Boolean Indexed Models and Wheeler's Conjecture

By

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Abstract

Wheeler conjectured in [16] that if a theory has a model companion, then its universal Horn fragment has a model companion. This conjecture was made on several positive examples, see [6] and [8]. In these examples, models of the universal Horn fragments contain definable Boolean algebras. Wheeler's conjecture is shown to be false in [5] with an example that does not contain a Boolean algebra. We focus on finding a positive alternative to Wheeler's conjecture. This is discussed more fully in chapter 1.

In chapter 2, we construct a language $\mathcal{L}^{\mathrm{BA}}$ which permits an approximation of a model having an underlying Boolean algebra. This is closely related to work done by Weispfenning in [15]. We provide a seemingly trivial translation of an $\mathcal{L}$-theory Γ to a $\mathcal{L}^{\mathrm{BA}}$-theory $\Gamma^{\mathrm{BA}\mathbf{2}}$ such that the classes of models of these theories are essentially the same.

In chapter 3, we examine products of models of $\Gamma^{\mathrm{BA}\mathbf{2}}$. This requires a more general translation of $\mathcal{L}$-sentences to $\mathcal{L}^{\mathrm{BA}}$-sentences. We provide two translations of $\mathcal{L}$-sentences to this context: one associated with Kripke forcing, and a second translation which is essentially Boolean forcing, which we call Γ^{BA}.

In chapter 4, we show that the Boolean translation associates with each $\mathcal{L}$-sentence an ideal on the Boolean algebra. We then construct $\mathcal{L}$-models out of models of Γ^{BA} that preserve the Π^0_2 subset of Γ. Using this, we show that Γ^{BA} is the universal Horn fragment of $\Gamma^{\mathrm{BA}\mathbf{2}}$.

In chapter 5, we extend Γ^{BA} to a theory Γ^{ABA} by requiring the underlying Boolean algebra to be atomless. We show that if a Π^0_2 theory Γ is model complete, then its

translated theory Γ^{ABA} is model complete.

In chapter 6, we show that models of Γ^{BA} embed into models of Γ^{ABA}. With the results of the previous chapters, this gives us a positive alternative to Wheeler's conjecture.

In chapter 7, we investigate the internal logic of the structures obtained. We show that the deductive power of the Boolean translation is strictly stronger than intuitionistic logic, but not as strong as classical logic.

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Chapter 1

Introduction

A common pursuit in model theory is the construction of model complete theories. Model complete theories are often very useful. For example, David Marker says in [10, page 111], "Model-completeness and quantifier-elimination have many applications in real algebraic geometry." In support of this idea, Annalisa Marcja and Carlo Toffalori state in [9, page 86] that "In fact, Algebra inspires the notion of model completeness . . . ", and ". . . some developments in Model Theory do produce a significant progress in Algebra; indeed some alternative elegant proofs of the celebrated Hilbert Nullstellensatz, or of the Hilbert Seventeenth Problem, and, more notably, the solution of Artin's Conjecture on p-adic fields witness these fruitful contributions." Often these model complete theories are the model companions of well-known theories or theories that are easy to describe. One well-known model theory result of interest to us is the following.

Proposition 1.1 *Let Γ be a theory such that Γ has a model companion. Then the universal fragment $\Gamma_\forall$ of Γ has the same model companion.*

So the model companion of a theory is determined by the universal fragment of the theory, and different model complete theories must have different universal fragments.

Recall that a sentence φ is equivalent to a universal Horn sentence if and only if it is preserved under submodels of products of models satisfying φ. In 1978, William Wheeler made the following conjecture in [16].

Conjecture 1.2 (Wheeler's Conjecture) *Let Γ be a theory such that Γ has a model companion Γ^*. Then the universal Horn fragment Γ_{UH} of Γ has a model companion.*

This conjecture was made on a number of known examples. For instance, the theory of commutative rings without nilpotent elements is the universal Horn fragment of the theory of commutative integral domains. The latter has as its model companion the theory of algebraically closed fields. The former was shown to have a model companion by Lipschitz and Saracino. The method used in [6] was then employed by MacIntyre for the case of linearly ordered integral domains; see [8]. To make his proof work, MacIntyre had to alter the language of ordered rings: in place of the order predicate $\leq$, he introduced two binary functions for the maximum and minimum of pairs of elements. This change is trivial in the case of linearly ordered rings. However, the universal Horn fragment of linearly ordered integral domains, over this language, changes to a theory of commutative rings with a distributive lattice ordering. The importance of this language change is discussed more below.

Wheeler's conjecture is false. In [5], Glass and Pierce proved the following result.

Theorem 1.3 *The theory of linearly ordered abelian groups has a model companion, but the universal Horn fragment of this theory does not have a model companion. If we replace the order predicate with maximum and minimum, then the theory of linearly ordered abelian groups still has a model companion and the universal Horn fragment still has no model companion.*

There is a significant difference between the examples of Lipschitz and Saracino and MacIntyre on the one hand, and the example of Glass and Pierce on the other. Recall that for a theory Γ, models of the universal Horn fragment Γ_{UH} correspond to submodels of products of models of Γ. In products of integral domains and products of ordered integral domains, the set of idempotents form a Boolean algebra. Hence, these product models have a Boolean algebra contained within them. This is not the case with ordered abelian groups.

With this in mind, our work is motivated by trying to find a uniform first-order approximation of the methods used in the examples of integral domains and ordered integral domains. We begin by making a trivial change to the original language $\mathcal{L}$ of the companionable theory Γ. The new language $\mathcal{L}^{\mathrm{BA}}$ is obtained by replacing predicates in $\mathcal{L}$ with function symbols in $\mathcal{L}^{\mathrm{BA}}$ that map to the two-element Boolean algebra.

Definition 1.4 *We let* $\mathbf{2}$ *be the nondegenerate Boolean algebra consisting of two elements,* 0 *and* 1.

Our companionable theory Γ is replaced with a theory $\Gamma^{\mathrm{BA}\mathbf{2}}$ over $\mathcal{L}^{\mathrm{BA}}$ such that models of Γ trivially correspond to models of $\Gamma^{\mathrm{BA}\mathbf{2}}$.

Proposition 1.5 *Let* Γ *be an* $\mathcal{L}$*-theory. Then* Γ *is companionable if and only if* $\Gamma^{\mathrm{BA}\mathbf{2}}$ *is companionable and* Γ *admits quantifier-elimination if and only if* $\Gamma^{\mathrm{BA}\mathbf{2}}$ *admits quantifier-elimination.*

In our context, the domains of models over $\mathcal{L}^{\mathrm{BA}}$ contain three disjoint parts. The first is the underlying Boolean algebra. The second part is the structure. The structure of a model over $\mathcal{L}^{\mathrm{BA}}$ corresponds to a model over $\mathcal{L}$. The final part is the chaff, which consists of all elements that are neither Boolean nor structural. In general, chaff has little importance other than the fact that it exists.

The following result is our positive solution to an alternative to Wheeler's conjecture.

Theorem 1.6 *Let* Γ *be a companionable* $\mathcal{L}$*-theory. Then the universal Horn fragment of* $\Gamma^{\mathrm{BA}\mathbf{2}}$ *has a model companion.*

Theorem 1.6 is a variation of a major result of Weispfenning's, see [15]. Weispfenning uses a two-sorted language over which he develops theories with many similarities to our theories $\Gamma^{\mathrm{BA}\mathbf{2}}$ and the theory Γ^{ABA} mentioned below. His work is a generalization of the model companion result found in [6]. Weispfenning's major result is different from MacIntyre's generalization in [8]. MacIntyre uses aspects of Boolean sheaves, while Weispfenning codes these sheaves in first order logic.

Theorem 1.6 is proved in several steps. In defining the axioms $\Gamma^{\mathrm{BA}\mathbf{2}}$, we also introduce the subtheory Γ^{BA}, where the major difference between these axiom sets is that Γ^{BA} allows for the underlying Boolean algebra to be any Boolean algebra, not just the two-element Boolean algebra. The first major step in proving the above theorem is the following result.

Theorem 1.7 *Let* Γ *be a universal* $\mathcal{L}$*-theory. Then the universal Horn fragment of* $\Gamma^{\mathrm{BA}\mathbf{2}}$ *is* Γ^{BA}*.*

Again, this is similar to a result of Weispfenning. However, our results are carried out in a more general fashion. One major difference is that Weispfenning works only with structural elements which exist over the entire Boolean algebra. We include partial structural elements; that is, structural elements which exist over Boolean elements that are not 0 or 1. Another difference with the work of Weispfenning is our translation. If all structural elements exist over the whole Boolean algebra, then our axiomatizations of Γ^{BA} are the same. However, the presence of partial structural elements requires a more general axiomatization of Γ^{BA}. In our context, we associate with an $\mathcal{L}$-sentence an ideal on the Boolean algebra.

By Theorem 1.7, it is sufficient for our main result to find a model companion for Γ^{BA}. We introduce an extension Γ^{ABA} of Γ^{BA}. We axiomatize Γ^{ABA} by adding to Γ^{BA} the following three axioms:

- The underlying Boolean algebra is atomless.
- There are infinitely many chaff elements.
- There exists an element of full extent.

With this, we show the following.

Theorem 1.8 *Let Γ be a model complete $\mathcal{L}$-theory. Then Γ^{ABA} is model-complete.*

With this result, we get new model complete theories which are distinct from the original theory.

Our main work is completed with the following result.

Theorem 1.9 *Let Γ be a Π^0_2-theory. Then every model of Γ^{BA} embeds into a model of Γ^{ABA}. So if Γ is a universal theory that has a model companion Γ^*, then the universal Horn fragment of $\Gamma^{\mathrm{BA}\mathbf{2}}$ has $(\Gamma^*)^{\mathrm{ABA}}$ model companion.*

This Theorem shows that by internalizing a Boolean algebra we give the language enough power to axiomatize the theory of the existentially closed models.

We conclude by demonstrating the deductive power of models over $\mathcal{L}^{\mathrm{BA}}$. Using the intuitionistic sequent calculus, we show that intuitionistic derivability transfers to

Boolean indexed models. We also demonstrate that we are not able to get full classical derivability in our context. We do this by describing models of $\emptyset^{\mathrm{BA}}$ and $\mathcal{L}$-sentences over this model such that the ideal these sentences generate are not principal. We also give two sufficient conditions for a model to have all of its $\mathcal{L}$-sentences to generate principal ideals.

1.1 Notation

The following identifies the notation we use. If $\mathfrak{A}$ is a model, then A denotes the domain of $\mathfrak{A}$. It will be important to distinguish between intuitionistic proof and classical proof. We use $\vdash_i$ to represent intuitionistic derivation and $\vdash_c$ to denote classical derivation.

Chapter 2

Language and Axioms

2.1 Introduction

We begin with an arbitrary language $\mathcal{L}$ of predicate logic, and define a new language $\mathcal{L}^{\mathrm{BA}}$. Over $\mathcal{L}^{\mathrm{BA}}$, we introduce a theory $\emptyset^{\mathrm{BA}\mathbf{2}}$ so that for each model $\mathfrak{A}$ over $\mathcal{L}$, there is an associated model $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ over $\mathcal{L}^{\mathrm{BA}}$ such that $\mathfrak{A}$ and $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ are essentially the same model. Here, $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ is in essence a two-sorted model, with one sort being elements from $\mathfrak{A}$ and the other sort being elements of the two element Boolean algebra $\mathbf{2}$. We replace every n-ary relation over $\mathfrak{A}$ by its characteristic function on $(A^{\mathrm{BA}\mathbf{2}})^n$. We leave functions on $\mathfrak{A}$ unchanged. For each $\mathcal{L}$-sentence φ there is a straightforward translation to an $\mathcal{L}^{\mathrm{BA}}$-sentence $\varphi^{\mathrm{BA}\mathbf{2}}$ such that $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \varphi^{\mathrm{BA}\mathbf{2}}$. Every set of $\mathcal{L}$-sentences Γ has a corresponding set of $\mathcal{L}^{\mathrm{BA}}$-sentences $\Gamma^{\mathrm{BA}\mathbf{2}}$ extending $\emptyset^{\mathrm{BA}\mathbf{2}}$ such that Γ and $\Gamma^{\mathrm{BA}\mathbf{2}}$ axiomatize essentially the same theories.

2.2 Creating an appropriate language

Our choice of predicate logic language is based on [13]. In this paper, Scott enriches the usual predicate logic language with an existence predicate E and a unique-identity operator I. We avoid the use of I, but the predicate E plays a significant role. Since we use predicate logic in both intuitionistic and classical situations, all of our languages include $\{\top, \bot, \wedge, \vee, \to, \exists, \forall, =, \mathrm{E}\}$, besides the usual predicates and function symbols, including constant symbols. In particular, the translated languages $\mathcal{L}^{\mathrm{BA}}$ also contain the predicate E. We discuss the presence of E in more detail below.

In all theories, we allow functions to be partial. The predicate E is a unary existence predicate, that is, $\vdash_i \mathrm{E}(x) \leftrightarrow x = x$. So the predicate E essentially stands for the sort of all existing elements. This is useful in the context of partial functions. All predicates and functions are **strict**, i.e., all theories include axioms of the form $\mathrm{E}(f(\mathbf{x})) \to \mathrm{E}(\mathbf{x})$ and $P(\mathbf{x}) \to \mathrm{E}(\mathbf{x})$. Because functions may be partial, the reverse implication $\mathrm{E}(\mathbf{x}) \to \mathrm{E}(f(\mathbf{x}))$ need not hold.

Definition 2.1 *Given a language $\mathcal{L}$, we form the* **associated language** $\mathcal{L}^{\mathrm{BA}}$*, a new language with symbols for a Boolean algebra:*

- *Introduce a unary predicate* $\mathrm{BA}(x)$ *into* $\mathcal{L}^{\mathrm{BA}}$.
- *For every predicate $P(\mathbf{x}) \in \mathcal{L}$, introduce a new function symbol $[\![P(\mathbf{x})]\!]$. This includes the predicates $x = y$ and $\mathrm{E}(x)$. Note that the symbols $[\![]\!]$ do not have any meaning by themselves, but only in the context of $[\![P(\mathbf{x})]\!]$.*
- *We include all function symbols from $\mathcal{L}$.*
- *We introduce two binary function symbols: $x \restriction y$, and $x \oplus y$, and a constant symbol ϖ, which will be a "structural" element.*
- *We introduce function symbols $x \sqcup y$, $x \sqcap y$, and $-x$ as well as constant symbols* 0 *and* 1 *for the Boolean algebra.*

Having introduced $\mathcal{L}^{\mathrm{BA}}$, we now provide some explanation on the intended meaning of its function and constant symbols. In our context, the domain of an $\mathcal{L}^{\mathrm{BA}}$-model has two essential parts: an underlying Boolean algebra and a structure which lives above this Boolean algebra. The structure is a generalization of a model over $\mathcal{L}$, and the Boolean algebra generalizes the two element Boolean algebra of true and false. These two parts are disjoint, but interact through the functions $[\![P(\mathbf{x})]\!]$ and $x \upharpoonright y$. In general, models over $\mathcal{L}^{\mathrm{BA}}$ will have a third part, which we call chaff. While the presence of chaff is unavoidable, it has no properties except existence.

We now discuss the functions $[\![P(\mathbf{x})]\!]$. This is read as "the extent of P", that is, the Boolean elements where P holds for each $\mathbf{x}$. These functions map elements from the structure to Boolean elements. At this point we discuss $=$ and E. There are two places where $=$ and E are present in $\mathcal{L}^{\mathrm{BA}}$. The first place they appear is in the functions $[\![x = y]\!]$ and $[\![\mathrm{E}(x)]\!]$, which are special cases of $[\![P(\mathbf{x})]\!]$. These are functions from the structure of a model to the Boolean algebra. Every structural element will have a unique Boolean element over which it lives, which we call the extent of that element. The function $[\![\mathrm{E}(x)]\!]$ takes as input a structural element and outputs its extent. There are certain contexts where structural elements do not exist over the full Boolean algebra, that is, there is a structural element where the extent of that element is neither 0 nor 1. For two distinct Boolean elements y and z, the inverse image of y and z are disjoint. Thus, the Boolean elements partition the structure under this inverse mapping. The function $[\![x = y]\!]$ takes as input two structural elements and outputs the Boolean element over which the two structural elements are equal.

The second place where $=$ and E occur in $\mathcal{L}^{\mathrm{BA}}$ is as the usual predicates, that is $x = y$ and $\mathrm{E}(x)$ assign true or false to elements from the domain of a model. As predicates, $=$

and E have as input all elements, whether structural, Boolean, or chaff. The functions $[\![x = y]\!]$ and $[\![\mathrm{E}(x)]\!]$ are different from but analogous to the predicates $x = y$ and $\mathrm{E}(x)$. These functions act only on structural elements, and if the structure of an $\mathcal{L}^{\mathrm{BA}}$-model is a generalization of an $\mathcal{L}$-model, then $[\![x = y]\!]$ and $[\![\mathrm{E}(x)]\!]$ are generalizations of the $\mathcal{L}$-predicates $=$ and E.

We briefly discuss the constant symbol ϖ. This symbol will represent a structural element which only exists above 0. We show below that this is the unique element that exists over 0. In some sense, this element corresponds to being "undefined". For example, suppose that in a model over $\mathcal{L}$, a particular function is undefined on some set of elements. Then, in the translation of this model to a model over $\mathcal{L}^{\mathrm{BA}}$, this function, when acting on this set of elements, maps to ϖ.

Our language $\mathcal{L}^{\mathrm{BA}}$ is similar to one used by Weispfenning in [15]. Here our sort BA corresponds to his B-sort, and our functions $[\![P(\mathbf{x})]\!]$ correspond to his functions v_R. He also refers to structural elements as $\mathcal{L}$-terms. Like in our context, his functions v_R map $\mathcal{L}$-terms to the B-sort. He also includes functions from $\mathcal{L}$ as functions from $\mathcal{L}$-terms to $\mathcal{L}$-terms.

2.2.1 Axiomatizations for the basic theories

We list axiomatizations for $\emptyset^{\mathrm{BA}}$ and $\emptyset^{\mathrm{BA}\mathbf{2}}$. Unless otherwise noted, all variables are understood to be universally quantified. We begin with axioms for the sort BA:

Ba1 $\mathrm{BA}(0) \wedge \mathrm{BA}(1)$

Ba2 $\mathrm{BA}(x) \wedge \mathrm{BA}(y) \rightarrow \mathrm{BA}(x \sqcap y)$

Ba3 $\mathrm{E}(x \sqcap y) \rightarrow \mathrm{BA}(x) \wedge \mathrm{BA}(y)$

Ba4 $\mathrm{BA}(x) \wedge \mathrm{BA}(y) \rightarrow \mathrm{BA}(x \sqcup y)$

Ba5 $\mathrm{E}(x \sqcup y) \rightarrow \mathrm{BA}(x) \wedge \mathrm{BA}(y)$

Ba6 $\mathrm{BA}(x) \rightarrow \mathrm{BA}(-x)$

Ba7 $\mathrm{E}(-x) \rightarrow \mathrm{BA}(x)$

These axioms imply that $x \sqcap y$ and $x \sqcup y$, and $-x$ are total functions on BA, the first two from BA $\times$ BA to BA, and the last from BA to BA. Next, we include the Boolean algebra axioms:

Ba8 $\mathrm{BA}(x) \rightarrow (x \sqcap x = x) \wedge (x \sqcup x = x)$

Ba9 $\mathrm{BA}(x) \wedge \mathrm{BA}(y) \rightarrow x \sqcap y = y \sqcap x$

Ba10 $\mathrm{BA}(x) \wedge \mathrm{BA}(y) \rightarrow x \sqcup y = y \sqcup x$

Ba11 $\mathrm{BA}(x) \wedge \mathrm{BA}(y) \wedge \mathrm{BA}(z) \rightarrow (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$

Ba12 $\mathrm{BA}(x) \wedge \mathrm{BA}(y) \wedge \mathrm{BA}(z) \rightarrow (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$

Ba13 $\mathrm{BA}(x) \rightarrow (x \sqcap 1 = x) \wedge (x \sqcup 0 = x)$

Ba14 $\mathrm{BA}(x) \wedge \mathrm{BA}(y) \rightarrow (x \sqcup (x \sqcap y) = x) \wedge (x \sqcap (x \sqcup y) = x)$

Ba15 $\mathrm{BA}(x) \wedge \mathrm{BA}(y) \wedge \mathrm{BA}(z) \rightarrow ((x \sqcap (y \sqcup z)) = (x \sqcap y) \sqcup (x \sqcap z)) \wedge ((x \sqcup (y \sqcap z)) = (x \sqcup y) \sqcap (x \sqcap z))$

Ba16 $\mathrm{BA}(x) \rightarrow (x \sqcup -x = 1) \wedge (x \sqcap -x = 0)$

Ba17 $1 = 0 \rightarrow \bot$

Thus, the model will contain a nondegenerate Boolean algebra. We provide the following definitions related to the Boolean algebra:

Definition 2.2 *1. We introduce a relation $x \trianglelefteq y$ on the Boolean algebra that holds precisely when $x \sqcup y = y$. Equivalently, $x \trianglelefteq y$ holds if $x \sqcap y = x$;*

2. If we have a Boolean algebra A, and $B \subseteq A$, we let $\langle B \rangle$ be the Boolean algebra generated by the elements from B.

3. For a model $\mathfrak{A}$, we define $\mathrm{BA}(A)$ to be the set $\{a \in A : \mathfrak{A} \models \mathrm{BA}(a)\}$.

Next, we introduce structural elements which are disjoint from the Boolean algebra:

So1 $\mathrm{BA}(x) \wedge \mathrm{E}([\![\mathrm{E}(x)]\!]) \rightarrow \bot$

Again, $[\![\mathrm{E}(x)]\!]$ is a function which only takes structural elements as its input. Thus, $\mathrm{E}([\![\mathrm{E}(x)]\!])$ holds only for structural elements.

Definition 2.3 *For a model $\mathfrak{A}$, we define the* **structure**, *denoted $\mathrm{ST}(A)$, to be the set $\{a \in A : \mathfrak{A} \models \mathrm{E}([\![\mathrm{E}(a)]\!])\}$. For convenience, we use $\mathrm{ST}(x)$ as shorthand for $\mathrm{E}([\![\mathrm{E}(x)]\!])$.*

Thus, So1 implies that the Boolean algebra BA is disjoint from the structural elements ST. Below, we axiomatize that the structural elements exist above the Boolean algebra.

We now list the axioms for the functions $[\![P(\mathbf{x})]\!]$ for each $P(x_0, \ldots, x_{n-1})$ in $\mathcal{L}$ of arity n.

Pr1 $\mathrm{E}([\![P(\mathbf{x})]\!]) \rightarrow \bigwedge_{i<n} \mathrm{ST}(x_i)$

Pr2 $(\bigwedge_{i<n} \mathrm{ST}(x_i)) \rightarrow \mathrm{BA}([\![P(\mathbf{x})]\!])$

These two axioms state that the functions $[\![P(\mathbf{x})]\!]$ are from ST^n to BA. In particular, this implies that the function $[\![\mathrm{E}(x)]\!]$ is a map from ST to BA, and $[\![x = y]\!]$ is from $\mathrm{ST} \times \mathrm{ST}$ to BA.

Definition 2.4 *1. For a model $\mathfrak{A}$ and $a \in \mathrm{ST}(A)$, we define the* **extent of** *a to be the Boolean element $[\![\mathrm{E}(a)]\!]$.*

2. For a model $\mathfrak{A}$ and an element $p \in \mathrm{BA}(A)$, we define A_p to be the set $\{a \in A : \mathfrak{A} \models [\![\mathrm{E}(a)]\!] = p\}$, that is, A_p is the set of structural elements with extent p.

Pr3 $\mathrm{E}([\![P(\mathbf{x})]\!]) \to ([\![P(\mathbf{x})]\!] \trianglelefteq \prod_{i<n}[\![\mathrm{E}(x_i)]\!])$

Recall that a function is strict if $\mathrm{E}(f(\mathbf{x})) \to \mathrm{E}(\mathbf{x})$, and a predicate is strict if $P(\mathbf{x}) \to \mathrm{E}(\mathbf{x})$. The previous axiom gives us that the strictness axioms for predicates are internalized within the new theory.

We now list the axioms for function symbols f from $\mathcal{L}$:

Fn1 $\mathrm{E}(f(\mathbf{x})) \to (\bigwedge_{i<n} \mathrm{ST}(x_i))$

Fn2 $(\bigwedge_{i<n} \mathrm{ST}(x_i)) \to \mathrm{ST}(f(\mathbf{x}))$

These axioms state that f is a map from ST^n to ST.

Fn3 $\mathrm{E}(f(\mathbf{x})) \to ([\![\mathrm{E}(f(\mathbf{x}))]\!] \trianglelefteq \prod_{i<n}[\![\mathrm{E}(x_i)]\!])$

This axiom states the strictness of f is internalized in the new theory.

We now produce the axioms for the restriction function $x \upharpoonright y$. We use $\mathbf{x} \upharpoonright y$ as shorthand for $x_0 \upharpoonright y, x_1 \upharpoonright y, \ldots, x_{n-1} \upharpoonright y$:

Rs1 $\mathrm{E}(x \upharpoonright y) \to \mathrm{ST}(x) \wedge \mathrm{BA}(y)$

Rs2 $\mathrm{ST}(x) \wedge \mathrm{BA}(y) \to \mathrm{ST}(x \restriction y)$

These two axioms state that $x \restriction y$ is a function from $\mathrm{ST} \times \mathrm{BA}$ to ST.

Rs3 $\mathrm{ST}(x) \to x \restriction [\![\mathrm{E}(x)]\!] = x$

Rs4 $\mathrm{ST}(x) \wedge \mathrm{BA}(y) \wedge \mathrm{BA}(z) \to ((x \restriction y) \restriction z) = x \restriction (y \sqcap z)$

Rs5 $\mathrm{E}([\![P(\mathbf{x})]\!]) \wedge \mathrm{BA}(y) \to [\![P(\mathbf{x} \restriction y)]\!] = [\![P(\mathbf{x})]\!] \sqcap y$

Rs6 $\mathrm{E}(f(\mathbf{x})) \wedge \mathrm{BA}(y) \to f(\mathbf{x} \restriction y) = f(\mathbf{x}) \restriction y$

Rs7 $\mathrm{E}([\![x = y]\!]) \to (x \restriction [\![x = y]\!]) = (y \restriction [\![x = y]\!])$

By Rs5, for a model $\mathfrak{A}$, if $a \in A_p$ and $q \in \mathrm{BA}(A)$, then $a \restriction q \in A_{p \sqcap q}$.

We next include the axioms for the piecing together function $x \oplus y$. To explain this function, we take a model $\mathfrak{A}$ and elements $a \in A_p$ and $b \in A_q$ such that $a \restriction (p \sqcap q) = b \restriction (p \sqcap q)$. Then $a \oplus b$ is the element in $A_{p \sqcup q}$ such that $(a \oplus b) \restriction p = a$ and $(a \oplus b) \restriction q = b$. We show below that this element is unique. We axiomatize this as follows:

Pt1 $\mathrm{E}(x \oplus y) \to \mathrm{ST}(x) \wedge \mathrm{ST}(y)$

Thus, only structural elements can be pieced together.

