If  $W_I^A$  enumerates a number at stage  $s$ , then we declare that

$V_{|W_{I,s}^A|}$  contains exactly  $s$  elements. Otherwise we declare that  $V_i$  contains at least  $s$  elements for each  $i > |W_{I,s}^A|$ .

This guarantees that  $T_{I,F}$  is complete, and the analysis above the construction already shows that in fact  $T_{I,F}$  is uncountably categorical if and only if  $W_F^A$  is finite (there are finitely many  $j$  so that  $U_j$  splits  $U_{j-1}$ ) and  $W_I^A$  is infinite (there are infinitely many  $V_i$  which are declared to be finite; since we do so in order, this means all of them are finite).  $\square$

## Part II

# Convex Orderability

## Chapter 4

# Introduction to Convex Orderability

### 4.1 Notation

Throughout Chapters 4 to 6, we will be considering linear orders, denoted by  $<$ ,  $\triangleleft$ ,  $\blacktriangleleft$ , etc. Since we may be considering multiple orders on the same set, we adopt the convention of prefixing order terminology with the order to which we refer. For example, we say  $<$ -convex or  $\triangleleft$ -convex rather than simply convex. We similarly label intervals with the relevant order, thus  $(a, b)_{<} = \{x \mid a < x < b\}$ , and  $[c, d]_{\triangleleft} = \{x \mid c \triangleleft x \triangleleft d\}$ .

By a  $<$ -**convex** set  $X$  in  $(M, <)$  we mean specifically that if  $a < c < b$  and  $a, b \in X$ , then  $c \in X$  as well. In contrast with an  $<$ -interval, a  $<$ -convex set may not have  $<$ -endpoints in  $M$ .

Any reference to topology here is defined in terms of the orders, and will therefore be prefaced with the relevant order as well, thus we speak of  $<$ -open sets or  $\triangleleft$ -open sets. This refers to the usual order topology in the home sort, and the product topology in higher dimensions, so for an order  $(M, <)$ , the intervals  $(a, b)_{<}$  form a basis of open sets in  $M$ , and products of the form  $\prod_{i=0}^d (a_i, b_i)_{<}$  form a basis of open sets in  $M^d$ .

For any sequence  $(A_i, <_i)_{i \in I}$  of orders indexed by another order  $(I, <)$ , we write  $\sum_I (A_i, <_i)$  for the order on the disjoint union of the  $A_i$  given by  $a < b$  if either  $(a, b \in A_i \text{ and } a <_i b)$  or  $(a \in A_i \text{ and } b \in A_j \text{ for some } i < j)$ . For finite sums, we write, for example,  $(A_0, <_0) + (A_1, <_1)$ . Note that this sum is not commutative. When  $(A_i, <_i) \cong (A, <_A)$  for each  $i \in I$ , we write  $(A, <_A) \times (I, <)$  for  $\sum_{i \in I} (A_i, <_i)$ . Equivalently,  $(A_i, <_A) \times (I, <)$  has underlying set  $A \times I$ , and is ordered reverse-lexicographically first by  $<$  and then by  $<_A$ . So, for example,  $\mathbb{Q} \times 2$  is densely ordered (it's just two side-by-side copies of  $\mathbb{Q}$ ), while every element of  $2 \times \mathbb{Q}$  has either a predecessor or a successor.

One piece of language in particular should be noted. When we refer to an order as dense, it can either mean that it is dense in an ambient topological space, or that it is itself a dense linear order. For example, in  $\mathbb{Q} \times \mathbb{Q}$ , the set  $\{0\} \times \mathbb{Q}$  is a dense linear order, but is not dense in  $\mathbb{Q} \times \mathbb{Q}$ . We adopt the convention of stating that a set  $Y \subseteq X$  is  **$<$ -densely ordered** when we mean to say that  $(Y, < \upharpoonright Y)$  is a dense linear order, and by contrast using the phrase  **$<$ -dense in  $X$**  to indicate that we are referring to how  $Y$  is embedded in the larger order  $(X, <)$ .

Similarly, we say a set  $Y \subseteq X$  is  **$<$ -discretely ordered** if each point of  $Y$  is  $(< \upharpoonright Y)$ -isolated, and we say

that  $Y$  is **<-discrete in  $X$**  if each point of  $Y$  is isolated in the subspace topology induced by the  $<$ -topology on  $(X, <)$ . For example, in  $\mathbb{Q} \times \mathbb{Q}$ ,  $\{0\} \times \mathbb{Q}$  is  $<$ -discrete in  $\mathbb{Q} \times \mathbb{Q}$ , but is not  $<$ -discretely ordered.

In Chapter 6 we will make reference to functions  $f: M \rightarrow \overline{M}$  definable  $\mathcal{M} = (M, <, \dots)$ .  $\overline{M}$  is the “definable completion” of the linear order  $\mathcal{M}$ , a multi-sorted structure with one definable sort for every  $\mathcal{M}$ -interpretable extension of  $(M, <)$  to a linear order  $(M', <')$  so that  $M$  is  $<'$ -dense in  $M'$ . Each such extension uniquely embeds into the Dedekind completion of  $(M, <)$ . In this sense, therefore, we think of  $\overline{M}$  as a suborder of the Dedekind Completion and identify elements of  $\overline{M}$  which have the same image under these embeddings.

## 4.2 An introduction to Convex Orderability

Convex orderability first arose in the context of VC-minimality:

**Definition 4.2.1 (Adler [1]).** A theory  $T$  is **VC-minimal** if there is a collection  $\Psi$  of formulas so that:

1. if  $\mathcal{M} \models T$  and  $X \subseteq M$  is definable, then  $X$  is a boolean combinations of instances of formulas in  $\Psi$ ,  
and
2. In any  $\mathcal{M} \models T$ , the instances of formulas in  $\Psi$  form a directed collection of subsets of  $M$ , i.e., if  $\varphi_a^{\mathcal{M}}$  and  $\psi_b^{\mathcal{M}}$  are instances of formulas from  $\Psi$ , then either  $\varphi_a^{\mathcal{M}}$  and  $\psi_b^{\mathcal{M}}$  are disjoint, or one contains the other.

The name VC-minimality comes from the fact that every directed family becomes a family of Vapnik-Chervonenkis dimension 1 (i.e., a chain of sets) after replacing some of the sets with their complements.

VC-minimality has shown promise as an extension of stability theory into the unstable domain. VC-minimality is a weakening of strong minimality, weak o-minimality, C-minimality, and D-minimality, but is still strong enough to imply dp-minimality. However, this original definition of VC-minimality is not obviously local, and working from this definition it can be rather difficult to show that a theory is not VC-minimal. Andrews and Guingona addressed this in [2] by finding a local characterization for VC-minimality, but before this result, Guingona and Laskowski defined convex orderability:

**Definition 4.2.2 (Guingona and Laskowski [10]).** Let  $\mathcal{M}$  be a structure. A definable family  $\varphi$  of subsets of  $M$  is said to be  **$n$ -convexly ordered** by an order  $\triangleleft$  on  $M$  if there is some finite  $n$  so that every instance of  $\varphi$  has at most  $n$  distinct  $\triangleleft$ -convex components. A structure  $\mathcal{M}$  is **convexly orderable** if there is an order  $\triangleleft$  on  $M$  (possibly not definable) which convexly orders every definable family of subsets of  $M$ . In this case, we also say that  $\triangleleft$  is a **convex ordering** of  $\mathcal{M}$ .

**Lemma 4.2.3 (Guingona and Laskowski [10]).** *Convex orderability is first-order; if  $\mathcal{N} \equiv \mathcal{M}$  and  $\mathcal{M}$  is convexly orderable, then  $\mathcal{N}$  is convexly orderable as well. In this case we say  $Th(\mathcal{M})$  is convexly orderable.*

Convex orderability is a weakening of VC-minimality. To show that a VC-minimal theory is convexly orderable, one need only remark that for any chain of sets, there is some order in which each of these sets is a ray. From the comment in Definition 4.2.1, the same is true for a directed family. On the other hand convex orderability is closed under reducts, while VC-minimality is not, so they are not equivalent.

Convex orderability is, however, close enough to VC-minimality that it was used in [8] to prove that all VC-minimal ordered groups are divisible and abelian, and to give a condition on abelian groups equivalent to VC-minimality. Research has moved away from convex orderability, however in favor of dp-smallness:

**Definition 4.2.4 (Guingona [9]).** A theory  $T$  is **dp-small** if there are no  $\mathcal{M} \models T$ , definable subsets  $X_0, X_1, X_2, \dots \subseteq M$  and uniformly definable subsets  $A_0, A_1, A_2, \dots \subseteq M$  such that for every  $i, j \in \omega$  there is some  $b_{i,j}$  so that  $b_{i,j} \in X_\ell \cap A_k$  if and only if  $i = \ell$  and  $j = k$ . We say  $\mathcal{M}$  is dp-small if  $Th(\mathcal{M})$  is dp-small.

In the same paper, Guingona showed that every convexly orderable theory is dp-small, and noted that the proofs from [8] generalize to dp-small structures. Furthermore, Johnson showed in [12] that every dp-small field is in fact real closed or algebraically closed, and therefore also VC-minimal in the field language.

On the other hand, dp-smallness is not equivalent to convex orderability, for example:

**Example 4.2.5.** Let  $\mathcal{M} = (M, E, A_{i,j})_{i < j < \omega}$  where  $E$  is an equivalence relation on  $M$  (so, for  $a \in M$ , the set  $E_a$  is the  $E$ -class of  $a$ ), and the  $A_{i,j}$  are disjoint sets covering  $M$  so that for  $i, j, i', j' \in \omega$ :

1. for  $a \in M$ , if  $A_{i,j} \cap E_a$  is nonempty, then it is infinite;
2. for  $a \in M$ , both  $A_{i,j} \cap E_a$  and  $A_{i',j'} \cap E_a$  are nonempty if and only if  $j = j'$ ; and
3. each  $A_{i,j}$  intersects infinitely many  $E$ -classes.

Then  $\mathcal{M}$  is dp-small but not convexly orderable.

*Proof.* The theory of  $\mathcal{M}$  admits quantifier elimination, so dp-smallness is a straightforward result.

Suppose toward a contradiction that  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ . Given  $n < \omega$ , we will show that there is an  $E$ -class with at least  $n$  distinct  $\triangleleft$ -convex components. Each of the sets  $A_{i,n}$  for  $i < n$  has finitely many  $\triangleleft$ -convex components, and they intersect infinitely many  $E$ -classes, so in particular, there is some  $E$ -class  $E_a$  which intersects  $A_{i,n}$  for each  $i < n$  but does not contain any  $\triangleleft$ -endpoints of any  $\triangleleft$ -convex components of these sets. Therefore  $E_a \cap A_{i,n}$  contains at least one  $\triangleleft$ -convex component of  $E_a$  for each  $i$ , and it follows that  $E_a$  has at least  $n$  distinct  $\triangleleft$ -convex components. Since we can show that there is an  $E$ -class with at



least  $n$  distinct  $\triangleleft$ -convex components for each  $n$ , and the  $E$ -classes are uniformly definable, this contradicts the assumption that  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ .  $\square$

The idea of Example 4.2.5 is that the structure  $\mathcal{M}$  has finite approximations to the configuration from Definition 4.2.4. In this sense, convex orderability can be thought of as requiring uniform dp-smallness.

While convex orderability extends the same list of minimality notions that VC-minimality does, the most notable is perhaps weak o-minimality because the definitions are similar. In fact, the definition of weak o-minimality can be stated as follows:

**Definition 4.2.6 (Dickmann [6]).** We say the theory of a structure  $\mathcal{M}$  is **weakly o-minimal** if  $\mathcal{M}$  has a definable convex ordering.

This implies that in fact an understanding of sets in the convex ordering might lead to some sort of classification of definable sets along the line of weakly o-minimal cell decomposition. On the other hand, this relationship between convex orderability and weak o-minimality can be misleading, as the following example clarifies:

**Example 4.2.7.** Let  $\mathcal{N} = (\omega, E)$  where  $\mathcal{N} \models nEm$  if and only if  $\lfloor \log_2(n) \rfloor = \lfloor \log_2(m) \rfloor$  (so  $E$  is an equivalence relation).  $\mathcal{N}$  is convexly orderable, but if  $\triangleleft$  is a convex ordering of  $\mathcal{N}$ , then  $\mathcal{N}_{\triangleleft} = (\mathcal{N}, \triangleleft)$  is not convexly orderable, so in particular  $\mathcal{N}_{\triangleleft}$  is not weakly o-minimal.

*Proof.* Given an order  $\triangleleft$  on  $\omega$  and definable families  $\varphi$  and  $\psi$  of subsets of  $\omega$ :

1. If  $\varphi$  is  $n$ -convexly ordered by  $\triangleleft$ , then  $\neg\varphi$  is  $(n + 1)$ -convexly ordered by  $\triangleleft$ .
2. If  $\varphi$  is  $n$ -convexly ordered by  $\triangleleft$  and  $\psi$  is  $m$ -convexly ordered by  $\triangleleft$ , then  $\varphi \vee \psi$  is  $(m + n)$ -convexly ordered by  $\triangleleft$ .

From these two facts and quantifier elimination in  $\mathcal{N}$ , it follows that any order which convexly orders the  $E$ -classes is a convex ordering of  $\mathcal{N}$ . In particular, the  $E$ -classes are all convex in the usual order on  $\omega$ , so it is a convex ordering of  $\mathcal{N}$ .

Now fix a convex ordering  $\triangleleft$  of the structure  $\mathcal{N}$ . Since each  $E$ -class in  $\mathcal{N}$  is finite, each is discretely ordered by  $\triangleleft$  with a least element. So, if  $\mathcal{N}'_{\triangleleft}$  is an elementary extension of  $\mathcal{N}_{\triangleleft}$  with infinite  $E$ -classes  $A_0, A_1, A_2, \dots$ , then each  $A_i$  is discretely ordered by  $\triangleleft$  with a least element. For  $i \in \omega$ , let  $X_i$  be the set of all  $x \in \mathcal{N}'_{\triangleleft}$  which are the  $i$ -th element of their  $E$ -class in the  $\triangleleft$  order. Each  $X_i$  is definable in  $\mathcal{N}'_{\triangleleft}$ . The  $X_i$  are disjoint, the  $A_j$  are disjoint, and for any  $i, j \in \omega$ , we have  $X_i \cap A_j = \{b_{i,j}\}$ . So, in fact the  $X_i, A_j$ , and  $b_{i,j}$  are as in Definition 4.2.4, and it follows that  $\mathcal{N}'_{\triangleleft}$  (and thus  $\mathcal{N}_{\triangleleft}$ ) are not dp-small, and thus not convexly orderable.  $\square$

So, there is a difference between a structure being convexly orderable and being the reduct of a weakly o-minimal structure. Even if a structure is the reduct of a weakly o-minimal structure, not every convex ordering is a witness to this fact. In fact, convex orderings can make arbitrary subsets of the structure convex:

**Lemma 4.2.8.** *Fix an order  $\triangleleft$  on  $M$ . We define the  $\triangleleft$ -rank of an order on  $M$  to be the largest possible rank so that:*

1.  $\triangleleft$  and  $\triangleright$  have  $\triangleleft$ -rank 1,
2. for any set  $X \subseteq M$ , and any order  $<$  of  $\triangleleft$ -rank  $n$ , the order  $<^X$  where  $(M, <^X) = (X, <) + (M \setminus X, <)$  has  $\triangleleft$ -rank at most  $2n$ , and
3. the order  $<$  is  $\triangleleft$ -rank at most  $(2n + 1)m$  if  $(M, <) = \sum_{X \in M/E} (X, <_X)$  where  $E$  is an equivalence relation on  $M$  and there is a  $\triangleleft$ -rank  $m$  order  $\blacktriangleleft$  so that the  $E$ -classes are  $\blacktriangleleft$ -convex,  $M/E$  is ordered by  $\blacktriangleleft/E$ , and for every  $X \in M/E$ , the order  $<_X$  is  $(\blacktriangleleft \upharpoonright X)$ -rank  $n$ .

By this definition, orders which don't match these descriptions have  $\triangleleft$ -rank  $\infty$ . If  $\triangleleft$  is a convex order of  $\mathcal{M}$ , then all orders on  $M$  with finite  $\triangleleft$ -rank are convex orders of  $\mathcal{M}$ .

*Proof.* We show that if  $X$  is  $\triangleleft$ -convex and  $<$  is  $\triangleleft$ -rank  $n$ , then  $X$  has at most  $n$  distinct  $<$ -convex components by following the above:

1. Any  $\triangleleft$ -convex set is also  $\triangleright$ -convex, so the  $\triangleleft$ -rank 1 case is trivial.
2. If  $<$  has  $\triangleleft$ -rank  $n$ , then by inductive hypothesis, every  $\triangleleft$ -convex set has at most  $n$  distinct  $<$ -convex components. If  $U$  is  $<$ -convex, then both  $U \cap X$  and  $U \setminus X$  are  $<^X$ -convex, so  $U$  has at most 2 distinct  $<^X$ -convex components. So, each  $\triangleleft$ -convex set has at most  $n$  distinct  $<$ -convex components, which in turn have at most 2  $<^X$ -convex components, and it follows each  $\triangleleft$ -convex set has at most  $2n$  distinct  $<^X$ -convex components.
3. For the third case, let  $U$  be any  $\blacktriangleleft$ -convex set. Then define  $U^\circ = \{x \in U \mid U \supseteq E_x\}$ .  $U^\circ$  is  $\blacktriangleleft$ -convex, and since  $</E = \blacktriangleleft/E$ , it follows that  $U^\circ$  is also  $<$ -convex. Now, for any  $a \in U \setminus U^\circ$ , we know that  $E_a \cap U$  is  $(\blacktriangleleft \upharpoonright E_a)$ -convex, and so by inductive hypothesis has at most  $n$  distinct  $(< \upharpoonright E_a)$ -convex components. There are at most two such  $E$ -classes since they must bound  $U$  either above or below. So  $U \setminus U^\circ$  has at most  $2n$  distinct  $<$ -convex components, and  $U^\circ$  is  $<$ -convex. It follows that  $U$  has at most  $2n + 1$  distinct  $<$ -convex components, as desired.  $\square$

This brings up the two main questions which motivate the work in Chapters 5 and 6:

**Question 4.2.9.** Which convexly orderable structures are reducts of weakly o-minimal structures?

**Question 4.2.10.** Is there a canonical choice of convex ordering for a given convexly orderable structure?

Chapters 5 and 6 focus on ordered convexly orderable structures in part because one might expect a ranked order to be a better choice of canonical order for Question 4.2.10. They also represent a direct attempt at answering Question 4.2.9; in Example 4.2.7 we saw that it was possible for a convexly orderable structure to fail to be convexly orderable after making the convex ordering definable. Since a weakly o-minimal structure has a definable convex ordering, we make the following definition:

**Definition 4.2.11.** A structure  $\mathcal{M}$  is  *$n$ -times convexly orderable* (denoted  $\text{CO}_n$ ) if there are orders  $\triangleleft_1, \triangleleft_2, \dots, \triangleleft_n$  on  $M$  (possibly not definable) so that  $(\mathcal{M}, \triangleleft_1, \triangleleft_2, \dots, \triangleleft_i)$  is convexly ordered by  $\triangleleft_{i+1}$  for each  $i < n$ . In particular, Example 4.2.7 is an example of a structure which is  $\text{CO}_1$  but not  $\text{CO}_2$ .

Since the convex ordering is already definable in a weakly o-minimal structure, adding it to the structure does not change the definable sets, so the reduct of a weakly o-minimal structure is  $\text{CO}_n$  for every  $n$ . On the other hand, it is unknown whether the converse is true. It seems possible, given the results in Chapters 5 and 6 that in fact there is some  $n$  so that every  $\text{CO}_n$  structure is the reduct of a weakly o-minimal structure.

### 4.3 Observations on Convex Orderability

In this section, we make some observations on convex orderability which will be referred to later. The first important observation is that convex orderability is closed under substructure in two senses:

**Lemma 4.3.1.** *If  $\triangleleft$  is a convex ordering of  $\mathcal{M}$  and  $\mathcal{N}$  is a substructure of  $\mathcal{M}$  so that for any family  $\mathcal{F}$  of subsets of  $N$  definable in  $\mathcal{N}$ , there is a family  $\mathcal{G}$  of sets definable in  $\mathcal{M}$  so that for  $X \in \mathcal{F}$  there is  $X' \in \mathcal{G}$  such that  $X = X' \cap N$ , then  $\triangleleft \upharpoonright N$  is a convex ordering of  $\mathcal{N}$ .*

**Corollary 4.3.2.** *If  $\mathcal{M}$  is convexly orderable and  $\mathcal{N}$  is a substructure of  $\mathcal{M}$  with universe definable in  $\mathcal{M}$ , then  $\mathcal{N}$  is convexly orderable.  $\square$*

This result also reverses:

**Lemma 4.3.3.** *If  $\mathcal{N}_0, \dots, \mathcal{N}_n$  are substructures of  $\mathcal{M}$  with  $M = \bigcup_i N_i$  and for every family  $\mathcal{F}$  of subsets of  $M$  definable in  $\mathcal{M}$ , the collection  $\mathcal{F}_i = \{X \cap N_i \mid X \in \mathcal{F}\}$  is uniformly definable in  $\mathcal{N}_i$  for each  $i$ , then if  $\triangleleft_i$  is a convex ordering of  $\mathcal{N}_i$  for each  $i$ , it follows that  $\mathcal{M}$  is convexly ordered by  $(M, \triangleleft) = \sum_i (N_i, \triangleleft_i)$ .*

**Corollary 4.3.4.** *If  $\mathcal{N}_0, \dots, \mathcal{N}_n$  are substructures of  $\mathcal{M}$  which are definable in  $\mathcal{M}$  and  $M = \bigcup_i N_i$ , then if  $\triangleleft_i$  is a convex ordering of  $\mathcal{N}_i$  for each  $i$ ,  $\mathcal{M}$  is convexly ordered by  $(M, \triangleleft) = \sum_i (N_i, \triangleleft_i)$ .  $\square$*

We finish this chapter with one last example of a condition separating convex orderability from dp-minimality:

**Example 4.3.5.** Let  $\mathcal{M}$  be any structure and  $S = (b_\sigma)_{\sigma \in 2^{<\omega}} \subseteq M$ . If there is a formula  $\varphi(x; y)$  so that  $\mathcal{M} \models \varphi(b_\sigma; b_\tau)$  if and only if  $\sigma \prec \tau$ , then  $\mathcal{M}$  is not convexly orderable. In particular, the theory of a tree is dp-minimal (Simon [17]) but if it embeds  $2^{<\omega}$ , then it is not convexly orderable.

*Proof.* Let  $\triangleleft$  be any ordering of  $M$ . For any finite partition of a binary tree, one of the sets in the partition contains a complete binary subtree. So, either one of the definable sets  $\varphi(a, x)^{\mathcal{M}}$  where  $a \in \mathcal{M}$  has infinitely many  $\triangleleft$ -convex components (so  $\triangleleft$  is not a convex ordering), or we can pass to a subtree with the property that if  $\sigma \in 2^{<\omega}$ , then the set  $A_\sigma = \{b_{\sigma \frown \tau} \mid \tau \in 2^{<\omega}\}$  is  $\triangleleft$ -convex. Then by the same partitioning idea (and possibly reversing  $\triangleleft$ ), we can pass to a subtree so that one of the following cases holds:

1. We have  $b_{\sigma \frown 0} \triangleleft b_\sigma \triangleleft b_{\sigma \frown 1}$  for all  $\sigma \in 2^{<\omega}$ . In this case, since  $A_\sigma$  is  $\triangleleft$ -convex, we see that

$$b_\epsilon \triangleleft b_{10} \triangleleft b_1 \triangleleft b_{1^2 0} \triangleleft b_{1^3} \triangleleft \cdots \triangleleft b_{1^{n0}} \triangleleft b_{1^n} \triangleleft b_{1^{n+10}} \cdots$$

2.  $b_\sigma$  is  $\triangleleft$ -below both  $b_{\sigma \frown 0}$  and  $b_{\sigma \frown 1}$  for all  $\sigma \in 2^{<\omega}$ . Relabeling the tree if necessary, we can therefore assume that  $b_\sigma \triangleleft b_{\sigma \frown 0} \triangleleft b_{\sigma \frown 1}$  for all  $\sigma \in 2^{<\omega}$  and, since  $A_\sigma$  is  $\triangleleft$ -convex, it follows that

$$b_\epsilon \triangleleft b_0 \triangleleft b_1 \triangleleft b_{10} \triangleleft b_{1^2} \triangleleft \cdots \triangleleft b_{1^{n-10}} \triangleleft b_{1^n} \triangleleft b_{1^{n0}} \triangleleft \cdots$$

By compactness we can find, in some elementary extension  $(\mathcal{M}', \triangleleft)$  of  $(\mathcal{M}, \triangleleft)$ , an element  $c$  satisfying any limit of  $\left(\text{tp}_{(\mathcal{M}, \triangleleft)}(b_{1^n}/S)\right)_{n \in \omega}$ . But then  $\mathcal{M}' \models \varphi(b_{1^n}; c) \wedge \neg \varphi(b_{1^{n0}}; c)$  so  $\varphi_c$  has infinitely many  $\triangleleft$ -convex components, and  $\triangleleft$  is therefore not a convex ordering.  $\square$

## Chapter 5

# Convex Orderings of Ordered Structures

### 5.1 Decompositions of Orders

We consider a specific instance of Question 4.2.10. Namely, if  $\mathcal{M} = (M, <, \dots)$  is a linearly ordered structure and  $\mathcal{M}$  is convexly orderable, then is there a convex ordering on  $\mathcal{M}$  which is somehow a straightforward modification of  $<$  as in the following examples?

**Example 5.1.1.** Let  $\mathcal{M} = (\mathbb{Z}, <, f)$ , where  $f(x) = 2\lfloor \frac{x}{2} \rfloor$ . The structure  $\mathcal{M}$  is convexly ordered by  $<^{2\mathbb{Z}}$ .

**Example 5.1.2.** Let  $\mathcal{M} = (\mathbb{R}, <, f)$ , where  $f(x) = x$  when  $x$  is irrational or  $|x| > 1$ , and  $f(x) = -x$  otherwise. The structure  $\mathcal{M}$  is convexly ordered by  $<^{\mathbb{Q}}$ .

These examples both use construction 2 from Lemma 4.2.8. In fact, the constructions in Lemma 4.2.8 guarantee that for any  $<$ -rank  $n$  order, every  $<$ -interval has at most  $n$  convex components. This is a necessary condition for a convex ordering, so one guess might be that there is some finite  $<$ -rank order which is a convex ordering. This is not the most helpful conjecture, however, since finite  $<$ -rank orders can be quite nasty. On the other hand, Example 5.1.1 is of a structure with a discrete linear order, which is convexly ordered by a discrete order. This hints at the following “reflection principle”:

**Theorem 5.2.14.** *If  $\mathcal{M} = (M, <, \dots)$  is discretely ordered and convexly orderable, then  $\mathcal{M}$  has a discrete convex ordering.*

Similarly, Example 5.1.2 hints that we might be able to do the same for a dense linear order, however, it is possible for a densely ordered convexly orderable structure to name a discretely ordered set:

**Example 5.1.3.** Let  $\mathcal{M} = (\mathbb{R}, <, \lfloor \cdot \rfloor)$  where  $\lfloor \cdot \rfloor$  is the usual floor function. Then  $\mathcal{M}$  is convexly ordered by  $<^{\mathbb{Z}}$ .

And in fact, it is easy to see that there is therefore no direct analogue for Theorem 5.2.14 for densely-ordered structures:

**Lemma 5.1.4.** *Suppose  $\triangleleft$  is a convex ordering of the linear order  $\mathcal{M} = (M, <, \dots)$  and that there is some  $X \subseteq M$  which is definable in  $\mathcal{M}$ ,  $<$ -discretely ordered and infinite. Then for any  $n$ , there is some element of  $\mathcal{M}$  which has an  $n$ -th  $\triangleleft$ -successor.*

*Proof.* We consider the family  $\mathcal{F} = (X \cap [b, c]_{<})_{b, c \in X}$ . Since these sets are uniformly definable, there is some  $m$  so that each set in  $\mathcal{F}$  has at most  $m$  distinct  $\triangleleft$ -convex components. Since  $X$  is  $<$ -discretely ordered and infinite, we can find  $a \in X$  so that  $a$  has an  $nm$ -th ( $< \upharpoonright X$ )-successor  $b$ . Then  $[a, b]_{<} \cap X$  has exactly  $(nm + 1)$  elements, and by the pigeonhole principle, at least  $n + 1$  of them are in the same  $\triangleleft$ -convex component of  $[a, b]_{<} \cap X$ . So there is a finite  $\triangleleft$ -convex set with at least  $n + 1$  elements. Since every finite order is discrete, it follows that there is an element with an  $n$ -th  $\triangleleft$ -successor, as desired.  $\square$

Beyond this, not every convexly orderable order is dense or discrete:

**Example 5.1.5.** Let  $\mathcal{M} = (2 \times \mathbb{Q}, <)$ . The structure  $\mathcal{M}$  is convexly ordered by  $<^{\{0\} \times \mathbb{Q}}$

Notice that in this example, we actually convexly order  $\mathcal{M}$  by splitting it into two pieces which are each  $<$ -densely ordered. Our goal for the remainder of this section is to show that in fact such a partition is possible for any dp-small order.

**Theorem 5.1.12.** *If  $\mathcal{M} = (M, <, \dots)$  is a dp-small linear order, then  $M$  has a 0-definable partition into finitely many sets which are either  $<$ -densely ordered or  $<$ -discretely ordered.*

We delay the proof so as to give some definitions and lemmas first:

**Definition 5.1.6.** We say that the sets  $(B_i \subseteq M^d)_{i \in I}$  are **definably separated** if there are definable sets  $(A_i \subseteq M^d)_{i \in I}$  so that  $A_i \supseteq B_i$  for each  $i$ , and the  $A_i$  are pairwise disjoint. Note that the  $A_i$  in this definition need not be uniformly definable.

**Lemma 5.1.7.** *Let  $\mathcal{M} = (M, <, \dots)$  be a linear order and suppose there are definably separated infinite sets  $B_0, B_1, B_2, \dots \subseteq M$  so that for every  $i$  and every  $a, b \in B_{i+1}$  with  $a < b$  there is some  $c$  in  $B_i$  so that  $a < c < b$ . Then  $\mathcal{M}$  is not dp-small.*

*Proof.* Suppose that the sets  $A_0, A_1, A_2, \dots$  witness that  $B_0, B_1, B_2, \dots$  are definably separated and fix  $n$ .

Since  $B_n$  is infinite, we can choose elements  $b_{0,n} < b_{1,n} < b_{2,n} < \dots < b_{n^2,n}$  of  $B_n$ . Now, by assumption, there are elements  $b_{0,n-1} < b_{1,n-1} < \dots < b_{n^2-1,n-1}$  of  $B_{n-1}$  so that  $b_{i,n} < b_{i,n-1} < b_{i+1,n}$  for  $i < n^2$ . From there we choose  $b_{0,n-2} < b_{1,n-2} < \dots < b_{n^2-2,n-2}$  with  $b_{i,n-1} < b_{i,n-2} < b_{i+1,n-1}$ , and so on until we have  $b_{i,j}$  for  $j \leq n$  and  $i \leq n^2 - (n - j)$  so that  $b_{i,j+1} < b_{i,j} < b_{i+1,j+1}$  when  $j < n$  and  $i \leq n^2 - (n - j)$ .

Let  $a_{i,j} = b_{ni+j,j}$  for  $i, j < n$  and define  $X_i = (b_{ni,n}, b_{ni+n,n})_{<}$ . Then  $a_{i,j} \in X_k \cap A_\ell$  if and only if  $i = k$  and  $j = \ell$  where the  $A_i$  are the definable separating sets and the  $X_i$  are  $<$ -intervals, so uniformly definable

(with defining formula which does not depend on  $n$ ). Since we can find this configuration, for any  $n < \omega$ , it follows that by compactness (possibly changing to an elementary superstructure  $\mathcal{M}'$  of  $\mathcal{M}$ ), we can find  $a_{i,j}$  for  $i, j \in \omega$  and uniformly definable  $X_0, X_1, \dots$  so that  $a_{i,j} \in X_k \cap A_\ell$  if and only if  $i = k$  and  $j = \ell$ . This is a witness to non-dp-smallness of  $\mathcal{M}$  as in Definition 4.2.4.  $\square$

The following uniform version for convexly orderable structures will be useful later:

**Porism 5.1.8.** *Let  $\mathcal{M} = (M, <, \dots)$  be a convexly orderable linear order. Then there is some  $N$  so that there are no definably separated infinite sets  $B_0, B_1, B_2, \dots, B_N$  so that for every  $i$  and every  $a, b \in B_{i+1}$  with  $a < b$  there is some  $c$  in  $B_i$  so that  $a < c < b$ .*

*Proof.* Fix  $\triangleleft$  a convex ordering of  $\mathcal{M}$ , and let  $N$  be the maximum number of  $\triangleleft$ -convex components of a  $<$ -interval. Now, for  $B_0, B_1, \dots, B_N$  as in the statement, we define  $X_i, A_j$  and  $a_{i,j}$  as in Lemma 5.1.7. Since the  $X_i$  is always a  $<$ -interval, we conclude that the bound on the number of  $A_i$  is uniform, as in Example 4.2.5. As in the proof there, we can use  $X_i, A_j$ , and  $a_{i,j}$  to show that one of the  $X_i$  has  $N + 1$  distinct  $\triangleleft$ -convex components, but then this contradicts the definition of  $N$ .  $\square$

**Definition 5.1.9.** The  $<$ -**discrete equivalence relation**  $\equiv \overset{\circ}{<}$  is the equivalence relation defined by  $a \equiv \overset{\circ}{<} b$  if and only if the closed  $<$ -interval between  $a$  and  $b$  is  $<$ -discretely ordered. In particular,  $<$  is a dense order if and only if  $\equiv \overset{\circ}{<}$  is equality, and  $<$  is a discrete order if and only if  $\equiv \overset{\circ}{<}$  has only one equivalence class. Note, that  $\equiv \overset{\circ}{<}$  is definable in  $\mathcal{M} = (M, <, \dots)$ .