Pt2 $\mathrm{E}(x \oplus y) \to x \restriction [\![\mathrm{E}(y)]\!] = y \restriction [\![\mathrm{E}(x)]\!]$

Thus, the only elements which can be pieced together are those which are equal on their shared extent.

Pt3 $\mathrm{E}(x \oplus y) \to [\![\mathrm{E}(x \oplus y)]\!] = [\![\mathrm{E}(x)]\!] \sqcup [\![\mathrm{E}(y)]\!]$

Thus, the piecing element has extent equal to the join of the extents of the individual elements.

Pt4 $\mathrm{E}(x \oplus y) \rightarrow ((x \oplus y) \restriction [\![\mathrm{E}(x)]\!] = x) \wedge ((x \oplus y) \restriction [\![\mathrm{E}(y)]\!] = y)$

Thus, the piecing element restricted to the extent of an individual element is equal to that individual element.

Pt5 $y \restriction [\![\mathrm{E}(x)]\!] = x \restriction [\![\mathrm{E}(y)]\!] \rightarrow \mathrm{E}(x \oplus y)$

Thus, $x \oplus y$ exists for elements that are equal over their shared extent.

We conclude the axiomatization of $\emptyset^{\mathrm{BA}}$ with some axioms on particular extents.

Ex1 $[\![\top]\!] = 1$

Ex2 $[\![\bot]\!] = 0$

Ex3 $\mathrm{ST}(x) \rightarrow [\![x = x]\!] = [\![\mathrm{E}(x)]\!]$

Ex4 $[\![\mathrm{E}(\varpi)]\!] = 0$

This concludes the axioms $\emptyset^{\mathrm{BA}}$. Note that this axiom set has many similarities to the two-sorted system of [15]. Our Fn axioms correspond with Weispfenning's description that $\mathcal{L}$-functions have only L-terms as their domains, and these functions themselves are L-terms. Our Pr axioms are equivalent to his statement that v_R have L-terms as their domain, and themselves are B-terms. There are some marked differences, however. In [15], all structural elements are only allowed to have full extent. In our case, we include elements that can have as extent any Boolean element. As such, we require Rs axioms to provide us with the needed structure. Also, Weispfenning does not include piecing together in his base theory, but instead includes it in extensions of his base theory. We include it for technical reasons, which we discuss later.

We label the theory axiomatized by $\emptyset^{\mathrm{BA}}$ as $\mathrm{Th}(\emptyset^{\mathrm{BA}})$. We first note that each of the axioms of $\emptyset^{\mathrm{BA}}$ has a well-known form:

Definition 2.5 *A sentence is universal Horn if it is equivalent to a conjunction of sentences of the form* $\forall\mathbf{x}(\alpha_1(\mathbf{x}) \wedge \ldots \alpha_n(\mathbf{x}) \rightarrow \alpha_0(\mathbf{x}))$*, where* $\alpha_i(\mathbf{x})$ *is atomic for each* i*. For a set of sentences* Γ*, we label* $(\Gamma)_{\mathrm{UH}}$ *as the set of universal Horn sentences derivable from* Γ*.*

As $\perp$ is an atom in our language, universal Horn sentences include the negation of conjunctions of atoms. Note that each axiom of $\emptyset^{\mathrm{BA}}$ is universal Horn.

Definition 2.6 *We call models of* $\emptyset^{\mathrm{BA}}$ **Boolean indexed models***.*

We are now ready to define the axiom set $\emptyset^{\mathrm{BA}\mathbf{2}}$.

The following axiom states that BA equals **2**:

Ba18 $\mathrm{BA}(x) \rightarrow x = 0 \vee x = 1$

The following axioms states that all elements are either Boolean or structural elements:

So2 $\mathrm{E}(x) \rightarrow \mathrm{BA}(x) \vee \mathrm{ST}(x)$

Definition 2.7

1. *We define the axiom set* $\emptyset^{\mathrm{BA}\mathbf{2}}$ *to be the set of sentences* $\emptyset^{\mathrm{BA}} \cup \{\mathrm{Ba18}, \mathrm{So2}\}$*.*
2. *We call models of* $\emptyset^{\mathrm{BA}\mathbf{2}}$ **simple Boolean indexed models***.*

We now discuss our inclusion of $\oplus$ as a function symbol. We demonstrate below that $\emptyset^{\mathrm{BA}}$ is the universal Horn fragment of $\emptyset^{\mathrm{BA}\mathbf{2}}$. If $x \oplus y$ is not a function symbol in $\mathcal{L}^{\mathrm{BA}}$, then we would need to define it as an abbreviation for

$$Iz.((z \restriction [\![\mathrm{E}(x)]\!] = x) \wedge (z \restriction [\![\mathrm{E}(y)]\!] = y) \wedge ([\![\mathrm{E}(z)]\!] = [\![\mathrm{E}(x)]\!] \sqcup [\![\mathrm{E}(y)]\!]))$$

where $Iz.$ is the existence operator used in [13]. With this term, we can no longer axiomatize piecing together by universal Horn axioms. We include $x \oplus y$ as a function symbol in the language and as a consequence, piecing together becomes part of the universal Horn fragment of $\emptyset^{\mathrm{BA}\mathbf{2}}$.

2.3 Simple Boolean indexed models

We now show that simple Boolean indexed models are essentially models over $\mathcal{L}$. We begin by translating $\mathcal{L}$-formulas to $\mathcal{L}^{\mathrm{BA}}$-formulas:

Definition 2.8 *1. For an $\mathcal{L}$-formula φ, we define the translated $\mathcal{L}^{\mathrm{BA}}$-formula $\varphi^{\mathrm{BA}\mathbf{2}}$ inductively as follows:*

If φ is atomic, of the form $P(\mathbf{t})$, then $\varphi^{\mathrm{BA}\mathbf{2}}$ is $[\![P(\mathbf{t})]\!] = 1$. In particular, $(\mathrm{E}(x))^{\mathrm{BA}\mathbf{2}}$ is $[\![\mathrm{E}(x)]\!] = 1$ and $(x = y)^{\mathrm{BA}\mathbf{2}}$ is $[\![x = y]\!] = 1$.

If φ is $\psi \wedge \theta$, then $\varphi^{\mathrm{BA}\mathbf{2}}$ is $\psi^{\mathrm{BA}\mathbf{2}} \wedge \theta^{\mathrm{BA}\mathbf{2}}$.

If φ is $\psi \vee \theta$, then $\varphi^{\mathrm{BA}\mathbf{2}}$ is $\psi^{\mathrm{BA}\mathbf{2}} \vee \theta^{\mathrm{BA}\mathbf{2}}$.

If φ is $\psi \rightarrow \theta$, then $\varphi^{\mathrm{BA}\mathbf{2}}$ is $\psi^{\mathrm{BA}\mathbf{2}} \rightarrow \theta^{\mathrm{BA}\mathbf{2}}$.

If φ is $\forall x\psi$, then $\varphi^{\mathrm{BA}\mathbf{2}}$ is $\forall x([\![\mathrm{E}(x)]\!] = 1 \rightarrow \psi(x)^{\mathrm{BA}\mathbf{2}})$.

If φ is $\exists x\psi$, then $\varphi^{\mathrm{BA}\mathbf{2}}$ is $\exists x([\![\mathrm{E}(x)]\!] = 1 \wedge \psi(x)^{\mathrm{BA}\mathbf{2}})$.

2. For a set Γ of $\mathcal{L}$-sentences, we define $\Gamma^{\mathrm{BA}\mathbf{2}}$ to be the set $\{\gamma^{\mathrm{BA}\mathbf{2}} : \gamma \in \Gamma\} \cup \emptyset^{\mathrm{BA}\mathbf{2}}$.

We thus have a new translation of sentences Γ over $\mathcal{L}$ into a set of sentences $\Gamma^{\mathrm{BA}\mathbf{2}}$ over $\mathcal{L}^{\mathrm{BA}}$. Specifically, if Γ is empty, we get $\emptyset^{\mathrm{BA}\mathbf{2}}$. Note that $\top$ translates to $[\![\top]\!] = 1$, which, by Ex1, is equivalent to $\top$. Similarly, $\perp$ translates to $[\![\perp]\!] = 1$, which, by Ex2 and Ba17, is equivalent to $\perp$.

We first show that the translation of a quantifier-free formula has the following general form:

Lemma 2.9 *Suppose φ is an $\mathcal{L}$-formula. Let φ' be the $\mathcal{L}^{\mathrm{BA}}$ formula gotten by replacing each atom δ that appears in φ with $([\![\delta]\!] = 1)$. Then φ' is equivalent to $\varphi^{\mathrm{BA}\mathbf{2}}$.*

Proof. This holds a straightforward induction on the complexity of φ, along with the fact that for an $\mathcal{L}$-formula ψ, $\forall x\psi$ is equivalent to $\forall x(\mathrm{E}(x) \rightarrow \psi)$, and $\exists x\psi$ is equivalent to $\exists x(\mathrm{E}(x) \wedge \psi)$. $\dashv$

We now show how to translate a model $\mathfrak{A}$ over $\mathcal{L}$ into a model $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ over $\mathcal{L}^{\mathrm{BA}}$. This translation has the property that, for any set Γ of $\mathcal{L}$-sentences, $\mathfrak{A} \models \Gamma$ if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}\mathbf{2}}$:

Definition 2.10 *Let $\mathfrak{A}$ be a $\mathcal{L}$-model. We define the translated model $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ as follows:*

- *We set $\mathrm{BA}(A^{\mathrm{BA}\mathbf{2}}) = \mathbf{2}$, with the functions $\sqcup$, $\sqcap$, and $-$ defined as usual in the two element Boolean algebra.*
- *For each $a \in A$, we introduce a new element $\hat{a}$ into $A^{\mathrm{BA}\mathbf{2}}$, and set $[\![\mathrm{E}(\hat{a})]\!]$ equal to 1. We let $\hat{A}$ be the set $\{\hat{a} : a \in A\}$. Then $\mathrm{ST}(A^{\mathrm{BA}\mathbf{2}})$ is $\hat{A} \cup \{\varpi\}$. As we explain below, ϖ will take the place of a function being undefined. We also set $[\![\mathrm{E}(\varpi)]\!]$ equal to 0.*

- *For each n-ary predicate $P \in \mathcal{L}$ and $\mathbf{a} \in (\hat{A})^n$, we set $[\![P(\hat{\mathbf{a}})]\!]$ equal to 1 if and only if $\mathfrak{A} \models P(\mathbf{a})$. If $\varpi \in \mathbf{b}$, we set $[\![P(\mathbf{b})]\!]$ equal to 0. We only define $[\![P(\mathbf{x})]\!]$ for tuples from $\mathrm{ST}(A^{\mathrm{BA}\mathbf{2}})$.*

- *For each function $f \in \mathcal{L}$ and $\hat{\mathbf{a}}, \hat{b} \in (\hat{A})$, we set $f(\hat{\mathbf{a}})$ equal to $\hat{b}$ if and only if $\mathfrak{A} \models f(\mathbf{a}) = b$. If $f(\mathbf{a})$ is undefined, then we set $f(\hat{\mathbf{a}})$ equal to ϖ. If $\varpi \in \mathbf{b}$, we set $f(\mathbf{b})$ equal to ϖ. We only define $f(\mathbf{x})$ for elements in $\mathrm{ST}(A^{\mathrm{BA}\mathbf{2}})$.*

- *We define $x \restriction y$ only if $x \in \mathrm{ST}(A^{\mathrm{BA}\mathbf{2}})$, and $y \in \mathrm{BA}(A^{\mathrm{BA}\mathbf{2}})$. We set $x \restriction 1$ equal to x and $x \restriction 0$ equal to ϖ.*

- *We define $x \oplus y$ if and only if either x is ϖ and $y \in \mathrm{ST}(A^{\mathrm{BA}\mathbf{2}})$, if y is ϖ and $x \in \mathrm{ST}(A^{\mathrm{BA}\mathbf{2}})$, or $x = y$ and $x \in \mathrm{ST}(A^{\mathrm{BA}\mathbf{2}})$. In the first two cases, we set $x \oplus \varpi$ and $\varpi \oplus x$ equal to x, and in the last case, we set $x \oplus y$ equal to x.*

- *Finally, we set $[\![\top]\!]$ equal to 1, and $[\![\bot]\!]$ equal to 0.*

We now have that this translated model satisfies the axioms in $\emptyset^{\mathrm{BA}\mathbf{2}}$:

Lemma 2.11 *Suppose $\mathfrak{A}$ is a $\mathcal{L}$-model. Then $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \emptyset^{\mathrm{BA}\mathbf{2}}$.*

Proof. This is a simple exercise on the axioms listed. $\dashv$

Clearly, for an $\mathcal{L}$-model $\mathfrak{A}$, the translated model $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ is very similar to $\mathfrak{A}$. To make this statement more concrete, we first define $(\varphi(\mathbf{a}))^{\mathrm{BA}\mathbf{2}}$ to be $\varphi^{\mathrm{BA}\mathbf{2}}(\hat{\mathbf{a}})$. With this, we get the following:

Lemma 2.12 *Suppose $\mathfrak{A}$ is a $\mathcal{L}$-model, and let $\varphi(\mathbf{a})$ be a $\mathcal{L}(A)$-sentence. Then $\mathfrak{A} \models \varphi(\mathbf{a})$ if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \varphi^{\mathrm{BA}\mathbf{2}}(\hat{\mathbf{a}})$.*

Proof. We prove this by induction on φ. If φ is atomic, then by the translation above, the result is obvious. We now suppose the result holds for ψ and θ.

Suppose φ is $\psi \wedge \theta$. Then $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \models \psi$ and $\mathfrak{A} \models \theta$. By induction, this holds if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \psi^{\mathrm{BA}\mathbf{2}}$ and $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \theta^{\mathrm{BA}\mathbf{2}}$. This holds if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \varphi^{\mathrm{BA}\mathbf{2}}$.

Suppose φ is $\psi \vee \theta$. Then $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \models \psi$ or $\mathfrak{A} \models \theta$. By induction, this holds if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \psi^{\mathrm{BA}\mathbf{2}}$ or $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \theta^{\mathrm{BA}\mathbf{2}}$. This holds if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \varphi^{\mathrm{BA}\mathbf{2}}$.

Suppose φ is $\psi \to \theta$. Then $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \models \psi$ implies $\mathfrak{A} \models \theta$. By induction, this holds if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \psi^{\mathrm{BA}\mathbf{2}}$ implies $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \theta^{\mathrm{BA}\mathbf{2}}$. This holds if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \varphi^{\mathrm{BA}\mathbf{2}}$.

Suppose φ is $\forall x\psi$. Then $\mathfrak{A} \models \varphi$ if and only if, for every $a \in A$, $\mathfrak{A} \models \psi(a)$. By induction, this holds if and only if, for every $\hat{a} \in A^{\mathrm{BA}\mathbf{2}}$ such that if $\mathfrak{A} \models [\![\mathrm{E}(\hat{a})]\!] = 1$, $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \psi(\hat{a})^{\mathrm{BA}\mathbf{2}}$. This holds if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \varphi^{\mathrm{BA}\mathbf{2}}$.

Finally, suppose φ is $\exists x\psi$. Then $\mathfrak{A} \models \varphi$ if and only if there exists an $a \in A$ with $\mathfrak{A} \models \psi(a)$. By induction, this holds if and only if there exists $\hat{a} \in \hat{A}$ with $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models [\![\mathrm{E}(\hat{a})]\!] = 1$ and $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \psi(\hat{a})^{\mathrm{BA}\mathbf{2}}$. This holds if and only if $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \varphi^{\mathrm{BA}\mathbf{2}}$. $\dashv$

With this, we get that $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ satisfies all of $\Gamma^{\mathrm{BA}\mathbf{2}}$:

Corollary 2.13 *Suppose* $\mathfrak{A}$ *is an* $\mathcal{L}$*-model and* Γ *is a set of* $\mathcal{L}(A)$ *sentences with* $\mathfrak{A} \models \Gamma$. *Then* $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}\mathbf{2}}$.

Proof. This follows from Lemma 2.11 and Lemma 2.12. $\dashv$

We now show how to get matching $\mathcal{L}$-structures from models of $\emptyset^{\mathrm{BA}\mathbf{2}}$:

Definition 2.14 *If* $\mathfrak{B} \models \emptyset^{\mathrm{BA}\mathbf{2}}$*, then we define* $\mathfrak{B}_1$ *as the following* $\mathcal{L}$*-model:*

The domain B_1 *is* $\{b \in B : \mathfrak{B} \models [\![\mathrm{E}(b)]\!] = 1\}$.

For predicates P, $\mathfrak{B}_1 \models P(\mathbf{b})$ *if and only if* $\mathfrak{B} \models 1 \trianglelefteq [\![P(\mathbf{b})]\!]$.

For functions symbols f, $\mathfrak{B}_1 \models f(\mathbf{a}) = b$ *if and only if* $\mathfrak{B} \models f(\mathbf{a}) = b$.

Note that the map sending B to B_1 is essentially a projection operator. We now explicitly show the connection between the original model $\mathfrak{A}$ and the model $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$:

Lemma 2.15 *Let* $\mathfrak{A}$ *be a* $\mathcal{L}$*-model and let* $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ *be its translation. Then* $(\mathfrak{A}^{\mathrm{BA}\mathbf{2}})_1 \cong \mathfrak{A}$.

Proof. Recall that for each element $a \in A$, there is a corresponding element $\hat{a} \in A^{\mathrm{BA}\mathbf{2}}$ with $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models [\![\mathrm{E}(\hat{a})]\!] = 1$. Thus, we let $F : \mathfrak{A} \to (\mathfrak{A}^{\mathrm{BA}\mathbf{2}})_1$ by $a \mapsto \hat{a}$. This is clearly one-to-one and onto. Now, suppose $\mathfrak{A} \models P(\mathbf{a})$. Then $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models 1 \trianglelefteq [\![P(\hat{\mathbf{a}})]\!]$, so that $\mathfrak{A}_1^{\mathrm{BA}\mathbf{2}} \models P(\hat{\mathbf{a}})$. Now if $\mathfrak{A} \not\models P(\mathbf{a})$, then $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \not\models 1 \trianglelefteq [\![P(\hat{\mathbf{a}})]\!]$, so that $\mathfrak{A}_1^{\mathrm{BA}\mathbf{2}} \not\models P(\hat{\mathbf{a}})$. Finally, suppose $\mathfrak{A} \models f(\mathbf{a}) = b$. Then $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models f(\hat{\mathbf{a}}) = \hat{b}$, so that $\mathfrak{A}_1^{\mathrm{BA}\mathbf{2}} \models f(\hat{\mathbf{a}}) = \hat{b}$. $\dashv$

Thus, we can think of $\mathfrak{A}^{\mathrm{BA}\mathbf{2}}$ as the model $\mathfrak{A}$, but instead of sentences being true or false, we say that sentences have extent 1 or extent 0, respectively.

Next, we show that Lemma 2.15 has a corresponding result for models of $\Gamma^{\mathrm{BA}\mathbf{2}}$:

Lemma 2.16 *Let* $\mathfrak{B} \models \emptyset^{\mathrm{BA}\mathbf{2}}$*. Then* $(\mathfrak{B}_1)^{\mathrm{BA}\mathbf{2}} \cong \mathfrak{B}$.

Proof. Each element $b \in B_1$ has a corresponding element $\hat{b} \in B^{\mathrm{BA}\mathbf{2}}$. Thus, we set $F : \mathfrak{B} \to (\mathfrak{B}_1)^{\mathrm{BA}\mathbf{2}}$ with $F(\varpi) = \varpi$, $F(b) = \hat{b}$ for $b \in B_1$, $F(0) = 0$, and $F(1) = 1$. Clearly this map is one-to-one and onto. Clearly this map preserves predicates E and BA. Also, $F(x \oplus y) = F(x) \oplus F(y)$, and $F(x \upharpoonright y) = F(x) \upharpoonright F(y)$. Next, suppose $\mathfrak{B} \models f(\mathbf{a}) = b$ for $f \in \mathcal{L}$. Then $\mathfrak{B}_1 \models f(\mathbf{a}) = b$, so that $(\mathfrak{B}_1)^{\mathrm{BA}\mathbf{2}} \models f(\hat{\mathbf{a}}) = \hat{b}$. Now, suppose $\mathfrak{B} \models [\![P(\mathbf{b})]\!] = 1$. Then $\mathfrak{B}_1 \models P(\mathbf{b})$, so that $(\mathfrak{B}_1)^{\mathrm{BA}\mathbf{2}} \models [\![P(\hat{\mathbf{b}})]\!] = 1$. A similar proof holds if $\mathfrak{B} \models [\![P(\mathbf{b})]\!] = 0$. Thus, for all functions, $F(f(\mathbf{b})) = f(F(\mathbf{b}))$. $\dashv$

Corollary 2.17 *Let Γ be a set of $\mathcal{L}$-sentences and suppose $\mathfrak{B} \models \Gamma^{\mathrm{BA}\mathbf{2}}$. Then $\mathfrak{B}_1 \models \Gamma$.*

Proof. By Lemma 2.16, $(\mathfrak{B}_1)^{\mathrm{BA}\mathbf{2}} \cong \mathfrak{B}$, so that $(\mathfrak{B}_1)^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}\mathbf{2}}$. By Lemma 2.12, $\mathfrak{B}_1 \models \Gamma$. $\dashv$

Thus, there is a one-to-one correspondence between models of Γ and $\Gamma^{\mathrm{BA}\mathbf{2}}$.

We now show that if Γ and Δ are classically equivalent sets of sentences, then $\Gamma^{\mathrm{BA}\mathbf{2}}$ and $\Delta^{\mathrm{BA}\mathbf{2}}$ are classically equivalent.

Lemma 2.18 *Let $\Gamma \cup \{\gamma\}$ be a set of $\mathcal{L}$-sentences. Then $\Gamma \vdash_c \gamma$ if and only if $\Gamma^{\mathrm{BA}\mathbf{2}} \vdash_c \gamma^{\mathrm{BA}\mathbf{2}}$.*

Proof. Suppose $\Gamma \vdash_c \gamma$, and let $\mathfrak{B}$ be an arbitrary model of $\Gamma^{\mathrm{BA}\mathbf{2}}$. By Corollary 2.17, $\mathfrak{B}_1 \models \Gamma$. Thus, $\mathfrak{B}_1 \models \gamma$. Since $\mathfrak{B} \cong (\mathfrak{B}_1)^{\mathrm{BA}\mathbf{2}}$, we get that $\mathfrak{B} \models \gamma^{\mathrm{BA}\mathbf{2}}$. As $\mathfrak{B}$ was arbitrary, we get $\Gamma^{\mathrm{BA}\mathbf{2}} \models \gamma^{\mathrm{BA}\mathbf{2}}$. Conversely, suppose $\Gamma^{\mathrm{BA}\mathbf{2}} \vdash_c \gamma^{\mathrm{BA}\mathbf{2}}$, and let $\mathfrak{A} \models \Gamma$. Then $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}\mathbf{2}}$, so $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \gamma^{\mathrm{BA}\mathbf{2}}$. Hence $(\mathfrak{A}^{\mathrm{BA}\mathbf{2}})_1 \models \gamma$. As $\mathfrak{A} \cong (\mathfrak{A}^{\mathrm{BA}\mathbf{2}})_1$, we are done. $\dashv$

Thus, models of Γ and models of $\Gamma^{\mathrm{BA}\mathbf{2}}$ are essentially the same. The following is now obvious:

Proposition 2.19 *Let Γ be a set of $\mathcal{L}$-sentences. Then Γ has a model companion if and only if $\Gamma^{\mathrm{BA}\mathbf{2}}$ has a model companion. If Γ is model complete, then $\Gamma^{\mathrm{BA}\mathbf{2}}$ is model complete. Finally, Γ admits quantifier-elimination if and only if $\Gamma^{\mathrm{BA}\mathbf{2}}$ admits quantifier-elimination.*

Proof. The first two claims are straight-forward. The last claim follows from the observation that if $\varphi(x)$ is an $\mathcal{L}^{\mathrm{BA}}$-formula, then $\exists x \varphi(x)$ is equivalent to $\exists x(\mathrm{ST}(x) \wedge \varphi(x)) \vee \varphi(0) \vee \varphi(1)$. ⊣

Chapter 3

Boolean Indexed Models and Intuitionistic Logic

3.1 Introduction

In defining $\emptyset^{\mathrm{BA}\mathbf{2}}$, we also defined the subset $\emptyset^{\mathrm{BA}}$. We begin by describing models of $\emptyset^{\mathrm{BA}}$ which have elements that are neither Boolean nor structural. We then produce some basic properties satisfied by $\emptyset^{\mathrm{BA}}$. We then show that each model of $\emptyset^{\mathrm{BA}}$ has an associated Kripke model, where the underlying poset corresponds to the Boolean algebra. Similarly to the case for $\emptyset^{\mathrm{BA}\mathbf{2}}$, we introduce two new translations of each $\mathcal{L}$-sentence φ to an $\mathcal{L}^{\mathrm{BA}}$-sentence. The first translation we name φ^{BA}. In the presence of $\emptyset^{\mathrm{BA}\mathbf{2}}$, the sentences $\varphi^{\mathrm{BA}\mathbf{2}}$ and φ^{BA} are equivalent. The second translation we name $\varphi^{\mathrm{BA}}_{K}$. This second translation is closely related to intuitionistic forcing.

3.2 Clean and non-clean models

The set $\emptyset^{\mathrm{BA}}$ does not contain the axiom So2. Thus, if $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, there may be elements in A that are neither Boolean nor structural.

Definition 3.1 *Let $\mathfrak{A}$ be a model of $\emptyset^{\mathrm{BA}}$.*

1. *We call* $\mathfrak{A}$ **clean** *if it satisfies* So2. *If* $\mathfrak{A}$ *does not satisfy* So2, *we say it is* **non-clean**.

2. *We call an element* a *such that* $\mathfrak{A} \models \neg(\mathrm{ST}(a) \vee \mathrm{BA}(a))$ **chaff**. *We let* $\mathrm{CH}(A)$ *represent the chaff of* $\mathfrak{A}$.

We begin by showing that in working with models of $\emptyset^{\mathrm{BA}}$, we may ignore the presence of chaff.

Proposition 3.2 *Suppose* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$. *Then* A *is the disjoint union of* $\mathrm{ST}(A)$, $\mathrm{BA}(A)$, *and* $\mathrm{CH}(A)$.

Proof. Clearly these sets union to A. By So1, and the fact that $a \in \mathrm{CH}(A)$ if and only if $\mathfrak{A} \models \neg\,\mathrm{BA}(a) \wedge \neg\,\mathrm{ST}(a)$, these sets are disjoint. $\dashv$

With this, we give the following definition:

Definition 3.3 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$.

1. *We set* $\mathrm{BA}(\mathfrak{A})$ *to be the model over language* $\{\sqcap, \sqcup, -, 0, 1\}$ *with domain* $\mathrm{BA}(A)$, *that is,* $\mathrm{BA}(\mathfrak{A})$ *is the model consisting just of the underlying Boolean algebra.*

2. *As all* $\mathcal{L}^{\mathrm{BA}}$*-functions are only defined on* $\mathrm{ST}(A) \cup \mathrm{BA}(A)$, *we let* $\mathfrak{A}^\circ$ *be the submodel of* $\mathfrak{A}$ *with domain* $\mathrm{ST}(A) \cup \mathrm{BA}(A)$.

Proposition 3.4 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$. *Then* $\mathfrak{A}^\circ$ *is the largest clean submodel of* $\mathfrak{A}$, *and* $\mathfrak{A}^\circ \models \emptyset^{\mathrm{BA}}$.