**Lemma 5.1.10.** *If  $\mathcal{M} = (M, <, \dots)$  is a dp-small linear order, then there is some  $n$  so that there are only finitely many  $a \in M$  so that  $a$  has an  $n$ -th  $<$ -successor but no  $<$ -predecessor. Similarly, there are only finitely many  $a \in M$  so that  $a$  has an  $n$ -th  $<$ -predecessor but no  $<$ -successor.*

*Proof.* We only need to prove the first statement; the second is equivalent since  $<$ -predecessors are  $>$ -successors and vice versa. Assume toward a contradiction that for each  $n$  there are infinitely many elements of  $M$  which have an  $n$ -th  $<$ -successor but no  $<$ -predecessor. Since dp-smallness is elementary, we can assume without loss of generality (passing to an elementary extension of  $\mathcal{M}$  if necessary) that there are elements  $a_0 < a_1 < a_2 < \dots$  which each have  $n$ -th  $<$ -successor for each  $n$  but have no  $<$ -predecessor. Let  $X_i = (a_i, a_{i+1})_{<}$ , let  $A_i$  be the set of elements which have an  $i$ -th  $<$ -predecessor but no  $(i + 1)$ -th  $<$ -predecessor, and let  $a_{i,j}$  be the  $j$ -th  $<$ -successor of  $a_i$ . Then  $a_{i,j} \in X_k \cap A_\ell$  exactly when  $i = k$  and  $j = \ell$ , i.e.,  $(X_i)_{i \in \omega}, (A_j)_{j \in \omega}$ , and  $(a_{i,j})_{i,j \in \omega}$  are a witness to non-dp-smallness as in Definition 4.2.4, which contradicts the hypothesis that  $\mathcal{M}$  is dp-small.  $\square$

**Corollary 5.1.11.** *If  $\mathcal{M} = (M, <, \dots)$  is a dp-small linear order, then there is some  $n$  so that if  $a \in M$  has a finite  $\equiv \overset{\circ}{<}$ -class, then it has an  $\equiv \overset{\circ}{<}$ -class of at most  $n$  elements.*  $\square$

We are now prepared to prove our Theorem on decompositions of dp-small orders:

**Theorem 5.1.12.** *If  $\mathcal{M} = (M, <, \dots)$  is a dp-small linear order, then  $M$  has a 0-definable partition into finitely many sets which are either  $<$ -densely ordered or  $<$ -discretely ordered.*

*Proof.* We will construct a tree of definable sets  $\{A_\sigma \mid \sigma \in T \subseteq 2^{<\omega}\}$  so that  $A_\varepsilon = M$ , and for  $\sigma \in T$  which is not a leaf,  $A_{\sigma \smallfrown 0}$  and  $A_{\sigma \smallfrown 1}$  form a partition of  $A_\sigma$ . To simplify notation throughout the proof, we will write  $\equiv_\sigma^\circ$  for the equivalence relation  $\equiv_{< \upharpoonright A_\sigma}^\circ$ . Given  $A_\sigma$  in the tree we define  $A_{\sigma \smallfrown 0}$  as follows (in each case  $A_{\sigma \smallfrown 1} = A_\sigma \setminus A_{\sigma \smallfrown 0}$ ):

1. If there are infinitely many  $\equiv_\sigma^\circ$ -classes which have at least 2 elements and have a  $<$ -least element, then we let  $A_{\sigma \smallfrown 0}$  be the collection of  $<$ -least elements of  $\equiv_\sigma^\circ$ -classes of size at least 2.
2. Otherwise, if there are infinitely many  $\equiv_\sigma^\circ$ -classes which have at least 2 elements and have a  $<$ -greatest element, then we let  $A_{\sigma \smallfrown 0}$  be the collection of  $<$ -greatest elements of  $\equiv_\sigma^\circ$ -classes of size at least 2.
3. Otherwise, if there are some infinite  $\equiv_\sigma^\circ$ -classes in  $A_\sigma$  and some finite  $\equiv_\sigma^\circ$ -classes in  $A_\sigma$ , then we let  $A_{\sigma \smallfrown 0}$  be the elements in infinite  $\equiv_\sigma^\circ$ -classes. This is 0-definable by Corollary 5.1.11.
4. Otherwise, if  $A_\sigma$  has some  $\equiv_\sigma^\circ$ -classes of size 1 and some  $\equiv_\sigma^\circ$ -classes of size at least 2, then we let  $A_{\sigma \smallfrown 0}$  be the elements which are  $<$ -least in their  $\equiv_\sigma^\circ$ -class.
5. Otherwise, if there is more than one  $\equiv_\sigma^\circ$ -class and there is a  $<$ -least  $\equiv_\sigma^\circ$ -class, then let  $A_{\sigma \smallfrown 0}$  be the  $<$ -least  $\equiv_\sigma^\circ$ -class.
6. Finally, if none of the above cases hold, then  $\sigma$  is a leaf and both  $A_{\sigma \smallfrown 0}$  and  $A_{\sigma \smallfrown 1}$  are undefined. In this case, either:
  - (a) There is an  $\equiv_\sigma^\circ$ -class of size 1. Then since case 4 fails above it follows that every  $\equiv_\sigma^\circ$ -class has size 1, i.e., that  $A_\sigma$  is  $<$ -densely ordered.
  - (b) There is no  $\equiv_\sigma^\circ$ -class of size 1. If  $a \not\equiv_\sigma^\circ b$  then  $A_\sigma \cap [a, b]_{<}$  is not  $<$ -discretely ordered, so there is  $c \in A_\sigma \cap [a, b]_{<}$  which either has no  $(< \upharpoonright A_\sigma)$ -successor or no  $(< \upharpoonright A_\sigma)$ -predecessor. In particular, since cases 1 and 2 fail above, it follows that there are finitely many  $\equiv_\sigma^\circ$ -classes. So, there is a  $<$ -least  $\equiv_\sigma^\circ$ -class. Since case 5 above fails, it follows that there is only one  $\equiv_\sigma^\circ$ -class, i.e., that  $A_\sigma$  is  $<$ -discretely ordered.

So, it suffices to show that the tree  $T$  is finite, in which case  $M$  can be 0-definably partitioned into the sets  $(A_\sigma)_{\sigma \text{ is a leaf of } T}$ , which are each  $<$ -densely ordered or  $<$ -discretely ordered. We first note that these operators are designed to grow  $\equiv_\sigma^\circ$ -classes:



**Claim 5.1.13.** *If  $a, b \in A_{\sigma \frown i}$  with  $a < b$  and  $a \equiv_{\sigma}^{\circ} b$ , then  $a \equiv_{\sigma \frown i}^{\circ} b$ .*

*Proof.* In cases 1, 2 and 4,  $b$  is not the  $<$ -least element of its  $\equiv_{\sigma}^{\circ}$ -class and  $a$  is not the  $<$ -greatest element of its  $\equiv_{\sigma}^{\circ}$ -class, so we see that  $i = 1$ . Any  $c \in [a, b]_{<} \cap A_{\sigma}$  is also neither the  $<$ -least nor the  $<$ -greatest element of its  $\equiv_{\sigma}^{\circ}$ -class, and so is also in  $A_{\sigma \frown 1}$ . It follows that  $[a, b]_{<} \cap A_{\sigma \frown i} = [a, b]_{<} \cap A_{\sigma}$  is  $<$ -discretely ordered, so  $a \equiv_{\sigma \frown i}^{\circ} b$  as desired.

In cases 3 and 5, the entire  $\equiv_{\sigma}^{\circ}$ -class of  $b$  is contained in  $A_{\sigma \frown i}$ . Since  $\equiv_{\sigma}^{\circ}$ -classes are  $<$ -convex in  $A_{\sigma}$ , it follows that again  $[a, b]_{<} \cap A_{\sigma \frown i} = [a, b]_{<} \cap A_{\sigma}$  is  $<$ -discretely ordered, and  $a \equiv_{\sigma \frown i}^{\circ} b$  as desired. ■

Now there are two cases we can show directly do not occur on an infinite path:

**Claim 5.1.14.** *If  $\sigma$  satisfies case 4, then  $\sigma \frown 0$  and  $\sigma \frown 1$  are leaves.*

*Proof.* Since there are finite  $\equiv_{\sigma}^{\circ}$ -classes and case 3 failed, we know that every  $\equiv_{\sigma}^{\circ}$ -class is finite, and since case 1 failed, only finitely many of the  $\equiv_{\sigma}^{\circ}$ -classes with more than one element have a least element, i.e., there are finitely many  $\equiv_{\sigma}^{\circ}$ -classes with more than one element. It follows that there are only finitely many elements in  $A_{\sigma \frown 1}$ , so  $A_{\sigma \frown 1}$  is discretely ordered, i.e.,  $\sigma \frown 1$  is a leaf.

We will show that  $A_{\sigma \frown 0}$  is  $<$ -densely ordered, i.e., for any  $a, b \in A_{\sigma \frown 0}$  with  $a < b$  we need to show that there is  $c' \in A_{\sigma \frown 0}$  so that  $a < c' < b$ . Let  $a'$  be the  $<$ -greatest element of the  $\equiv_{\sigma}^{\circ}$ -class of  $a$ . Since  $a \not\equiv_{\sigma}^{\circ} b$  (they can't both be the  $<$ -least element of the same  $\equiv_{\sigma}^{\circ}$ -class) and  $\equiv_{\sigma}^{\circ}$ -classes are convex, it follows that  $a' < b$  and  $a' \not\equiv_{\sigma}^{\circ} b$ . Thus there is some  $c \in A_{\sigma} \cap [a', b]_{<}$  so that  $a' \not\equiv_{\sigma}^{\circ} c \not\equiv_{\sigma}^{\circ} b$ . Let  $c'$  be the  $<$ -least element of the  $\equiv_{\sigma}^{\circ}$ -class of  $c$ , then  $a \leq a' < c' < b$  and  $c' \in A_{\sigma \frown 0}$  as desired. ■

**Claim 5.1.15.** *If  $\sigma$  satisfies case 5, then  $T$  is finite above  $\sigma$ .*

*Proof.* Since  $A_{\sigma \frown 0}$  is an  $\equiv_{\sigma}^{\circ}$ -class, it follows from Claim 5.1.13 that  $A_{\sigma \frown 0}$  has a single  $\equiv_{\sigma \frown 0}^{\circ}$ -class, so it is  $<$ -discretely ordered. From Claim 5.1.13,  $A_{\sigma \frown 1}$  satisfies case 5 or case 6, but there are fewer  $\equiv_{\sigma \frown 1}^{\circ}$ -classes than there are  $\equiv_{\sigma}^{\circ}$ -classes. This process can therefore only continue for finitely many levels above before we reach  $A_{\sigma \frown 1^n}$  which has a unique  $\equiv_{\sigma}^{\circ}$ -class and is therefore  $<$ -discretely ordered. ■

And so any infinite path would have to consist only of cases 1 to 3. We can further restrict the possibilities in these cases:

**Claim 5.1.16.** *If  $\sigma$  satisfies case 3, then  $T$  is finite above  $\sigma \frown 0$ .*

*Proof.* In this case, each  $\equiv_{\sigma \frown 0}^{\circ}$ -class is a union of infinite  $\equiv_{\sigma}^{\circ}$ -classes by Claim 5.1.13. The  $<$ -least (or  $<$ -greatest) element of any  $\equiv_{\sigma \frown 0}^{\circ}$ -class is therefore the  $<$ -least (or  $<$ -greatest) element of an infinite  $\equiv_{\sigma}^{\circ}$ -class. There are finitely many such by Corollary 5.1.11. It follows that  $\sigma \frown 0$  doesn't satisfy cases 1 to 4, and so either  $\sigma \frown 0$  is itself a leaf or it satisfies case 5 and we can apply Claim 5.1.15. ■

**Claim 5.1.17.** *If  $\sigma$  satisfies case 1 or case 2, then for  $a, b \in A_{\sigma \frown 1}$  with  $a < b$ , if  $a \not\equiv_{\sigma}^{\circ} b$ , then  $a \not\equiv_{\sigma \frown 1}^{\circ} b$  (so  $\equiv_{\sigma \frown 1}^{\circ}$  and  $\equiv_{\sigma}^{\circ}$  agree on  $A_{\sigma \frown 1}$ )*

*Proof.* We assume that there is some  $c \in [a, b]_{<} \cap A_{\sigma}$  which has no  $<$ -successor in  $A_{\sigma}$  (otherwise there is some  $c \in (a, b]_{<} \cap A_{\sigma}$  with no predecessor; this case follows the same argument). If  $c \in A_{\sigma \frown 1}$ , then let  $c' = c$ , otherwise  $c$  is the  $<$ -greatest element of its  $\equiv_{\sigma}^{\circ}$ -class, which has more than one element, so it has a  $<$ -predecessor in  $A_{\sigma}$  which is then  $c' \in [a, b]_{<} \cap A_{\sigma \frown 1}$ . We will show that  $c'$  has no  $<$ -successor in  $A_{\sigma \frown 1}$ .

Now, assume that  $c', d \in A_{\sigma \frown 1}$  with  $c' < d$ . We need to show that  $(c', d)_{<} \cap A_{\sigma \frown 1}$  is nonempty. We know  $c < d$ , and since  $c$  has no  $<$ -successor in  $A_{\sigma}$ , we can choose  $e, f, g \in A_{\sigma}$  so that  $c' \leq c < e < f < g < d$ . There are four possibilities:

1. If  $f \in A_{\sigma \frown 1}$ , then we are done.
2. If  $f \equiv_{\sigma}^{\circ} c'$ , then  $e$  is not the  $<$ -least or  $<$ -greatest element of its  $\equiv_{\sigma}^{\circ}$ -class, so  $e \in (c', d)_{<} \cap A_{\sigma \frown 1}$ .
3. If  $f \equiv_{\sigma}^{\circ} d$ , then by the same argument  $g \in (c', d)_{<} \cap A_{\sigma \frown 1}$ .
4. Finally, if  $c' \not\equiv_{\sigma}^{\circ} f \not\equiv_{\sigma}^{\circ} d$ , then since  $f \notin A_{\sigma \frown 1}$ , we know there is  $f' \equiv_{\sigma}^{\circ} f$  so that  $f' \in A_{\sigma \frown 1}$ . Since  $\equiv_{\sigma}^{\circ}$ -classes are convex, it follows that  $c' < f' < d$  as desired.

Since  $a \leq c' < b$  and  $c'$  has no  $<$ -successor in  $A_{\sigma \frown 1}$ , it follows that  $a \not\equiv_{\sigma \frown 1}^{\circ} b$ . ■

**Claim 5.1.18.** *There are no  $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \dots$  in  $T$  so that each  $\sigma_i$  satisfies case 1 or case 2 and  $\sigma_{i+1} \succeq \sigma_i \frown 0$  for each  $i$ .*

*Proof.* Assume toward a contradiction that there are, and let  $Y_i = A_{\sigma_i \frown 1}$ . Fix  $a, b \in Y_{i+1}$  with  $a < b$ . We have  $a$  and  $b$  are both in  $A_{\sigma_i \frown 0}$ . If  $\sigma_i$  satisfies case 1 then  $a$  and  $b$  are each the  $<$ -least elements of their  $\equiv_{\sigma_i}^{\circ}$ -class and if we let  $c$  be the  $<$ -successor of  $a$  in  $A_{\sigma_i}$ , then  $a < c < b$  and  $c \in Y_i$ . Similarly, if  $\sigma_i$  satisfies case 2 then  $a$  and  $b$  are each the  $<$ -greatest elements of their  $\equiv_{\sigma_i}^{\circ}$ -class and if we let  $c$  be the  $<$ -predecessor of  $b$  in  $A_{\sigma_i}$ , then  $a < c < b$  and  $c \in Y_i$ .

So, the sets  $Y_0, Y_1, Y_2, \dots$  are definable and disjoint, and between every two elements of  $Y_{i+1}$  is an element of  $Y_i$ . This violates Lemma 5.1.7. ■

**Claim 5.1.19.** *There are no  $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \dots$  in  $T$  each satisfying case 3.*

*Proof.* Assume toward a contradiction that there are. We can assume without loss of generality that any  $\tau$  so that  $\sigma_i \prec \tau \prec \sigma_{i+1}$  satisfies case 1 or case 2 (by Claims 5.1.14 and 5.1.15) and that any  $\tau, \tau'$  so that  $\sigma_i \preceq \tau \prec \tau' \preceq \sigma_{i+1}$ , satisfy  $\tau' \succeq \tau \frown 1$  (by Claims 5.1.16 and 5.1.18).

Let  $Y_i$  be the set of  $a \in A_{\sigma_{i+1}}$  so that  $a$  is the  $<$ -least element of its  $\equiv_{\sigma_i}^\circ$ -class in  $A_{\sigma_{i+1}}$ , and  $a$  is not the  $<$ -least element of its  $\equiv_{\sigma_{i+1}}^\circ$ -class, and the  $\equiv_{\sigma_{i+1}}^\circ$ -class of  $a$  is infinite. The set  $Y_i$  is definable and since  $A_{\sigma_{i+1}}$  is made up of elements of finite  $\equiv_{\sigma_i}^\circ$ -classes (since  $\sigma_{i+1} \succ \sigma_i \frown 1$ ). It follows that every infinite  $\equiv_{\sigma_{i+1}}^\circ$ -class intersects  $Y_i$  infinitely often. There is at least one such, since  $\sigma_{i+1}$  satisfies case 3, so  $Y_i$  is infinite.

Let  $a, b \in Y_{i+1}$  with  $a < b$ , then  $a \not\equiv_{\sigma_{i+1}}^\circ b$  since  $b$  is the  $<$ -least element of its  $\equiv_{\sigma_{i+1}}^\circ$ -class in  $A_{\sigma_{i+2}}$ , and  $a < b$ . By Claim 5.1.17, if  $a \equiv_{\sigma_{i+2}}^\circ b$  then  $a \equiv_{\sigma_{i+1} \frown 1}^\circ b$ , and since  $a \not\equiv_{\sigma_{i+1}}^\circ b$ , it follows that  $[a, b]_{<} \cap A_{\sigma_{i+1}} \neq [a, b]_{<} \cap A_{\sigma_{i+1} \frown 1}$ . So, there is some infinite  $\equiv_{\sigma_{i+1}}^\circ$ -class contained in  $(a, b)_{<}$ . Then in particular,  $Y_i$  intersects  $(a, b)_{<}$ . Thus the sets  $Y_0, Y_1, Y_2, \dots$  are definable and disjoint, and between every two elements of  $Y_{i+1}$  is an element of  $Y_i$ . This violates Lemma 5.1.7.  $\blacksquare$

Let  $\tau \in 2^\omega$  be an infinite path through  $T$ . From Claims 5.1.14, 5.1.15 and 5.1.19, we may assume (passing to  $A_\sigma$  for some  $\sigma$  if necessary) that if  $\sigma \prec \tau$  then  $\sigma$  satisfies case 1 or case 2. From Claim 5.1.17 (passing to  $A_\sigma$  for some  $\sigma$  if necessary), we can assume that  $\tau = 1^\omega$ , so in particular, by Claim 5.1.17, it follows that  $\equiv_{1^n}^\circ$  and  $\equiv_{1^m}^\circ$  agree on  $A_{1^m}$  when  $m > n$ . Since case 2 never changes the  $<$ -initial segment of an  $\equiv_{1^n}^\circ$ -class, it follows that if  $1^n$  satisfies case 2, then so does  $1^{n+1}$  (because in particular it still doesn't satisfy case 1).

So, we may assume that either  $1^n$  satisfies case 1 for each  $n$  or  $1^n$  satisfies case 2 for each  $n$ . Assume it is the former (the arguments are analogous). There are infinitely many  $\equiv_{1^n}^\circ$ -classes with  $<$ -least element. Given  $n \in \omega$ , suppose that  $a_n$  is the  $<$ -least element of an  $\equiv_{1^n}^\circ$ -class. Then since the  $\equiv_{1^{n-1}}^\circ$ -class of  $a$  has at most one more element than the  $\equiv_{1^n}^\circ$ -class of  $a$ , it follows that the  $\equiv_{1^{n-1}}^\circ$ -class of  $a$  has a  $<$ -least element which is not in  $A_{1^n}$ , and is therefore the  $<$ -predecessor of  $a$  in  $A_{1^{n-1}}$ . Repeating this argument, we find that the  $\equiv_{1^n}^\circ$ -class of  $a$  has a  $<$ -least element which is the  $n$ -th  $<$ -predecessor of  $a$ . Thus there are infinitely many  $\equiv_{1^n}^\circ$ -classes with  $<$ -least element which has an  $n$ -th  $<$ -successor. Since this is true for every  $n \in \omega$ , this contradicts Lemma 5.1.10.  $\square$

So, Theorem 5.1.12 gives us a basic idea of how to separate orders into “discrete parts” and “dense parts”. It is sometimes easier to think of the structure in one discrete piece and one dense piece:

**Corollary 5.1.20.** *If  $\mathcal{M} = (M, <, \dots)$  is a  $dp$ -small linear order, then there are some linear order  $<^{dd}$  on  $M$ , definable with parameters from  $\mathcal{M}$ , which has finite  $<$ -rank, and some  $x \in M$  so that  $(-\infty, x]_{<^{dd}}$  is  $<^{dd}$ -discretely ordered, and  $(x, \infty)_{<^{dd}}$  is  $<^{dd}$ -densely ordered.*

*Proof.* Let  $A_0, A_1, \dots, A_n$  be the partition given by Theorem 5.1.12. Without loss of generality, each  $A_i$  is either  $<$ -discretely ordered or  $<$ -densely ordered without  $<$ -endpoints (if  $A_i$  is  $<$ -densely ordered and has a  $<$ -endpoint, then the  $<$ -endpoint is 0-definable, so we can split  $A_i$  into  $A_i \setminus \{a\}$ , which is  $<$ -densely ordered and has one  $<$ -endpoint fewer than  $A_i$ , and  $\{a\}$  which is  $<$ -discretely ordered).

If none of the  $A_i$  is  $<$ -discretely ordered, then we can choose  $a \in A_i$  for some  $A_i$  which is  $<$ -densely ordered, and split  $A_i$  into  $A_i \setminus \{a\}$  and  $\{a\}$  (this requires a single parameter  $a$  in the definition).

Now, if  $A_i$  is  $<$ -discretely ordered and has no  $<$ -endpoints, we can choose  $a \in A_i$  which has a  $<$ -predecessor and a  $<$ -successor. Then we can split  $A_i$  into  $A_i \cap (-\infty, a]_{<}$  and  $A_i \cap [a, \infty)_{<}$ , which are both  $<$ -discretely ordered and have a  $<$ -endpoint. Thus by allowing  $A_i$  to be defined with parameters, we may assume that each  $A_i$  is either  $<$ -discretely ordered with at least one endpoint, or  $<$ -densely ordered with no  $<$ -endpoints.

Now, without loss of generality, we label the  $A_i$  so that if  $A_i$  is  $<$ -discretely ordered and  $A_j$  is  $<$ -densely ordered, then  $i < j$ . In particular, let  $k$  be the minimum so that  $A_k$  is  $<$ -densely ordered. Then we see that  $(D, <_D) = \sum_{i=k}^n (A_i, < \upharpoonright A_i)$  is densely ordered, and  $<_D$  is built by repeated applications of Construction 2 from Lemma 4.2.8, so it has finite  $(< \upharpoonright D)$ -rank.

For the discrete portion of the order, note for  $i < k$  that if  $A_i$  is  $<$ -discretely ordered and has a  $<$ -least element but no  $<$ -greatest element, then we can replace  $(A_i, <)$  with  $(A_i, >)$  and get a  $<$ -discretely ordered set with  $<$ -greatest element and no  $<$ -least element. This is of course an example of Construction 1 from Lemma 4.2.8. So, we may assume that the discrete order  $(A_i, <_{A_i})$  is of finite  $(< \upharpoonright A_i)$ -rank and has a  $<_{A_i}$ -greatest element.

For  $i, j < k$  with  $i \neq j$ , if  $A_i$  has no  $<_{A_i}$ -least element and  $A_j$  has no  $<_{A_j}$ -least element, then we see that  $(B, <_B) = (A_i, >_{A_i}) + (A_j, <_{A_j})$  is discretely ordered. The order  $<_B$  has finite  $(< \upharpoonright B)$ -rank (it is an example of construction 3 from Lemma 4.2.8), and  $B$  has both a  $<_B$ -least and a  $<_B$ -greatest element, so we may assume that there is at most one  $i$  so that  $A_i$  has no  $<_{A_i}$ -least element.

Now, again assuming  $i, j < k$  with  $i \neq j$ , if  $A_i$  has a  $<_{A_i}$ -least element, then since  $A_j$  has a  $<_{A_j}$ -greatest element, it follows that  $(B, <_B) = (A_i, <_{A_i}) + (A_j, <_{A_j})$  is a discrete order and  $<_B$  has finite  $(< \upharpoonright B)$ -rank. Iterating this construction, we can paste the discrete orders together to get a single discrete order  $(B, <_B)$  with a greatest element.

Now,  $(M, <_M) = (B, <_B) + (D, <_D)$  is the desired order on  $M$ . □

This has an immediate application to  $\text{CO}_n$  structures:

**Corollary 5.1.21.** *If  $\mathcal{M}$  is  $\text{CO}_n$  for some  $n > 1$  then there are some convex ordering  $\triangleleft$  of  $\mathcal{M}$  and  $x \in M$  so that  $(\mathcal{M}, \triangleleft)$  is  $\text{CO}_{n-1}$ ,  $(-\infty, x]_{\triangleleft}$  is  $\triangleleft$ -discretely ordered, and  $(x, \infty)_{\triangleleft}$  is  $\triangleleft$ -densely ordered. □*

Since any definable set in a dp-small structure is itself dp-small, we can also decompose definable sets into dense and discrete parts. By compactness, this is in fact uniform for families of definable sets:

**Corollary 5.1.22.** *If  $\mathcal{M} = (M, <, \dots)$  is a dp-small linear order, and  $\mathcal{F}$  is a definable family of subsets of  $M$ , then there are finitely many families  $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$  so that each  $\mathcal{A}_i$  is either a family of  $<$ -densely*

ordered subsets of  $M$  or a family of  $<$ -discretely ordered subsets of  $M$  and for each  $X \in \mathcal{F}$  then there are sets  $A_i \in \mathcal{A}_i$  for each  $i$  so that  $X = \bigcup_{i=0}^n A_i$ .  $\square$

So, to understand dp-small orders, we need to understand discrete and dense orders, as well as their densely or discretely ordered subsets. In Section 5.2 we will consider the case of a discretely ordered structure, and prove our reflection principle Theorem 5.2.14. In Section 5.3 we will look more closely at the dense case, and see a modified analogue to Theorem 5.2.14 which applies to dense orders.

## 5.2 Discretely Ordered CO Structures

We now consider the case of a discrete convexly orderable order in more detail. Our primary goal in this section will be to prove our reflection principle Theorem 5.2.14, but we will also more generally discuss the families of sets definable in a discretely ordered structure. We begin with a simple observation on dp-small discrete orders:

**Example 5.2.1.** Let  $\mathcal{M} = (\mathbb{Z} \times 2, <, U)$ , where  $U$  is a unary predicate naming  $(\mathbb{Z} \times \{0\})$ . Then  $\mathcal{M}$  is weakly o-minimal and  $<$ -discretely ordered, but  $A = \mathbb{Z}^+ \times 2$  is a definable subset which is not  $<$ -discretely ordered (because in particular,  $(1, 1)$  is not the  $<$ -least element of  $A$ , but does not have a  $< \upharpoonright A$ -predecessor).

So, certainly, we must accept that even in a weakly o-minimal discretely ordered structure  $\mathcal{M}$ , not every definable subset of  $M$  will be discretely ordered. On the other hand, we want to say that the definable sets are essentially discrete. This line of thought inspires the following definition:

**Definition 5.2.2.** Let  $\mathcal{X} = (X, <)$  be a linear order. We say that  $\mathcal{X}$  is **pseudo-discrete** if there are only finitely many  $x \in X$  so that  $X$  has either no  $<$ -predecessor or no  $<$ -successor. We say that  $\mathcal{X}$  is **pseudo-dense** if there are only finitely many  $x \in X$  so that  $X$  has a  $<$ -predecessor or a  $<$ -successor. We say that  $\mathcal{M} = (M, <, \dots)$  is **fully pseudo-discrete (fully pseudo-dense)** if whenever  $\mathcal{N} \equiv \mathcal{M}$  and  $A \subseteq N$  is definable in  $\mathcal{N}$ , then  $A$  is  $<$ -pseudo-discretely ordered ( $<$ -pseudo-densely ordered).

In understanding pseudo-discrete orders it is useful to think in terms of  $<$ -successors. The following terminology will be helpful:

**Definition 5.2.3.** For any linear order  $(X, <)$  and  $x \in X$ , the  **$<$ -entourage of  $x$**  is the collection of  $n$ -th  $<$ -successors and  $n$ -th  $<$ -predecessors of  $x$  for all  $n \in \omega$ . We say  $(X, <)$  is **weakly pseudo-discrete** if only finitely many elements of  $X$  have finite  $<$ -entourage.

Weak pseudo-discreteness will be a useful stepping-stone in our proof of Theorem 5.2.14, but on a dp-small structure, this is really the same condition as pseudo-discreteness

**Lemma 5.2.4.** *If  $\mathcal{M} = (M, <, \dots)$  is a dp-small order, then  $(M, <)$  is pseudo-discrete if and only if it is weakly pseudo-discrete.*

*Proof.* If a  $<$ -entourage has least element  $a$ , then  $a$  has no  $<$ -predecessor. Since every finite  $<$ -entourage has a  $<$ -least element, clearly pseudo-discrete implies weakly pseudo-discrete. For the converse, we note that from Lemma 5.1.10, there are only finitely many elements of  $\mathcal{M}$  which have infinite  $<$ -entourage but have either no  $<$ -predecessor or no  $<$ -successor. It follows that there are finitely many elements with no  $<$ -predecessor or no  $<$ -successor if and only if there are finitely many such with finite  $<$ -entourage. So, weakly pseudo-discrete implies pseudo-discrete.  $\square$

So, we will instead be using weak pseudo-discreteness as a condition on the convex order. In this case it is a distinct condition from pseudo-discreteness, but it is still strong enough to imply that some convex order is discrete:

**Lemma 5.2.5.** *If  $\triangleleft$  is a weakly pseudo-discrete order, then there is a finite  $\triangleleft$ -rank discrete order.*

*Proof.* We can assume without loss of generality that each element of  $M$  has infinite  $\triangleleft$ -entourage, because there are only finitely elements which do not have infinite  $\triangleleft$ -entourages, and by construction 2 of Lemma 4.2.8 we can move these points into infinite  $\triangleleft$ -entourages. We will apply construction 3 of Lemma 4.2.8 to construct our discrete order. Say  $a \equiv b$  if  $a$  and  $b$  have the same  $\triangleleft$ -entourage. The  $\equiv$ -classes are convex, as desired. It will suffice to find a  $(\triangleleft \upharpoonright X)$ -ranked order on each  $\equiv$ -class  $X$  in which each element has a predecessor and successor. Since  $\equiv$ -classes are just  $\triangleleft$ -entourages, there are three possibilities for an  $\equiv$ -class  $X$ :

1.  $X$  has  $\triangleleft$ -order type  $\mathbb{Z}$ : No change is needed.
2.  $X$  has order-type  $\omega$ : We need to show that there is a  $<$ -ranked order on  $(\omega, <)$  with the desired properties. By moving the odds to the front of the order, we get a  $<$ -rank 2 order (construction 2) with order type  $\omega + \omega$ . Now, we can reverse the order on the first copy of  $\omega$  (construction 3) to get a  $<$ -ranked order with order type  $\mathbb{Z}$ .

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \dots \end{array} \Rightarrow \begin{array}{ccccccc} 1 & 3 & 5 & 7 & \dots & 0 & 2 & 4 & 6 & \dots \\ \bullet & \bullet & \bullet & \bullet & \dots & \circ & \circ & \circ & \circ & \dots \end{array} \Rightarrow \begin{array}{ccccccc} \dots & 7 & 5 & 3 & 1 & 0 & 2 & 4 & 6 & \dots \\ \dots & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \dots \end{array}$$

3.  $X$  has order-type  $\omega^*$ . In this case  $X$  has order-type  $\omega$  in the  $(\triangleleft \upharpoonright X)$ -rank 1 order  $\triangleright \upharpoonright X$ , so this reduces to the previous case.  $\square$

**Corollary 5.2.6.** *If  $\mathcal{M}$  has a weakly pseudo-discrete convex ordering, then  $\mathcal{M}$  has a discrete convex ordering.*  $\square$

So, our goal will be to show that every pseudo-discrete convexly orderable structure has a weakly pseudo-discrete convex ordering. We first show that the definable subsets of a pseudo-discrete convexly orderable structure are finite unions of periodic sets.

**Definition 5.2.7.** Let  $\mathcal{M} = (M, <, \dots)$ . For any  $X, A \subseteq M$ , where  $A$  is  $<$ -discretely ordered, we say that  $X$  is  **$<$ -periodic on  $A$  with period  $n$**  if, for any  $a \in A$  with  $n$ -th ( $< \upharpoonright A$ )-successor  $a'$ , we have  $a \in X$  if and only if  $a' \in X$ . We say that  $A$  is a  **$<$ -periodic component of  $X$**  if  $A$  is a maximal  $<$ -discretely ordered  $<$ -convex set on which  $X$  is periodic.

**Example 5.2.8.** One specific oddity of Definition 5.2.7 is that periodic components of a set are not necessarily disjoint. As an example, here are two ways to divide a sequence of ones and zeros into periodic pieces:

$$\dots 101101101|01010\dots = \dots 1011011|0101010\dots$$

Note that the “01” is part of both maximal periodic pieces, i.e., part of both periodic components. On the other hand, the overlap between two periodic components is always finite.

So, we want to show that definable subsets of a convexly orderable pseudo-discrete order have finitely many  $<$ -periodic pieces. We do this by first showing that the distance between elements of a definable set is uniformly short:

**Lemma 5.2.9.** *Suppose that  $\mathcal{M} = (M, <, \dots)$  is a convexly orderable pseudo-discrete order. Given a definable  $X \subseteq M$ , and  $a \in M$ , we define  $d_X(a)$  to be the least positive  $n$  so that the  $n$ -th successor of  $a$  is in  $X$  if and only if  $a$  is (if there is no such  $n$ , then  $d_X(a) = \infty$ ). There is some  $N$  (which doesn't depend on  $X$  and  $a$ ) so that for any  $X \subseteq M$  the set  $\{x \mid d_X(a) > N + 1\}$  is finite.*

*Proof.* If  $a$  has  $n$ -th  $<$ -successor for each  $n$  and  $d_X(a) = \infty$ , then the  $<$ -successor of  $a$  is an element of  $X$  which has an  $n$ -th ( $< \upharpoonright M \setminus X$ )-successor for each  $n$  but has no ( $< \upharpoonright M \setminus X$ )-predecessor. By pseudo-discreteness and Lemma 5.1.10, then, it follows that there are only finitely many  $a$  so that  $d_X(a) = \infty$  for any definable  $X \subseteq M$ . By compactness, it follows that there are only finitely many  $a$  so that  $d_X(a) > m$  for some  $m$  (which might depend on  $X$ ). So, showing that  $\{x \mid d_X(a) > N + 1\}$  is finite is equivalent to showing that  $d_X^{-1}(\{k\})$  is finite whenever  $k > N + 1$ .

Let  $N$  be the as in Porism 5.1.8, and assume there are some definable  $X \subseteq M$  and  $k > N + 1$  so that  $B_0 = d_X^{-1}(\{k\})$  is infinite. Let  $B_i$  be the set of  $i$ -th successors of  $B_0$ . By pseudo-discreteness and Lemma 5.1.10, since  $B_0$  is infinite,  $B_i$  is infinite for each  $i$  too.

Suppose  $a \in B_0$  and  $b$  is the  $j$ -th  $<$ -successor of  $a$  for some  $j \leq N$ . Then  $a \in X$  if and only if  $b \notin X$  since  $d_X(a) > j$ . Let  $b'$  be the successor of  $b$ . Then  $a \in X$  if and only if  $b' \notin X$  since  $d_X(a) > j + 1$ . It

follows that  $b \in X$  if and only if  $b' \in X$ , so  $d_X(b) = 1$ , i.e.,  $b \notin B_0$ . So the  $B_0, B_1, \dots, B_N$  are disjoint, and for  $a, b \in B_{i+1}$  with  $a < b$ , there is  $b' \in B_i$  so that  $b$  is the  $<$ -successor of  $b'$ , from which  $a < b' < b$ . This violates Porism 5.1.8.  $\square$

**Corollary 5.2.10.** *If  $\mathcal{M} = (M, <, \dots)$  is a convexly orderable pseudo-discrete order, then  $\mathcal{M}$  is fully pseudo-discrete.*

*Proof.* Fix definable  $X \subseteq M$ . By Lemma 5.2.9 for all but finitely many  $a \in X$  the  $k$ -th successor of  $a$  is in  $X$  for some  $k < N$ . In particular, this means that all but finitely many  $a \in X$  have  $(< \upharpoonright X)$ -successors, as desired.  $\square$

Now we use the fact that if  $X$  is a definable set, then so are sets that describe the “pattern” of  $X$  on a specific  $<$ -periodic component. We show that sets with repeating patterns are periodic:

**Lemma 5.2.11.** *Given  $\sigma = b_0 b_1 b_2 \dots \in 2^\omega$ , suppose there is some  $N$  so that for any  $i$ , there is  $0 < j \leq N$  so that  $b_i b_{i+1} b_{i+2} \dots b_{i+N} = b_{i+j} b_{i+1+j} b_{i+2+j} \dots b_{i+N+j}$ . Then  $\sigma$  is periodic with period  $\leq N$ .*

*Proof.* Let  $f(i)$  be any  $j$  with  $N \geq j > 0$  which satisfies the equation:

$$b_i b_{i+1} b_{i+2} \dots b_{i+N} = b_{i+j} b_{i+1+j} b_{i+2+j} \dots b_{i+N+j}$$

Since  $f$  takes finitely many values, there is some value  $\tau$  so that  $f(i) = \tau$  infinitely often. We will show that if  $f(i+1) = \tau$ , we can assume that  $f(i) = \tau$  as well. Since for every  $i_0$  there is some  $i_1 > i_0$  with  $f(i_1) = \tau$ , this is sufficient to show that we can assume  $f(i) = \tau$  for all  $i$ , and therefore that  $b_i = b_{i+\tau}$  for all  $i$ , i.e., that  $\sigma$  is periodic with period dividing  $\tau$ .

By definition of  $f(i)$ , we have  $b_i = b_{i+f(i)}$ , and since  $i+1 \leq i+f(i) < i+1+N$ , it follows by definition of  $f(i+1) = \tau$  that  $b_{i+f(i)} = b_{i+f(i)+\tau}$ , so we have  $b_i = b_{i+\tau+f(i)}$ . Finally, since  $i < i+\tau \leq i+N$ , it follows from the definition of  $f(i)$  that  $b_{i+\tau} = b_{i+\tau+f(i)}$ , so we have  $b_i = b_{i+\tau}$ .