Proof. Clearly $\mathfrak{A}^\circ$ is clean. Further, if $\mathfrak{A}' \subseteq \mathfrak{A}$ is another clean submodel, then for all $a \in A', \mathfrak{A} \models \mathrm{BA}(a) \vee \mathrm{ST}(a)$. Thus $\mathfrak{A}' \subseteq \mathfrak{A}$. As $\emptyset^{\mathrm{BA}}$ contains only universal axioms, the last claim follows. ⊣

For a given model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$ and an $\mathcal{L}^{\mathrm{BA}}$-sentence φ, we let φ° to be the usual relativization of φ to the submodel $\mathfrak{A}^\circ$. For convenience, we reproduce the definition below:

Definition 3.5 *Define φ° as follows:*

- *If φ is atomic, then φ° is simply φ.*
- *If φ is $\psi \wedge \theta$, then φ° is $\psi^\circ \wedge \theta^\circ$.*
- *If φ is $\psi \vee \theta$, then φ° is $\psi^\circ \vee \theta^\circ$.*
- *If φ is $\psi \to \theta$, then φ° is $\psi^\circ \to \theta^\circ$.*
- *If φ is $\exists x \psi$, then φ° is $\exists x((\mathrm{ST}(x) \vee \mathrm{BA}(x)) \wedge \psi^\circ)$.*
- *If φ is $\forall x \psi$, then φ° is $\forall x((\mathrm{ST}(x) \vee \mathrm{BA}(x)) \to \psi^\circ)$.*

The following result is obvious from this translation:

Lemma 3.6 *For an $\mathcal{L}^{\mathrm{BA}}$-sentence φ, we have $\mathrm{So2} \vdash_i \varphi \leftrightarrow \varphi^\circ$ and $\vdash_i \varphi^\circ \leftrightarrow (\varphi^\circ)\circ$.*

With this, we show how φ is related to φ°.

Lemma 3.7 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and φ be an $\mathcal{L}^{\mathrm{BA}}(A^\circ)$ sentence. Then the following are equivalent:*

1. *$\mathfrak{A} \models \varphi^\circ$;*

2. $\mathfrak{A}^\circ \models \varphi^\circ$*;*

3. $\mathfrak{A}^\circ \models \varphi$*.*

Proof. 1 and 2 are equivalent by the usual properties of relativization. 2 and 3 are equivalent by Lemma 3.6. $\dashv$

We now show that for our purposes, the submodel relation between models of $\emptyset^{\mathrm{BA}}$ is essentially the same as the submodel relation between models of $\emptyset^{\mathrm{BA}} \cup \{\mathrm{So2}\}$.

Proposition 3.8 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be models of* $\emptyset^{\mathrm{BA}}$*.*

1. *If* $\mathfrak{A}$ *embeds into* $\mathfrak{B}$*, then* $\mathfrak{A}^\circ$ *embeds into* $\mathfrak{B}^\circ$*.*
2. *Suppose* $\mathfrak{A}^\circ$ *embeds into* $\mathfrak{B}^\circ$*. Let* $\mathfrak{C}$ *be the model extending* $\mathfrak{B}^\circ$ *with* $\mathfrak{C}^\circ = \mathfrak{B}^\circ$ *and* $\mathrm{CH}(C) = \mathrm{CH}(A)$*. Then* $\mathfrak{A}$ *embeds into* $\mathfrak{C}$*, and* $\mathfrak{C} \models \emptyset^{\mathrm{BA}}$*.*

Proof. Part 1 is obvious. For 2, $\mathfrak{A}$ clearly embeds in $\mathfrak{C}$. The last claim follows from the fact that $\mathfrak{B} \models \emptyset^{\mathrm{BA}}$. $\dashv$

Thus, for models $\mathfrak{A}$ and $\mathfrak{B}$ of $\emptyset^{\mathrm{BA}}$, up to a minor change in the chaff of $\mathfrak{B}$, $\mathfrak{A}$ embeds into $\mathfrak{B}$ if and only if $\mathfrak{A}^\circ$ embeds into $\mathfrak{B}^\circ$. In proofs that a model $\mathfrak{A}$ embeds into a model of a particular theory, it suffices to show that $\mathfrak{A}^\circ$ embeds into a model of that particular theory. This will prove useful in finding the universal Horn fragments of certain theories extending $\emptyset^{\mathrm{BA}}$.

3.3 Basic properties of Boolean indexed models

We now show some basic properties which are derivable from $\emptyset^{\mathrm{BA}}$. For tuples $\mathbf{x}$ and $\mathbf{y}$ of the same length, we let $[\![\mathbf{x} = \mathbf{y}]\!]$ stand for $[\![x_0 = y_0]\!] \sqcap \ldots \sqcap [\![x_{n-1} = y_{n-1}]\!]$.

Lemma 3.9 *The following are consequences of* $\emptyset^{\mathrm{BA}}$*.*

1. $(\mathrm{E}([\![P(\mathbf{x}, y_1, \mathbf{z})]\!]) \wedge \mathrm{E}([\![P(\mathbf{x}, y_2, \mathbf{z})]\!])) \to ([\![y_1 = y_2]\!] \sqcap [\![P(\mathbf{x}, y_1, \mathbf{z})]\!] \trianglelefteq [\![P(\mathbf{x}, y_2, \mathbf{z})]\!])$.

2. $(\mathrm{E}(f(\mathbf{x}, y_1, \mathbf{z})) \wedge \mathrm{BA}([\![y_1 = y_2]\!])) \to f(\mathbf{x}, y_1 \restriction [\![y_1 = y_2]\!], \mathbf{z}) = f(\mathbf{x}, y_2 \restriction [\![y_1 = y_2]\!], \mathbf{z})$.

3. $\mathrm{ST}(x \restriction y) \to ([\![\mathrm{E}(x \restriction y)]\!] = [\![\mathrm{E}(x)]\!] \sqcap y)$.

4. $(\mathrm{ST}(x) \wedge \mathrm{BA}(y)) \to x \restriction y = (x \restriction ([\![\mathrm{E}(x)]\!] \sqcap y))$.

5. $[\![\mathrm{E}(x)]\!] \trianglelefteq y \to x \restriction y = x$.

6. $(\mathrm{E}(f(x_0, \ldots, x_{n-1})) \wedge \mathrm{BA}(y)) \to f(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1}) = f(\mathbf{x}) \restriction y$.

7. $(\mathrm{E}(f(x_0, \ldots, x_{n-1})) \wedge \bigwedge_{i<n} \mathrm{BA}(y_i)) \to f(x_0 \restriction y_0, \ldots, x_{n-1} \restriction y_{n-1}) = f(\mathbf{x}) \restriction \bigsqcap_{i<n} y_i$.

8. $(\mathrm{ST}(\mathbf{x}) \wedge \mathrm{BA}(y)) \to [\![P(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1})]\!] = [\![P(\mathbf{x})]\!] \sqcap y$.

9. $(([\![\mathrm{E}(x_1)]\!] = y_1 \sqcup y_2) \wedge ([\![\mathrm{E}(x_2)]\!] = y_1 \sqcup y_2) \wedge (x_1 \restriction y_1 = x_2 \restriction y_1) \wedge (x_1 \restriction y_2 = x_2 \restriction y_2)) \to x_1 = x_2$.

10. $((x_1 \restriction y_1 = x_2 \restriction y_1) \wedge (x_1 \restriction y_2 = x_2 \restriction y_2)) \to x_1 \restriction (y_1 \sqcup y_2) = x_2 \restriction (y_1 \sqcup y_2)$.

11. $\mathrm{E}(\mathbf{x} \oplus \mathbf{y}) \to f(\mathbf{x} \oplus \mathbf{y}) = f(\mathbf{x}) \oplus f(\mathbf{y})$.

12. $\mathrm{E}(f(\mathbf{x}) \restriction (y \sqcup z)) \to (f(\mathbf{x}) \restriction (y \sqcup z)) = (f(\mathbf{x}) \restriction y) \oplus (f(\mathbf{x}) \restriction z)$.

13. $\mathrm{E}(\mathbf{x} \oplus \mathbf{y}) \to [\![P(\mathbf{x} \oplus \mathbf{y})]\!] = [\![P(\mathbf{x})]\!] \sqcup [\![P(\mathbf{y})]\!]$.

14. $[\![\mathrm{E}(x)]\!] = 0 \to x = \varpi$.

15. $\mathrm{ST}(x) \to x \oplus x = x$.

16. $\mathrm{E}(x \oplus y) \to x \oplus y = y \oplus x$.

17. $(\mathrm{E}(x \oplus y) \wedge \mathrm{E}(y \oplus z) \wedge \mathrm{E}(x \oplus z)) \to (x \oplus y) \oplus z = x \oplus (y \oplus z)$.

Proof. For 1, by axiom Rs7, $y_1 \restriction [\![y_1 = y_2]\!] = y_2 \restriction [\![y_1 = y_2]\!]$. Thus

$$[\![P((\mathbf{x}, y_1, \mathbf{z}) \restriction [\![y_1 = y_2]\!])]\!] = [\![P((\mathbf{x}, y_2, \mathbf{z}) \restriction [\![y_1 = y_2]\!])]\!].$$

By axiom Rs5, this is equal to $[\![P(\mathbf{x}, y_2, \mathbf{z})]\!] \sqcap [\![y_1 = y_2]\!] \trianglelefteq [\![P(\mathbf{x}, y_2, \mathbf{z})]\!]$.

2 follows from axiom Rs7 and Fn2.

3 is a special case of Rs5.

For 4, by Rs3 and Rs4, $x \restriction ([\![\mathrm{E}(x)]\!] \sqcap y) = (x \restriction [\![\mathrm{E}(x)]\!]) \restriction y = x \restriction y$.

For 5, $x = x \restriction [\![\mathrm{E}(x)]\!] = x \restriction ([\![\mathrm{E}(x)]\!] \sqcap y) = x \restriction y$.

For 6, by axiom Fn3, $[\![\mathrm{E}(f(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1}))]\!] \trianglelefteq y$. Thus,

$$f(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1}) = (f(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1})) \restriction y = f(\mathbf{x} \restriction y) = f(\mathbf{x}) \restriction y$$

where the first equality follows from part 5, and the last two equalities hold by Rs6.

7 follows immediately from 6.

For 8, with Pr3 we have $[\![P(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1})]\!] \trianglelefteq y$, so that

$$[\![P(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1})]\!] = [\![P(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1})]\!] \sqcap y$$

Using Rs5,

$$[\![P(x_0, \ldots, x_i \restriction y, \ldots, x_{n-1})]\!] \sqcap y = [\![P(x_0 \restriction y, \ldots, x_i \restriction y, \ldots, x_{n-1} \restriction y)]\!]$$

. Again, by Rs5, this is equal to $[\![P(\mathbf{x})]\!] \sqcap y$.

For 9, we have that $[\![x_1 = x_2]\!] \sqcap y_1 = [\![x_1 \restriction y_1 = x_2 \restriction y_1]\!]$ by Rs5. Similarly, $[\![x_1 = x_2]\!] \sqcap y_2 = [\![x_1 \restriction y_2 = x_2 \restriction y_2]\!]$. Thus, $([\![x_1 = x_2]\!] \sqcap y_1) \sqcup ([\![x_1 = x_2]\!] \sqcap y_2) = [\![x_1 = x_2]\!] \sqcap (y_1 \sqcup y_2) = [\![x_1 \restriction (y_1 \sqcup y_2) = x_2 \restriction (y_1 \sqcup y_2)]\!]$. But since $x_1 \restriction y_1 = x_2 \restriction y_1$,

$[\![x_1 = x_2]\!] \sqcap y_1$ is $[\![\mathrm{E}(x_1)]\!] \sqcap y_1$. Similarly, $[\![x_1 = x_2]\!] \sqcap y_2 = [\![\mathrm{E}(x_1)]\!] \sqcap y_2$. Thus, $[\![x_1 = x_2]\!] \sqcap (y_1 \sqcup y_2) = ([\![x_1 = x_2]\!] \sqcap y_1) \sqcup ([\![x_1 = x_2]\!] \sqcap y_2) = ([\![\mathrm{E}(x_1)]\!] \sqcap y_1) \sqcup ([\![\mathrm{E}(x_1)]\!] \sqcap y_2) = [\![\mathrm{E}(x_1)]\!] \sqcap (y_1 \sqcup y_2) = [\![\mathrm{E}(x_1)]\!]$, where the first and third equality hold by Ba15. A similar argument gives us that $[\![x_1 = x_2]\!] \sqcap (y_1 \sqcup y_2) = [\![\mathrm{E}(x_2)]\!]$. Thus $x_1 = x_1 \restriction [\![\mathrm{E}(x_1)]\!] = x_1 \restriction [\![x_1 = x_2]\!] = x_2 \restriction [\![x_1 = x_2]\!] = x_2$.

10 follows with part 9 applied to the elements $x_1 \restriction (([\![\mathrm{E}(x_2)]\!] \sqcap y_1) \sqcup ([\![\mathrm{E}(x_2)]\!] \sqcap y_2))$ and $x_2 \restriction (([\![\mathrm{E}(x_1)]\!] \sqcap y_1) \sqcup ([\![\mathrm{E}(x_1)]\!] \sqcap y_2))$.

11 follows with parts 2 and 6.

12 follows with 10.

13 follows with parts 1 and 8.

For 14, suppose we have $[\![\mathrm{E}(x)]\!] = 0$. Now, $0 = [\![x = \varpi]\!] \sqcap 0$, since $[\![x = \varpi]\!]$ is a Boolean element. By axiom Rs5, $[\![x = \varpi]\!] \sqcap 0 = [\![x \restriction 0 = \varpi \restriction 0]\!]$. But, by axiom Rs3 and Ex4, $x \restriction 0 = x$ and $\varpi \restriction 0 = \varpi$. As a result, $[\![x \restriction 0 = \varpi \restriction 0]\!] = [\![x = \varpi]\!]$, so that $0 = [\![x = \varpi]\!]$. Thus, $x = x \restriction 0 = x \restriction [\![x = \varpi]\!] = \varpi \restriction [\![x = \varpi]\!] = \varpi \restriction 0 = \varpi$.

For 15, note that $\mathrm{E}([\![\mathrm{E}(x)]\!])$ implies $\mathrm{E}(x \oplus x)$, and $[\![\mathrm{E}(x \oplus x)]\!] = [\![\mathrm{E}(x)]\!]$. The result then follows by part 9.

16 and 17 both follow with part 9. $\dashv$

By Lemma 3.9, we have the following definition:

Definition 3.10 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$.*

1. *If a and b are such that $\mathfrak{A} \models \mathrm{E}(a \oplus b)$, we say a and b are* **pieceable**.

2. *By parts 15, 16m and 17, if $\{a_i : i < n\}$ is a set of pairwise pieceable elements, we write $\oplus_{i<n} a_i$ for $a_0 \oplus a_1 \oplus \ldots a_{n-1}$.*

3. *For tuples* $\mathbf{a}$ *and* $\mathbf{b}$ *of the same length such that* a_i *and* b_i *are pieceable for all* i, *we write* $\mathbf{a} \oplus \mathbf{b}$ *for the tuple* $a_0 \oplus b_0, \ldots, a_{n-1} \oplus b_{n-1}$.

3.4 The associated Kripke model of a Boolean indexed model

We show that for each Boolean indexed model there is an associated Kripke model.

Definition 3.11 *Let* $\mathfrak{A}$ *be a Boolean indexed model.*

1. *For* $p \in \mathrm{BA}(A)$, *recall that* A_p *is the set* $\{a \in \mathrm{ST}(A) \ : \ p = [\![\mathrm{E}(a)]\!]\}$, *that is, the set of elements that have extent* p.

2. *For* $p \in \mathrm{BA}(A)$, *we set* $\mathfrak{A}_p$ *to be the* $\mathcal{L}$*-structure on domain* A_p *as follows:*

   - *For every nullary predicate* P, *set* $\mathfrak{A}_p \models P$ *if and only if* $\mathfrak{A} \models p \trianglelefteq [\![P]\!]$.
   - *For every atom of positive arity* $P(t_0(\mathbf{x}), \ldots, t_{n-1}(\mathbf{x}))$ *in* $\mathcal{L}$ *and* $\mathbf{a} \in A_p$, $\mathfrak{A}_p \models P(t_0(\mathbf{a}), \ldots, t_{n-1}(\mathbf{a}))$ *if and only if* $\mathfrak{A} \models [\![P(t_0(\mathbf{a}), \ldots, t_{n-1}(\mathbf{a}))]\!] = p$. *Again, note this includes the functions* $[\![x = y]\!]$ *and* $[\![\mathrm{E}(x)]\!]$.
   - *For every function* $f(\mathbf{x})$ *in* $\mathcal{L}$ *and elements* $\mathbf{a}, a'$ *in* A_p, *set* $\mathfrak{A}_p \models f(\mathbf{a}) = a'$ *if and only if* $\mathfrak{A} \models f(\mathbf{a}) = a'$.

   We call this structure the **node structures at** p.

3. *Given* $p, q \in \mathrm{BA}(A)$ *with* $q \trianglelefteq p$, *we define a map* $\pi^p_q : \mathfrak{A}_p \to \mathfrak{A}_q$ *by* $\pi^p_q(a) = a \restriction q$.

We defined A_p to be set of elements that have extent exactly equal to p so that if $q \trianglelefteq p$, then the map π^p_q sends elements $a \in A_p$ to a unique image in A_q. We now prove that these maps act as functors.

Lemma 3.12 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$. For every $p \in \mathrm{BA}(A)$, π^p_p is the identity map, and if $r \trianglelefteq q \trianglelefteq p$, then $\pi^q_r \circ \pi^p_q = \pi^p_r$. Further, each π^p_q is a morphism.*

Proof. The first two claims follow from Rs3 and Rs4. Suppose $\mathfrak{A}_p \models P(\mathbf{a})$, where $\mathbf{a} \in A_p$. Then $\mathfrak{A} \models [\![P(\mathbf{a})]\!] = p$. By Rs5, $\mathfrak{A} \models [\![P(\mathbf{a} \restriction q)]\!] = q$. Thus, $\mathfrak{A}_q \models P(\pi^p_q(\mathbf{a}))$. Further, if $\mathfrak{A}_p \models f(\mathbf{a}) = a'$, then $\mathfrak{A} \models f(\mathbf{a}) = a'$. By Rs6, $\mathfrak{A} \models f(\mathbf{a} \restriction q) = a' \restriction q$. Thus, $f(\pi^p_q(\mathbf{a})) = \pi^p_q(f(\mathbf{a}))$. $\dashv$

Before describing the associated Kripke model, we include one last result on the morphisms π^p_q.

Lemma 3.13 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and let $p \in \mathrm{BA}(A)$, and suppose $\mathfrak{A} \models q \sqcup r = p$. Then the map $\langle \pi^p_q, \pi^p_r \rangle : \mathfrak{A}_p \to \mathfrak{A}_q \times \mathfrak{A}_r$ is an embedding. Additionally, if $q \sqcap r = 0$, then $\langle \pi^p_q, \pi^p_r \rangle$ is an isomorphism.*

Proof. By Lemma 3.12, it suffices to prove that for a set of elements $\mathbf{a} \in A_p$, if $\mathfrak{A}_q \times \mathfrak{A}_r \models P(\langle \pi^p_q, \pi^p_r \rangle(\mathbf{a}))$, then $\mathfrak{A}_p \models P(\mathbf{a})$. Thus, suppose $\mathfrak{A}_q \times \mathfrak{A}_r \models P(\langle \pi^p_q, \pi^p_r \rangle(\mathbf{a}))$. Then $\mathfrak{A}_q \models P(\mathbf{a} \restriction q)$ and $\mathfrak{A}_r \models P(\mathbf{a} \restriction r)$. Thus, $\mathfrak{A} \models [\![P(\mathbf{a} \restriction q)]\!] = q$ and $\mathfrak{A} \models [\![P(\mathbf{a} \restriction r)]\!] = r$. With Rs5, $\mathfrak{A} \models q \trianglelefteq [\![P(\mathbf{a})]\!]$ and $\mathfrak{A} \models r \trianglelefteq [\![P(\mathbf{a})]\!]$. As $[\![\mathrm{E}(\mathbf{a})]\!]$ is a Boolean element, we have that $p = q \sqcup r \trianglelefteq [\![P(\mathbf{a})]\!]$. Since $[\![\mathrm{E}(\mathbf{a})]\!] = p$, we have that $\mathfrak{A} \models [\![P(\mathbf{a})]\!] = p$. Thus $\mathfrak{A}_p \models P(\mathbf{a})$. A similar argument applies to atoms of the form $f(\mathbf{a}) = b$. Finally, if $q \sqcap r = 0$, then the embedding is clearly onto, so it is an isomorphism. $\dashv$

We are now ready to define the **associated Kripke model** $K(\mathfrak{A})^-$ for a Boolean indexed model $\mathfrak{A}$. For a definition of Kripke models, see [14, pages 77–80]. The underlying poset of $K(\mathfrak{A})^-$ is $\mathrm{BA}(A) \setminus \{0\}$, with $p \leq q$ if and only if $\mathfrak{A} \models q \trianglelefteq p$. Above each Boolean element p is the structure $\mathfrak{A}_p$. For all $p \leq q$ we have the morphism π^p_q. We note that the poset ordering $\leq$ has the reverse ordering of $\trianglelefteq$, so that there is a morphism which sends us from $\mathfrak{A}_p$ to $\mathfrak{A}_q$. As a consequence, for an atomic sentence δ over $\mathcal{L}(A_p)$, we have $p \Vdash \delta$ if and only if $\mathfrak{A}_p \models \delta$. Note that in our formulation, rather than equality being a congruence relation, it is true equality. This is different from the common intuitionistic use of equality just as a congruence relation. This is a standard variation on the definitions of [14, pages 77–80].

For technical reasons, we expand this Kripke model as follows: expand the underlying poset to all Boolean elements, including 0. Above 0, we put the structure $\mathfrak{A}_0$, which has domain $\{\varpi\}$, and $0 \Vdash \delta$ for all atoms δ, including $\bot$. We name this expansion $K(\mathfrak{A})$. This expansion is minimal by the following:

Proposition 3.14 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and let $p \in \mathrm{BA}(A)$ with $p \neq 0$. Let φ be an $\mathcal{L}(A_p)$-sentence. Then $(K(\mathfrak{A})^-, p) \Vdash \varphi$ if and only if $(K(\mathfrak{A}), p) \Vdash \varphi$.*

Proof. We proceed by induction on the complexity of φ. The only non-trivial cases are for implication and universal quantification. For implication, let φ be $\psi \to \theta$, where the result holds for ψ and θ. Suppose $(K(\mathfrak{A}), p) \Vdash \varphi$. Then for any $q \geq p$ where $(K(\mathfrak{A}), q) \Vdash \psi$, $(K(\mathfrak{A}), q) \Vdash \theta$. Then for any $q \neq 0$ with $p \leq q$, if $(K(\mathfrak{A})^-, q) \Vdash \psi$, then $(K(\mathfrak{A})^-, q) \Vdash \theta$. Thus $(K(\mathfrak{A})^-, p) \Vdash \varphi$. For the other direction, suppose $(K(\mathfrak{A})^-, p) \Vdash \varphi$, and let $p \leq q$. If $q \neq 0$ and $(K(\mathfrak{A}), q) \Vdash \psi$, then by induction we get $(K(\mathfrak{A}), q) \Vdash \theta$. If $q = 0$, then $(K(\mathfrak{A}), 0) \Vdash \bot$. So $(K(\mathfrak{A}), 0) \Vdash \varphi$. The case for universal quantification is also straightforward. $\dashv$

3.5 Translations to the associated language

For an $\mathcal{L}$-formula φ, we have already introduced a translation of φ to an $\mathcal{L}^{\mathrm{BA}}$-formula $\varphi^{\mathrm{BA}\mathbf{2}}$. In that context, a true sentence is associated with the Boolean element 1, and a false sentence is associated with 0. However, in a model with a larger Boolean algebra, there may be many different Boolean elements. Further, we can have structural elements which have an extent properly between 0 and 1. In the context of $\emptyset^{\mathrm{BA}}$, the translation $\varphi^{\mathrm{BA}\mathbf{2}}$ is no longer sufficient, as we can have sentences which do not have a truth value of 1, but also do not have a truth value of 0. We must generalize the translation $\varphi^{\mathrm{BA}\mathbf{2}}$ to a translation that will take into account that there are many different Boolean values.

We associate with a given sentence the set of Boolean elements which in some sense live below that sentence. We accomplish this with two new translations. One is φ^{BA}, which is obtained as follows: we introduce a translation of the form $y \trianglelefteq [\![\varphi]\!]$, where y is thought of as a Boolean element below φ. The translation φ^{BA} then is defined as $1 \trianglelefteq [\![\varphi]\!]$. The other translation, φ_K^{BA}, is related to the Kripke models of the last section.

3.5.1 The Boolean translation

We first introduce the notation $y \trianglelefteq [\![\varphi]\!]$, where y is a Boolean element and φ is an $\mathcal{L}$-formula. In the next chapter, we shall show that the elements y that satisfy $y \trianglelefteq [\![\varphi]\!]$ form an ideal. We then introduce the translation φ^{BA} based on this notation. This translation is a generalization of $\varphi^{\mathrm{BA}\mathbf{2}}$.

Definition 3.15 *Let φ be an $\mathcal{L}$-formula.*

1. *We define $y \trianglelefteq^0 [\![\varphi]\!]$ for all $\mathcal{L}$-formulas φ inductively as follows:*

   - *For an atomic sentence φ, which is of the form $P(\mathbf{t})$, define $y \trianglelefteq^0 [\![P(\mathbf{t})]\!]$ as $y \sqcap [\![P(\mathbf{t})]\!] = y$. Again, this includes the functions $[\![x = y]\!]$ and $[\![\mathrm{E}(x)]\!]$.*
   - *For conjunction: $y \trianglelefteq^0 [\![\varphi \wedge \psi]\!]$ is defined as $y \trianglelefteq^0 [\![\varphi]\!] \wedge y \trianglelefteq^0 [\![\psi]\!]$.*
   - *For disjunction: $y \trianglelefteq^0 [\![\varphi \vee \psi]\!]$ is defined as $\exists y_1, y_2((y_1 \sqcup y_2 = y) \wedge y_1 \trianglelefteq^0 [\![\varphi]\!] \wedge y_2 \trianglelefteq^0 [\![\psi]\!])$.*
   - *For implication: $y \trianglelefteq^0 [\![\varphi \to \psi]\!]$ is defined as $\forall z((z \trianglelefteq y \wedge z \trianglelefteq^0 [\![\varphi]\!]) \to z \trianglelefteq^0 [\![\psi]\!])$ (note that $z \trianglelefteq y$ is already defined for Boolean elements as $z \sqcap y = z$).*
   - *For the existential case: $y \trianglelefteq^0 [\![\exists x \varphi(x)]\!]$ is defined as $\exists x([\![\mathrm{E}(x)]\!] = y \wedge y \trianglelefteq^0 [\![\varphi(x)]\!])$.*
   - *For the universal case: $y \trianglelefteq^0 [\![\forall x \varphi(x)]\!]$ is defined as $\forall x(([\![\mathrm{E}(x)]\!] \trianglelefteq y) \to [\![\mathrm{E}(x)]\!] \trianglelefteq^0 [\![\varphi(x)]\!])$ (note that $[\![\mathrm{E}(x)]\!] \trianglelefteq y$ is defined as $[\![\mathrm{E}(x)]\!] \sqcap y = [\![\mathrm{E}(x)]\!]$, as $[\![\mathrm{E}(x)]\!]$ is a Boolean element).*

2. *We define $y \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ as $y \trianglelefteq^0 [\![\varphi(\mathbf{x})]\!] \wedge \mathrm{BA}(y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$.*

We include the last two conjuncts in the definition of $\trianglelefteq$ so that a sentence cannot be made vacuously true by putting elements of the incorrect sort into the various predicates and functions. This definition might seem unnecessarily cumbersome. For example, it appears as if we could have defined $y \trianglelefteq^0 [\![\varphi \to \psi]\!]$ as $y \sqcap [\![\varphi]\!] \trianglelefteq^0 [\![\psi]\!]$. However, we do not know that, for arbitrary $\mathcal{L}$-formulas φ, that $[\![\varphi]\!]$ is a Boolean element. In section 7.3, we show there are models of $\emptyset^{\mathrm{BA}}$ and formulas φ such that $[\![\varphi]\!]$ is indeed not a Boolean element. In the next chapter we describe in more detail formulas where $[\![\varphi]\!]$ is a Boolean element. We also show in section 7.4 such contexts where $[\![\varphi]\!]$ is a Boolean element for every $\mathcal{L}$-formula φ.