By definition of  $f(i+1) = \tau$ , we have:

$$b_{i+1} b_{i+2} b_{i+3} \dots b_{i+N} = b_{i+1+\tau} b_{i+2+\tau} b_{i+3+\tau} \dots b_{i+N+\tau}$$

Which, along with  $b_i = b_{i+\tau}$  tells us that  $f(i) = \tau$  works.  $\square$

And we combine these results to show that we have periodicity within a  $<$ -entourage:



**Lemma 5.2.12.** *If  $\mathcal{M} = (M, <, \dots)$  is a convexly orderable pseudo-discrete order, then there is some  $N$  so that for any definable  $X \subseteq M$  and  $a \in M$  with  $<$ -entourage  $Y$  there are  $b, c \in Y$  with  $b \leq a \leq c$  so that  $X$  is periodic with period  $n \leq N$  on  $(-\infty, b)_{<} \cap Y$ , and  $X$  is periodic with period  $n' \leq N$  on  $(c, \infty)_{<} \cap Y$ .*

*Proof.* Fix  $N$  as in Lemma 5.2.9. Let  $S$  be the  $<$ -successor function. For  $x, y \in M$  we will say  $x \sim y$  if,  $S^j(x) \in X \leftrightarrow S^j(y) \in X$  for every  $j \leq N$ . Then  $\sim$  is a definable equivalence relation with at most  $2^N$  equivalence classes, so by Lemma 5.2.9 we can choose  $b, c \in Y$  with  $b \leq a \leq c$  so that if  $x \in Y \setminus [b, c]_{<}$ , then there is  $j \leq N$  so that  $x \sim S^j(x)$ .

Now, if  $(c, \infty)_{<} \cap Y$  is finite, then we could have chosen  $c$  to be its maximum element, in which case we are done. On the other hand, if  $(c, \infty)_{<} \cap Y$  is infinite, then we define  $b_j = 1$  if  $S^{j+1}(c) \in X$ , and  $b_j = 0$  if  $S^{j+1}(c) \notin X$ . For each  $j \in \omega$ , there is some  $k < N$  so that  $S^{j+1}(c) \sim S^{j+1+k}(c)$ , i.e., so that  $b_{j+\ell} = b_{j+\ell+k}$  for each  $\ell < N$ . It follows from Lemma 5.2.11 that  $b_0 b_1 b_2 b_3 \dots$  is periodic with period  $n' < N$ , as desired. The argument for  $(-\infty, b)_{<} \cap Y$  is analogous.  $\square$

Compactness then tells us that we have periodicity overall:

**Corollary 5.2.13.** *If  $\mathcal{M} = (M, <, \dots)$  is a convexly orderable pseudo-discrete order, then there is some  $N$  so that if  $X \subseteq M$  is definable, then it has finitely many  $<$ -periodic components, and has period dividing  $N!$  on each.*

*Proof.* Fix  $N$  as in Lemma 5.2.12. Let  $B_0$  be the collection of  $a$  so that  $a \in X$  but the  $N!$ -th successor of  $a$  is not or vice versa. It is enough to show that this is finite.

By Lemma 5.2.12, each  $<$ -entourage intersects  $B_0$  finitely. The set  $B_0$  is definable. Let  $C_0$  be the collection of  $a \in B_0$  so that  $a$  is the  $<$ -greatest element of  $B_0$  in its  $<$ -entourage. Let  $B_0^\circ = B_0$ , and for each  $i > 0$  we let  $C_{i+1}$  be the collection of  $<$ -successors of elements of  $C_i$ , let  $B_{i+1}$  be the collection of  $<$ -successors of elements of  $B_i^\circ$ , and let  $B_{i+1}^\circ = B_{i+1} \setminus B_0$ . Then  $B_i^\circ \supseteq C_i$ , and the  $B_i^\circ$  are disjoint, i.e., the  $B_i^\circ$  witness that the  $C_i$  are definably separated. For  $a, b \in C_{i+1}$  with  $a < b$ , we know  $b$  is the  $<$ -successor of some  $b' \in C_i$ , and thus  $a < b' < b$ . It follows from Porism 5.1.8 that  $C_i$  is finite for some  $i$ , and so by Corollary 5.1.11,  $C_0$  is finite, and it follows that  $B_0$  is finite.  $\square$

We now have a strong enough understanding of the definable subsets of a convexly orderable discrete order to prove the reflection principle:

**Theorem 5.2.14.** *If  $\mathcal{M} = (M, <, \dots)$  is discretely ordered and convexly orderable, then  $\mathcal{M}$  has a discrete convex ordering.*

*Proof.* Let  $N$  be as given by Lemma 5.2.12. For  $a, b \in M$ , we say that  $a \sim b$  if there are exactly  $N!k - 1$  elements  $<$ -between  $a$  and  $b$  for some  $k \in \omega$ . Note that if  $a < b < c$ , and there are  $r$  elements between  $a$  and

$b$  and there are  $s$  elements between  $b$  and  $c$ , then there are  $r + s + 1$  elements between  $a$  and  $c$  (the  $+1$  is for  $b$  itself). So,  $\sim$  is an equivalence relation on  $M$ . Let  $f: M \rightarrow M$  be a choice function for this equivalence relation.

Now, given  $\triangleleft$  a convex ordering on  $M$ , we say  $a \triangleleft_f b$  if  $f(a) \triangleleft f(b)$  or if  $f(a) = f(b)$  and  $a < b$ . Clearly  $\triangleleft_f$  is an order on  $M$ , and if  $a \in M$  has infinitely many  $<$ -successors (or  $<$ -predecessors), then  $a$  has infinitely many  $\triangleleft_f$ -successors (or  $\triangleleft_f$ -predecessors). So, since  $\mathcal{M}$  is  $<$ -weakly pseudo-discrete, it follows that  $\mathcal{M}$  is  $\triangleleft_f$ -weakly pseudo-discrete. So by Corollary 5.2.6, if  $\triangleleft_f$  is a convex ordering of  $\mathcal{M}$ , then we are done.

Suppose that  $\triangleleft_f$  is not a convex ordering of  $\mathcal{M}$ , then (possibly passing to some elementary extension  $(\mathcal{M}', \triangleleft', f')$  of  $(\mathcal{M}, \triangleleft, f)$ ) there is some definable  $X \subseteq M$  and some strictly  $\triangleleft_f$ -monotone sequence  $a_0, a_1, a_2, \dots$  so that  $a_i \in X$  if and only if  $i$  is even. By Corollary 5.2.13 we may assume that the  $a_i$  are all in the same  $<$ -periodic component of  $X$ , so if  $a_i \sim a_j$ , then  $a_i \in X$  if and only if  $a_j \in X$ . So, it follows that  $a_i \approx a_{i+1}$  for each  $i$  and, since  $\sim$ -classes are  $\triangleleft_f$ -convex,  $a_i \approx a_j$  for  $i \neq j$ .

By Corollary 5.2.13, we may assume that for each  $i$ , we have  $f(a_i) \in X$  if and only if  $a_i \in X$ . Then since the  $f(a_i)$  are distinct, we have a strictly  $\triangleleft_f$ -monotone sequence  $f(a_0), f(a_1), f(a_2), \dots$  so that  $f(a_i) \in X$  if and only if  $i$  is even. But then  $f(a_0), f(a_1), f(a_2), \dots$  is a  $\triangleleft$ -monotone sequence by definition of  $\triangleleft_f$ , which contradicts the assumption that  $\triangleleft$  is a convex ordering.  $\square$

In Theorem 5.2.14 it is sufficient to the hypothesis that  $\mathcal{M}$  be a pseudo-discrete order, rather than a discrete order. However, consider the following example:

**Example 5.2.15.** Let  $\mathcal{M} = ((\mathbb{Z} + 1) \times \mathbb{Z}, <)$ .  $\mathcal{M}$  is convexly orderable, and is not pseudo-discrete, but  $\mathcal{M}$  defines the discrete order  $<^{\mathbb{Z} \times \mathbb{Z}}$  on  $M$ , so  $\mathcal{M}$  has a discrete convex ordering.

As in this example, the most general way to apply Theorem 5.2.14 is to first apply Corollary 5.1.20 to make the order discrete if possible. In this sense, pseudo-discreteness is actually a stronger condition than necessary.

### 5.3 Dense Orders

We now consider the analogue of Theorem 5.2.14 for densely ordered structures. We will prove this analogue by successive approximations, beginning by showing that monotone sequences in the original order can't be too far from monotone in the convex order:

**Definition 5.3.1.** Given orders  $<$  and  $\triangleleft$  on  $X$ , we say that a  $\triangleleft$ -monotone sequence  $a_0, a_1, a_2, \dots, a_k$  is a  **$<$ -zigzag in  $\triangleleft$**  if either  $a_{2i} < a_{2j+1}$  for all  $i, j$ , or  $a_{2i} > a_{2j+1}$  for all  $i, j$ .

**Lemma 5.3.2.** *If  $\mathcal{M} = (M, <, \dots)$  is a linear order and  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ , then there is some  $N$  so that no  $<$ -zigzag in  $\triangleleft$  has length  $2N$ .*

*Proof.* Let  $N$  be the maximum number of  $\triangleleft$ -convex components of a  $<$ -interval, and suppose  $a_0, a_1, \dots, a_{2N}$  is a  $<$ -zigzag in  $\triangleleft$ . Let  $a_i$  and  $a_j$  be the  $<$ -minimum and  $<$ -maximum of  $\{a_0, a_2, a_4, \dots, a_{2N}\}$  respectively. Then  $a_k \in [a_i, a_j]_{<}$  exactly when  $k$  is even. For  $k < k' \leq N$ ,  $a_{2k+1}$  is  $\triangleleft$  between  $a_{2k}$  and  $a_{2k'}$  and is not in  $[a_i, a_j]_{<}$ , so the elements  $a_0, a_2, a_4, \dots, a_{2N}$  are each in different  $\triangleleft$ -components of  $[a_i, a_j]_{<}$ . Thus  $[a_i, a_j]_{<}$  has at least  $N + 1$  distinct  $\triangleleft$ -convex components, which contradicts the choice of  $N$ .  $\square$

Now, we can use this to show that a convex ordering of a dense order has dense pieces.

**Lemma 5.3.3.** *Suppose  $\mathcal{M} = (M, <, \dots)$  is a linear order and  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ . If  $X \subseteq M$  is infinite and  $<$ -densely ordered then there is an infinite  $(\triangleleft \upharpoonright X)$ -interval  $I \subseteq X$  which is  $\triangleleft$ -densely ordered.*

*Proof.* Assume not, then any infinite  $(\triangleleft \upharpoonright X)$ -interval of  $X$  contains a point with a  $\triangleleft$ -successor. Choose such a point  $a_0$  and call its successor  $b_0$ . Now, given such a pair  $a_i, b_i$ , the  $<$ -interval between  $a_i$  and  $b_i$  is definable and intersects  $X$  infinitely, so it contains some infinite  $(\triangleleft \upharpoonright X)$ -interval. We can therefore pick  $a_{i+1}, b_{i+1}$  in the  $(\triangleleft \upharpoonright X)$ -interval between  $a_i$  and  $b_i$  so that  $b_{i+1}$  is the  $\triangleleft$ -successor of  $a_{i+1}$ .

By switching  $a_i$  and  $b_i$  as necessary, we have  $a_0 < a_1 < a_2 < a_3 < \dots < b_3 < b_2 < b_1 < b_0$  where  $b_i$  may be either the  $\triangleleft$ -predecessor of  $a_i$  (if we switched them) or  $\triangleleft$ -successor of  $a_i$  (if we didn't switch them). By the pigeonhole principle, one of these two cases happens infinitely often so (reversing  $\triangleleft$  if necessary) we can once again assume that  $b_i$  is the  $\triangleleft$ -successor of  $a_i$  for each  $i$ . Applying Ramsey's theorem, we can also assume that  $a_0, a_1, a_2, \dots$  is  $\triangleleft$ -monotone. Since  $b_i$  is the  $\triangleleft$ -successor of  $a_i$  for each  $i$ , if  $a_0, a_1, a_2, \dots$  is  $\triangleleft$ -increasing, then  $a_0, b_0, a_1, b_1, a_2, b_2, \dots$  is  $\triangleleft$ -increasing as well, and if  $a_0, a_1, a_2, \dots$  is  $\triangleleft$ -decreasing, then  $b_0, a_0, b_1, a_1, b_2, a_2, \dots$  is  $\triangleleft$ -decreasing as well.

Either way, we have found an infinite  $<$ -zigzag in  $\triangleleft$ , which is a contradiction by Lemma 5.3.2.  $\square$

This combines with our results on discrete linear orders to show that dense and discrete are truly a dichotomy on convexly orderable linear orders.

**Corollary 5.3.4.** *If  $\mathcal{M} = (M, <, \dots)$  is a convexly orderable linear order, and  $M$  is  $<$ -discretely ordered, then for any definable order  $\triangleleft$  on  $M$ , we can definably partition  $M$  into finitely many  $\triangleleft$ -discretely ordered sets.*  $\square$

**Corollary 5.3.5.** *If  $\mathcal{M} = (M, <, \dots)$  is a  $\text{CO}_n$  pseudo-discrete order for some  $n > 1$ , then there is some discrete convex ordering of  $\mathcal{M}$  so that  $(\mathcal{M}, \triangleleft)$  is  $\text{CO}_{n-1}$ .*

*Proof.* By Corollary 5.1.21 there are some order  $\triangleleft$  and  $x \in M$  so that  $(M, \triangleleft)$  is  $\text{CO}_{n-1}$ ,  $(-\infty, x]_{\triangleleft}$  is  $\triangleleft$ -discretely ordered, and  $(x, \infty)_{\triangleleft}$  is  $\triangleleft$ -densely ordered without endpoints. If  $(x, \infty)_{\triangleleft}$  isn't empty, then in fact it is an infinite definable  $\triangleleft$ -densely ordered set. By Corollary 5.3.4,  $(x, \infty)_{\triangleleft}$  cannot be  $<$ -pseudo-discrete, which contradicts Corollary 5.2.10.  $\square$

Now, we show the converse, that if the convex ordering is dense, then this can be traced back to the original order:

**Definition 5.3.6.** We say that orders  $<$  and  $\triangleleft$  are **compatible** on a set  $X$  if either for every  $a, b \in X$  we have  $a < b$  if and only if  $a \triangleleft b$ , or for every  $a, b \in X$  we have  $a < b$  if and only if  $b \triangleleft a$ .

**Lemma 5.3.7.** *Suppose  $\mathcal{M} = (M, <, \dots)$  is a linear order, and  $\triangleleft$  is a convex order of  $\mathcal{M}$ . If there is some infinite  $X \subseteq M$  which is  $\triangleleft$ -densely ordered, then there is some infinite  $Y \subseteq X$  so that  $Y$  is  $\triangleleft$ -densely ordered and  $<$  and  $\triangleleft$  are compatible on  $Y$ .*

*Proof.* Since  $(X, \triangleleft)$  is infinite and densely ordered by  $\triangleleft$ , we can choose  $x_\sigma \in X$  for  $\sigma \in 2^{<\omega}$  so that when  $\tau \succeq \sigma \frown 0$  we have  $x_\tau \triangleleft x_\sigma$  and when  $\tau \succeq \sigma \frown 1$  we have  $x_\sigma \triangleleft x_\tau$ . By Milliken's tree theorem, we can pass to a subtree so that if we fix  $i \in 2$ , then either  $x_{\sigma \frown i \frown \tau} < x_\sigma$  for every  $\sigma, \tau \in 2^{<\omega}$ , or  $x_{\sigma \frown i \frown \tau} > x_\sigma$  for every  $\sigma, \tau \in 2^{<\omega}$ .

In the case where  $x_0 < x_\varepsilon < x_1$ , it follows that  $x_\tau < x_\sigma$  for  $\tau \succeq \sigma \frown 0$ , and  $x_\sigma < x_\tau$  for  $\tau \succeq \sigma \frown 1$ . Thus  $<$  and  $\triangleleft$  are the same on  $Y = \{x_\sigma \mid \sigma \in 2^{<\omega}\}$ , and we are done. Analogously, in the case where  $x_1 < x_\varepsilon < x_0$ , then similarly we can show that  $<$  and  $\triangleleft$  are the reverse of each other on  $Y$ .

Now, if  $x_0, x_1 < x_\varepsilon$ , then let  $A$  be the set of  $\sigma$  so that for any  $\tau \succeq \sigma \frown 1$ , there is some  $\eta \succeq \sigma \frown 0$  such that  $x_\eta < x_\tau$ . There are two possibilities:

1. There is a sequence  $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \dots$  disjoint from  $A$  with  $\sigma_{i+1} \succeq \sigma_i \frown 0$ . By definition of  $A$ , we can choose  $\tau_i \succeq \sigma_i \frown 1$  for each  $i \in \omega$  so that  $x_{\tau_i} < x_{\sigma_j}$  for each  $j \in \omega$ . By definition of the  $x_\sigma$ , we see that  $x_{\tau_0} \triangleright x_{\sigma_0} \triangleright x_{\tau_1} \triangleright x_{\sigma_1} \triangleright x_{\tau_2} \triangleright x_{\sigma_2} \triangleright \dots$ , so this is an infinite  $<$ -zigzag in  $\triangleleft$ , which contradicts Lemma 5.3.2.
2. If there is no such sequence, it is because  $A$  contains an entire cone in  $2^{<\omega}$ . We may assume that  $A = 2^{<\omega}$ . For any  $k$ , we can choose  $\tau_0, \tau_1, \tau_2, \dots, \tau_{N-1}$  so that  $\tau_i \succ 1^i \frown 0$  and so that we have  $x_{\tau_0} < x_{\tau_1} < x_{\tau_2} < \dots < x_{\tau_{N-1}} < x_{1^{N-1}} < x_{1^{N-2}} < \dots < x_1 < x_\varepsilon$ . By definition of the  $x_\sigma$ , we have  $x_{\tau_0} \triangleleft x_\varepsilon \triangleleft x_{\tau_1} \triangleleft x_1 \triangleleft x_{\tau_2} \triangleleft x_{1^2} \triangleleft \dots \triangleleft x_{\tau_{N-1}} \triangleleft x_{1^{N-1}}$ , so this is a  $<$ -zigzag of length  $2N$  in  $\triangleleft$ . Since  $N$  is arbitrary, this contradicts Lemma 5.3.2.  $\square$

Note, of course, that  $X$  and  $Y$  are not necessarily definable (in  $\mathcal{M}$  or in  $(\mathcal{M}, \triangleleft)$ ) so either or both could be subsets of a set which is  $<$ -or  $\triangleleft$ -discretely ordered. We must therefore refine these results to find more useful compatible sets. Our final reflection principle for dense orders will be as follows:

**Theorem 5.3.11.** *Suppose  $\mathcal{M} = (M, <, \dots)$  is a linear order, and  $\triangleleft$  is a convex order of  $\mathcal{M}$ . Then for  $X \subseteq M$  which is  $<$ -densely ordered there is some  $N$  so that for any  $(< \upharpoonright X)$ -interval  $I$ , there are a  $(< \upharpoonright X)$ -interval  $J \subseteq I$ , and a partition  $Y_0, Y_1, Y_2, \dots, Y_{N-1}$  of  $J$  into sets  $Y_i$  on which  $<$  and  $\triangleleft$  are compatible so that each  $Y_i$  is  $<$ -dense in  $J$  and  $(\triangleleft \upharpoonright X)$ -convex.*

We begin by making the compatible sets  $\triangleleft$ -dense in a  $\triangleleft$ -interval:

**Lemma 5.3.8.** *Suppose  $\mathcal{M} = (M, <, \dots)$  is a linear order, and  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ . Then for any  $<$ -densely ordered infinite  $X \subseteq M$  there are  $I \subseteq X$ , an infinite  $(\triangleleft \upharpoonright X)$ -interval which is  $\triangleleft$ -densely ordered, and  $Y \subseteq I$  which is  $(\triangleleft \upharpoonright X)$ -dense in  $I$  so that  $\triangleleft$  and  $<$  are compatible on  $Y$ .*

*Proof.* Assume toward a contradiction that this is false. By Lemma 5.3.3, there is a  $(\triangleleft \upharpoonright X)$ -interval  $I_0$  which is  $\triangleleft$ -densely ordered. We build  $a_i, b_i, c_i, I_i$ , and  $Y_i$  for each  $i \in \omega$  so that:

1.  $I_i$  is a  $\triangleleft$ -densely ordered  $\triangleleft \upharpoonright X$ -interval,
2.  $Y_i \subseteq I_i$  is  $\triangleleft$ -densely ordered,
3. the orders  $\triangleleft$  and  $<$  are compatible on  $Y_i$ ,
4.  $a_i, b_i, c_i \in Y_i$  with  $a_i \triangleleft c_i \triangleleft b_i$ , and
5. either  $I_{i+1} \subseteq (c_i, b_i)_{\triangleleft}$  and  $I_{i+1}$  is contained in the  $<$ -interval between  $a_i$  and  $c_i$ , or  $I_{i+1} \subseteq (a_i, c_i)_{\triangleleft}$  and  $I_{i+1}$  is contained in the  $<$ -interval between  $c_i$  and  $b_i$ .

Given  $I_i$  there is, by Lemma 5.3.7 some  $Y_i \subseteq I_i$  which is  $\triangleleft$ -densely ordered and on which  $<$  and  $\triangleleft$  are compatible. We choose a maximal such  $Y_i$ . The next step of the construction is written assuming  $<$  and  $\triangleleft$  are the same on  $Y_i$ . If not, replace  $<$  with  $>$  in the construction. The rest is the same.

Choose  $a_i \triangleleft b_i$  in  $Y_i$ . The interval  $(a_i, b_i)_{<}$  contains  $Y_i \cap (a_i, b_i)_{\triangleleft}$ , which is infinite and  $\triangleleft$ -densely ordered, so since  $(a_i, b_i)_{<}$  has finitely many  $(\triangleleft \upharpoonright X)$ -convex components, it contains some infinite  $(\triangleleft \upharpoonright X)$ -interval  $J \subseteq (a_i, b_i)_{\triangleleft}$ . Since we assumed that the lemma is false, we may assume without loss of generality that  $J \cap Y_i = \emptyset$ . By Lemma 5.3.7 we can choose some  $Z \subseteq J$ , densely ordered by  $\triangleleft$ , on which  $\triangleleft$  and  $<$  are compatible. Choose  $x \in Z$  which is not a  $<$ -endpoint. By the maximality of  $Y_i$ ,  $<$  and  $\triangleleft$  are not compatible on  $Y_i \cup \{x\}$ , so there is some  $c_i \in Y_i$  so that either  $c_i \triangleleft x$  and  $c_i > x$ , or  $c_i \triangleright x$  and  $c_i < x$ .

We consider the case where  $c_i \triangleleft x$  (the other case is analogous). We have  $c_i > z$  for any  $z \in Z \cap (a_i, x)_{<}$  by compatibility of  $\triangleleft$  and  $<$  on  $Z$ , and this set is infinite since  $x$  is not a  $<$ -endpoint of  $Z$ . On the other hand  $c_i \triangleleft z$  for any  $z \in J$  since  $J$  is  $(\triangleleft \upharpoonright X)$ -convex and does not contain  $c_i$ . So,  $(a_i, c_i)_{<} \supseteq (a_i, x)_{<}$  contains an infinite  $(\triangleleft \upharpoonright X)$ -interval  $I_{i+1} \subseteq (c_i, b_i)_{\triangleleft}$ .

Now, by the pigeonhole principle, we can assume that either  $<$  and  $\triangleleft$  match on  $Y_i$  always. If not, replace  $<$  with  $>$  throughout the rest of the proof, but the argument is otherwise the same. We may also assume that either  $I_{i+1} \subseteq (a_i, c_i)_{<} \cap (c_i, b_i)_{\triangleleft}$  always, in which case  $a_0 \triangleleft c_0 \triangleleft a_1 \triangleleft c_1 \triangleleft \dots$  is an infinite zigzag, or that  $I_{i+1} \subseteq (c_i, b_i)_{<} \cap (a_i, c_i)_{\triangleleft}$  always, in which case  $b_0 \triangleright c_0 \triangleright b_1 \triangleright c_1 \triangleright \dots$  is an infinite zigzag. Either case contradicts Lemma 5.3.2.  $\square$

One thing worth noting with Lemma 5.3.8 is that given a uniformly definable collection of sets ( $<$ -intervals), we have found a set in which these sets make some sort of uniform sense in terms of  $\triangleleft$ . However,  $\triangleleft$  is a convex ordering, so we expect definable structure to correspond to  $\triangleleft$ -convex sets. So, our next step is to show that in fact we can require our compatible set to be  $\triangleleft$ -convex as well:

**Lemma 5.3.9.** *Suppose  $\mathcal{M} = (M, <, \dots)$  is a linear order, and  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ . Then for any  $<$ -densely ordered infinite  $X \subseteq M$  there is some infinite  $(\triangleleft \upharpoonright X)$ -densely ordered  $(\triangleleft \upharpoonright X)$ -interval in  $M$  on which  $<$  and  $\triangleleft$  are compatible.*

*Proof.* Assume toward a contradiction that there is no such  $(\triangleleft \upharpoonright X)$ -interval. By Lemma 5.3.8, there are some infinite  $(\triangleleft \upharpoonright X)$ -interval  $I \subseteq X$  and some  $Y$  which is  $\triangleleft$ -dense in  $I$  so that  $<$  and  $\triangleleft$  are compatible on  $Y$ . Let  $Y$  be maximal with this property. By assumption,  $Y$  cannot contain a  $(\triangleleft \upharpoonright X)$ -interval, so  $Y$  and  $I \setminus Y$  are both  $\triangleleft$ -dense in  $I$ .

Let  $N$  be the maximum number of  $\triangleleft$ -convex components of a  $<$ -interval. Then for any sequence of elements  $y_0 < y_1 < y_2 < y_3 < \dots < y_N$  in  $Y$ , there must be some  $i$  so that  $y_i$  and  $y_{i+1}$  are in the same  $(\triangleleft \upharpoonright X)$ -convex component of  $[y_0, y_N]_{<}$ . Since  $I \setminus Y$  intersects the  $(\triangleleft \upharpoonright X)$ -interval between  $y_i$  and  $y_{i+1}$ , it follows that  $[y_0, y_N]_{<}$  contains some point of  $I \setminus Y$  which is  $\triangleleft$ -between  $y_0$  and  $y_N$ . So, we can choose  $Z \subseteq I \setminus Y$  so that for  $y, y' \in Y$  with  $y < y'$ , there is some  $z \in Z$  so that  $y < z < y'$  and  $z$  is  $\triangleleft$ -between  $y$  and  $y'$ . We can in fact choose  $Z$  so that if  $z, z' \in Z$  with  $z < z'$  there is some  $y \in Y$  so that  $z < y < z'$ . (Say that  $z \sim z'$  if there is no such  $y$ , and choose a single representative of each  $\sim$ -class). It follows that  $\triangleleft$  and  $<$  are compatible on  $Z$ . Furthermore  $<$  and  $\triangleleft$  are the same on  $Y$  if and only if they are the same on  $Z$ .

We assume that  $<$  and  $\triangleleft$  are in fact the same on  $Y$  (otherwise, replace  $<$  with  $>$  in the rest of the argument). Now, fix  $a \in Z$ . If  $b \in Y$  with  $a \triangleleft b$ , then  $(a, b)_{\triangleleft}$  intersects  $Y$  infinitely, so it intersects  $(-\infty, b)_{<}$  infinitely. Since  $\triangleleft$  is a convex ordering and  $(-\infty, b)_{<}$  is definable, it follows that some  $\triangleleft$ -subinterval of  $(a, b)_{\triangleleft}$  is contained in  $(-\infty, b)_{<}$ . Then there is  $c \in Z \cap (-\infty, b)_{<} \cap (a, b)_{\triangleleft}$ . In particular,  $a \triangleleft c$  implies

$a < c < b$ . By an analogous argument, if  $b \in Y$  with  $a \triangleleft b$ , then  $b < a$ . In other words,  $\triangleleft$  and  $<$  are compatible on  $Y \cup \{b\}$ , which contradicts our choice of  $Y$ .  $\square$

Now, we can infer that such a compatible set should be somewhere dense in the original structure too:

**Lemma 5.3.10.** *Suppose  $\mathcal{M} = (M, <, \dots)$  is a linear order, and  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ . Then for any  $<$ -densely ordered infinite  $X \subseteq M$  there are some infinite  $(< \upharpoonright X)$ -interval  $I$ , and some  $(\triangleleft \upharpoonright X)$ -interval  $J$  so that  $<$  and  $\triangleleft$  are compatible on  $J$  and  $J$  is  $<$ -dense in  $I$ .*

*Proof.* We assume toward a contradiction that there are no such  $I$  and  $J$ . Let  $I_0 = X$ . Given  $I_i$  an infinite  $(< \upharpoonright X)$ -interval, we have a finite coloring of  $I_i$  by  $(\triangleleft \upharpoonright X)$ -convex components. Since  $I_i$  is  $<$ -densely ordered, it follows that one of the colors is somewhere dense. So, possibly replacing  $I_i$  with a  $(< \upharpoonright X)$ -subinterval as necessary, we may assume that there is a  $(\triangleleft \upharpoonright X)$ -interval  $K \subseteq I_i$  which is  $<$ -densely ordered.

By Lemma 5.3.9, there is some infinite, densely ordered  $(\triangleleft \upharpoonright K)$ -subinterval  $J_i$  of  $K$  so that  $<$  and  $\triangleleft$  are compatible on  $J_i$  (and of course  $J_i$  is also a  $(\triangleleft \upharpoonright X)$ -interval). We choose  $a_i, b_i \in J_i$  with  $a_i < b_i$ . By assumption, we may let  $I_{i+1}$  be any  $(< \upharpoonright X)$ -subinterval of  $(a_i, b_i)_{< \upharpoonright X}$  which does not intersect  $J_i$ .

From this construction, we have  $a_i < b_j$  for all  $i, j \in \omega$ , and since the  $(\triangleleft \upharpoonright X)$ -interval between  $a_i$  and  $b_i$  is contained in  $J_i$  we know that it doesn't contain  $a_j$  or  $b_j$  for  $i \neq j$ . In other words either  $a_i, b_i \triangleleft a_j, b_j$  or  $a_i, b_i \triangleright a_j, b_j$ . By the pigeonhole principle we may assume that either  $a_0, b_0, a_1, b_1, \dots$  is  $\triangleleft$ -monotone, or that  $b_0, a_0, b_1, a_1, \dots$  is  $\triangleleft$ -monotone. Either way, this is an infinite zigzag, which contradicts Lemma 5.3.2.  $\square$

This finally allows us to prove our main reflection principle for densely ordered structures:

**Theorem 5.3.11.** *Suppose  $\mathcal{M} = (M, <, \dots)$  is a linear order, and  $\triangleleft$  is a convex order of  $\mathcal{M}$ . Then for  $X \subseteq M$  which is  $<$ -densely ordered there is some  $N$  so that for any  $(< \upharpoonright X)$ -interval  $I$ , there are a  $(< \upharpoonright X)$ -interval  $J \subseteq I$ , and a partition  $Y_0, Y_1, Y_2, \dots, Y_{N-1}$  of  $J$  into sets  $Y_i$  on which  $<$  and  $\triangleleft$  are compatible so that each  $Y_i$  is  $<$ -dense in  $J$  and  $(\triangleleft \upharpoonright X)$ -convex.*

*Proof.* We can build a sequence  $I_0 \supsetneq I_1 \supsetneq \dots$  of  $(< \upharpoonright X)$ -intervals and  $Y_0, Y_1, Y_2, Y_3, \dots$  so that the  $Y_i$  are disjoint  $(\triangleleft \upharpoonright X)$ -intervals on which  $\triangleleft$  and  $<$  match, and  $Y_i$  is  $<$ -dense in  $I_i$  for each  $i$ . We begin by any  $I_0$  and  $Y_0$  with these properties. There is some such by Lemma 5.3.10. Now, assuming that we have  $I_i$  and  $Y_i$  for  $i < n$ . If  $\bigcup_{i=0}^{n-1} Y_i$  contains any  $<$ -subinterval of  $I_{n-1}$ , then they satisfy the theorem so we are done. Otherwise  $A = I_{n-1} \setminus Y_0 \cup Y_1 \cup Y_2 \cup \dots \cup Y_{n-1}$ , is  $<$ -dense in  $I_{n-1}$ , so we apply Lemma 5.3.10 to get an infinite  $(< \upharpoonright A)$ -interval  $I$  and a  $(\triangleleft \upharpoonright A)$ -interval  $Y_n \subseteq I$  so that  $Y_n$  is  $<$ -dense in  $I$  and  $<$  and  $\triangleleft$  match on  $Y_n$ . Since  $Y_n$  is an infinite  $(\triangleleft \upharpoonright A)$ -interval and  $A$  has at most  $n+1$  distinct  $(\triangleleft \upharpoonright X)$ -convex components, there is some  $(\triangleleft \upharpoonright X)$ -convex subset of  $Y_n$ . In particular, we may assume that  $Y_n$  is in fact a  $(\triangleleft \upharpoonright X)$ -interval. Let

$I_n$  be the  $<$ -convex hull of  $I$ . Since  $I$  is a  $(< \upharpoonright A)$ -interval and  $A$  is  $(< \upharpoonright X)$ -dense in  $I_{n-1}$ , it follows that  $I_n$  is a  $(< \upharpoonright X)$ -subinterval of  $I_{n-1}$ , and since  $Y_n$  is  $<$ -dense in  $I$  and  $I$  is  $<$ -dense in  $I_n$ , it follows that  $Y_n$  is  $<$ -dense in  $I_n$  as desired.

Now, if the construction continues infinitely, we derive a contradiction, since for  $a, b \in Y_N$  with  $a < b$ , only  $(\triangleleft \upharpoonright X)$ -interior points of  $Y_i$  are in  $(a, b)_{<} \cap Y_i$  for each  $0 \leq i \leq N$ , and it follows that  $Y_i$  contains a  $(\triangleleft \upharpoonright X)$ -convex component of  $(a, b)_{<}$  for  $0 \leq i \leq N$ . Thus we can find for any  $N$  a  $(< \upharpoonright X)$ -interval with more than  $N$  distinct  $(\triangleleft \upharpoonright N)$ -convex components, which contradicts the assertion that  $\triangleleft$  is a convex ordering. Note, in fact, that the number of  $Y_i$  in our decomposition is therefore no greater than the number of  $\triangleleft$ -convex components of a  $<$ -interval, which is bounded.  $\square$

We use this theorem in Chapter 6 to give a partial characterization of the definable sets in a convexly orderable linear order. For this reason we make the following definition:

**Definition 5.3.12.** Let  $\mathcal{M} = (M, <, \dots)$  be a linear order convexly ordered by  $\triangleleft$ . Then we say that the sets  $Y_0, Y_1, \dots, Y_{N-1}$  are a **match partition** of  $X$  if the  $Y_i$  partition  $X$ , each  $Y_i$  is  $<$ -dense in  $X$ , each  $Y_i$  is  $\triangleleft$ -convex, and for each  $i$ , the orders  $<$  and  $\triangleleft$  are compatible on  $Y_i$ .