We can now define our translation φ^{BA}.

Definition 3.16 *1. For an $\mathcal{L}$-formula φ, we define φ^{BA} to be the $\mathcal{L}^{\mathrm{BA}}$-formula $1 \trianglelefteq [\![\varphi]\!]$.*

2. For a set of $\mathcal{L}$-sentences Γ, we define Γ^{BA} to be the set $\{\gamma^{\mathrm{BA}} : \gamma \in \Gamma\} \cup \emptyset^{\mathrm{BA}}$.

The use of $\trianglelefteq$ is sometimes cumbersome, as it requires us to verify that all elements are of the correct type. That is, in order to show that $\mathfrak{A} \models p \trianglelefteq [\![\varphi(\mathbf{x})]\!]$, we have to verify that p is a Boolean element and $\mathbf{x}$ are all structural elements. With this in mind, we introduce a translation $\trianglelefteq^1$ which is similar to $\trianglelefteq$. We show below that $\trianglelefteq^1$ and $\trianglelefteq$ are equivalent, but $\trianglelefteq^1$ is easier to verify than $\trianglelefteq$. In the next chapter there are many results which, given various assumptions, have as a conclusion $y \trianglelefteq [\![\varphi]\!]$. In such cases it is easier to prove $y \trianglelefteq^1 [\![\varphi]\!]$ than $y \trianglelefteq [\![\varphi]\!]$.

Definition 3.17 *We inductively define $y \trianglelefteq^1 [\![\varphi]\!]$ as follows:*

- *For the atomic, conjunction, disjunction, and existential case, we define $y \trianglelefteq^1 [\![\varphi]\!]$ the same way as $y \trianglelefteq^0 [\![\varphi]\!]$.*
- *For implication: $y \trianglelefteq^1 [\![\varphi \to \psi]\!]$ is defined as $\mathrm{BA}(y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i) \wedge \forall z((z \trianglelefteq y \wedge z \trianglelefteq^1 [\![\varphi]\!]) \to z \trianglelefteq^1 [\![\psi]\!])$.*
- *For the universal case: $y \trianglelefteq^1 [\![\forall x \varphi(x)]\!]$ is defined as $\mathrm{BA}(y) \wedge \forall x(([\![\mathrm{E}(x)]\!] \trianglelefteq y) \to [\![\mathrm{E}(x)]\!] \trianglelefteq^1 [\![\varphi(x)]\!])$.*

We now show that the translations $y \trianglelefteq [\![\varphi]\!]$ and $y \trianglelefteq^1 [\![\varphi]\!]$ are equivalent.

Lemma 3.18 *Let φ be an $\mathcal{L}$-formula. Then $\emptyset^{\mathrm{BA}} \vdash_i y \trianglelefteq [\![\varphi]\!] \leftrightarrow y \trianglelefteq^1 [\![\varphi]\!]$.*

Proof. It is straightforward to show that $\emptyset^{\mathrm{BA}} \vdash_i y \trianglelefteq [\![\varphi]\!] \rightarrow y \trianglelefteq^1 [\![\varphi]\!]$. Thus, it suffices to show $\emptyset^{\mathrm{BA}} \vdash_i y \trianglelefteq^1 [\![\varphi]\!] \rightarrow y \trianglelefteq [\![\varphi]\!]$. Now, as $y \trianglelefteq [\![\varphi]\!]$ is defined $y \trianglelefteq^0 [\![\varphi]\!] \wedge \mathrm{BA}(y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$, we must show that $y \trianglelefteq^1 [\![\varphi]\!]$ implies $\mathrm{BA}(y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$. Thus, suppose $y \trianglelefteq^1 [\![\varphi]\!]$. We proceed by induction on the complexity of φ.

Suppose φ is atomic. Then φ is of the form $P(\mathbf{t})$. Then $y \trianglelefteq^1 [\![\varphi]\!]$ is $y \sqcap [\![\varphi]\!] = y$. By axiom Ba3, that means that $\mathrm{BA}(y) \wedge \mathrm{BA}([\![\varphi]\!])$. By Pr1, we get $\bigwedge_{i<n} \mathrm{ST}(x_i)$.

We now suppose the result holds for ψ and γ.

Suppose φ is $\psi \wedge \gamma$. Then $y \trianglelefteq^1 [\![\varphi]\!]$ is interpreted as $y \trianglelefteq^1 [\![\psi]\!] \wedge y \trianglelefteq^1 [\![\gamma]\!]$. By induction, we get $\mathrm{BA}(y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$.

Suppose φ is $\psi \vee \gamma$. Then $y \trianglelefteq^1 [\![\varphi]\!]$ is interpreted as $\exists y_1, y_2(y_1 \sqcup y_2 = y \wedge y_1 \trianglelefteq^1 [\![\psi]\!] \wedge y_2 \trianglelefteq^1 [\![\theta]\!])$. By axiom Ba4, we get $\mathrm{BA}(y)$. By induction, we also get $\bigwedge_{i<n} \mathrm{ST}(x_i)$.

Suppose φ is $\psi \rightarrow \gamma$. Then $y \trianglelefteq^1 [\![\varphi]\!]$ is interpreted as $\mathrm{BA}(y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i) \wedge \forall z((z \trianglelefteq y \wedge z \trianglelefteq^1 [\![\varphi]\!]) \rightarrow z \trianglelefteq^1 [\![\psi]\!])$. Clearly then $\mathrm{BA}(y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$ holds.

Suppose φ is $\forall x \psi$. Then $y \trianglelefteq^1 [\![\varphi]\!]$ is interpreted as $\mathrm{BA}(y) \wedge \forall x(([\![\mathrm{E}(x)]\!] \trianglelefteq y) \rightarrow [\![\mathrm{E}(x)]\!] \trianglelefteq^1 [\![\varphi(x)]\!])$. Clearly $\mathrm{BA}(y)$ then holds. Now, as $\emptyset^{\mathrm{BA}} \vdash_i [\![\mathrm{E}(\varpi)]\!] = 0 \trianglelefteq y$, by supposition, we get that $[\![\mathrm{E}(\varpi)]\!] = 0 \trianglelefteq^1 [\![\varphi(\varpi)]\!]$. By induction, $0 \trianglelefteq [\![\varphi]\!]$ holds. Thus, we get that $\emptyset^{\mathrm{BA}} \vdash_i \bigwedge_{i<n} \mathrm{ST}(x_i)$.

Finally, suppose φ is $\exists x \psi$. Then $y \trianglelefteq^1 [\![\varphi]\!]$ is interpreted as $\exists x([\![\mathrm{E}(x)]\!] = y \wedge y \trianglelefteq^1 [\![\psi(x)]\!])$. By axioms Pr1 and Pr2, we have that $\mathrm{BA}([\![\mathrm{E}(x)]\!])$, hence $\mathrm{BA}(y)$ holds. Since $y \trianglelefteq^1 [\![\psi(x)]\!]$, by induction we get $\emptyset^{\mathrm{BA}} \vdash_i \bigwedge_{i<n} \mathrm{ST}(x_i)$. $\dashv$

Thus, we get that $y \trianglelefteq [\![\varphi]\!]$ and $y \trianglelefteq^1 [\![\varphi]\!]$ are equivalent. In general, we use the notation $y \trianglelefteq [\![\varphi]\!]$, but prove $y \trianglelefteq^1 [\![\varphi]\!]$. The usefulness of $\trianglelefteq^1$ will become clear in the next chapter.

Since $\neg\varphi$ is $\varphi \to \bot$, the formula $y \trianglelefteq^0 [\![\neg\varphi]\!]$ becomes $\forall z((z \trianglelefteq y \wedge z \trianglelefteq^0 [\![\varphi]\!]) \to z \trianglelefteq [\![\bot]\!])$. In the presence of $\emptyset^{\mathrm{BA}}$, $[\![\bot]\!] = 0$, so that $\emptyset^{\mathrm{BA}}$ proves that $y \trianglelefteq^0 [\![\neg\varphi]\!]$ is equivalent to $\forall z((z \trianglelefteq y \wedge z \trianglelefteq^0 [\![\varphi]\!]) \to z = 0)$.

We now show that φ^{BA} is a generalization of the translation $\varphi^{\mathrm{BA}\mathbf{2}}$:

Proposition 3.19 *Let φ be an $\mathcal{L}$-formula. Then $\emptyset^{\mathrm{BA}\mathbf{2}} \vdash_i \varphi^{\mathrm{BA}\mathbf{2}} \leftrightarrow \varphi^{\mathrm{BA}}$.*

Proof. Use that 0 and 1 are the only Boolean elements. $\dashv$

3.5.2 The Kripke translation

We now introduce the notation $y \trianglelefteq_K [\![\varphi]\!]$ similar to $y \trianglelefteq [\![\varphi]\!]$. We then define the translation φ_K^{BA} based on this. This translation is essentially identical to intuitionistic forcing as discussed in 3.4.

Definition 3.20 *1. We define $y \trianglelefteq_K^0 [\![\varphi]\!]$ for all $\mathcal{L}$-formulas inductively as follows:*

- *For an atomic sentence φ, which is of the form $P(\mathbf{t})$, define $y \trianglelefteq_K^0 [\![P(\mathbf{t})]\!]$ as $y \sqcap [\![P(\mathbf{t})]\!] = y$.*
- *For conjunction: $y \trianglelefteq_K^0 [\![\varphi \wedge \psi]\!]$ is defined as $y \trianglelefteq_K^0 [\![\varphi]\!] \wedge y \trianglelefteq_K^0 [\![\psi]\!]$.*
- *For disjunctions, we define $y \trianglelefteq_K^0 [\![\varphi \vee \psi]\!]$ as $y \trianglelefteq_K^0 [\![\varphi]\!] \vee y \trianglelefteq_K^0 [\![\psi]\!]$.*
- *For implication: $y \trianglelefteq_K^0 [\![\varphi \to \psi]\!]$ is defined as $\forall z((z \trianglelefteq y \wedge z \trianglelefteq_K^0 [\![\varphi]\!]) \to z \trianglelefteq_K^0 [\![\psi]\!])$.*

- *For the existential case:* $y \trianglelefteq_K^0 [\![\exists x\varphi(x)]\!]$ *is defined as* $\exists x([\![\mathrm{E}(x)]\!] = y \wedge y \trianglelefteq_K^0 [\![\varphi(x)]\!])$.
- *For the universal case:* $y \trianglelefteq_K^0 [\![\forall x\varphi(x)]\!]$ *is defined as* $\forall x(([\![\mathrm{E}(x)]\!] \trianglelefteq y) \rightarrow [\![\mathrm{E}(x)]\!] \trianglelefteq_K^0 [\![\varphi(x)]\!])$.

2. *We define* $y \trianglelefteq_K [\![\varphi(\mathbf{x})]\!]$ *as* $y \trianglelefteq_K^0 [\![\varphi(\mathbf{x})]\!] \wedge \mathrm{BA}(y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$.

3. *For an* $\mathcal{L}$*-formula* φ*, we define* φ^{BA} *to be the* $\mathcal{L}^{\mathrm{BA}}$*-formula* $1 \trianglelefteq_K [\![\varphi]\!]$.

4. *For a set* Γ *of* $\mathcal{L}$*-sentences, we define* Γ_K^{BA} *to be the set* $\{\gamma_K^{\mathrm{BA}} : \gamma \in \Gamma\} \cup \emptyset^{\mathrm{BA}}$.

We note that the only difference between $\trianglelefteq$ and $\trianglelefteq_K$ came in the disjunction case. When defining forcing in a Kripke model, one of the main differences comes in the universal case. However, we defined $y \trianglelefteq [\![\forall x\varphi]\!]$ to say that if an element lives below y, the extent of that element is below the extent of φ when that element is substituted in φ. Thus, our definition of $y \trianglelefteq [\![\forall x\varphi]\!]$ already matches the definition of forcing. We demonstrate this below.

With this we get the following result:

Proposition 3.21 *1. Let* φ *be an* $\mathcal{L}$*-sentence in which disjunction does not occur. Then* $y \trianglelefteq_K [\![\varphi]\!]$ *and* $y \trianglelefteq [\![\varphi]\!]$ *are the same formula.*

2. Let Γ *be a set of* $\mathcal{L}$*-sentences such that disjunction does not occur in any* $\gamma \in \Gamma$*. Then* $\Gamma_K^{\mathrm{BA}} = \Gamma^{\mathrm{BA}}$.

Proof. Part 1 follows from the definitions of $\trianglelefteq_K$ and $\trianglelefteq$. Part 2 follows from part 1. $\dashv$

We now show that φ_K^{BA} is also a generalization of the translation $\varphi^{\mathrm{BA}\mathbf{2}}$:

Proposition 3.22 *Let φ be an $\mathcal{L}$-formula. Then* $\emptyset^{\mathrm{BA}\mathbf{2}} \vdash_i \varphi^{\mathrm{BA}\mathbf{2}} \leftrightarrow \varphi_K^{\mathrm{BA}}$.

Proof. Again, this follows from the fact that 0 and 1 are the only Boolean elements. $\dashv$

We now explicitly show how $\trianglelefteq_K$ relates to intuitionistic logic:

Proposition 3.23 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, $p \in \mathrm{BA}(A)$, *and* φ *an* $\mathcal{L}(A_p)$*-sentence. Then* $p \trianglelefteq_K [\![\varphi]\!]$ *if and only if* $(K(\mathfrak{A}), p) \Vdash \varphi$.

Proof. The forcing definition in [14, pages 77–80] and the inductive definition of $\trianglelefteq_K$ are essentially the same. The result then follows by a straight-forward induction on the complexity of φ. $\dashv$

We get an immediate corollary:

Corollary 3.24 *Let* $\Gamma \cup \{\varphi\}$ *be a set of* $\mathcal{L}$*-sentences, with* $\Gamma \vdash_i \varphi$. *Then* $\Gamma_K^{\mathrm{BA}} \vdash_c 1 \trianglelefteq_K [\![\varphi]\!]$.

Proof. First, $\Gamma_K^{\mathrm{BA}} \vdash_c 1 \trianglelefteq_K [\![\gamma]\!]$ for all $\gamma \in \Gamma$. By Proposition 3.23, $1 \Vdash \gamma$ for each γ. By Kripke's completeness theorem, $1 \Vdash \varphi$. Again, by Proposition 3.23, $\Gamma_K^{\mathrm{BA}} \vdash_c 1 \trianglelefteq_K [\![\varphi]\!]$. $\dashv$

Thus, $\trianglelefteq_K$ preserves intuitionistic derivability. In section 7.2, we will show that $\trianglelefteq$ also preserves intuitionistic derivability. This proof is not as straightforward.

Chapter 4

Point Models and Universal Horn Fragments

Let Γ be a set of universal $\mathcal{L}$-sentences. For each $\gamma \in \Gamma$, we have that γ^{BA} is equivalent to a universal Horn sentence over $\emptyset^{\mathrm{BA}}$. If ψ and θ are universal $\mathcal{L}$-sentences, then ψ and θ are classically equivalent if and only if ψ^{BA} and θ^{BA} are classically equivalent over $\emptyset^{\mathrm{BA}}$. Further, we show that $\Gamma^{\mathrm{BA}\mathbf{2}}$ is a universal theory, and the universal Horn fragment $(\Gamma^{\mathrm{BA}\mathbf{2}})_{\mathrm{UH}}$ of $\Gamma^{\mathrm{BA}\mathbf{2}}$ is axiomatized by Γ^{BA}. Also, Γ^{BA} and $(\Gamma_{\mathrm{UH}})^{\mathrm{BA}}$ axiomatize the same theory. In particular, the universal Horn fragment of $\emptyset^{\mathrm{BA}\mathbf{2}}$ is axiomatized by simply removing the axioms Ba18 and So2.

We do this by first proving some technical lemmas which show that each $\mathcal{L}$-formula has a corresponding ideal on the Boolean algebra. We introduce a generalization of the node structures $\mathfrak{A}_p$. With this, we are able to show that if Γ and Δ are Π^0_2 theories which are classically equivalent, then Γ^{BA} and Δ^{BA} are classically equivalent.

4.1 Local and pointwise truth of $\mathcal{L}$-sentences

For an $\mathcal{L}$-sentence φ, we consider two ways in which φ can be true in a model $\mathfrak{A}$ of $\emptyset^{\mathrm{BA}}$. The first is of the form $p \trianglelefteq [\![\varphi]\!]$, which we call local truth over p. The second is of the form $\mathfrak{A}_p \models \varphi$, which we call pointwise truth at p. We begin by showing that for an $\mathcal{L}$-sentence, local truth corresponds with ideals on the underlying Boolean algebra. Next, we generalize the notion of pointwise truth from elements of the Boolean algebra to filters on the Boolean algebra. Finally, we connect local truth of Π^0_2 formulas with their pointwise truth.

4.1.1 Local truth of $\mathcal{L}$-sentences

Before showing that $\mathcal{L}$-formulas correspond to ideals on the Boolean algebra, we need to develop some properties. We first introduce a new notation:

Definition 4.1 *1. Let φ be an $\mathcal{L}$-formula. Then we write $z \trianglelefteq [\![\varphi(x \restriction y)]\!]$ for the substitution $z \trianglelefteq [\![\varphi(x)]\!]\ [x/(x \restriction y)]$.*

2. Let φ be an $\mathcal{L}$-formula. Then we write $z \trianglelefteq [\![\varphi(x \oplus y)]\!]$ for the substitution $z \trianglelefteq [\![\varphi(x)]\!]\ [x/(x \oplus y)]$.

Note that Definition 4.1 may lead to ambiguities if the restriction function symbol $x \restriction y$ or the piecing together function symbol $x \oplus y$ already occur in $\mathcal{L}$. In our context, we will not encounter this problem.

Lemma 4.2 *Let $\varphi(\mathbf{x})$ be an $\mathcal{L}$-formula. Then $\emptyset^{\mathrm{BA}} \vdash_i z \trianglelefteq y \rightarrow (z \trianglelefteq [\![\varphi(\mathbf{x})]\!] \leftrightarrow z \trianglelefteq [\![\varphi(\mathbf{x} \restriction y)]\!])$.*

Proof. We proceed by induction on the complexity of φ. If φ is atomic, then the result holds by Rs5 and Rs6.

Suppose φ is $\psi \wedge \theta$. Then $z \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ means that $z \trianglelefteq [\![\psi(\mathbf{x}) \wedge \theta(\mathbf{x})]\!]$, or $z \trianglelefteq [\![\psi(\mathbf{x})]\!]$ and $z \trianglelefteq [\![\theta(\mathbf{x})]\!]$. By induction, this holds if and only if $z \trianglelefteq [\![\psi(\mathbf{x}\restriction y)]\!]$ and $z \trianglelefteq [\![\theta(\mathbf{x}\restriction y)]\!]$, or $z \trianglelefteq [\![\varphi(\mathbf{x}\restriction y)]\!]$.

Suppose φ is $\psi \vee \theta$. Then $z \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ is interpreted as $\exists z_1, z_2(z_1 \sqcup z_2 = z \wedge z_1 \trianglelefteq [\![\psi(\mathbf{x})]\!] \wedge z_2 \trianglelefteq [\![\theta(\mathbf{x})]\!])$. Now, $z_1 \trianglelefteq y$ and by induction, $z_1 \trianglelefteq [\![\psi(\mathbf{x})]\!]$ if and only if $z_1 \trianglelefteq [\![\psi(\mathbf{x}\restriction y)]\!]$. Similarly, $z_2 \trianglelefteq [\![\theta(\mathbf{x})]\!]$ if and only if $z_2 \trianglelefteq [\![\theta(\mathbf{x}\restriction y)]\!]$. Hence, this is equivalent to $\exists z_1, z_2(z_1 \sqcup z_2 = z \wedge z_1 \trianglelefteq [\![\psi(\mathbf{x}\restriction y)]\!] \wedge z_2 \trianglelefteq [\![\theta(\mathbf{x}\restriction y)]\!])$. Thus, $z \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ is equivalent to $z \trianglelefteq [\![\varphi(\mathbf{x}\restriction y)]\!]$.

Suppose φ is $\psi \rightarrow \theta$ and $z \trianglelefteq y$. Note that if $z \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ holds, then $\mathrm{BA}(z) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$ also holds, and $z \trianglelefteq [\![\varphi(\mathbf{x}\restriction y)]\!]$ implies the same. Now, $z \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ is $z \trianglelefteq [\![\psi(\mathbf{x}) \rightarrow \theta(\mathbf{x})]\!]$, which, by the discussion above, becomes $\forall w \trianglelefteq z(w \trianglelefteq [\![\psi(\mathbf{x})]\!] \rightarrow w \trianglelefteq [\![\theta(\mathbf{x})]\!])$. Then $w \trianglelefteq z$ implies $w \trianglelefteq y$, so by induction, this holds if and only if $\forall w \trianglelefteq z(w \trianglelefteq [\![\psi(\mathbf{x}\restriction y)]\!] \rightarrow w \trianglelefteq [\![\theta(\mathbf{x}\restriction y)]\!])$. Thus, $z \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ is equivalent to $z \trianglelefteq [\![\varphi(\mathbf{x}\restriction y)]\!]$. Note, as $\neg\psi$ is defined as $\psi \rightarrow \bot$, this case shows the Lemma holds when φ is $\neg\psi$.

Suppose φ is $\forall x \psi(x)$ and $z \trianglelefteq y$. Again, note that $z \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ implies $\mathrm{BA}(z)$, as does $z \trianglelefteq [\![\varphi(\mathbf{x}\restriction y)]\!]$. Then, as in the implication case, $z \trianglelefteq [\![\forall x \psi(\mathbf{x}, x)]\!]$ becomes $\forall x([\![\mathrm{E}(x)]\!] \trianglelefteq z \rightarrow [\![\mathrm{E}(x)]\!] \trianglelefteq [\![\psi(\mathbf{x}, x)]\!])$. Now, $[\![\mathrm{E}(x)]\!] \trianglelefteq z$ implies $[\![\mathrm{E}(x)]\!] \trianglelefteq y$, so by induction, this is equivalent to $\forall x([\![\mathrm{E}(x)]\!] \trianglelefteq z \rightarrow [\![\mathrm{E}(x)]\!] \trianglelefteq [\![\varphi(\mathbf{x}\restriction y, x\restriction y)]\!])$. Thus, $z \trianglelefteq [\![\varphi(\mathbf{x}, x)]\!]$ is equivalent to $z \trianglelefteq [\![\varphi(\mathbf{x}\restriction y, x\restriction y)]\!]$.

Finally, suppose φ is $\exists x \psi(x)$. Then $z \trianglelefteq [\![\varphi(\mathbf{x}, x)]\!]$ translates to $\exists x([\![\mathrm{E}(x)]\!] = z \wedge z \trianglelefteq [\![\psi(\mathbf{x}, x)]\!])$. By induction, this is equivalent to $\exists x([\![\mathrm{E}(x)]\!] = z \wedge z \trianglelefteq [\![\psi(\mathbf{x}\restriction y, x\restriction y)]\!])$, so that $z \trianglelefteq [\![\varphi(\mathbf{x}, x)]\!]$ is equivalent to $z \trianglelefteq [\![\varphi(\mathbf{x}\restriction y, x\restriction y)]\!]$. $\dashv$

We get two corollaries from this result:

Corollary 4.3 $\emptyset^{\text{BA}} \vdash_i y \trianglelefteq [\![\varphi(\mathbf{x})]\!] \leftrightarrow y \trianglelefteq [\![\varphi(\mathbf{x}\restriction y)]\!]$

Proof. Obvious from Lemma 4.2. $\dashv$

Corollary 4.4 *Let $\mathfrak{A}$ be a model of $\emptyset^{\text{BA}}$, $\varphi(\mathbf{x})$ an $\mathcal{L}$-formula, and suppose* **a, b** *and p are elements such that $\mathfrak{A} \models \mathbf{b}\restriction p = \mathbf{a}\restriction p$. Then $\mathfrak{A} \models p \trianglelefteq [\![\varphi(\mathbf{b})]\!] \leftrightarrow p \trianglelefteq [\![\varphi(\mathbf{a})]\!]$.*

Proof. By the previous corollary, $p \trianglelefteq [\![\varphi(\mathbf{a})]\!]$ if and only if $p \trianglelefteq [\![\varphi(\mathbf{a}\restriction p)]\!]$ if and only if $p \trianglelefteq [\![\varphi(\mathbf{b}\restriction p)]\!]$ if and only if $p \trianglelefteq [\![\varphi(\mathbf{b})]\!]$. $\dashv$

We now show that $\trianglelefteq$ as a Boolean relation and $\trianglelefteq$ as a relation between Boolean elements and extents of formulas are compatible.

Lemma 4.5 *Let φ be an $\mathcal{L}$-formula. Then $\emptyset^{\text{BA}} \vdash_i z \trianglelefteq y \wedge y \trianglelefteq [\![\varphi]\!] \to z \trianglelefteq [\![\varphi]\!]$.*

Proof. We proceed by an induction on the complexity of φ. If φ is atomic, then $y \trianglelefteq [\![\varphi]\!]$ means that $y \sqcap [\![\varphi]\!] = y$. Since $z \sqcap y = z$, we have $z \sqcap [\![\varphi]\!] = (z \sqcap y) \sqcap [\![\varphi]\!] = z \sqcap (y \sqcap [\![\varphi]\!]) = z \sqcap y = z$, so that $z \trianglelefteq [\![\varphi]\!]$.

Suppose φ is $\psi \wedge \theta$. Then $y \trianglelefteq [\![\varphi]\!]$ is $y \trianglelefteq [\![\psi \wedge \theta]\!]$, so $y \trianglelefteq [\![\psi]\!]$ and $y \trianglelefteq [\![\theta]\!]$. By induction, if $z \trianglelefteq y$, then $z \trianglelefteq [\![\psi]\!]$ and $z \trianglelefteq [\![\theta]\!]$, so $z \trianglelefteq [\![\psi \wedge \theta]\!]$.