Theorem 5.3.11 then, can be thought of as saying that every infinite densely ordered set in a convexly orderable structure has a  $<$ -subinterval with a match partition. The main use of a match partition is that in a match partitioned interval, the  $\triangleleft$ -convex sets are somewhere  $<$ -dense. We have the following:

**Lemma 5.3.13.** *Suppose that  $\mathcal{M} = (M, <, \dots)$  is a dense linear order and  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ . If  $Y_0, Y_1, Y_2, \dots, Y_{N-1}$  is a match partition of some  $<$ -interval  $I$ , then for any definable set  $X$  there is some infinite  $<$ -interval  $J \subseteq I$  so that for each  $i$  either  $Y_i \cap J \subseteq X$  or  $Y_i \cap J \cap X = \emptyset$ .*

*Proof.* Fix  $i$ . We show that we can satisfy the conclusion for  $Y_i$ . It follows that we can do so for all  $Y_j$  by iterating finitely many times (just replace  $I$  with  $J$  and use the same argument). There are two possibilities. The first is that  $X$  intersects  $Y_i$  infinitely, in which case  $X \cap Y_i$  contains a  $\triangleleft$ -interval since both  $X$  and  $Y_i$  have finitely many  $\triangleleft$ -convex components. In this case, we choose  $a, b \in I$  with  $a < b$  so that the  $\triangleleft$ -interval between  $a$  and  $b$  is contained in  $X \cap Y_i$ , and let  $J = (a, b)_{<}$  so that  $J \cap X \supseteq Y_i$ . The second possibility is that  $X$  only intersects  $Y_i$  finitely, in which case, since  $Y_i$  is infinite and  $\triangleleft$ -densely ordered, we can choose  $a, b \in Y_i$  so that  $a < b$  and no element of  $X$  is in the  $\triangleleft$ -interval between them. Then we let  $J = (a, b)_{<}$  so that  $J \cap Y_i \cap X = \emptyset$ .  $\square$

Of course, match partitioned sets are not definable *a priori*, but since we can detect  $<$ -dense definable sets, we have the following (weaker) versions of Theorem 5.3.11 in only the language of  $\mathcal{M}$ :



**Lemma 5.3.14.** *Suppose that  $\mathcal{M} = (M, <, \dots)$  is a dense linear order,  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ , and  $I$  is an infinite  $<$ -interval match partitioned by  $Y_0, Y_1, \dots, Y_{N-1}$ . Then for any definable family  $\mathcal{F}$  of subsets of  $M$ , there is some  $n \in \omega$  so that if  $X \in \mathcal{F}$  and  $a_0 < a_1 < a_2 < \dots < a_n$  is a sequence of elements of  $X \cap J$ , then for some  $i < n$ , and  $\ell < N$ , we have  $X \supseteq Y_\ell \cap (a_i, a_{i+1})_{<}$ .*

*Proof.* Given a definable family  $\mathcal{F}$  of subsets of  $M$ , there is some  $k$  so that each  $X \in \mathcal{F}$  has at most  $k$  distinct  $\triangleleft$ -convex components. Let  $n = kN$ . Then, if  $X \in \mathcal{F}$  and  $a_0 < a_1 < a_2 < \dots < a_n$  is a sequence of elements of  $X \cap J$ , then by the pigeonhole principle, there are  $i < j < n$  and  $\ell < N$  so that  $a_i$  and  $a_j$  are in the same  $\triangleleft$ -convex component of  $X$  and  $a_i, a_j \in Y_\ell$ , so the  $\triangleleft$ -interval between  $a_i$  and  $a_j$  is infinite, and contained in  $X \cap Y_\ell$ , in particular this set is exactly  $(a_i, a_j)_{<} \cap Y_\ell$ .  $\square$

**Porism 5.3.15.** *If  $\mathcal{M} = (M, <, \dots)$  is a convexly orderable dense linear order, then for every infinite  $<$ -interval  $I \subseteq M$ , there is an infinite  $<$ -interval  $J \subseteq I$  so that for any definable family  $\mathcal{F}$  of subsets of  $M$ , there is some  $n \in \omega$  (which depends only on  $\mathcal{F}$ , not on  $I$ ) so that if  $X \in \mathcal{F}$  and  $a_0 < a_1 < a_2 < \dots < a_n$  is a sequence of elements of  $X \cap J$ , then  $X$  is dense in  $(a_i, a_{i+1})_{<}$  for some  $i < n$ .*

*Proof.* By Theorem 5.3.11  $I$  contains some match partitioned  $<$ -interval  $J$ , so we can apply the argument of Lemma 5.3.14. We simply note that since the number of  $\triangleleft$ -convex components of a  $<$ -interval is finitely bounded, so is the number of sets in a match partition. So,  $N$  and  $k$  in Lemma 5.3.14 is bounded independently of the match partitioned  $<$ -interval we choose, which means that  $n = Nk$  is as well.  $\square$

**Lemma 5.3.16.** *Suppose that  $\mathcal{M} = (M, <, \dots)$  is a dense linear order,  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ ,  $I$  is a nonempty  $<$ -open interval, and  $Y_0, Y_1, \dots, Y_{N-1}$  is a match partition of  $I$ . Then if  $a \in I$  is a  $<$ -lower-limit of some definable  $X \subseteq M$ , it follows that there are  $b > a$  and  $\ell < N$  so that  $X \supseteq (a, b)_{<} \cap Y_\ell$ .*

*Proof.* Assume that  $a \in I$  is a  $<$ -lower-limit of a definable set  $X \subseteq M$  but  $X \cap Y_\ell$ . Then  $a$  is a  $<$ -lower-limit of  $X \cap Y_\ell$  for some  $\ell$ . Then we build a  $<$ -descending sequence  $b_0 > c_0 > b_1 > c_1 > b_2 > c_2 > \dots > a$  as follows: Begin with any  $b_0 \in Y_\ell \cap (a, \infty)_{<}$ . Given  $b_i$ , if  $X \supseteq (a, b_i)_{<} \cap Y_\ell$  then we are done. Otherwise, choose some  $c_i \in (a, b_i)_{<} \cap Y_\ell \setminus X$ . Given  $c_i$ , since  $a$  is a  $<$ -lower-limit of  $X \cap Y_\ell$ , we can choose  $b_{i+1} \in (a, c_i)_{<} \cap Y_\ell \cap X$ . Since  $\triangleleft$  and  $<$  match on  $Y_\ell \cap (a, \infty)_{<}$ , it follows that this sequence is  $\triangleleft$ -monotone, but it alternates in and out of the definable set  $X$ , which contradicts the assertion that  $\triangleleft$  convexly orders  $\mathcal{M}$ .  $\square$

**Corollary 5.3.17.** *If  $\mathcal{M} = (M, <, \dots)$  is a convexly orderable dense linear order, then for any  $<$ -interval  $I \subseteq M$  there is a  $<$ -interval  $J \subseteq I$  so that for every definable  $X$  and any  $<$ -lower-limit  $a$  of  $X$  contained in  $J$ , there is some  $b > a$  so that  $X$  is  $<$ -dense in  $(a, b)_{<}$ .  $\square$*

## Chapter 6

# Monotonicity in Ordered Convexly Orderable Structures

### 6.1 Introduction and Notation

As remarked in Chapter 4, convex orderability intentionally mimics the definition of a weakly o-minimal theory. In fact (Definition 4.2.6), a structure with weakly o-minimal theory is exactly one with a distinguished definable convex ordering  $<$ . One of the most important features of weak o-minimality is that it admits a cell decomposition theorem:

**Theorem 6.1.1 (Macpherson [15]).** *Let  $\mathcal{M}$  be a model of a weakly o-minimal theory, let  $d > 0$  and let  $X$  be a definable subset of  $M^d$ . Then there is a definable partition of  $X$  into cells  $C_0, C_1, \dots, C_{M-1}$  so that for  $0 \leq i \leq M-1$  there exist  $d_i \leq d$  and a projection  $\pi_i: M^d \rightarrow M^{d_i}$  so that  $\pi_i(C_i)$  is open in  $M^{d_i}$  and  $\pi_i: C_i \rightarrow \pi_i(C_i)$  is a homeomorphism.*

Where a *cell* is defined as follows:

**Definition 6.1.2.** A **weakly o-minimal cell** in  $\mathcal{M}$  is a subset of  $M^d$  where  $d > 0$  defined as follows:

1. A 1-cell is a definable  $<$ -convex subset of  $M$ .
2. A set  $X \subseteq M^{d+1}$  is a  $(d+1)$ -cell if there is a  $d$ -cell  $Y \subseteq M^d$  so that  $X_z = \emptyset$  for any  $z \notin Y$  and  $(X_y)_{y \in Y}$  is a definable family of nonempty  $<$ -convex sets.

These results are used in [15] to show that topological dimension is well behaved in weakly o-minimal theories. The topological dimension of a finite union of definable sets is the maximum of the dimensions of the individual sets, it is preserved by definable bijections, and if algebraic closure has the exchange property, then algebraic dimension is the same as topological dimension. Key to these results is showing that weakly o-minimal theories have monotonicity:

**Definition 6.1.3 (Aref'ev [4]).** We say that  $\mathcal{M} = (M, <, \dots)$  **has monotonicity** if for every definable function  $f: D \subseteq M \rightarrow \overline{M}$ , there is some partition of  $D = \text{Dom}(f)$  into definable sets  $X$  and  $I_1, \dots, I_m$  so that  $X$  is finite, each  $I_i$  is  $<$ -convex, and for  $x \in I_i$ , and for fixed  $i$ , either  $f$  is strictly  $<$ -increasing in a  $<$ -neighborhood of each  $x \in I_i$ , or  $f$  is strictly  $<$ -decreasing in a  $<$ -neighborhood of each  $x \in I_i$ , or  $f$  is constant in a  $<$ -neighborhood of each  $x \in I_i$ , where the case depends only on  $i$ , not on  $x$ .

More recently, Simon and Walshberg [18] proved that if a dp-minimal structure has a uniform structure (in the topological sense), then definable multi-functions are continuous almost everywhere. They employ this result to show that in such structures topological dimension, algebraic dimension, and dp-rank all agree on definable sets and are definable in families.

Convexly orderable structures need not admit cell-decomposition. Our goal in this chapter is more modest: to give our own approximation to the monotonicity theorem for convexly orderable structures.

Throughout the chapter, we will assume that  $\mathcal{M} = (M, <, \dots)$  is a dense linear order and that  $\triangleleft$  is a convex ordering of  $\mathcal{M}$ . By a  $<$ -open box, we mean a non-empty set of the form  $U = \prod_{i=0}^{d-1} (a_i, b_i)_{<}$ . We call  $d$  the dimension of  $U$  (this is unambiguous as such a set has the same dp-rank, topological dimension, and algebraic dimension). We say that  $\prod_{i=0}^{d-1} I_i$  is match partitioned if each  $I_i$  is match partitioned. By a **grid** we mean a definable set of the form  $G = \prod_{i=0}^{d-1} X_i$  where each  $X_i \subseteq M$ . If  $X_i$  has at least  $k$  elements for each  $i < d$ , then we say  $\prod_{i=0}^{d-1} X_i$  is a  $k \times d$ -**grid**.

## 6.2 Density and Grids

Our main goal in this section is to show that graphs of definable functions in  $\mathcal{M}$  cannot be  $<$ -dense anywhere. We do so by showing that a set which is  $<$ -dense in some  $<$ -open box must contain a grid  $<$ -dense in some  $<$ -open box. In particular, the graph of a function cannot contain a  $2 \times d$ -grid, and so cannot be  $<$ -dense in any  $<$ -open box. We rely on the following pigeonhole principle for grids:

**Lemma 6.2.1 (Pigeonhole principle for grids).** *For any  $k, \ell, d \in \omega$  there is some  $n_{k, \ell, d} \in \omega$  so that any  $\ell$ -coloring of an  $n_{k, \ell, d} \times d$ -grid contains a monochrome  $k \times d$ -subgrid.*

*Proof.* In the case where  $d = 1$ , this is the usual pigeonhole principle. Now, if the statement holds when  $d = r$ , we simply need to show that it holds for  $d = r + 1$  as well. There are  $\binom{n_{k, \ell, r}}{k}^r$  distinct  $k \times r$ -subgrids of an  $(n_{k, \ell, r}) \times r$ -grid, so  $n_{k, \ell, r+1} = k \ell \binom{n_{k, \ell, r}}{k}^r$ , should work. In any  $n_{k, \ell, r+1} \times r+1$  grid  $G = A \times B$  where  $B \subseteq M$ , we can pick an  $n_{k, \ell, r} \times r$ -grid  $C \subseteq A$ . Fix any  $\ell$ -coloring of  $G$ . Then for each  $b \in B$ , there is by inductive hypothesis some  $C_b \subseteq C$  so that  $C_b \times \{b\}$  is monochrome. By the pigeonhole principle, we can find  $B_0 \subseteq B$  with  $|B_0| = k \ell$  so that  $C_b = C_{b'} = D$  for any  $b, b' \in B_0$ . Again by the pigeonhole principle, we can find

$B_1 \subseteq B_0$  with  $|B_1| = k$  so that  $D \times \{b\}$  and  $D \times \{b'\}$  are the same color for  $b, b' \in B_1$ . But then  $D \times B_1$  is the desired  $k^{\times(r+1)}$ -grid.  $\square$

As we're going to be parsing sets up into  $\prec$ -grids, the following will also be useful:

**Lemma 6.2.2.** *Let  $N$  be as in Theorem 5.3.11. If  $A_0, A_1, A_2, \dots, A_{N^d}$  are grids  $\prec$ -dense in the same  $\prec$ -open box  $U \subseteq M^d$ , then they are not pairwise disjoint.*

*Proof.* We can assume without loss of generality that  $U = \prod_{i=0}^{d-1} I_i$  where  $I_i$  is match partitioned by  $Y_{i,0}, Y_{i,1}, \dots, Y_{i,n_i}$  and  $n_i < N$ . By Lemma 5.3.13 we can further assume that each  $A_r$  is equal to a product  $\prod_{i=0}^{d-1} Y_{i,f(i)}$  for some  $f$ . Since  $f \in N^d$ , it follows by the pigeonhole principle that two of the  $A_i$  are the same.  $\square$

**Theorem 6.2.3.** *Let  $\mathcal{F}$  be a definable family of subsets of  $M^d$ . Then there is some definable family  $\mathcal{G}$  of grids so that:*

1. *if  $X \in \mathcal{F}$  is  $\prec$ -dense in the  $\prec$ -open box  $U$ , then  $X \cap U \supseteq G$  for some grid  $G \in \mathcal{G}$  which is  $\prec$ -dense in some  $\prec$ -open box  $V \subseteq U$ , and*
2. *there is some  $k \in \omega$  so that, for any  $X \in \mathcal{F}$  and match partitioned  $\prec$ -open box  $U$ , if  $X \cap U$  contains a  $k^{\times d}$ -grid  $A$ , then  $X \cap U \supseteq G$  for some grid  $G \in \mathcal{G}$  which is  $\prec$ -dense in some  $\prec$ -open box  $V \subseteq U$ .*

*Proof.* We proceed by induction on  $d$ . In the  $d = 1$  case, we simply let  $\mathcal{G}$  be the collection of sets  $X \cap U$  where  $U \subseteq M$  is a  $\prec$ -open box. If  $X$  is  $\prec$ -dense in the  $\prec$ -interval  $U$ , then certainly  $X \cap U$  is  $\prec$ -dense in  $U$ , so point 1 holds. For point 2, this is just a direct application of Porism 5.3.15.

Now, assume that the statement holds for  $d = r$ . We must show that it holds for  $d = r + 1$ . Consider the sets  $B_a^I = \{c \in M^r \mid X_c \cap I = X_a \cap I\}$  where  $I$  is a  $\prec$ -interval and  $a \in M^r$ . The sets  $B_a^I$  form a uniformly definable family  $\mathcal{F}_0$ , and so by inductive hypothesis, we can choose a corresponding family  $\mathcal{G}_0$  and  $k_0 \in \omega$  as in the statement of the theorem.

1. Suppose that  $X \in \mathcal{F}$  is  $\prec$ -dense in a match partitioned  $\prec$ -open box  $U = \prod_{i=0}^{d-1} I_i$ . Then we define  $U_1 = \prod_{i=1}^{d-1} I_i$  and let  $Y_0, Y_1, \dots, Y_{N-1}$  be the match partition of  $I_0$ . By Lemma 5.3.13, for any  $a \in U_1$  we can find some  $J \subseteq I_0$  so that:

$$P(a, J): \text{ for each } i < N \text{ either } X_a \supseteq Y_i \cap J \text{ or } X_a \cap Y_i \cap J = \emptyset.$$

By successive applications of Lemma 5.3.13, we can therefore find an  $n^{\times r}$ -grid  $G \subseteq U_1$  and an  $J \subseteq I_0$  so that, for each  $a \in G$ , the set  $X_a \cap I_0$  is nonempty and  $P(a, J)$  holds. The map which sends  $a \in G$

to the collection of  $i < N$  so that  $X_a \supseteq Y_i \cap J$  is a  $2^N$ -coloring of  $G$ . Since we can make  $n$  arbitrarily large, then by the pigeonhole principle for grids we can assume, without loss of generality, that for each  $a, a' \in G$  and  $i < N$  we have  $X_a \cap Y_i \cap J = X_{a'} \cap Y_i \cap J$ .

In particular, we can find  $a_0 \in U_1$  with  $X_{a_0} \cap I_0 \neq \emptyset$  and  $J_0 \subseteq I_0$  so that  $B_{a_0}^{J_0} \cap U_1$  contains a  $k_0^{\times r}$ -grid. Since  $U_1$  is match partitioned, there is some  $G_0 \in \mathcal{G}_0$  so that  $G_0 \subseteq B_{a_0}^{J_0} \cap U_1$  and  $G_0$  is  $<$ -dense in some  $<$ -open box  $V_0 \subseteq U_1$ .

Now, without loss of generality we can choose  $V_0$  so that either  $G_0 \supseteq V_0$  or  $V_0 \setminus G$  is  $<$ -dense in  $V_0$ . Similarly, without loss of generality we can assume that one of the following cases holds:

- (a)  $X_{a_0}$  is  $<$ -dense in  $J_0$ , in which case  $X_{a_0} \times G \subseteq X$  is  $<$ -dense in  $J_0 \times V_0$ .
- (b)  $X_{a_0}$  is not  $<$ -dense in  $J_0$ , so without loss of generality,  $X_{a_0} \cap J_0 = \emptyset$ , and for  $a \in G_0$  we have  $X_a \cap J_0 = X_{a_0} \cap J_0 = \emptyset$ . In this case, we repeat the argument replacing  $U$  with  $J_0 \times V$  to find a  $a_1$ ,  $J_1 \subseteq J_0$ ,  $G_1 \in \mathcal{G}_0$  (with  $G_1$  necessarily disjoint from  $G_0$ ), and  $V_1 \subseteq V$ . If necessary (i.e., if we still aren't in case 1a), we can repeat until we have  $a_i$ ,  $J_i$ ,  $G_i$ , and  $V_i$  for  $i < N^d$ , with the  $G_i$  disjoint. Then by Lemma 6.2.2, we conclude that there is some  $<$ -open box  $V \subseteq U_1$  so that  $\bigcup_{i=0}^{N^d-1} G_i \supseteq V$ . But then  $X \cap (J_n \times V) = \bigcup_{i=0}^{N^d-1} (X_{a_i} \times G_i) \cap (J_n \times V_n) = \emptyset$ , which contradicts the assertion that  $X$  is  $<$ -dense in  $U$ .

So, by this argument, it suffices to let  $\mathcal{G} = \{X_a \times G \mid a \in M, G \in \mathcal{G}_0\}$ .

2. Now, suppose that  $X \in \mathcal{F}$  and  $U = \prod_{i=0}^{d-1} I_i$  is a match partitioned  $<$ -open box. Let  $U_1$  be as before. By Porism 5.3.15 there is some  $n$  so that for any  $a \in U_1$  if  $b_0 < b_1 < b_2 < \dots < b_{n-1}$  are in  $X_a$ , then,  $X_a \cap (b_i, b_{i+1})_{<}$  is  $<$ -dense in  $(b_i, b_{i+1})_{<}$  for some  $i < n-1$ . So, suppose that there are  $b_0 < b_1 < \dots < b_{n-1}$  in  $I_0$  and an  $\ell^{\times r}$ -grid  $G_0 \subseteq U_1$  so that for each  $a \in G_0$  and each  $i < n$  we have  $b_i \in X_a$ . Then we can choose for  $a \in G_0$  some  $f(a) < n$  so that  $X_a \cap (b_{f(a)}, b_{f(a)+1})_{<}$  is  $<$ -dense in  $(b_{f(a)}, b_{f(a)+1})_{<}$ .  $f$  is an  $n$ -coloring of  $G_0$ , so by the pigeonhole principle for grids, if  $\ell$  is large enough there are an  $m^{\times r}$ -grid  $G_1 \subseteq G_0$  and an  $i$  so that  $X_a \cap (b_i, b_{i+1})_{<}$  is  $<$ -dense in  $(b_i, b_{i+1})_{<}$  for each  $a \in G_1$ .

Let  $B_{b,c}$  be the set of  $a \in M^r$  so that  $X_a \cap (b, c)_{<}$  is  $<$ -dense in  $(b, c)_{<}$ . By inductive hypothesis there are a definable family  $\mathcal{G}_1$  of grids and  $m \in \omega$  so that if  $B_{b,c} \cap U_1$  contains a  $m^{\times r}$ -grid then there are  $G \in \mathcal{G}_1$ , and some  $<$ -open box  $V_1 \subseteq U_1$  so that  $G \subseteq X \cap V_1$  and  $G$  is  $<$ -dense in  $V_1$ . By the argument above, if  $X \cap U$  contains an  $\ell^{\times r}$ -grid then there is some  $i < n$  so that  $B_{b_i, b_{i+1}} \cap U_1$  contains an  $m^{\times r}$ -grid. So there are a  $<$ -open box  $V_1$  and grid  $G \in \mathcal{G}_1$  so  $G \subseteq B_{b_i, b_{i+1}} \cap V_1$ . Since  $X_a$  is  $<$ -dense

in  $(b_i, b_{i+1})_{<}$  for every  $a \in G$  and  $G$  is  $<$ -dense in  $V_1$ , it follows that  $X$  is  $<$ -dense in  $(b_i, b_{i+1})_{<} \times V_1$  which is sufficient by part 1 of the theorem.  $\square$

**Corollary 6.2.4.** *For any  $<$ -open box  $U \subseteq M^d$  there are no definable disjoint sets  $D_0, D_1, D_2, \dots, D_{N^d}$  which are each  $<$ -dense in  $U$ . In fact, if  $D_0, D_1, \dots, D_{N^d}$  are each  $<$ -dense in  $U$ , then there are some  $<$ -open box  $V \subseteq U$ , some grid  $G$  which is  $<$ -dense in  $V$ , and some  $i, j$  so that  $G \cap V \subseteq D_i \cap D_j$ .*

*Proof.* From Theorem 6.2.3, we may assume that each  $D_i$  is a grid, and we conclude from Lemma 6.2.2 that for any  $<$ -open  $V \subseteq U$  there are  $i$  and  $j$  so that  $D_i \cap V$  and  $D_j \cap V$  are not disjoint. In particular, one of the intersections  $D_i \cap D_j$  must be somewhere dense in  $U$ . Applying Theorem 6.2.3 again, it contains a grid  $G$  which is  $<$ -dense in some  $<$ -open  $V \subseteq U$  as desired.  $\square$

One immediate result of this proof is that convexly orderable orders have the nowhere-dense graph property.

**Corollary 6.2.5.** *Suppose  $f: M^d \rightarrow \overline{M}$  is definable. Then the graph of  $f$  is nowhere  $<$ -dense in  $M^n \times \overline{M}$ .*

*Proof.* Assume toward a contradiction that some definable function  $f: M^d \rightarrow \overline{M}$  is  $<$ -dense in the  $<$ -open box  $U = \prod_{i=0}^{d-1} I_i \times J$ . Then we can find disjoint  $<$ -open intervals  $J_0, J_1, \dots, J_{N^d}$  contained in  $J$  (where  $N$  is as in Theorem 5.3.11). Then the sets  $f^{-1}(J_i)$  for  $0 \leq i \leq N^d$  are disjoint, definable, and dense in  $\prod_{i=0}^{d-1} I_i$ , which contradicts Corollary 6.2.4.  $\square$

We also get the following topological consequence:

**Corollary 6.2.6.** *If  $X$  is definable and somewhere  $<$ -dense, then  $X$  has  $\triangleleft$ -interior.*  $\square$

Unfortunately, there is no immediate converse:

**Example 6.2.7.** Let  $\mathcal{M} = (\mathbb{Q}^2, <, U)$ , where  $U$  is a unary predicate naming  $\{0\} \times \mathbb{Q}$ . Then  $\mathcal{M}$  is convexly ordered by  $<^U$ . In particular, notice that  $U$  must have  $\triangleleft$ -interior in any convex ordering  $\triangleleft$  of  $\mathcal{M}$ , but is nowhere  $<$ -dense.

Example 6.2.7 shows that in fact discrete sets are not the only barrier to a cell decomposition theorem for convexly orderable dense linear orders. In particular, the main issue appears to be definable infinite sets of  $<$ -isolated points.

## 6.3 Monotonicity

We end our exploration of convexly orderable linear orders by proving a monotonicity result for convexly orderable structures. We start with functions  $f: M \rightarrow \overline{M}$ .

**Lemma 6.3.1.** *For any definable function  $f: D \rightarrow \overline{M}$  where  $D$  is  $<$ -dense in an infinite  $<$ -open interval  $U \subseteq M$  there are some infinite  $<$ -open interval  $V \subseteq U$  and definable  $G$  which is  $<$ -dense in  $V$  and on which  $f$  is either constant or strictly  $<$ -monotone.*

*Proof.* We assume without loss of generality that  $U$  is match partitioned. For any  $d \in D$ , we define the sets  $A_d = f^{-1}((f(d), \infty)_{<})$ ,  $B_d = f^{-1}((-\infty, f(d))_{<})$ , and  $C_d = f^{-1}(f(d))$ . If  $C_d$  is somewhere  $<$ -dense, then we are done, as this gives an infinite  $<$ -dense grid on which  $f$  is  $<$ -constant. If there is no such  $C_d$ , then for each  $d \in D$  then either  $d$  is a  $<$ -lower-limit of  $A_d$  or  $d$  is a  $<$ -lower-limit of  $B_d$ . By Lemma 5.3.16, there is some  $d' > d$  so that  $A_d$  and  $B_d$  are each either  $<$ -dense in  $(d, d')_{<}$  or disjoint from  $(d, d')_{<}$ . So we can choose, for any  $k$ , a sequence  $d_0 < d_1 < \dots < d_k$  of elements of  $D$  and some  $a > d_k$  so that for each  $i$  the sets  $A_{d_i}$  and  $B_{d_i}$  are each either  $<$ -dense in  $(d_i, a)_{<}$  or disjoint from  $(d_i, a)_{<}$ . Since  $k$  is arbitrarily large, by Lemma 5.3.14 we can define  $V$  and  $G \subseteq D$  which is  $<$ -dense in  $V$  so that:

1.  $V \subseteq U$  is a  $<$ -open interval,
2.  $G \subseteq D$  is definable and  $<$ -dense in  $D$ , and
3.  $A_d$  and  $B_d$  are each either  $<$ -dense in or disjoint from  $(d, \infty)_{<} \cap V \cap D$ , for any  $d \in G$ .

Without loss of generality, we can replace 3 with:

- 3a. Either  $A_d$  is  $<$ -dense in  $(d, \infty)_{<} \cap V \cap D$  for each  $d \in G$  or  $A_d$  is disjoint from  $(d, \infty)_{<} \cap V \cap D$  for each  $d \in G$ .
- 3b. Either  $B_d$  is  $<$ -dense in  $(d, \infty)_{<} \cap V \cap D$  for each  $d \in G$  or  $B_d$  is disjoint from  $(d, \infty)_{<} \cap V \cap D$  for each  $d \in G$ .

There are three cases to consider:

1.  $A_d$  is disjoint from  $(d, \infty)_{<} \cap V \cap D$  for each  $d \in G$ . Then for  $d < d'$  in  $G$  we have  $d' \in B_d$ , so  $f(d') < f(d)$ , i.e.,  $f$  is strictly  $<$ -decreasing on  $G$ .
2.  $B_d$  is disjoint from  $(d, \infty)_{<} \cap V \cap D$  for each  $d \in G$ . Then for  $d < d'$  in  $G$  we have  $d' \in A_d$ , so  $f(d') > f(d)$ , i.e.,  $f$  is strictly  $<$ -increasing on  $G$ .
3.  $A_d$  and  $B_d$  are both  $<$ -dense in  $(d, \infty)_{<} \cap V \cap D$  for each  $d \in G$ . Choose  $d_0 \in G$  and define the sets  $D_0 = A_{d_0} \cap (d_0, \infty)_{<} \cap D$  and  $U_0 = (d_0, \infty)_{<} \cap V$ . We repeat the argument replacing  $U$  with  $U_0$  and  $D$  with  $D_0$ . If we keep returning to this case, then eventually, we will have built  $d_0, d_1, d_2, \dots, d_N$ , and  $V$  so that for each  $i < N$  we have  $d_{i+1} \in A_{d_i}$ , i.e.,  $f(d_0) < f(d_1) < f(d_2) < \dots < f(d_N)$ , and  $A_{d_i} \cap B_{d_{i+1}}$  is  $<$ -dense in  $V$  for each  $i < N$ . This contradicts Lemma 6.2.2.  $\square$

**Corollary 6.3.2.** *For any definable function  $f: U \rightarrow \overline{M}$  where  $U$  is a  $<$ -open interval, there are some  $<$ -open interval  $V \subseteq U$  and disjoint  $<$ -open intervals  $I_0, \dots, I_n$  so that for each  $i$ :*

1.  $V \subseteq \bigcup_{i=0}^n f^{-1}(I_i)$ ,
2.  $f^{-1}(I_i)$  is  $<$ -dense in  $V$ , and
3.  $f$  is  $<$ -strictly monotone or constant on  $f^{-1}(I_i)$ .

*Proof.* By repeated applications of Lemma 6.3.1, we can find a  $<$ -open interval  $V \subseteq U$  and disjoint definable  $G_0, G_1, \dots, G_n$  so that each  $G_i$  is  $<$ -dense in  $V$  and  $f$  is  $<$ -monotone on each  $G_i$ . By Lemma 6.2.2, there is a uniform bound on the number of such sets we can find, so we can assume that  $G_0 \cup G_1 \cup G_2 \cup \dots \cup G_n \supseteq V$ . For the rest, we simply need the  $<$ -convex hulls of the sets  $f(G_i \cap V)$  to be disjoint. Suppose there are  $i, j$  so that for every  $<$ -open  $V' \subseteq V$ , the  $<$ -convex hulls of  $f(G_i \cap V')$  and  $f(G_j \cap V')$  intersect. Then if  $a < b < c$  with  $a, c \in G_i \cap V$  and  $b \in G_j \cap V$  we conclude that  $f(b)$  is  $<$ -between  $f(a)$  and  $f(c)$ . Otherwise, by strict monotonicity of  $f$  on  $G_j \cap V$ , and since  $G_j$  is  $<$ -dense in  $(a, c)_<$  we can find  $b'$  so that  $a < b' < c$  and the  $<$ -interval between  $f(b')$  and  $f(b)$  is disjoint from the  $<$ -interval between  $f(a)$  and  $f(c)$ , a direct contradiction to the assumption on  $G_i$  and  $G_j$ . So,  $f$  is  $<$ -strictly monotone on  $G_i \cup G_j$  and we simply replace  $G_i$  and  $G_j$  in our collection with  $G_i \cup G_j$ .  $\square$

For the higher dimensional analogue to this lemma, we will need to clarify the meaning of “ $<$ -monotone”.

**Definition 6.3.3.** For any grid  $G \subseteq M^d$ , we say that  $f: G \rightarrow \overline{M}$  is  **$<$ -simply monotone in direction  $\sigma$**  (where  $\sigma \in 3^d$ ) if one of the following holds:

1.  $d = 1$ ,  $f$  is  $<$ -strictly decreasing, and  $\sigma = 2$
2.  $d = 1$ ,  $f$  is constant, and  $\sigma = 0$
3.  $d = 1$ ,  $f$  is  $<$ -strictly increasing, and  $\sigma = 1$
4.  $d = r + 1$ ,  $G = \prod_{i=0}^r G_i \subseteq M^{r+1}$  and for each  $n \leq r$  and  $g \in G_n$  the map:

$$f_g^n : (a_0, a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_r) \mapsto f(a_0, a_1, \dots, a_{n-1}, g, a_{n+1}, \dots, a_r)$$

with domain  $\prod_{i \neq n} G_i$  is  $<$ -simply monotone with direction  $\sigma_0 \sigma_1 \dots \sigma_{n-1} \sigma_{n+1} \dots \sigma_r$ .

Our final monotonicity result for convexly orderable dense linear orders is as follows

**Theorem 6.3.4.** *For any definable function  $f: U \rightarrow \overline{M}$  where  $U \subseteq M^d$  is a  $<$ -open box, there are some  $<$ -open box  $V \subseteq U$  and disjoint grids  $G_0, G_1, \dots, G_n$  so that  $V \subseteq \bigcup_{i=0}^n G_i$  and  $f$  is  $<$ -simply monotone on  $G_i$  and  $G_i$  is  $<$ -dense in  $V$  for each  $i$ .*



*Proof.* If  $d = 1$ , then this is simply Corollary 6.3.2. Now assume the theorem holds for  $d = r$ . We show that it holds for  $d = r + 1$  as well. Assume without loss of generality that  $U = \prod_{i=0}^r I_i$  is match partitioned. Fix  $n \leq r$ .—

By inductive hypothesis, we can find, for any  $x \in I_n$ , a  $<$ -open box  $V_x \subseteq \prod_{i \neq n} I_i$ , a  $<$ -open grid  $G_x \subseteq V_x$  which is  $<$ -dense in  $V_x$  and  $\sigma_x \in 3^r$  so that  $f_x^n$  is  $<$ -simply monotone on  $G_x$  in direction  $\sigma_x$ . We can choose  $x_0, x_1, x_2, \dots$  so that  $V_{x_{i+1}} \subseteq V_{x_i}$  for each  $i$ , and by the pigeonhole principle we can assume that  $\sigma_{x_i} = \sigma_{x_j} = \sigma$  for  $i, j \in \omega$ . Then  $G_{x_i}$  is  $<$ -dense in  $V_{x_{kN^d}}$  for  $i \leq kN^d$ , so by applying Corollary 6.2.4 and the pigeonhole principle we find some  $<$ -open box  $V \subseteq V_{x_{kN^d}}$ , some grid  $G$  which is  $<$ -dense in  $V$ , and  $i_0, i_1, \dots, i_k$  so that  $G \subseteq \bigcup_{j=0}^k G_{x_{i_j}}$ .

So, using the uniformity of Corollary 6.2.4 (which comes from the uniformity of Theorem 6.2.3) there is some definable family  $\mathcal{G}$  of grids so that for any  $k$  we can find  $<$ -open box  $V = \prod_{i \neq n} V_i \subseteq \prod_{i \neq n} I_i$  and  $\prod_{i \neq n} G_i = G \in \mathcal{G}$  so that  $G$  is  $<$ -dense in  $V$ , and the set  $B_G^V$  of  $x$  so that  $f_x^n$  is  $<$ -simply monotone on  $G \cap V$  in direction  $\sigma$  contains at least  $k$  points of  $I_n$ . The sets  $B_G^V$  are uniformly definable so by Porism 5.3.15 we can assume that  $B_G^V$  is  $<$ -dense in some  $<$ -open interval  $V_n \subseteq I_n$ .

We may repeat the same argument for each  $n \leq r$  to get a  $<$ -open box  $V \subseteq U$  and a grid  $G$  which is  $<$ -dense in  $V$  so that  $f$  satisfies condition 4 for  $<$ -simple monotonicity. By repeating this argument, we can get grids  $G_0, G_1, \dots, G_n$  so that  $f$  is  $<$ -simply monotone on each  $G_i$ , each  $G_i$  is  $<$ -dense in  $V$ , and  $V \subseteq \bigcup_i G_i$ , as desired.  $\square$

# Bibliography

- [1] Hans Adler, *Theories controlled by formulas of Vapnik-Chervonenkis codimension 1*, 2008. Accessed on the author's webpage at <http://www.logic.univie.ac.at/~adler/docs/vcm.pdf> on 9 March 2018.
- [2] Uri Andrews and Vincent Guingona, *A local characterization of VC-minimality*, Proceedings of the American Mathematical Society **144** (2016), no. 5, 2241–2256.
- [3] Uri Andrews and Tamvana Makuluni, *The index set of uncountably categorical theories*, Israel Journal of Mathematics **196** (August 2013), no. 1, 491–497.
- [4] R. Aref'ev, *On the monotonicity property of weakly o-minimal structures.*, Algebra and model theory ii, 1997, pp. 8–15 (in Russian).
- [5] J. T. Baldwin and A. H. Lachlan, *On Strongly Minimal Sets*, Journal of Symbolic Logic **36** (March 1971), no. 1, 79–96. MR286642
- [6] M. A. Dickmann, *Elimination of Quantifiers for Ordered Valuation Rings*, The Journal of Symbolic Logic **52** (1987), no. 1, 116–128.
- [7] M. M. Erimbetov, *Complete theories with 1-cardinal formulas*, Algebra and Logic **14** (May 1975), no. 3, 151–158.
- [8] Joseph Flenner and Vincent Guingona, *Convexly orderable groups and valued fields*, The Journal of Symbolic Logic **79** (March 2014), no. 01, 154–170.
- [9] Vincent Guingona, *On VC-minimal fields and dp-smallness*, Archive for Mathematical Logic **53** (August 2014), no. 5-6, 503–517.
- [10] Vincent Guingona and Michael C. Laskowski, *On VC-minimal theories and variants*, Archive for Mathematical Logic **52** (November 2013), no. 7-8, 743–758.
- [11] Wilfrid Hodges, *Model theory*, Encyclopedia of mathematics and its applications, Cambridge University Press, Cambridge; New York, 1993.
- [12] Will Johnson, *On dp-minimal fields*, arXiv:1507.02745 [math] (July 2015). arXiv: 1507.02745.
- [13] Asher M. Kach and Antonio Montalbán, *Undecidability of the theories of classes of structures*, The Journal of Symbolic Logic **79** (December 2014), no. 04, 1001–1019.
- [14] Steffen Lempp and Theodore A. Slaman, *The complexity of the index sets of  $\aleph_0$ -categorical theories and of Ehrenfeucht theories*, Advances in Logic, 2007.
- [15] Dugald Macpherson, David Marker, and Charles Steinhorn, *Weakly o-minimal structures and real closed fields*, Transactions of the American Mathematical Society **352** (2000), no. 12, 5435–5483.
- [16] A. Yu Ol'shanskiĭ, *Groups of bounded period with subgroups of prime order*, Algebra and Logic **21** (September 1982), no. 5, 369–418.
- [17] Pierre Simon, *On dp-minimal ordered structures*, The Journal of Symbolic Logic **76** (June 2011), no. 02, 448–460.
- [18] Pierre Simon and Erik Walsberg, *Tame Topology over Dp-minimal Structures*, arXiv:1509.08484 [math] (September 2015). arXiv: 1509.08484.