Suppose φ is $\psi \vee \theta$. Then $y \trianglelefteq \varphi$ is $y \trianglelefteq [\![\psi \vee \theta]\!]$, which translates to there exist y_1 and y_2 such that $y_1 \sqcup y_2 = y$ and $y_1 \trianglelefteq [\![\psi]\!]$ and $y_2 \trianglelefteq [\![\theta]\!]$. Suppose $z \trianglelefteq y$, and let $y_1' = y_1 \sqcap z$ and $y_2' = y_2 \sqcap z$. Then $y_1' \trianglelefteq y_1$ and $y_2' \trianglelefteq y_2$, so, by induction, $y_1' \trianglelefteq [\![\psi]\!]$ and $y_2' \trianglelefteq [\![\theta]\!]$. Further, $y_1' \sqcup y_2' = z$ by axiom Ba15. Thus $z \trianglelefteq [\![\psi \vee \theta]\!]$.

Suppose φ is $\psi \to \theta$. First, note that $z \trianglelefteq y \wedge y \trianglelefteq [\![\varphi]\!]$ implies $\text{BA}(y) \wedge \bigwedge_{i<n} \text{ST}(x_i)$. So $y \trianglelefteq [\![\varphi]\!]$ becomes $\forall w \trianglelefteq y(w \trianglelefteq [\![\psi]\!] \to w \trianglelefteq [\![\theta]\!])$. Suppose that $z \trianglelefteq y$. We need to

show that $z \trianglelefteq [\![\varphi]\!]$, that is, $\forall w \trianglelefteq z(w \trianglelefteq [\![\psi]\!] \to w \trianglelefteq [\![\theta]\!])$. Now, if $w \trianglelefteq z$, then certainly $w \trianglelefteq y$. So if $w \trianglelefteq [\![\psi]\!]$, then $w \trianglelefteq [\![\theta]\!]$. Thus, $z \trianglelefteq [\![\varphi]\!]$.

Suppose φ is $\forall x\psi(x)$. Again, note that $z \trianglelefteq y \wedge y \trianglelefteq [\![\varphi]\!]$ implies $\mathrm{BA}(y)$. Then $y \trianglelefteq [\![\varphi]\!]$ becomes $\forall x([\![\mathrm{E}(x)]\!] \trianglelefteq y \to [\![\mathrm{E}(x)]\!] \trianglelefteq [\![\psi(x)]\!])$. We need to show that $\forall x([\![\mathrm{E}(x)]\!] \trianglelefteq z \to [\![\mathrm{E}(x)]\!] \trianglelefteq [\![\psi(x)]\!])$. If $z \trianglelefteq y$, and if x is such that $[\![\mathrm{E}(x)]\!] \trianglelefteq z$, then $[\![\mathrm{E}(x)]\!] \trianglelefteq y$. By supposition, $z \trianglelefteq [\![\psi(x)]\!]$.

Finally, suppose that φ is $\exists x\psi(x)$. Then $y \trianglelefteq [\![\varphi]\!]$ translates to $\exists x([\![\mathrm{E}(x)]\!] = y \wedge y \trianglelefteq [\![\psi(x)]\!])$. If x is such an element, then by induction, $z \trianglelefteq [\![\psi(x)]\!]$. By Corollary 4.3, this implies that $z \trianglelefteq [\![\psi(x \restriction z)]\!]$, and so $z \trianglelefteq [\![\exists x\psi(x)]\!]$. $\dashv$

We now show that, for an $\mathcal{L}$-formula φ, the set of Boolean elements that are below the extent of φ is closed under join.

Lemma 4.6 *If φ is an $\mathcal{L}$-formula, then $\emptyset^{\mathrm{BA}} \vdash_i y \trianglelefteq [\![\varphi]\!] \wedge z \trianglelefteq [\![\varphi]\!] \to y \sqcup z \trianglelefteq [\![\varphi]\!]$.*

Proof. We again proceed by induction on φ. If φ is atomic, then $[\![\varphi]\!]$ has a Boolean value, so that the result is obvious.

If φ is $\psi \wedge \theta$, and both $y \trianglelefteq [\![\varphi]\!]$ and $z \trianglelefteq [\![\varphi]\!]$, then $y \trianglelefteq [\![\psi]\!]$, $z \trianglelefteq [\![\psi]\!]$, $y \trianglelefteq [\![\theta]\!]$ and $z \trianglelefteq [\![\theta]\!]$, so by induction, $y \sqcup z \trianglelefteq [\![\psi]\!]$ and $y \sqcup z \trianglelefteq [\![\theta]\!]$. Thus $y \sqcup z \trianglelefteq [\![\psi \wedge \theta]\!]$.

If φ is $\psi \vee \theta$, and $y \trianglelefteq [\![\varphi]\!]$, there exist y_1, y_2 such that $y = y_1 \sqcup y_2$, $y_1 \trianglelefteq [\![\psi]\!]$, and $y_2 \trianglelefteq [\![\theta]\!]$. Let z_1, z_2 be similar elements for z. Then, by induction $y_1 \sqcup z_1 \trianglelefteq [\![\psi]\!]$ and $y_2 \sqcup z_2 \trianglelefteq [\![\psi]\!]$. Then $y_1 \sqcup z_1 \sqcup y_2 \sqcup z_2 = y \sqcup z \trianglelefteq [\![\psi \vee \theta]\!]$.

If φ is $\psi \to \theta$, then $y \trianglelefteq [\![\varphi]\!] \wedge z \trianglelefteq [\![\varphi]\!]$ implies $\mathrm{BA}(y) \wedge \mathrm{BA}(z) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$. Now, suppose $y \trianglelefteq [\![\varphi]\!]$ and $z \trianglelefteq [\![\varphi]\!]$. Let $p \trianglelefteq y \sqcup z$, and suppose $p \trianglelefteq [\![\psi]\!]$. Then, $p \sqcap y \trianglelefteq p$, so, by Lemma 4.5, $p \sqcap y \trianglelefteq [\![\psi]\!]$. Since $y \trianglelefteq [\![\varphi]\!]$, we have that $p \sqcap y \trianglelefteq [\![\theta]\!]$. Similarly, we have that $p \sqcap z \trianglelefteq [\![\psi]\!]$. Thus, by induction, $(p \sqcap y) \sqcup (p \sqcap z) = p \trianglelefteq [\![\theta]\!]$, so $y \sqcup z \trianglelefteq [\![\varphi]\!]$.

If φ is $\forall x\psi(x)$, then, again, $y \trianglelefteq [\![\varphi]\!] \wedge z \trianglelefteq [\![\varphi]\!]$ implies that $\mathrm{BA}(y) \wedge \mathrm{BA}(z)$. Now, suppose $y \trianglelefteq [\![\varphi]\!]$ and $z \trianglelefteq [\![\varphi]\!]$. Let a be an element such that $[\![\mathrm{E}(a)]\!] \trianglelefteq y \sqcup z$. Now, we have that $[\![\mathrm{E}(a \restriction y)]\!] \trianglelefteq [\![\psi(a \restriction y)]\!]$ and $[\![\mathrm{E}(a \restriction z)]\!] \trianglelefteq [\![\psi(a \restriction z)]\!]$ by Lemma 4.5 and Corollary 4.3. By Corollary 4.3, we have that $[\![\mathrm{E}(a)]\!] \sqcap y \trianglelefteq [\![\psi(a)]\!]$ and $[\![\mathrm{E}(a)]\!] \sqcap z \trianglelefteq [\![\psi(a)]\!]$. By induction, we have that $([\![\mathrm{E}(a)]\!] \sqcap y) \sqcup ([\![\mathrm{E}(a)]\!] \sqcap z) = [\![\mathrm{E}(a)]\!] \trianglelefteq [\![\psi(a)]\!]$, and so $y \sqcup z \trianglelefteq [\![\varphi]\!]$.

Finally, suppose φ is $\exists x\psi(x)$. If both $y \trianglelefteq [\![\varphi]\!]$ and $z \trianglelefteq [\![\varphi]\!]$, then there exists a_1 where $[\![\mathrm{E}(a_1)]\!] = y$ and $y \trianglelefteq [\![\psi(a_1)]\!]$, and a similar element a_2 where $[\![\mathrm{E}(a_2)]\!] = z$ and $z \trianglelefteq [\![\psi(a_2)]\!]$. Let a be the unique element where $[\![\mathrm{E}(a)]\!] = y \sqcup (z \sqcap -y)$, and $a \restriction y = a_1$ and $a \restriction (z \sqcap -y) = a_2$, that is $a = (a_1 \restriction y) \oplus (a_2 \restriction (-y \sqcap z))$. Then $[\![\mathrm{E}(a)]\!] \sqcap y \trianglelefteq [\![\psi(a)]\!]$ and $[\![\mathrm{E}(a)]\!] \sqcap (z \sqcap -y) \trianglelefteq [\![\psi(a)]\!]$. By induction, $([\![\mathrm{E}(a)]\!] \sqcap y) \sqcup ([\![\mathrm{E}(a)]\!] \sqcap (z \sqcap -y)) = [\![\mathrm{E}(a)]\!] \trianglelefteq [\![\psi(a)]\!]$. Thus $y \sqcup z \trianglelefteq [\![\varphi]\!]$. $\dashv$

Next we show that, in some sense, we can piece together pieceable formulas.

Corollary 4.7 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and let $\mathbf{a}, \mathbf{b} \in \mathrm{ST}(A)$, $p, q \in \mathrm{BA}(A)$, and φ be an $\mathcal{L}(\mathrm{ST}(A))$-formula. Suppose $p \trianglelefteq [\![\varphi(\mathbf{a})]\!]$, $q \trianglelefteq [\![\varphi(\mathbf{b})]\!]$, and $\mathbf{a}$ and $\mathbf{b}$ can be pieced together. Then $p \sqcup q \trianglelefteq [\![\varphi(\mathbf{a} \oplus \mathbf{b})]\!]$.*

Proof. By Corollary 4.3, we have that $p \trianglelefteq [\![\varphi(\mathbf{a} \oplus \mathbf{b})]\!]$ and $q \trianglelefteq [\![\varphi(\mathbf{a} \oplus \mathbf{b})]\!]$. By Lemma 4.6, we have that $p \sqcup q \trianglelefteq [\![\varphi(\mathbf{a} \oplus \mathbf{b})]\!]$. $\dashv$

We now show that for any $\mathcal{L}$-sentence φ, we have that 0 is below the extent of φ.

Lemma 4.8 *Let $\varphi(\mathbf{x})$ be a $\mathcal{L}$-formula. Then $\emptyset^{\mathrm{BA}} \vdash_i \bigwedge_{i<n} \mathrm{ST}(x_i) \rightarrow 0 \trianglelefteq [\![\varphi(\mathbf{x})]\!]$.*

Proof. We suppose $\bigwedge_{i<n} \mathrm{ST}(x_i)$. We proceed by induction on the complexity of φ. Now, by Corollary 4.3, $0 \trianglelefteq [\![\varphi(\mathbf{x})]\!]$ if and only if $0 \trianglelefteq [\![\varphi(\mathbf{x} \restriction 0)]\!]$, so we may assume all elements appearing in φ are copies of ϖ.

If φ is atomic, then $[\![\varphi]\!]$ is a Boolean element, so $0 \trianglelefteq [\![\varphi]\!]$.

If φ is $\psi \wedge \theta$, where $0 \trianglelefteq [\![\psi]\!]$ and $0 \trianglelefteq [\![\theta]\!]$, then $0 \trianglelefteq [\![\psi \wedge \theta]\!]$.

If φ is $\psi \vee \theta$, where $0 \trianglelefteq [\![\psi]\!]$ and $0 \trianglelefteq [\![\theta]\!]$, then $0 \trianglelefteq [\![\psi \vee \theta]\!]$.

If φ is $\psi \rightarrow \theta$, and if $y \trianglelefteq 0$ and $y \trianglelefteq [\![\psi]\!]$, then $y = 0$, so by induction, $y \trianglelefteq [\![\theta]\!]$.

If φ is $\forall x \psi(x)$, then, by induction, $0 \trianglelefteq [\![\psi(\varpi)]\!]$, so $0 \trianglelefteq [\![\forall x \psi(x)]\!]$.

Finally, if φ is $\exists x \psi(x)$, then by induction, $0 \trianglelefteq [\![\psi(\varpi)]\!]$, so $0 \trianglelefteq [\![\exists x \psi(x)]\!]$. $\dashv$

Definition 4.9 *Recall that in a Boolean algebra A, an* **ideal** *is a subset $I \subseteq A$ such that:*

- $0 \in I$.
- *If $p \in I$ and $q \trianglelefteq p$, then $q \in I$.*
- *If $p, q \in I$, then $p \sqcup q \in I$.*

Combining Lemma 4.5, Lemma 4.6, and Lemma 4.8, we get the following result.

Theorem 4.10 *If $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$ and φ an $\mathcal{L}(\mathrm{ST}(A))$-sentence, then $\{p : \mathfrak{A} \models p \trianglelefteq [\![\varphi]\!]\}$ is an ideal.*

Thus, $\mathcal{L}$-formulas correspond to ideals on the underlying Boolean algebra. This allows us to express certain properties involving local truth in terms of ideals.

Definition 4.11 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$.*

1. *Suppose $p \in \mathrm{BA}(A)$. Then $(p]$ is the principal ideal $\{q \in \mathrm{BA}(A) : q \trianglelefteq p\}$.*
2. *For an $\mathcal{L}(\mathrm{ST}(A))$-sentence φ, with Lemma 4.10, we denote the ideal of Boolean elements q such that $\mathfrak{A} \models q \trianglelefteq [\![\varphi]\!]$ as $(\varphi]$.*

3. *If* φ *and* ψ *are* $\mathcal{L}(\mathrm{ST}(A))$*-sentences, then we write* $[\![\varphi]\!] \trianglelefteq [\![\psi]\!]$ *for* $(\varphi] \subseteq (\psi]$*.*

4. *If* $p \in \mathrm{BA}(A)$*, then we say* $[\![\varphi]\!] = p$ *if* $(\varphi] = (p]$*. So* $\mathfrak{A} \models [\![\varphi]\!] = p$ *exactly when* $\mathfrak{A} \models \forall z(z \trianglelefteq [\![\varphi]\!] \leftrightarrow z \trianglelefteq p)$*.*

As we shall see in section 7.3, $(\varphi]$ need not be a principal ideal. We give a characterization of when an $\mathcal{L}$-formula corresponds with a principal ideal.

Definition 4.12 *Let* φ *be an* $\mathcal{L}$*-formula. We define* $[\![\varphi(\mathbf{x})]\!] \trianglelefteq y$ *as* $\forall z(z \trianglelefteq [\![\varphi(\mathbf{x})]\!] \rightarrow z \trianglelefteq y) \wedge \bigwedge_{i<n} \mathrm{ST}(x_i)$*.*

The following proposition easily follows from the above definitions.

Proposition 4.13 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$ *and* φ *be an* $\mathcal{L}(\mathrm{ST}(A))$*-sentence. Then* $\mathfrak{A} \models p = [\![\varphi]\!]$ *is equivalent to* $\mathfrak{A} \models p \trianglelefteq [\![\varphi]\!] \wedge [\![\varphi]\!] \trianglelefteq p$*.*

We now provide some explicit correspondences between $\mathcal{L}$-formulas and ideals.

Proposition 4.14 *Let* φ *and* ψ *be* $\mathcal{L}(\mathrm{ST}(A))$*-sentences, and* $p \in \mathrm{BA}(A)$*.*

1. $\mathfrak{A} \models p \trianglelefteq [\![\varphi]\!]$ *if and only if* $(p] \subseteq (\varphi]$ *if and only if* $p \in (\varphi]$*.*

2. $\mathfrak{A} \models [\![\varphi]\!] \trianglelefteq p$ *if and only if* $(\varphi] \subseteq (p]$*.*

3. $(\varphi] \subseteq (\psi]$ *if and only if* $\mathfrak{A} \models [\![\varphi \rightarrow \psi]\!] = 1$*.*

Proof. For 1, suppose $p \trianglelefteq [\![\varphi]\!]$. By Theorem 4.10, $(p] \subseteq (\varphi]$. Now suppose $(p] \subseteq (\varphi]$. By the definition of $(\varphi]$ we have $p \trianglelefteq [\![\varphi]\!]$. 2 immediately follows from the definition of $[\![\varphi]\!] \trianglelefteq p$. For 3, suppose $(\varphi] \subseteq (\psi]$. We need to show that $1 \trianglelefteq [\![\varphi \rightarrow \psi]\!]$. Suppose $q \trianglelefteq 1 \wedge q \trianglelefteq [\![\varphi]\!]$. Since $(\varphi] \subseteq (\psi]$, we have that $q \in (\psi]$. By part 1, this implies that

$q \trianglelefteq [\![\psi]\!]$. Conversely, suppose $1 \trianglelefteq [\![\varphi \to \psi]\!]$. Thus, for all $q \trianglelefteq 1$, if $q \trianglelefteq [\![\varphi]\!]$, then $q \trianglelefteq [\![\psi]\!]$. By part 1, this implies that $(\varphi] \subseteq (\psi]$. ⊣

We now discuss, for a model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$ and an $\mathcal{L}(\mathrm{ST}(A))$-sentence φ, when $(\varphi]$ is a principal ideal.

Definition 4.15 *1. We call an $\mathcal{L}(\mathrm{ST}(A))$-sentence φ whose ideal is principal over $\mathfrak{A}$ a* **discrete sentence**. *Equivalently, φ is discrete if $\mathfrak{A} \models [\![\varphi]\!] = p$ for some p in $\mathrm{BA}(A)$.*

2. For an $\mathcal{L}^{\mathrm{BA}}$-formula $\theta(x)$ and a discrete sentence φ, we define $\theta([\![\varphi]\!])$ to be $\theta(p)$, where $(\varphi] = (p]$.

For discrete sentences, we can interchange $[\![\varphi]\!]$ and the Boolean element which gives the same principal ideal. Below are some illustrations of its use.

Lemma 4.16 *Let $\mathfrak{A}$ be a model of $\emptyset^{\mathrm{BA}}$. Then the set of $\mathcal{L}(\mathrm{ST}(A))$-sentences φ that are discrete is closed under conjunction, disjunction, and implication.*

Proof. Suppose φ is $\psi \wedge \theta$, where $p = [\![\psi]\!]$ and $q = [\![\theta]\!]$. Then $p \sqcap q = [\![\psi \wedge \theta]\!]$: certainly $p \sqcap q \trianglelefteq [\![\psi \wedge \theta]\!]$. For the other direction, suppose $r \trianglelefteq [\![\varphi \wedge \psi]\!]$. Then $r \trianglelefteq [\![\varphi]\!] = p$ and $r \trianglelefteq [\![\psi]\!] = q$, so that $r \trianglelefteq (p \sqcap q)$. Thus $[\![\varphi \wedge \psi]\!] \trianglelefteq p \sqcap q$.

Suppose φ is $\psi \vee \theta$, where $p = [\![\psi]\!]$ and $q = [\![\theta]\!]$. Then $p \sqcup q = [\![\psi \vee \theta]\!]$: clearly $p \sqcup q \trianglelefteq [\![\psi \vee \theta]\!]$ by Lemma 4.5. For the other direction, suppose $r \trianglelefteq [\![\psi \vee \theta]\!]$. Then there exists r_1, r_2 such that $r_1 \trianglelefteq [\![\psi]\!] = p$, $r_2 \trianglelefteq [\![\theta]\!] = q$, and $r_1 \sqcup r_2 = r$. Since $r_1 \trianglelefteq p$ and $r_2 \trianglelefteq q$, then $r = r_1 \sqcup r_2 \trianglelefteq p \sqcup q$.

Finally, suppose φ is $\psi \to \theta$, where $p = [\![\psi]\!]$ and $q = [\![\theta]\!]$. We show that $(-p) \sqcup q = [\![\psi \to \theta]\!]$. First, $(-p) \sqcup q \trianglelefteq [\![\psi \to \theta]\!]$: if $r \trianglelefteq (-p) \sqcup q$ and $r \trianglelefteq [\![\psi]\!]$, then $r \trianglelefteq$

$((-p) \sqcup q) \sqcap p = p \sqcap q$, so that $r \trianglelefteq [\![\theta]\!]$ by Lemma 4.5. For the other direction, we show that $[\![\psi \to \theta]\!] \trianglelefteq (-p) \sqcup q$. Suppose $r \trianglelefteq [\![\psi \to \theta]\!]$. Then $\forall s \trianglelefteq r(s \trianglelefteq p \to s \trianglelefteq q)$. Let $s = r \sqcap (p \sqcap (-q))$. Then $s \trianglelefteq r$ and $s \trianglelefteq p$. Thus, $s \trianglelefteq q$. But since $s \trianglelefteq (-q)$, we have that $s = 0$. Thus $r \sqcap (p \sqcap (-q)) = 0$, so $r \trianglelefteq -(p \sqcap (-q)) = (-p) \sqcup q$. $\dashv$

We get two immediate corollaries from this Lemma.

Corollary 4.17 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$ and φ, ψ be $\mathcal{L}(\mathrm{ST}(A))$-sentences. If φ and ψ are discrete, then $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \sqcap [\![\psi]\!]$, $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \sqcup [\![\psi]\!]$, and $[\![\varphi \to \psi]\!] = -[\![\varphi]\!] \sqcup [\![\psi]\!]$.*

Proof. Immediate from the proof of Lemma 4.16. $\dashv$

The second corollary shows that the set of discrete sentences is nonempty.

Corollary 4.18 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$. Then its set of discrete sentences contains all quantifier-free sentences.*

Proof. If φ is atomic, then by Pr2 $[\![\varphi]\!]$ is a Boolean element. Since quantifier-free sentences are created from conjunctions, disjunctions, and implications of atomic sentences, we are done by Lemma 4.16. $\dashv$

4.1.2 Pointwise truth of $\mathcal{L}$-sentences

In order to discuss pointwise truth, we first review some standard properties of filters and ultrafilters on Boolean algebras.

Definition 4.19 1. *Recall that in a Boolean algebra, a* **filter** *is a subset $F \subseteq \mathrm{BA}(A)$ such that:*

- $1 \in F$.
- *If* $p \in F$ *and* $p \trianglelefteq q$, *then* $q \in F$.
- *If* $p \in F$ *and* $q \in F$, *then* $p \sqcap q \in F$.

2. *A filter is a* **proper** *filter if* $0 \notin F$. *In our context, all filters are proper.*

3. *An* **ultrafilter** U *on* $\mathrm{BA}(A)$ *is a filter such that for all* $u \in \mathrm{BA}(A)$, *either* $u \in U$ *or* $-u \in U$.

4. *If* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, *we let* $\mathcal{U}_{\mathfrak{A}}$ *be the set of all ultrafilters on* $\mathrm{BA}(A)$.

5. *A* **prime** *filter is a filter such that if* $v \sqcup u \in U$, *then* $v \in U$ *or* $u \in U$.

6. *For an element* $p \in \mathrm{BA}(A)$, *we define* $[p] = \{U \in \mathcal{U}_{\mathfrak{A}} : p \in U\}$.

In our context, prime filters and ultrafilters are identical, for suppose U is an ultrafilter, and suppose $v \sqcup u \in U$, and $v \notin U$. Since U is an ultrafilter, $-v \in U$. Thus $(-v) \sqcap (v \sqcup u) = (-v) \sqcap u \in U$, so that $u \in U$. Conversely, suppose U is a prime filter. Then $u \sqcup -u = 1 \in U$, so that either $u \in U$ or $-u \in U$.

It is a well known result that for each nonzero Boolean element p, there exists an ultrafilter containing p. For the reader's convenience, we include the following Lemma:

Lemma 4.20 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, *with* p, $q \in \mathrm{BA}(A)$. *If* $p \ntrianglelefteq q$, *then there exists* $U \in \mathcal{U}_{\mathfrak{A}}$ *such that* $p \in U$ *and* $q \notin U$.

Proof. If $p \ntrianglelefteq q$, then $p \sqcap -q \neq 0$. So there is $U \in [p \sqcap -q]$. Then $p \in U$ and $q \notin U$. $\dashv$

Now, the sets $[p]$ for each $p \in \mathrm{BA}(A)$ form the basis of a topology on $\mathcal{U}_{\mathfrak{A}}$. We now reproduce some fundamental properties of that topology.

Lemma 4.21 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$. *Then the topology on* $\mathcal{U}_{\mathfrak{A}}$ *with basis* $\{[p] : \mathfrak{A} \models \mathrm{BA}(p)\}$ *is a compact, 0-dimensional, Hausdorff space.*

Proof. Assume we have elements $P \subseteq \mathrm{BA}(A)$ with $\bigcup_{p \in P}[p] = [1]$, but no finite subset of the p_i cover the space. That is, for every subset $F \in [P]^{<\omega}$, we have that $[\bigsqcup_{p \in F} p] = \bigcup_{p \in F}[p] \neq [1]$. Thus, $\bigsqcap_{p \in F} -p \neq 0$, otherwise $\bigsqcup_{p \in F} p = 1$. Hence, the collection $\{[-p] : p \in P\}$ has the finite intersection property, so there is an ultrafilter U which contains each of $-p$ for $p \in P$. But then $U \notin \bigcup_{p \in P}[p] = [1]$, a contradiction.

Since both $[p]$ and $[-p]$ are basis elements of the topology, it is clearly 0-dimensional. Finally, let U_1, U_2 be distinct ultrafilters in $\mathcal{U}_{\mathfrak{A}}$. So we may suppose there is $p \in \mathrm{BA}(A)$ such that $p \in U_1$ and $p \notin U_2$. Since U_2 is an ultrafilter, we must have that $-p \in U_2$. Thus $U_1 \in [p]$ and $U_2 \in [-p]$. As $[p]$ and $[-p]$ are disjoint, the space is Hausdorff. $\dashv$

For a model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, we now relate filters on $\mathrm{BA}(A)$ to structural elements.

Definition 4.22 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, *and let* F *be a filter on* $\mathrm{BA}(A)$.

1. *For elements* $a, b \in \mathrm{ST}(A)$ *we define* $a \sim_F b$ *if there exists* $p \in F$ *such that* $\mathfrak{A} \models [\![a = b]\!] = p$. *For tuples* $\mathbf{a}, \mathbf{b}$ *of length* n, *we say* $\mathbf{a} \sim_F \mathbf{b}$ *if* $a_i \sim_F b_i$ *for all* $i < n$.

2. *We define* $\mathrm{ST}(A)_F$ *to be the set* $\{a \in \mathrm{ST}(A) : \mathfrak{A} \models p = [\![\mathrm{E}(a)]\!]$ *for some* $p \in F\}$.

One easily sees $\sim_F$ is an equivalence relation on $\mathrm{ST}(A)_F$: it is clearly reflexive and symmetric. For transitivity, suppose $a \sim_F b$ and $b \sim_F c$. Then $\mathfrak{A} \models [\![a = b]\!] = p$ and $\mathfrak{A} \models [\![b = c]\!] = q$ with $p, q \in U$. Then $\mathfrak{A} \models p \sqcap q \trianglelefteq [\![a = c]\!]$. Since F is a filter and $p \sqcap q \in F$, we have that $[\![a = c]\!] \in F$, so $a \sim_F c$. This allows us to make the following definition.

Definition 4.23 *1. We define A_F to be the set of equivalence classes of elements on $\mathrm{ST}(A)_F$ modulo $\sim_F$.*

2. If $a \in \mathrm{ST}(A)_F$, we write a_F for the equivalence class of a in A_F.

Note that an element $a \in \mathrm{ST}(A)$ is in an equivalence class if and only if $a \in \mathrm{ST}(A)_F$. In particular, ϖ is not in any equivalence class in A_F for any filter F.

We are now ready to introduce point models.

Definition 4.24 *We define a* **point model** $\mathfrak{A}_F$ *as follows:*

- *The domain of $\mathfrak{A}_F$ is A_F.*
- *For a predicate $P(\mathbf{x})$ and elements $\mathbf{a_F} \in (A_F)$, we define $\mathfrak{A}_F \models P(\mathbf{a}_F)$ if there exists $p \in F$ such that $\mathfrak{A} \models [\![P(\mathbf{a})]\!] = p$.*
- *For a function f, and elements $\mathbf{a}_F$, b_F, we define $\mathfrak{A}_U \models f(\mathbf{a}_F) = b_F$ if there exists $p \in F$ such that $\mathfrak{A} \models [\![f(\mathbf{a}) = b]\!] = p$.*

Note that for a sentence φ over $\mathcal{L}(\mathrm{ST}(A)_F)$, we have $[\![\varphi]\!] \in F$ if and only if there exists $p \in F$ such that $\mathfrak{A} \models [\![\varphi]\!] = p$. We now show that the above definition is consistent:

Lemma 4.25 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and let F be a filter on $\mathrm{BA}(A)$.*

1. *Suppose $\mathbf{a}, \mathbf{b} \in (A_F)$ with $\mathbf{a} \sim_F \mathbf{b}$. Then $[\![P(\mathbf{a})]\!] \in F$ if and only if $[\![P(\mathbf{b})]\!] \in F$.*
2. *Suppose $\mathbf{a}, \mathbf{a}' \in (A_F)$ with $\mathbf{a} \sim_F \mathbf{a}'$, and $b, b' \in A_F$ with $b \sim_F b'$. Then $[\![f(\mathbf{a}) = b]\!] \in F$ if and only if $[\![f(\mathbf{a}') = b']\!] \in F$.*

Proof. For part 1, suppose $[\![P(\mathbf{a})]\!] \in F$. Let $p = [\![\mathbf{a} = \mathbf{b}]\!]$. Since $p \in F$, $p \sqcap [\![P(\mathbf{a})]\!] \in F$. By Corollary 4.4, we have that $p \sqcap [\![P(\mathbf{a})]\!] \trianglelefteq [\![P(\mathbf{b})]\!]$. Since F is a filter, we have that $[\![P(\mathbf{b})]\!] \in F$. The reverse direction is similar.

For part 2, suppose $[\![f(\mathbf{a}) = b]\!] \in F$. Let $p = [\![\mathbf{a} = \mathbf{a}']\!]$ and $q = [\![b = b']\!]$, with p, $q \in F$. Then $p \sqcap q \sqcap [\![f(\mathbf{a}) = b]\!] \in F$. By Lemma 3.9.2, we have that $p \sqcap q \sqcap [\![f(\mathbf{a}) = b]\!] \trianglelefteq [\![f(\mathbf{a}') = b']\!]$. Thus $[\![f(\mathbf{a}') = b']\!] \in F$. The reverse direction is similar. $\dashv$

We now show that point models generalize node structures.

Proposition 4.26 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and let F be a filter on* BA(A). *If F is a principal filter with generator p, then $\mathfrak{A}_F \cong \mathfrak{A}_p$.*

Proof. One easily shows that the mapping $f : A_F \to A_p$ defined by $f(a_F) = a \restriction p$ is well-defined and an isomorphism. $\dashv$

Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, F a filter on BA(A), and φ an $\mathcal{L}(\mathrm{ST}(A)_F)$-sentence. Then φ is called a **positive** F**-sentence** if $[\![\varphi]\!] \in F$ implies $\mathfrak{A}_F \models \varphi$, and φ is called a **negative** F**-sentence** if $\mathfrak{A}_F \models \varphi$ implies $[\![\varphi]\!] \in F$. For the remainder of this subsection, we consider only the case where F is an ultrafilter U.

Definition 4.27 *For an ultrafilter U, we call* U**-sentences** *those $\mathcal{L}(\mathrm{ST}(A)_U)$-sentences φ such that $\mathfrak{A}_U \models \varphi$ if and only if $[\![\varphi]\!] \in U$, that is, all sentences that are both positive and negative U-sentences.*

We now show that U-sentences are much like discrete sentences.

Lemma 4.28 *Let $\mathfrak{A}$ be a model of $\emptyset^{\mathrm{BA}}$ and U an ultrafilter on* BA(A). *Then the set of U-sentences is closed under conjunction, disjunction, and implication.*

Proof. Let ψ and θ be U-sentences. Note that if $[\![\psi]\!] \in U$ and $[\![\theta]\!] \in U$, then ψ and θ are discrete formulas. By Corollary 4.17, we know that $[\![\psi \wedge \theta]\!] = [\![\psi]\!] \sqcap [\![\theta]\!]$, $[\![\psi \vee \theta]\!] = [\![\psi]\!] \sqcup [\![\theta]\!]$, and $[\![\psi \wedge \theta]\!] = -[\![\psi]\!] \sqcup [\![\theta]\!]$. We shall use these facts throughout the proof.

Suppose φ is $\psi \wedge \theta$. Then $\mathfrak{A}_U \models \varphi$ if and only if $\mathfrak{A}_U \models \psi$ and $\mathfrak{A}_U \models \theta$. By induction, this holds if and only if $[\![\psi]\!] \in U$ and $[\![\theta]\!] \in U$. Since U is an ultrafilter, this holds if and only if $[\![\psi]\!] \sqcap [\![\theta]\!] = [\![\psi \wedge \theta]\!] \in U$.

Suppose φ is $\psi \vee \theta$. Then $\mathfrak{A}_U \models \varphi$ if and only if $\mathfrak{A}_U \models \psi$ or $\mathfrak{A}_U \models \theta$. By induction, this holds if and only if $[\![\psi]\!] \in U$ or $[\![\theta]\!] \in U$. As U is prime, this holds if and only if $[\![\psi]\!] \sqcup [\![\theta]\!] = [\![\psi \vee \theta]\!] \in U$.

Suppose φ is $\psi \rightarrow \theta$. Let $q = [\![\psi]\!]$ and $r = [\![\theta]\!]$. Now, suppose $\mathfrak{A}_U \models \varphi$, so that if $\mathfrak{A}_U \models \psi$, then $\mathfrak{A}_U \models \theta$. We need to show that $[\![\psi \rightarrow \theta]\!] = (-q) \sqcup r \in U$. Now, if $q \in U$, then by induction, $\mathfrak{A}_U \models \psi$, so that $\mathfrak{A}_U \models \theta$. By induction, $r \in U$. Since U is a filter, we get that $\mathfrak{A} \models (-q) \sqcup r \in U$. If $q \notin U$, then, since U is an ultrafilter, we have that $(-q) \in U$. Thus, $(-q) \sqcup r \in U$.

Conversely, suppose $[\![\varphi]\!] = (-q) \sqcup r \in U$. We need to show that $\mathfrak{A}_U \models \psi \rightarrow \theta$. So suppose $\mathfrak{A}_U \models \psi$. By induction, this means that $q \in U$. Thus, $q \sqcap ((-q) \sqcup r) = (q \sqcap (-q)) \sqcup (q \sqcap r) = q \sqcap r \in U$. Thus, $r \in U$, so, by induction, $\mathfrak{A}_U \models \theta$. ⊣

We now show that the set U-sentences is nonempty.

Corollary 4.29 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and $U \in \mathcal{U}_{\mathfrak{A}}$. Then the set of U-sentences contains the set of quantifier free $\mathcal{L}(\mathrm{ST}(A)_U)$-sentences.*

Proof. If φ is atomic, then $\mathfrak{A}_U \models \varphi$ if and only if $[\![\varphi]\!] \in U$ by the definition of $\mathfrak{A}_U$ above. Since quantifier-free sentences are constructed using atomic sentences and conjunction, disjunction, and implication, the result follows. $\dashv$

We conclude this subsection by showing that clean models of $\emptyset^{\mathrm{BA}}$ essentially are subdirect products of simple Boolean indexed models. Recall that simple Boolean indexed models are, up to a trivial translation, the same as $\mathcal{L}$-models. Thus, clean models of $\emptyset^{\mathrm{BA}}$ can be thought of as submodels of products of models over $\mathcal{L}$.

Definition 4.30 *Let $\mathfrak{A}$ be a clean model of $\emptyset^{\mathrm{BA}}$, and let $U \in \mathcal{U}_{\mathfrak{A}}$. We define the map $\pi_U : \mathfrak{A} \longrightarrow (\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}}$ as follows:*

$$\pi_U(x) = \begin{cases} \hat{x}_U & \text{if } \mathfrak{A} \models \mathrm{ST}(x) \text{ and } [\![\mathrm{E}(x)]\!] \in U \\ \varpi & \text{if } \mathfrak{A} \models \mathrm{ST}(x) \text{ and } [\![\mathrm{E}(x)]\!] \notin U \\ 1 & \text{if } \mathfrak{A} \models \mathrm{BA}(x) \text{ and } x \in U \\ 0 & \text{if } \mathfrak{A} \models \mathrm{BA}(x) \text{ and } x \notin U \end{cases}$$

We now show this map is a morphism.

Lemma 4.31 *Let $\mathfrak{A}$ be a clean model of $\emptyset^{\mathrm{BA}}$, and let $U \in \mathcal{U}_{\mathfrak{A}}$. Then π_U is an onto morphism.*

Proof. If $\mathfrak{A} \models \mathrm{E}(x)$, then $(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}} \models \mathrm{E}(\pi_U(x))$. Also, if $\mathfrak{A} \models \mathrm{BA}(x)$, then clearly $(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}} \models \mathrm{BA}(\pi_U(x))$.

We now discuss functions. By the definition of point model, we have $[\![P(\pi_U(\mathbf{a}))]\!] = \pi_U([\![P(\mathbf{a})]\!])$. Similarly, $f(\pi_U(\mathbf{a})) = \pi_U(f(\mathbf{a}))$. Further, $\pi_U(x \restriction y) = \pi_U(x) \restriction \pi_U(y)$, and $\pi_U(x \oplus y) = \pi_U(x) \oplus \pi_U(y)$. The result clearly holds for Boolean functions. Finally, the map is clearly onto. $\dashv$

We now show that the morphisms π_U, when collected, constitute an embedding.

Lemma 4.32 *Let $\mathfrak{A}$ be a clean model of $\emptyset^{\mathrm{BA}}$. Let $\mathfrak{B}$ be the model $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}}$. Then the map $\langle\pi_U\rangle_{U\in\mathcal{U}_{\mathfrak{A}}}$ embeds $\mathfrak{A}$ into $\mathfrak{B}$. Additionally, this map makes $\mathfrak{A}$ a subdirect product in $\mathfrak{B}$.*

Proof. As $\langle\pi_U\rangle$ is a product of of morphisms, it itself is a morphism. To see it is injective, let $a, b \in A$ with $\mathfrak{A} \models a \neq b$. The result is obvious if $a \in \mathrm{ST}(A)$ and $b \in \mathrm{BA}(A)$, or $b \in \mathrm{ST}(A)$ and $a \in \mathrm{BA}(A)$. First, suppose $a, b \in \mathrm{ST}(A)$. Let p be such that $\mathfrak{A} \models p = (-[\![a = b]\!])$. Note that $p \neq 0$. Now, either $[\![\mathrm{E}(a)]\!] \sqcap p \neq 0$ or $[\![\mathrm{E}(b)]\!] \sqcap p \neq 0$. Without loss of generality, let $[\![\mathrm{E}(a)]\!] \sqcap p \neq 0$. Let $U \in [[\![\mathrm{E}(a)]\!] \sqcap p]$. Now, if $p \sqcap [\![\mathrm{E}(b)]\!] = 0$, then $\pi_U(a) \neq \varpi$, while $\pi_U(b) = \varpi$. Thus, $\langle\pi_U\rangle(a) \neq \langle\pi_U\rangle(b)$. If $p \trianglelefteq [\![\mathrm{E}(a)]\!] \sqcap [\![\mathrm{E}(b)]\!]$, then $a_U \neq b_U$, so again, $\langle\pi_U\rangle(a) \neq \langle\pi_U\rangle(b)$. Now, suppose $a, b \in \mathrm{BA}(A)$. As $a \neq b$, at least one of $-a \sqcap b$ and $a \sqcap -b$ is nonzero. Without loss of generality, suppose $-a \sqcap b$ is nonzero. Let $V \in [-a \sqcap b]$. Then $\pi_V(a) = 0$ and $\pi_V(b) = 1$. Thus $\langle\pi_U\rangle(a) \neq \langle\pi_U\rangle(b)$.

Finally, we show that $\langle\pi_U\rangle$ is an embedding. First, $\mathfrak{B} \models \mathrm{E}(\langle\pi_U\rangle(a))$. Then clearly $\mathfrak{A} \models \mathrm{E}(a)$. Now, suppose $\mathfrak{A} \models \mathrm{BA}(\langle\pi_U\rangle(p))$. Then, for each $U \in \mathcal{U}_{\mathfrak{A}}$, $\mathfrak{A}_U \models \mathrm{BA}(\pi_U(a))$. Then $\mathfrak{A}_U \models \mathrm{BA}(a)$. As E, BA, and $=$ are the only predicates in $\mathcal{L}^{\mathrm{BA}}$, we get that $\langle\pi_U\rangle_{U\in\mathcal{U}_{\mathfrak{A}}}$ is an embedding. The last claim is obvious. $\dashv$

4.2 Local versus pointwise truth

In this section we connect local truth of universal formulas with their pointwise truth.

We begin by showing that a universal $\mathcal{L}$-formula φ has a straightforward translation $1 \trianglelefteq [\![\varphi]\!]$. Recall that for a tuple $\mathbf{x}$, $[\![\mathrm{E}(\mathbf{x})]\!]$ is shorthand for $\bigsqcap_{i<n}[\![\mathrm{E}(x_i)]\!]$:

Lemma 4.33 *Let φ be an $\mathcal{L}$-formula of the form $\forall\mathbf{x}\psi(\mathbf{x})$. Then $\emptyset^{\mathrm{BA}} \vdash_i 1 \trianglelefteq [\![\varphi]\!] \leftrightarrow \forall\mathbf{x}(\bigwedge_{i<n}\mathrm{ST}(x_i) \to [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\psi]\!])$.*

Proof. First, we describe the inductive translation of $1 \trianglelefteq^0 [\![\forall x_0 \ldots \forall x_{n-1}\psi(\mathbf{x})]\!]$. Applying the inductive schema, we get $\forall x_0([\![\mathrm{E}(x_0)]\!] \trianglelefteq 1 \to [\![\mathrm{E}(x_0)]\!] \trianglelefteq [\![\forall x_1 \ldots \forall x_{n-1}\psi(\mathbf{x})]\!])$. Applying the inductive schema again, we get

$$\forall x_0([\![\mathrm{E}(x_0)]\!] \trianglelefteq 1 \to \forall x_1([\![\mathrm{E}(x_1)]\!] \trianglelefteq [\![\mathrm{E}(x_0)]\!] \to [\![\mathrm{E}(x_1)]\!] \trianglelefteq [\![\forall x_2 \ldots \forall x_{n-1}\psi(\mathbf{x})]\!])).$$

This is equivalent to

$$\forall x_0 x_1(([\![\mathrm{E}(x_1)]\!] \trianglelefteq [\![\mathrm{E}(x_0)]\!] \wedge [\![\mathrm{E}(x_0)]\!] \trianglelefteq 1) \to [\![\mathrm{E}(x_1)]\!] \trianglelefteq [\![\forall x_0 \ldots \forall x_{n-1}\psi(\mathbf{x})]\!]).$$

Continuing in this manner, we get that $1 \trianglelefteq [\![\forall\mathbf{x}\psi]\!]$ is equivalent to

$$\forall\mathbf{x}([\![\mathrm{E}(x_{n-1})]\!] \trianglelefteq \ldots \trianglelefteq [\![\mathrm{E}(x_0)]\!] \trianglelefteq 1 \to [\![\mathrm{E}(x_{n-1})]\!] \trianglelefteq [\![\psi(\mathbf{x})]\!]).$$

We now have that $1 \trianglelefteq [\![\varphi]\!]$ is equivalent to

$$\forall\mathbf{x}([\![\mathrm{E}(x_{n-1})]\!] \trianglelefteq \ldots \trianglelefteq [\![\mathrm{E}(x_0)]\!] \trianglelefteq 1 \to [\![\mathrm{E}(x_{n-1})]\!] \trianglelefteq [\![\psi(\mathbf{x})]\!]).$$

To see that $\forall\mathbf{x}(\bigwedge_{i<n}\mathrm{ST}(x_i) \to [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\psi(\mathbf{x})]\!])$ implies $1 \trianglelefteq [\![\psi(\mathbf{x})]\!]$, let $\mathbf{x}$ be such that $[\![\mathrm{E}(x_{n-1})]\!] \trianglelefteq \ldots \trianglelefteq [\![\mathrm{E}(x_0)]\!] \trianglelefteq 1$. Then $[\![\mathrm{E}(\mathbf{x})]\!] = [\![\mathrm{E}(x_{n-1})]\!]$, so that by supposition, $[\![\mathrm{E}(x_{n-1})]\!] \trianglelefteq [\![\psi(\mathbf{x})]\!]$.

For the other direction, we suppose that $\forall\mathbf{x}([\![\mathrm{E}(x_{n-1})]\!] \trianglelefteq \ldots \trianglelefteq [\![\mathrm{E}(x_0)]\!] \trianglelefteq 1 \to [\![\mathrm{E}(x_{n-1})]\!] \trianglelefteq [\![\psi(\mathbf{x})]\!])$. Now, let $\mathbf{x}$ be such that $\mathrm{ST}(x_i)$ holds for each i. We need to show that $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\psi(\mathbf{x})]\!]$. Now, by Corollary 4.3, this holds if and only if $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\psi(\mathbf{x}\restriction[\![\mathrm{E}(\mathbf{x})]\!])]\!]$. Thus, it suffices to show the result holds for $\mathbf{x}\restriction[\![\mathrm{E}(\mathbf{x})]\!]$. But then, for each i, $[\![\mathrm{E}(x_i\restriction[\![\mathrm{E}(\mathbf{x})]\!])]\!] = [\![\mathrm{E}(\mathbf{x})]\!]$, so that $[\![\mathrm{E}(x_{n-1}\restriction[\![\mathrm{E}(\mathbf{x})]\!])]\!] = [\![\mathrm{E}(\mathbf{x})]\!]$. Then by supposition, $[\![\mathrm{E}(x_{n-1}\restriction[\![\mathrm{E}(\mathbf{x})]\!])]\!] = [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\psi(\mathbf{x})]\!]$. $\dashv$

Thus, we have a simpler translation for $1 \trianglelefteq [\![\varphi]\!]$ for universal sentences. This will be useful in demonstrating when $1 \trianglelefteq [\![\varphi]\!]$ holds in various models.

Theorem 4.34 *Let $\mathfrak{A}$ be a model of $\emptyset^{\mathrm{BA}}$, and let φ be a Π^0_2 sentence over $\mathcal{L}$. Then $\mathfrak{A} \models 1 \trianglelefteq [\![\varphi]\!]$ if and only if $\mathfrak{A}_U \models \varphi$ for every $U \in \mathcal{U}_{\mathfrak{A}}$.*

Proof. Let φ be $\forall \mathbf{x} \exists \mathbf{y} \psi(\mathbf{x}, \mathbf{y})$, where ψ is quantifier-free, and suppose $\mathfrak{A} \models 1 \trianglelefteq [\![\varphi]\!]$. This translates to $\forall \mathbf{x} \exists \mathbf{y} (\bigwedge_{i<n} \mathrm{ST}(x_i) \rightarrow [\![\mathrm{E}(\mathbf{y})]\!] = [\![\mathrm{E}(\mathbf{x})]\!] \wedge [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\psi(\mathbf{x}, \mathbf{y})]\!])$. Let $U \in \mathcal{U}_{\mathfrak{A}}$. We need to show that $\mathfrak{A}_U \models \forall \mathbf{x} \exists \mathbf{y} \psi(\mathbf{x}, \mathbf{y})$. To this end, let $\mathbf{a}_U \in A_U$ with $\mathbf{a} \in \mathrm{ST}(A)_U$. Let $u = [\![\mathrm{E}(\mathbf{a})]\!]$. By our supposition, there exists $\mathbf{b} \in (\mathrm{ST}(A)_U)$ such that $\mathfrak{A} \models u \trianglelefteq [\![\psi(\mathbf{a}, \mathbf{b})]\!]$ and $[\![\mathrm{E}(\mathbf{b})]\!] = u$. Thus $\mathbf{b}_U \in A_U$. Since $u \in U$, $u \trianglelefteq [\![\psi(\mathbf{a}, \mathbf{b})]\!]$, and ψ is quantifier-free, $\mathfrak{A}_U \models \psi(\mathbf{a}_U, \mathbf{b}_U)$ by Lemma 4.29.

Conversely, suppose that for each $U \in \mathcal{U}$, $\mathfrak{A}_U \models \forall \mathbf{x} \exists \mathbf{y} \psi(\mathbf{x}, \mathbf{y})$, and let $\mathbf{a} \in \mathrm{ST}(A)$. We need to find $\mathbf{b}$ such that $[\![\mathrm{E}(\mathbf{b})]\!] = [\![\mathrm{E}(\mathbf{a})]\!]$ and $[\![\mathrm{E}(\mathbf{a})]\!] \trianglelefteq [\![\psi(\mathbf{a}, \mathbf{b})]\!]$. Let $q = [\![\mathrm{E}(\mathbf{a})]\!]$. If $q = 0$, then let $\mathbf{b}$ be the tuple where $b_i = \varpi$ for each i. Then $0 \trianglelefteq [\![\psi(\mathbf{a}, \mathbf{b})]\!]$. If $q \neq 0$, let U be an ultrafilter with $q \in U$. By our supposition, there exist $\mathbf{b} \in \mathrm{ST}(A)_U$ such that $\mathfrak{A}_U \models \psi(\mathbf{a}_U, \mathbf{b}_U)$. Thus, $[\![\psi(\mathbf{a}, \mathbf{b})]\!] \in U$ by Corollary 4.29. Let $u = [\![\psi(\mathbf{a}, \mathbf{b})]\!]$, and note that by Corollary 4.3 we can replace $\mathbf{b}$ by $\mathbf{b} \upharpoonright u$. Thus, we have shown that for every ultrafilter U such that $q \in U$, there exists a pair $(\mathbf{b}, u)$ such that $[\![\mathrm{E}(\mathbf{b})]\!] = u$, $u \trianglelefteq [\![\psi(\mathbf{a}, \mathbf{b})]\!]$ and $U \in [u]$. Thus, the set $\{[u] : (\mathbf{b}, u)$ is such that $[\![\mathrm{E}(\mathbf{b})]\!] = u$ and $u \trianglelefteq [\![\psi(\mathbf{a}, \mathbf{b})]\!]\}$ is a cover of $[q]$. But since $[q]$ is closed, it is compact by Lemma 4.21. Thus, there exist finitely many elements $u_0, \ldots, u_{n-1}$ such that $[u_0] \cup \ldots \cup [u_{n-1}] = [q]$. But then $u_0 \sqcup \ldots \sqcup u_{n-1} = q$; otherwise, $q \sqcap -u_0 \sqcap \ldots \sqcap -u_{n-1}$ would be nonempty, so there would exist an ultrafilter V with $q \in V$, but $u_i \notin V$ for all i, a contradiction. Thus, $(\mathbf{b}_0, u_0), \ldots, (\mathbf{b}_{n-1}, u_{n-1})$ are such that $u_0 \sqcup \ldots \sqcup u_{n-1} = q$. Now, we have that

$u_i \trianglelefteq [\![\exists \mathbf{x}\psi(\mathbf{a},\mathbf{x})]\!]$ for all i. By Lemma 4.6, $\bigsqcup_{i<n} u_i = q \trianglelefteq [\![\exists \mathbf{x}\psi(\mathbf{a},\mathbf{x})]\!]$. Let $\mathbf{b}'$ be witness this. Then $[\![\mathrm{E}(\mathbf{b}')]\!] = q$ and $q \trianglelefteq [\![\psi(\mathbf{a},\mathbf{b}')]\!]$. $\dashv$

We now show that if Γ and Δ are classically equivalent Π^0_2-sentences, then Γ^{BA} and Δ^{BA} are classically equivalent.

Theorem 4.35 *Let γ and δ be Π^0_2 $\mathcal{L}$-formulas in prenex normal form. Then the following are equivalent:*

1. $\emptyset \vdash_c \gamma \to \delta$.

2. $\emptyset^{\mathrm{BA}} \vdash_c \gamma^{\mathrm{BA}} \to \delta^{\mathrm{BA}}$.

Proof. For 1 implies 2, let $\mathfrak{B} \models \emptyset^{\mathrm{BA}}$. Suppose $\mathfrak{B} \models 1 \trianglelefteq [\![\gamma]\!]$. By Theorem 4.34, we have that $\mathfrak{B}_U \models \gamma$ for each $U \in \mathcal{U}_{\mathfrak{B}}$. By supposition, we have that $\mathfrak{B}_U \models \delta$ for each U. By Theorem 4.34, we have that $\mathfrak{B} \models 1 \trianglelefteq [\![\delta]\!]$.

For the other direction, let $\mathfrak{A} \models \gamma$. Then $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \gamma^{\mathrm{BA}\mathbf{2}}$ by Lemma 2.12. By Proposition 3.19, $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \gamma^{\mathrm{BA}}$, so that $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \delta^{\mathrm{BA}}$. Again, by Proposition 3.19, $\mathfrak{A}^{\mathrm{BA}\mathbf{2}} \models \delta^{\mathrm{BA}\mathbf{2}}$, so $\mathfrak{A} \models \delta$ by Lemma 2.12. $\dashv$

Thus, given an theory with a set Γ of Π^0_2-axioms, Γ^{BA} is independent of the choice of Γ. Theorem 4.35 is an extension of [15, Lemma 3.1]. With this, we get an immediate corollary.

Corollary 4.36 *Let $\Gamma \cup \gamma$ be prenex Π^0_2 sentences. Then $\Gamma \vdash_c \gamma$ if and only if $\Gamma^{\mathrm{BA}} \vdash_c \gamma^{\mathrm{BA}}$.*

Proof. The result follows from Theorem 4.35 and compactness. $\dashv$

Before we prove the main result of this chapter, we prove that if φ is universal, then $1 \trianglelefteq [\![\varphi]\!]$ is universal Horn.

Lemma 4.37 *Let γ be a universal $\mathcal{L}$-formula. Then $\emptyset^{\mathrm{BA}}$ proves that γ^{BA} is equivalent to a universal Horn formula.*

Proof. By Theorem 4.35, we may assume that γ is a sentence of the form

$$\forall \mathbf{x}(\bigwedge_{o<p}(\bigwedge_{j<m} \delta_{oj} \to \bigvee_{k<l} \epsilon_{ok}))$$

where each δ_j and ϵ_k is atomic. By Lemma 4.33, $1 \trianglelefteq [\![\gamma]\!]$ is equivalent to

$$\forall \mathbf{x}((\bigwedge_{i<n} \mathrm{ST}(x_i)) \to [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\bigwedge_{o<p}(\bigwedge_{j<m} \delta_{oj} \to \bigvee_{k<l} \epsilon_{ok})]\!])$$

Now, as each δ_{oj} and ϵ_{ok} is atomic, $\bigwedge_{o<p}(\bigwedge_{j<m} \delta_{oj} \to \bigvee_{k<l} \epsilon_{ok})$ is discrete. Further, by Corollary 4.17, $[\![\bigwedge_{o<p}(\bigwedge_{j<m} \delta_{oj} \to \bigvee_{k<l} \epsilon_{ok})]\!]$ is equal to $\prod_{o<p}[\![\bigwedge_{j<m} \delta_{oj} \to \bigvee_{k<l} \epsilon_{ok}]\!]$. Applying Corollary 4.17 again, this is equal to $\prod_{o<p}(\bigsqcup_{j<m} -[\![\delta_{oj}]\!] \sqcup \bigsqcup_{k<l}[\![\epsilon_{ok}]\!])$. Thus, we have that

$$\forall \mathbf{x}((\bigwedge_{i<n} \mathrm{ST}(x_i)) \to [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\bigwedge_{o<p}(\bigwedge_{j<m} \delta_{oj} \to \bigvee_{k<l} \epsilon_{ok})]\!])$$

is equivalent to

$$\forall \mathbf{x}((\bigwedge_{i<n} \mathrm{ST}(x_i)) \to [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq \prod_{o<p}(\bigsqcup_{j<m} -[\![\delta_{oj}]\!] \sqcup \bigsqcup_{k<l}[\![\epsilon_{ok}]\!]))$$

As this is universal Horn, we are done. ⊣

We now show the main theorem for this chapter: for a set of universal $\mathcal{L}$-sentences, Γ^{BA} is the universal Horn fragment of $\Gamma^{\mathrm{BA}\mathbf{2}}$.

Theorem 4.38 *Let Γ be a set of universal $\mathcal{L}$-sentences. Then $(\Gamma^{\mathrm{BA}\mathbf{2}})_{\mathrm{UH}} = \Gamma^{\mathrm{BA}}$.*

Proof. All axioms in $\emptyset^{\mathrm{BA}}$ are universal Horn. By Lemma 4.37, we have that for each axiom $\gamma \in \Gamma$, $1 \trianglelefteq [\![\gamma]\!]$ is universal Horn over $\emptyset^{\mathrm{BA}}$. Thus, $\Gamma^{\mathrm{BA}} \subseteq (\Gamma^{\mathrm{BA}\mathbf{2}})_{\mathrm{UH}}$. Hence, we need only show that every model of Γ^{BA} embeds into a product of models of $\Gamma^{\mathrm{BA}\mathbf{2}}$. Thus, let $\mathfrak{A} \models \Gamma^{\mathrm{BA}}$. By Proposition 3.8, we may suppose that $\mathfrak{A}$ is clean. By Theorem 4.34, $\mathfrak{A}_U \models \gamma$ for each $\gamma \in \Gamma$, so that $(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}} \models \gamma^{\mathrm{BA}\mathbf{2}}$. By Proposition 3.19, $(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}} \models 1 \trianglelefteq [\![\gamma]\!]$. Thus, $(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}\mathbf{2}}$. Define $\mathfrak{B}$ to be the model $\Pi_{U \in \mathcal{U}_{\mathfrak{A}}} (\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}}$. Then $\mathfrak{B}$ is a product of models of $\Gamma^{\mathrm{BA}\mathbf{2}}$, and $\mathfrak{A}$ embeds into $\mathfrak{B}$ by Lemma 4.32 $\dashv$

When Γ is just the empty set of axioms, we get the following corollary:

Corollary 4.39 *The universal Horn fragment of* $\emptyset^{\mathrm{BA}\mathbf{2}}$ *is* $\emptyset^{\mathrm{BA}}$.

The last two results are variations on results of Weispfenning. For Corollary 4.39 see [15, page 257].

We conclude by showing that, for a set of universal sentences, Γ^{BA} is the same $(\Gamma_{\mathrm{UH}})^{\mathrm{BA}}$.

Theorem 4.40 *Let* Γ *be a set of universal* $\mathcal{L}$*-sentences. Then* Γ^{BA} *and* $(\Gamma_{\mathrm{UH}})^{\mathrm{BA}}$ *axiomatize the same theory.*

Proof. Clearly $(\Gamma_{\mathrm{UH}})^{\mathrm{BA}} \subseteq \Gamma^{\mathrm{BA}}$. Let $\mathfrak{A} \models (\Gamma_{\mathrm{UH}})^{\mathrm{BA}}$. As Γ^{BA} is universal, it suffices to show that $\mathfrak{A}$ embeds into a model Γ^{BA}. By Lemma 4.32, $\mathfrak{A}$ embeds into $\Pi_{U \in \mathcal{U}_{\mathfrak{A}}} (\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}}$, where each $\mathfrak{A}_U \models \Gamma_{\mathrm{UH}}$ by Theorem 4.34. Thus, each $\mathfrak{A}_U$ is a submodel of a product of models of Γ, that is, for each U, there is a set $I(U)$ such that $\mathfrak{A}_U \subseteq \Pi_{i \in I(U)} \mathfrak{B}_i$ and $\mathfrak{B}_i \models \Gamma$. By Corollary 2.13 and Proposition 3.19, each $(\mathfrak{B}_i)^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}}$. As Γ^{BA} is universal Horn, we get that $\Pi_{U \in \mathcal{U}_{\mathfrak{A}}} \Pi_{i \in I(U)} (\mathfrak{B}_i)^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}}$. Thus $\mathfrak{A}$ embeds into a model of Γ^{BA}. $\dashv$

Chapter 5

Model Complete Boolean Indexed Models

In the previous chapter, we showed that, for a set of universal $\mathcal{L}$-sentences Γ, the universal Horn fragment of $\Gamma^{\mathrm{BA}\mathbf{2}}$ is Γ^{BA}. As $\Gamma^{\mathrm{BA}\mathbf{2}}$ is just a trivial translation of the theory Γ, the theory Γ^{BA} is similar to the universal Horn fragment of Γ. Thus, if we can find a model companion for Γ^{BA}, this would be not unlike finding a model companion for Γ_{UH}. In this chapter, we list the axiom set $\emptyset^{\mathrm{ABA}}$ which extends $\emptyset^{\mathrm{BA}}$. For a set of $\mathcal{L}$-sentences Γ, we extend Γ^{BA} to an $\mathcal{L}^{\mathrm{BA}}$-theory Γ^{ABA} simply by including the new axioms from $\emptyset^{\mathrm{ABA}}$. We then show that if a set of Π^0_2-sentences Γ axiomatizes a model complete theory over $\mathcal{L}$, then Γ^{ABA} axiomatizes a model complete theory over $\mathcal{L}^{\mathrm{BA}}$.

5.1 Atomless Boolean indexed models

Recall that a Boolean algebra is **atomless** if, for every non-zero Boolean element, there exists a strictly smaller non-zero Boolean element. This motivates the following definition.

Definition 5.1 *We say a model* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$ *is atomless if* $\mathrm{BA}(A)$ *is atomless.*

Equivalently, a model $\mathfrak{A}$ of $\emptyset^{\mathrm{BA}}$ is atomless if it satisfies the following axiom.

Ba19 $\mathrm{BA}(x) \wedge x \neq 0 \to \exists y(y \trianglelefteq x \wedge y \neq x \wedge y \neq 0)$

The following axiom states that there exists an element of full extent.

Ex5 $\exists x([\![\mathrm{E}(x)]\!] = 1)$

Definition 5.2 *For a model $\mathfrak{A}$ of $\emptyset^{\mathrm{BA}}$, if $a \in \mathrm{ST}(A)$ is such that $\mathfrak{A} \models [\![\mathrm{E}(a)]\!] = 1$, then we say a is a* **global element**. *In other words, the set A_1 defined in Definition 2.4 is nonempty.*

Thus, a model $\mathfrak{A}$ has a global element if and only if it satisfies Ex5.

As shorthand, recall that $\mathrm{CH}(x)$ is $\neg(\mathrm{BA}(x) \vee \mathrm{ST}(x))$. We also use as shorthand $n < |\,\mathrm{CH}\,|$ to be the obvious axiomatization that there are at least $n+1$ chaff elements. The following is an infinite axiom schema that states that the chaff of a model is infinite.

So3 $n < |\,\mathrm{CH}\,|$ for all $n \in \mathbb{N}$

We are now able to define the axiom set $\emptyset^{\mathrm{ABA}}$.

Definition 5.3

1. *We define $\emptyset^{\mathrm{ABA}}$ to be the set $\emptyset^{\mathrm{BA}} \cup \{\mathrm{Ba19}, \mathrm{Ex5}, \mathrm{So3}\}$.*
2. *We call a model of $\emptyset^{\mathrm{ABA}}$ an* **atomless Boolean indexed model**.
3. *For a set of $\mathcal{L}$-sentences Γ, we define the set Γ^{ABA} to be the set $\emptyset^{\mathrm{ABA}} \cup \{\gamma^{\mathrm{BA}} : \gamma \in \Gamma\}$.*

Equivalently, Γ^{ABA} is the set $\Gamma^{\mathrm{BA}} \cup \{\mathrm{Ba19}, \mathrm{Ex5}, \mathrm{So3}\}$.

As mentioned in the introduction, we show below that if a set Γ of Π^0_2-sentences axiomatizes a model complete $\mathcal{L}$-theory, then Γ^{ABA} axiomatizes a model complete theory. Recall that for a model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, $\mathfrak{A}^\circ$ is the largest clean submodel of $\mathfrak{A}$. We now show that in showing a model is existentially closed, it suffices to look at this clean submodel.

Lemma 5.4 *Let* $\mathfrak{A} \subseteq \mathfrak{A}$ *be models of* $\emptyset^{\mathrm{ABA}}$*. Then* $\mathfrak{A}$ *is existentially closed in* $\mathfrak{B}$ *if and only if* $\mathfrak{A}^\circ$ *is existentially closed in* $\mathfrak{B}^\circ$*.*

Proof. This immediately follows from the fact that if $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, then A is the disjoint union of $\mathrm{ST}(A)$, $\mathrm{BA}(A)$, and $\mathrm{CH}(A)$, plus the fact that the theory of the infinite set is model complete. $\dashv$

Note that $\mathcal{L}^{\mathrm{BA}}$-formulas can become very complicated with iterated functions. With this in mind, we recall a definition to simplify our formulas.

Definition 5.5 *Recall that an* $\mathcal{L}$*-formula* φ *is called* **term-reduced** *if the only atomic subformulas that appear in* φ *are* $P(\mathbf{x})$ *and* $f(\mathbf{x}) = y$*, where* P *is any predicate symbol and* f *is any function symbol.*

Note that, in particular, this includes $c = y$ where c is a constant. It is well-known that every formula is equivalent to a term-reduced formula, for example, see [3, chapter 8, section 1]. We now show that to prove a model is existentially closed, it suffices to show that the model is existentially closed with respect to term-reduced sentences.

Lemma 5.6 *Let* Γ *be a theory, and suppose that for all models* $\mathfrak{A} \subseteq \mathfrak{B}$ *of* Γ *and quantifier-free formulas* $\varphi(\mathbf{x}, \mathbf{y})$ *which are term-reduced, if* $\mathfrak{B} \models \varphi(\mathbf{a}, \mathbf{b})$*, where* $\mathbf{a} \in A$ *and* $\mathbf{b} \in B$*, then there exists* $\mathbf{a}' \in A$ *with* $\mathfrak{A} \models \varphi(\mathbf{a}, \mathbf{a}')$*. Then* Γ *is model complete.*

Proof. Let $\varphi(\mathbf{x}, \mathbf{y})$ be a quantifier-free formula, and suppose $\mathfrak{B} \models \varphi(\mathbf{a}, \mathbf{b})$. As in [3, chapter 8, section 1], φ is equivalent over $\emptyset^{\mathrm{ABA}}$ to a term-reduced existential formula, $\exists \mathbf{z} \psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Hence, for some elements $\mathbf{c} \in B$, we have that $\mathfrak{B} \models \psi(\mathbf{a}, \mathbf{b}, \mathbf{c})$. By our supposition, there are tuples $\mathbf{a}', \mathbf{a}'' \in A^{<\omega}$ such that $\mathfrak{A} \models \psi(\mathbf{a}, \mathbf{a}', \mathbf{a}'')$. Thus, $\mathfrak{A} \models \exists \mathbf{z} \psi(\mathbf{a}, \mathbf{a}', \mathbf{z})$, so that $\mathfrak{A} \models \varphi(\mathbf{a}, \mathbf{a}')$. $\dashv$

The results we present below require that our formulas be term-reduced. The proofs can be generalized to allow formulas of any complexity. However, we find that making use of term-reduced formulas makes the proofs simpler and easier to read.

5.2 Decomposition and Model Completeness

For the rest of the chapter we suppose that Γ is a model complete theory over $\mathcal{L}$. In this section we prove that Γ^{ABA} is model complete. By Lemma 5.4, we restrict ourselves to the clean submodels of Γ^{ABA}.

We prove that Γ^{ABA} is model complete by taking a quantifier-free $\mathcal{L}^{\mathrm{BA}}$-formula φ, models $\mathfrak{A} \subseteq \mathfrak{B}$ of Γ^{ABA}, and elements $\mathbf{a} \in \mathrm{ST}(A)$, $\mathbf{p} \in \mathrm{BA}(A)$, $\mathbf{b} \in \mathrm{ST}(B)$, $\mathbf{q} \in \mathrm{BA}(B)$. If $\varphi(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$ holds in $\mathfrak{B}$, we need to find elements $\mathbf{b}', \mathbf{q}'$ in A such that $\mathfrak{A} \models \varphi(\mathbf{a}, \mathbf{p}, \mathbf{b}', \mathbf{q}')$. As is usual in proving a theory is model complete, we show that it suffices to consider formulas φ of a particular form. In our context, we can also replace the elements $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$ with elements of a particular form. The special forms of the formula and elements is such that it allows us to consider $\mathfrak{B}$ as a product of models that are sufficiently similar to models of $\Gamma^{\mathrm{BA}\mathbf{2}}$ when it comes to properties of φ. As Γ is model complete, $\Gamma^{\mathrm{BA}\mathbf{2}}$ is model complete. We then find elements in these approximations of simple Boolean indexed models from which we can piece together the required elements $\mathbf{b}', \mathbf{q}'$.

We first introduce new notation in order to make proofs more accessible.

Definition 5.7 *Let $\mathcal{L}$ be a first-order language.*

1. *We define $\mathcal{A}\mathrm{t}(\mathbf{x})$ to be the set of term-reduced atoms from $\mathcal{L}$. We also define $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$ to be the set of term-reduced atoms and negations of term-reduced atoms.*

2. *For a finite set* $t \subseteq \mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*, we define* π_t *to be the formula which is the conjunction of all term-reduced atoms and negated term-reduced atoms that occur in* t*. We also call* t *a* **type***.*

3. *Given a set* t *of formulas, we write* $\mathcal{A}\mathrm{t}_t^{\pm}$ *for the collection* t *plus the negations of formulas in* t*. So if* $t = \mathcal{A}\mathrm{t}(\mathbf{x})$*, then* $\mathcal{A}\mathrm{t}_t^{\pm} = \mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*.*

The reader may recognize that these definitions are slight variants on definitions given in [4].

Our first lemma shows the special form of the elements. We provide a brief description of what this special form is. We begin with models $\mathfrak{A} \subseteq \mathfrak{B}$ and elements $\mathbf{a} \in \mathrm{ST}(A)$, $\mathbf{p} \in \mathrm{BA}(A)$, $\mathbf{b} \in \mathrm{ST}(B)$, and $\mathbf{q} \in \mathrm{BA}(B)$. From this we construct elements $\mathbf{c} \in \mathrm{ST}(A)$, $\mathbf{r} \in \mathrm{BA}(A)$, $\mathbf{d} \in \mathrm{ST}(B)$, $\mathbf{e} \in \mathrm{ST}(B)$, and $\mathbf{s} \in \mathrm{BA}(B)$. The elements $\mathbf{r}$, and $\mathbf{s}$ are partitions of the Boolean algebra $\mathrm{BA}(\mathfrak{B})$. Since $\mathbf{r}$ is a set of elements from $\mathrm{BA}(A)$, the partition $\mathbf{s}$ is a finer partition than $\mathbf{r}$. The elements $\mathbf{c}$ are restrictions of elements $\mathbf{a}$ to the elements $\mathbf{r}$. In some sense, the elements $\mathbf{c}$ are as fine a partition of $\mathbf{a}$ as the model $\mathfrak{A}$ can see. The set $\mathbf{d}$ is the finest partition of $\mathbf{a}$ that the model $\mathfrak{B}$ can see; since $\mathbf{c}$ is a set of elements from $\mathrm{ST}(A)$, the partition $\mathbf{d}$ is a finer partition than $\mathbf{c}$. The set of elements $\mathbf{e}$ is a partition of $\mathbf{b}$ in $\mathfrak{B}$. Finally, these partitions are as fine as the set t can tell, that is, for any atom in t with elements from $\mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ plugged in and for any element n from $\mathbf{r}$ or $\mathbf{s}$, the extent of the atom is either n or 0. The models over the elements $\mathbf{r}$ and $\mathbf{s}$ are the approximations of the simple Boolean indexed models mentioned above. We now proceed with the lemma.

Lemma 5.8 *Suppose we have* $\mathfrak{A} \subseteq \mathfrak{B}$*, both models of* Γ^{ABA}*, and elements* $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$*, where* $\mathbf{a} \in \mathrm{ST}(A), \mathbf{p} \in \mathrm{BA}(A), \mathbf{b} \in \mathrm{ST}(B), \mathbf{q} \in \mathrm{ST}(B)$*. Let* $t \subseteq \mathcal{A}\mathrm{t}(\mathbf{x})$ *be a finite type* *over* $\mathcal{L}$*. Then there exist elements* $\mathbf{c} \in \mathrm{ST}(A)$*;* $\mathbf{r} \in \mathrm{BA}(A)$*;* $\mathbf{d}, \mathbf{e} \in \mathrm{ST}(B)$*; and* $\mathbf{s} \in \mathrm{BA}(B)$ *that satisfy the following properties:*

1. $\mathbf{r}$ *forms a partition of* $\mathrm{BA}(\mathfrak{A})$ *and* $\mathbf{s}$ *forms a partition of* $\mathrm{BA}(\mathfrak{B})$*;*

2. *For every* $r \in \mathbf{r}$ *there exists* $\mathbf{s}_r \subseteq \mathbf{s}$ *such that* $\mathfrak{B} \models r = \bigsqcup_{s \in \mathbf{s}_r} s$*;*

3. *For every* $c \in \mathbf{c}$ *there exists* $r \in \mathbf{r}$ *such that* $\mathfrak{A} \models r = [\![\mathrm{E}(c)]\!]$*;*

4. *For every* $e \in \mathbf{e}$ *there exists* $s \in \mathbf{s}$ *such that* $\mathfrak{B} \models s = [\![\mathrm{E}(e)]\!]$*;*

5. *For every* $d \in \mathbf{d}$ *there exists* $s \in \mathbf{s}$ *such that* $\mathfrak{B} \models s = [\![\mathrm{E}(d)]\!]$*;*

6. *For every* $c \in \mathbf{c}$*, there exists* $\mathbf{d}_c \subseteq \mathbf{d}$ *such that* $\mathfrak{B} \models c = \bigoplus_{d \in \mathbf{d}_c} d$*. We write* $t_c(\mathbf{x_d})$ *for the* $\mathcal{L}^{\mathrm{BA}}$*-term* $\bigoplus_{d \in \mathbf{d}_c} x_d$*, so* $\mathfrak{B} \models c = t_c(\mathbf{d}_c)$*;*

7. *For every* $a \in \mathbf{a}$*, there exists* $\mathbf{c}_a \subseteq \mathbf{c}$ *such that* $\mathfrak{A} \models a = \bigoplus_{c \in \mathbf{c}_a} c$*. We write* $t_a(\mathbf{x_c})$ *for the* $\mathcal{L}^{\mathrm{BA}}$*-term* $\bigoplus_{c \in \mathbf{c}_a} x_c$*, so* $\mathfrak{A} \models a = t_a(\mathbf{c}_a)$*;*

8. *For every* $b \in \mathbf{b}$*, there exists* $\mathbf{e}_b \subseteq \mathbf{e}$ *such that* $\mathfrak{B} \models b = \bigoplus_{e \in \mathbf{e}_b} e$*. We write* $t_b(\mathbf{x_e})$ *for the* $\mathcal{L}^{\mathrm{BA}}$*-term* $\bigoplus_{e \in \mathbf{e}_b} x_e$*, so* $\mathfrak{B} \models b = t_b(\mathbf{e}_b)$*;*

9. *For every* $p \in \mathbf{p}$*, there exists* $\mathbf{r}_p \subseteq \mathbf{r}$ *such that* $\mathfrak{A} \models p = \bigsqcup_{r \in \mathbf{r}_p} r$*. We write* $t_p(\mathbf{x_r})$ *for the* $\mathcal{L}^{\mathrm{BA}}$*-term* $\bigsqcup_{r \in \mathbf{r}_p} x_r$*, so* $\mathfrak{A} \models p = t_p(\mathbf{r}_p)$*;*

10. *For every* $q \in \mathbf{q}$*, there exists* $\mathbf{s}_q \subseteq \mathbf{s}$ *such that* $\mathfrak{B} \models q = \bigsqcup_{s \in \mathbf{s}_q} s$*. We write* $t_q(\mathbf{x_s})$ *for the* $\mathcal{L}^{\mathrm{BA}}$*-term* $\bigsqcup_{s \in \mathbf{s}_q} x_s$*, so* $\mathfrak{B} \models q = t_q(\mathbf{s}_q)$*;*

11. *For all* $r \in \mathbf{r}$ *and* $\mathbf{f}' \subseteq \mathbf{c}$*, we have that* $\mathfrak{A} \models (r \trianglelefteq [\![\pi_t(\mathbf{f}')]\!]) \vee (r \sqcap [\![\pi_t(\mathbf{f}')]\!] = 0)$*, where we can replace* $\trianglelefteq$ *with* $=$ *if* π_t *has positive arity;*

12. *For all* $s \in \mathbf{s}$ *and* $\mathbf{f}' \subseteq (\mathbf{d} \cup \mathbf{e})$*, we have that* $\mathfrak{A} \models (s \trianglelefteq [\![\pi_t(\mathbf{f}')]\!]) \vee (s \sqcap [\![\pi_t(\mathbf{f}')]\!] = 0)$*, where we can replace* $\trianglelefteq$ *with* $=$ *if* π_t *has positive arity;*

13. *Each of* $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$ *can be expressed as a term with elements from* $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$*.*

14. *Each of* $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$ *can be expressed as a term with elements from* $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$*.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be models of Γ^{ABA}, and suppose $\mathbf{a} \in \mathrm{ST}(A)$, $\mathbf{p} \in \mathrm{BA}(A)$, $\mathbf{b} \in \mathrm{ST}(B)$, and $\mathbf{q} \in \mathrm{BA}(B)$. Note that we may assume that each of these tuples is nonempty, as we can add ϖ to either of $\mathbf{a}, \mathbf{b}$ and 0 to either of $\mathbf{p}, \mathbf{q}$.

As both t is finite, there are only finitely many $\mathcal{L}$-predicates P such that P appears in a formula in t. Similarly, there are only finitely many $\mathcal{L}$-function symbols which occur in t. With this, we consider the collection of $\mathcal{L}^{\mathrm{BA}}$-terms 0,1, x, $x \sqcap y$, $x \sqcup y$, $-x$, $x \restriction y$, $[\![\mathrm{E}(x)]\!]$, $[\![x = y]\!]$, $[\![P(\mathbf{x})]\!]$ for every predicate P which occurs in t, $f(\mathbf{x})$ for every function symbol that occurs in t, and $x \oplus y$. As a result, the term-reduced atomic formulas these terms generate are of the form $x \restriction y = z$, $[\![\mathrm{E}(x)]\!] = y$, $[\![P(\mathbf{x})]\!] = z$ for each P that occurs in t, $[\![x = y]\!] = z$, $x = y$, $x \sqcap y = z$, $x \sqcup y = z$, $-x - y$, $f(\mathbf{x}) = y$ for every function symbol f which occurs in t, $x \oplus y = z$, $\mathrm{BA}(x)$, $\mathrm{E}(x)$, $x = 0$, $x = 1$, and $x = \varpi$.

We take all the reduced terms listed above, and we replace all the free variables in those terms with each possible combination of elements from $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ so that the terms have no free variables. The result is a finite number of terms with elements from B. We call this set T. Let $S = \{w \in T : \mathfrak{B} \models \mathrm{BA}(w)\}$, that is, S is the set of terms of T such that the term defines a Boolean element in $\mathfrak{B}$. Now, as S is finite, recall that $\langle S \rangle$ is the finite Boolean algebra generated by elements from S. Note then that $\langle S \rangle$ is contained in $\mathrm{BA}(\mathfrak{B})$. Let $\mathbf{s} = \{s_i : i < n\}$ list the atoms of $\langle S \rangle$. Now, for each $s \in \mathbf{s}$, there is a term $t_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$ such that $\mathfrak{B} \models s_i = t_i(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$. Note that the terms t_i in general will not be in reduced form. In the same way, let $R = \{w \in T : \mathfrak{A} \models \mathrm{BA}(w)\}$, that is, R is the set of elements from T such that the term defines a Boolean element in $\mathrm{BA}(\mathfrak{A})$. Again, R is finite, so that $\langle R \rangle$ is a finite Boolean algebra contained in $\mathrm{BA}(\mathfrak{A})$. Let $\mathbf{r}$ list the atoms of $\langle R \rangle$. It is clear that the elements $\mathbf{r}, \mathbf{s}$ satisfy 1, 2, 9, and 10.

Now, for $r \in \mathbf{r}$, recall that $\mathbf{a} \restriction r$ is the tuple $\{a_i \restriction r\}$. We set $\mathbf{a}_r$ to be $(\mathbf{a} \restriction r) \setminus \{\varpi\}$, that is, $\mathbf{a}_r$ is the set of all $a_i \restriction r$ such that $\mathfrak{A} \models r \trianglelefteq [\![\mathrm{E}(a_i)]\!]$. Let $C = \bigcup_{r \in \mathbf{r}} \mathbf{a}_r$. Since $\mathbf{a}$ is finite and $\mathbf{r}$ is finite, we have that C is finite. Let $\mathbf{c}$ list the elements of C. We similarly define $\mathbf{a}_s$ for $s \in \mathbf{s}$. Let $D = \bigcup_{s \in \mathbf{s}} \mathbf{a}_s$. As $\mathbf{a}$ is finite, and $\mathbf{s}$ is finite, we again have that D is finite. Let $\mathbf{d}$ list the elements of D. Finally, we define $\mathbf{b}_s$ as before, and set E equal to $\bigcup_{s \in \mathbf{s}} \mathbf{b}_s$. Again, E is finite, so we let $\mathbf{e}$ enumerate the elements of E. It is clear that the elements $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$ satisfy 3, 4, 5, 6, 7, and 8.

We now show that 11 and 12 hold. It suffices to prove the result for a single atom $P(\mathbf{x})$ and a function $f(\mathbf{x}) = y$. We show it holds for a predicate. If P is nullary, the result is obvious. Let $\mathbf{f}' \subseteq \mathbf{c}$ be of the proper arity, and let $r \in \mathbf{r}$. If for some $f' \in \mathbf{f}'$, we have that $r \neq [\![\mathrm{E}(f')]\!]$, then, since r is an atom in $\langle R \rangle$, we have that $\mathfrak{A} \models r \sqcap [\![\mathrm{E}(f')]\!] = 0$, and so by strictness, $\mathfrak{A} \models r \sqcap [\![P(\mathbf{f}')]\!] = 0$. Thus, we may suppose that for all $f' \in \mathbf{f}'$, $\mathfrak{A} \models r = [\![\mathrm{E}(f')]\!]$. Now suppose $\mathfrak{A} \models r \sqcap [\![P(\mathbf{f}')]\!] \neq 0$. Since $[\![P(\mathbf{f}')]\!]$ is in $\langle R \rangle$, we have that $r \trianglelefteq [\![P(\mathbf{f}')]\!] \trianglelefteq [\![\mathrm{E}(\mathbf{f}')]\!] = r$. The argument for a function $f(\mathbf{x}) = y$ is a slight modification of this argument, so that 11 holds. A similar proof shows that 12 holds.

Finally, for part 13, each of $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$ can be written as terms over $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$ by parts 7, 8, 9, and 10. For part 14, it is obvious by the construction of $\mathbf{r}, \mathbf{s}$ given above that each of $\mathbf{r}, \mathbf{s}$ can be written as a term with elements from $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$. For an element $r \in \mathbf{r}$, we write $t_r(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$ for a term such that $\mathfrak{B} \models r = t_r(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$. Similarly, for $s \in \mathbf{s}$, we define $t_s(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$ to be a term such that $\mathfrak{B} \models s = t_s(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$. Note that each c, there are elements a_i, r_k such that $\mathfrak{B} \models c = a_i \restriction r$, so that $\mathbf{c}$ is clearly defined by the term as $x_i \restriction t_r$, with $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$ substituted in. Further, for each d, there are a_j, s such that $\mathfrak{B} \models d = a_j \restriction s$. Thus, we define $t_d(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$ to be $x_j \restriction t_s(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$, so that $\mathfrak{B} \models d = t_d(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$. Finally, for each e, there are b_i, s such that $\mathfrak{B} \models e = b_i \restriction s$. Thus, we define $t_e(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$ to be $z_i \restriction t_s(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$, so that $\mathfrak{B} \models e = t_e(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$. This proves part 14. $\dashv$

Thus, given two models $\mathfrak{A} \subseteq \mathfrak{B}$ of Γ^{ABA} and elements $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$, we can form a finite partition of $\mathrm{BA}(\mathfrak{B})$ which generates $\mathbf{p}, \mathbf{q}$. Further, there are structural elements over each element of that partition such that the structural elements can be glued together to form $\mathbf{a}, \mathbf{b}$. Finally, each atomic $\mathcal{L}$-formula, with these structural elements substituted in, has an extent of either 0 or one of the elements of the partition. We formalize this in a definition.

Definition 5.9 *Let $\mathfrak{A} \subseteq \mathfrak{B}$ be models of Γ^{ABA} with elements $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$, where $\mathbf{a} \in \mathrm{ST}(A), \mathbf{p} \in \mathrm{BA}(A), \mathbf{b} \in \mathrm{ST}(B), \mathbf{q} \in \mathrm{BA}(B)$, and let $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$ be a term-reduced quantifier-free $\mathcal{L}^{\mathrm{BA}}$-formula.*

1. *Let t be a finite set of atomic $\mathcal{L}$-formulas. We call the elements* $\mathbf{c}$, $\mathbf{r}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{s}$ *from Lemma 5.8 a* t**-decomposition** *of elements* $\mathbf{a}$, $\mathbf{p}$, $\mathbf{b}$, $\mathbf{q}$*; that is, the elements* $\mathbf{c}$, $\mathbf{r}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{s}$ *are a decomposition of* $\mathbf{a}$, $\mathbf{p}$, $\mathbf{b}$, $\mathbf{q}$ *if they are formed in the manner of Lemma 5.8.*

2. *We define the* $\mathcal{L}$**-part** *of φ to be the set of $\mathcal{L}$-predicates $P(\mathbf{x})$ such that $[\![P(\mathbf{x})]\!] = y$ occurs in φ together with the set of atomic formulas $f(\mathbf{x}) = y$ which occurs in φ, where $f(\mathbf{x})$ is an $\mathcal{L}$-function.*

3. *If t is the $\mathcal{L}$-part of a reduced-term $\mathcal{L}^{\mathrm{BA}}$-formula φ, we also call the t-decomposition a* φ**-decomposition**.

4. *For the tuple* $\mathbf{c}$*, we set* $\mathbf{x_c}$ *to be the tuple* $\mathbf{x}_{c_0}, \ldots, \mathbf{x}_{c_{n-1}}$*. We similarly define* $\mathbf{x_r}$*,* $\mathbf{x_d}$*,* $\mathbf{x_e}$*, and* $\mathbf{x_s}$*). Let* $t_a(\mathbf{x_c}), t_p(\mathbf{x_r}), t_b(\mathbf{x_e})$*, and* $t_q(\mathbf{x_s})$ *be the* $\mathcal{L}^{\mathrm{BA}}$*-terms defined in Lemma 5.8. For the tuple* $\mathbf{a}$*, we define* $\mathbf{t_a}(\mathbf{x_c})$ *to be the tuple* $\{t_{a_i}(\mathbf{x_c})\}$ *for each* $a_i \in \mathbf{a}$*, and similarly define* $\mathbf{t_p}(\mathbf{x_r})$*,* $\mathbf{t_b}(\mathbf{x_e})$*, and* $\mathbf{t_q}(\mathbf{x_s})$*. Now, let* $\psi(\mathbf{x_c}, \mathbf{x_r}, \mathbf{x_d}, \mathbf{x_e}, \mathbf{x_s})$ *be the* $\mathcal{L}^{\mathrm{BA}}$*-formula* $\varphi(\mathbf{t_a}(\mathbf{x_c}), \mathbf{t_p}(\mathbf{x_r}), \mathbf{t_b}(\mathbf{x_e}), \mathbf{t_q}(\mathbf{x_s}))$*. Then* ψ *is the* **decomposition of** φ*. We also say* $\psi(\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s})$ *is a* **decomposed sentence**.

Let the elements $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$ be a φ-decomposition. We define **the decomposition description of** $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$ to be the quantifier-free formula $\alpha(\mathbf{x_c}, \mathbf{x_r}, \mathbf{x_d}, \mathbf{x_e}, \mathbf{x_s})$ which says these variables are a φ-decomposition. This is a conjunction which includes the following:

- Both $\mathbf{x_r}$ and $\mathbf{x_s}$ are a partition of the Boolean algebra.
- If $r = [\![\mathrm{E}(c)]\!]$, then α contains $x_r = [\![\mathrm{E}(x_c)]\!]$.
- If $s = [\![\mathrm{E}(d)]\!]$, then α contains $x_s = [\![\mathrm{E}(x_d)]\!]$.
- If $s = [\![\mathrm{E}(e)]\!]$, then α contains $x_s = [\![\mathrm{E}(x_e)]\!]$.
- If $c = \bigoplus_{i \in I} d_i$, then α contains $x_c = \bigoplus_{i \in I} (x_d)_i$, that is, the i-th entry in the tuple $\mathbf{x_d}$.
- If $r \trianglelefteq [\![P(\mathbf{c})]\!]$, then α contains $x_r \trianglelefteq [\![P(\mathbf{x_c})]\!]$. If $r \sqcap [\![P(\mathbf{c})]\!] = 0$, then α contains $x_r \sqcap [\![P(\mathbf{x_c})]\!] = 0$.

- If $r \trianglelefteq [\![f(\mathbf{c}) = c']\!]$, then α contains $x_r \trianglelefteq [\![f(\mathbf{x_c}) = x_{c'}]\!]$. If $r \sqcap [\![f(\mathbf{c}) = c']\!] = 0$, then α contains $x_r \sqcap [\![f(\mathbf{x_c}) = x_{c'}]\!] = 0$.

This notation makes it easier to describe when we have a decomposition.

We now show that decomposed sentences are sufficient to show a model is existentially closed.

Lemma 5.10 *Suppose $\mathfrak{A} \subseteq \mathfrak{B}$ are models of Γ^{ABA}, and suppose that $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$ for all decomposed sentences; that is, for every decomposed sentence $\psi(\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s})$, where $\mathbf{c} \in \mathrm{ST}(A)$, $\mathbf{r} \in \mathrm{BA}(A)$, $\mathbf{d}, \mathbf{e} \in \mathrm{ST}(B)$, and $\mathbf{s} \in \mathrm{BA}(B)$, with $\mathfrak{B} \models \psi(\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s})$, then there exist $\mathbf{d}', \mathbf{e}' \in \mathrm{ST}(A)$ and $\mathbf{s}' \in \mathrm{BA}(A)$ with $\mathfrak{A} \models \psi(\mathbf{c}, \mathbf{r}, \mathbf{d}', \mathbf{e}', \mathbf{s}')$. Then $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be models of Γ^{ABA}, and suppose $\mathfrak{B} \models \varphi(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$, where φ is a quantifier-free formula, and $\mathbf{a} \in \mathrm{ST}(A)$, $\mathbf{p} \in \mathrm{BA}(A)$, $\mathbf{b} \in \mathrm{ST}(B)$, and $\mathbf{q} \in \mathrm{BA}(B)$. We need to find elements $\mathbf{b}' \in \mathrm{ST}(A)$ and $\mathbf{q}' \in \mathrm{BA}(A)$ such that $\mathfrak{A} \models \varphi(\mathbf{a}, \mathbf{p}, \mathbf{b}', \mathbf{q}')$. Let ψ be the decomposition of φ. Then $\mathfrak{B} \models \psi(\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s})$, and $\psi(\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s})$ is a quantifier-free sentence. By our assumption, we have that there exists $\mathbf{d}', \mathbf{e}', \mathbf{s}' \in A$ so that $\mathfrak{A} \models \psi(\mathbf{c}, \mathbf{r}, \mathbf{d}', \mathbf{e}', \mathbf{s}')$. Let $\mathbf{b}' = \mathbf{t_b}(\mathbf{e}')$, and let $\mathbf{q}' = \mathbf{t_q}(\mathbf{s}')$. Then $\mathfrak{A} \models \varphi(\mathbf{a}, \mathbf{p}, \mathbf{b}', \mathbf{q}')$, so that $\mathfrak{A}$ is existentially closed. $\dashv$

Thus, it suffices to prove that models of Γ^{ABA} are existentially closed with respect to decomposed sentences in order to prove that Γ^{ABA} is model complete. In some sense, decomposed sentences are the most basic of sentences, as they break down the structural elements and Boolean elements into their most basic components. In order to show that Γ^{ABA} is model complete, we need to find the form that decomposed sentences take. To this end, we prove the following Lemma:

Lemma 5.11 *Suppose $\mathfrak{A} \subseteq \mathfrak{B}$ are models of Γ^{ABA}, and let t be a finite subset of $\mathcal{A}\text{t}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ over $\mathcal{L}$. Let $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$ be the t-decomposition of $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$. If $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a quantifier-free $\mathcal{L}$-formula built from elements of t, then, for any elements $\mathbf{c}' \subseteq \mathbf{c}$, $\mathbf{d}' \subseteq \mathbf{d}$, $\mathbf{e}' \subseteq \mathbf{e}$ of the proper arity, there exists a set $I \subseteq \mathbf{s}$ such that $\mathfrak{B} \models [\![\varphi(\mathbf{c}', \mathbf{d}', \mathbf{e}')]\!] = \bigsqcup I$.*

Proof. We prove this by induction on the complexity of φ, and recall that if I is the empty set, then $\bigsqcup I = 0$. The case where φ is atomic or a negated atomic is by Lemma 5.8 part 12. We now suppose this holds for δ and γ. Now, as φ, δ and γ are quantifier-free, they are discrete. Thus, each of $[\![\varphi(\mathbf{c}', \mathbf{d}', \mathbf{e}')]\!]$, $[\![\delta(\mathbf{c}', \mathbf{d}', \mathbf{e}')]\!]$ and $[\![\gamma(\mathbf{c}', \mathbf{d}', \mathbf{e}')]\!]$ have Boolean values. We let s_φ, s_δ, and s_γ be these values, respectively.

Let φ is $\delta \wedge \gamma$. By induction, we have that $\mathfrak{B} \models \bigsqcup I = s_\delta$ and $\mathfrak{B} \models \bigsqcup J = s_\gamma$. Let K be $I \cap J$. Then by the translation schema, $\mathfrak{B} \models \bigsqcup K = s_\varphi$.

Suppose φ is $\delta \vee \gamma$. By induction, we have that $\mathfrak{B} \models \bigsqcup I = s_\delta$ and $\mathfrak{B} \models \bigsqcup J = s_\gamma$. Let $K := I \cup J$. Then by the translation schema, $\mathfrak{B} \models \bigsqcup K = s_\varphi$.

Finally, suppose φ is $\delta \to \gamma$. Now $s_\varphi = -s_\delta \sqcup s_\gamma$. By induction, we have that $\mathfrak{B} \models \bigsqcup I = s_\delta$ and $\mathfrak{B} \models \bigsqcup J = s_\gamma$. Now, as $\mathbf{s}$ is a partition of BA($\mathfrak{B}$), let I' list $\mathbf{s} \setminus I$. Then $\mathfrak{B} \models -s_\delta = \bigsqcup I'$. Let $K = I' \cup J$. Then $\mathfrak{B} \models s_\varphi = \bigsqcup K$.

As quantifier-free formulas are formed from atoms, negated atoms, conjunction, disjunction, and implication, we are done. $\dashv$

Thus, we get that the partition $\mathbf{s}$ is fine enough so that for any term-reduced quantifier-free sentence φ over the proper sublanguage of $\mathcal{L}$, we get that $[\![\varphi]\!]$ is a finite join of elements of $\mathbf{s}$. We now show a similar result for the smaller model $\mathfrak{A}$:

Corollary 5.12 *Suppose $\mathfrak{A} \subseteq \mathfrak{B}$ are models of Γ^{ABA}, and let t be a finite subset of*

$\mathcal{A}\mathrm{t}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ over $\mathcal{L}$. Let $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$ be the decomposition of $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$. If $\varphi(\mathbf{x})$ is a term-reduced quantifier-free formula, then, for any elements $r \in \mathbf{r}$ and $\mathbf{c}' \subseteq \mathbf{c}$, there exists a finite set $I \subseteq \mathbf{r}$ such that $\mathfrak{A} \models \bigsqcup I = [\![\varphi(\mathbf{c}')]\!]$.

Proof. The proof is similar to that of Lemma 5.11. $\dashv$

Now, in trying to prove that Γ^{ABA} is model complete, we need to find a general form for quantifier-free formulas over $\mathcal{L}^{\mathrm{BA}}$. This includes formulas of the form $s \trianglelefteq [\![\varphi]\!]$, where φ is a quantifier-free formula over $\mathcal{L}$. We now show that we can replace these formulas φ with a conjunction.

Lemma 5.13 *Suppose $\mathfrak{A} \subseteq \mathfrak{B}$ are models of Γ^{ABA}, let φ be a term-reduced quantifier-free $\mathcal{L}$-formula, let t be all the atoms which occur in φ, and let $\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s}$ be the t-decomposition of $\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q}$. Suppose that for some $s \in \mathbf{s}$, $\mathfrak{B} \models s \trianglelefteq [\![\varphi(\mathbf{c}, \mathbf{d}, \mathbf{e})]\!]$. Then there exists a largest finite term-reduced type $u \subseteq \mathcal{A}\mathrm{t}_t^{\pm}$ over $\mathcal{L}$ such that $\vdash_i \pi_u \rightarrow \varphi$ and $\mathfrak{B} \models s \trianglelefteq [\![\pi_u(\mathbf{c}, \mathbf{d}, \mathbf{e})]\!]$.*

Proof. Suppose $\mathfrak{B} \models s \trianglelefteq [\![\varphi(\mathbf{c}, \mathbf{d}, \mathbf{e})]\!]$. By Corollary 4.3, we have that $\mathfrak{B} \models s \trianglelefteq [\![\varphi(\mathbf{c}\restriction s, \mathbf{d}\restriction s, \mathbf{e}\restriction s)]\!]$. Note that $\mathbf{c}\restriction s \subseteq \mathbf{d}$, so we may suppose that the elements $\mathbf{c}$ appear in $\mathbf{d}$, that is, $\varphi(\mathbf{c}, \mathbf{d}, \mathbf{e})$ is of the form $\varphi(\mathbf{d}, \mathbf{e})$. By Lemma 5.8, we have that for each term reduced atom δ in t, either $\mathfrak{B} \models s \trianglelefteq [\![\delta(\mathbf{d}, \mathbf{e})]\!]$ or $\mathfrak{B} \models s \sqcap [\![\delta(\mathbf{d}, \mathbf{e})]\!] = 0$. We let u be the collection of all δ in t such that the former holds, together with the set of all $\neg\delta$ such that the latter holds. Obviously $\mathfrak{B} \models s \trianglelefteq [\![\pi_u(\mathbf{d}, \mathbf{e})]\!]$, and u is largest. To show $\vdash_i \pi_u \rightarrow \varphi$, we may suppose φ in disjunctive normal form $\bigvee_{i<n} \varphi_i$. Now, both φ and $\bigvee_{i<n} \varphi_i$ are discrete sentences, so they have the same Boolean extent. Thus, $s \trianglelefteq [\![\bigvee_{i<n} \varphi_i(\mathbf{d}, \mathbf{e})]\!]$, so by Lemma 5.11 there exist $\{s_i\}_{i<n} \subseteq \mathbf{s}$ such that $\mathfrak{B} \models \bigsqcup_{i<n} s_i = s$

and $s_i \trianglelefteq [\![\varphi_i(\mathbf{d}, \mathbf{e})]\!]$ for each i. But as s is a decomposed Boolean element itself, we have that $s = s_i$ for some i. Thus, every predicate and negated predicate which occurs in φ_i must occur in u. Thus $\vdash_i \pi_u \to \varphi_i$, so $\vdash_i \pi_u \to \varphi$. $\dashv$

We now show that these decomposed sentences may be taken to have a particular form.

Lemma 5.14 *Suppose* $\mathfrak{A} \subseteq \mathfrak{B}$ *are models of* Γ^{ABA}*, and let* $\psi(\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s})$ *be a decomposed sentence, and let* α *be the decomposition description. Then there exists a quantifier-free formula* $\theta(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}, \mathbf{w})$ *such that:*

1. $\emptyset^{\mathrm{BA}} \vdash_i \alpha(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}, \mathbf{w}) \to (\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}, \mathbf{w}) \leftrightarrow \theta(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}, \mathbf{w}))$

2. θ *is a disjunction of conjunctions of formulas and negations of formulas of the form* $x_i \restriction w_j = z_k$*,* $x_i = \bigoplus\{z_j : j \in J\}$*,* $y_i = \bigsqcup\{w_j : j \in J\}$*,* $y_i \trianglelefteq [\![\pi_u(\mathbf{x})]\!]$ *and* $w_i \trianglelefteq [\![\pi_v(\mathbf{z}, \mathbf{v})]\!]$*, where* $u \in \mathcal{A}\mathrm{t}_t^{\pm}$ *is a finite term-reduced type over* $\mathcal{L}$*, and* $v \in \mathcal{A}\mathrm{t}_t^{\pm}$ *is finite term-reduced type over* $\mathcal{L}$*.*

Proof. As ψ is decomposed, it is quantifier-free. We may assume that ψ is in disjunctive normal form. Thus, we may suppose that ψ is a conjunction. Now, the formulas $x_i \restriction w_j = z_k$, $x_i = \bigoplus\{z_j : j \in J\}$, and $y_i = \bigsqcup\{s_j : j \in J\}$ all simply reflect the fact that ψ is decomposed. For all $c \in \mathbf{c}, r \in \mathbf{r}$, we get that $c \restriction r = c$ or $c \restriction r = \varpi$. Thus, any instance of $x_i \restriction y_j$ can be replaced with either x_i or ϖ over α. The same holds for any instance of $z_i \restriction w_j$ or $v_i \restriction w_j$. Further, since for each c_i, s_j, there exists d_k such that $c_i \restriction s_j = d_k$, so that we can replace $x_i \restriction w_j$ with z_k over α. Thus, in ψ, the variables $\mathbf{x}, \mathbf{z}, \mathbf{w}$ can only occur in functions of the form $[\![\gamma(\mathbf{x}', \mathbf{z}', \mathbf{v}')]\!]$, where $\mathbf{x}' \subseteq \mathbf{x}$, $\mathbf{z}' \subseteq \mathbf{z}$, and $\mathbf{v}' \subseteq \mathbf{v}$. If, the variables $\mathbf{z}'$ and $\mathbf{v}'$ are empty for a given function $[\![\gamma(\mathbf{x}', \mathbf{z}', \mathbf{v}')]\!]$, then we simply call the function $[\![\gamma(\mathbf{x}')]\!]$. Further, by Corollary 5.12, over α we can replace any instance of $w_i \trianglelefteq [\![\gamma(\mathbf{x}')]\!]$ with $y_j \trianglelefteq [\![\gamma(\mathbf{x}')]\!]$ for some j. Now, if $\mathfrak{B} \models s \trianglelefteq [\![\gamma(\mathbf{c}', \mathbf{d}', \mathbf{e}')]\!]$, then $\mathfrak{B} \models s \trianglelefteq [\![\gamma(\mathbf{c}' \restriction s, \mathbf{d}', \mathbf{e}')]\!]$ by Corollary 4.3. But for each i, there exists a j such that $c_i' \restriction s$ is equal to d_j. Thus, any function $[\![\gamma(\mathbf{x}', \mathbf{z}', \mathbf{v}')]\!]$ can be replaced by a function of the form $[\![\gamma(\mathbf{z}', \mathbf{v}')]\!]$ over α.

With this, our problem reduces to classifying all functions with forms like $[\![P(\mathbf{t_a}(\mathbf{x_c}), \mathbf{t_b}(\mathbf{x_e}))]\!] = \mathbf{t_p}(\mathbf{x_r})$ or $f(\mathbf{t_a}(\mathbf{x_c}), \mathbf{t_b}(\mathbf{x_e})) = \mathbf{t_b}(\mathbf{x_e})$. Repeatedly apply Lemma 3.9, parts 11, 12, and 13 and that the sets $\mathbf{r}$ and $\mathbf{s}$ are atoms of the finite Boolean algebras $\langle R \rangle$ and $\langle S \rangle$, respectively. That is, we break the extents of the predicates and the functions into its smallest components. After these applications, we are left with a conjunction of formulas of the form $y \trianglelefteq [\![P(\mathbf{x})]\!]$, $w_i \trianglelefteq [\![\pi_v(\mathbf{z}, \mathbf{v})]\!]$, $f(\mathbf{x}) = x'$, $f(\mathbf{z}, \mathbf{v}) = z'$, and $f(\mathbf{z}, \mathbf{v}) = v'$. Obviously formulas of the form $y \trianglelefteq [\![P(\mathbf{x})]\!]$ are of the required form. We can also replace the functions $f(\mathbf{x}) = x'$ with formulas of the form $y \trianglelefteq [\![f(\mathbf{x}) = x']\!]$. To see this, if, for example, $f(\mathbf{c}) = c'$, then by the strictness of the functions f and as $\mathbf{r}$ are atoms of $\langle R \rangle$, this is equivalent to $r \trianglelefteq [\![f(\mathbf{c}) = c']\!]$. Note that $y \trianglelefteq [\![f(\mathbf{x}), = x']\!]$ is of the required form. Let θ be the result of applying the replacements described to ψ. The result then follows. $\dashv$

Recall that for a model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, an $\mathcal{L}(\mathrm{ST}(A))$-sentence φ is discrete if there is an element $p \in \mathrm{BA}(A)$ such that $\mathfrak{A} \models p = [\![\varphi]\!]$. Our next lemma shows that, for a model complete $\mathcal{L}$-theory Γ and a model $\mathfrak{A}$ of the translated theory Γ^{ABA}, the set of discrete sentences contains not only quantifier-free sentences but existential sentences as well.

Lemma 5.15 *Let Γ be a Π_2^0 axiomatization of a model complete $\mathcal{L}$-theory, and let $\mathfrak{A} \models \Gamma^{\mathrm{ABA}}$. Then the set of discrete sentences over Γ^{ABA} includes existential sentences.*

Proof. Let $\exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{a})$ be an existential $\mathcal{L}(\mathrm{ST}(A))$-sentence, where φ is quantifier-free. It suffices to show the result holds for the $\mathcal{L}(\mathrm{ST}(A))$-sentence $\mathrm{E}(\mathbf{a}) \wedge \exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{a})$. As Γ is model complete, $\Gamma \vdash_c \forall \mathbf{y}(\exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{y}) \leftrightarrow \forall \mathbf{x}(\neg\psi(\mathbf{x}, \mathbf{y})))$ for some quantifier-free ψ. Thus $\Gamma \vdash_c \forall \mathbf{y}(\exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{y}) \vee \exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y}))$ and $\Gamma \vdash_c \forall \mathbf{y}\mathbf{x}_1\mathbf{x}_2\neg(\varphi(\mathbf{x}_1, \mathbf{y}) \wedge \psi(\mathbf{x}_2, \mathbf{y}))$. By Corollary 4.36, we have that $\Gamma^{\mathrm{ABA}} \vdash_c 1 \trianglelefteq [\![\forall \mathbf{y}(\exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{y}) \vee \exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y}))]\!]$ and $\Gamma^{\mathrm{ABA}} \vdash_c 1 \trianglelefteq [\![\forall \mathbf{y}\mathbf{x}_1\mathbf{x}_2\neg(\varphi(\mathbf{x}_1, \mathbf{y}) \wedge \psi(\mathbf{x}_2, \mathbf{y}))]\!]$. The first sentence translates to $\forall \mathbf{y}(\mathrm{ST}(\mathbf{y}) \rightarrow [\![\mathrm{E}(\mathbf{y})]\!] \trianglelefteq [\![\exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{y}) \vee \exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y})]\!])$. Then $\mathfrak{A} \models [\![\mathrm{E}(\mathbf{a})]\!] \trianglelefteq [\![\exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{a}) \vee \exists \mathbf{x}\psi(\mathbf{x}, \mathbf{a})]\!]$, so there are p, $q \in \mathrm{BA}(A)$ such that $\mathfrak{A} \models p \sqcup q = [\![\mathrm{E}(\mathbf{a})]\!]$ and $p \trianglelefteq [\![\exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{a})]\!]$ and $q \trianglelefteq [\![\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{a})]\!]$.

We claim that $p = [\![\mathrm{E}(\mathbf{a}) \wedge \exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{a})]\!]$. To see this, let r be such that $r \trianglelefteq [\![\mathrm{E}(\mathbf{a})]\!]$ and $r \trianglelefteq [\![\exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{a})]\!]$. Then there exist elements $\mathbf{b}$ such that $\mathfrak{A} \models [\![\mathrm{E}(\mathbf{b})]\!] = r \wedge [\![\mathrm{E}(\mathbf{b})]\!] \trianglelefteq [\![\varphi(\mathbf{b}, \mathbf{a})]\!]$. Now, $r \sqcap q \trianglelefteq r$, so by Lemma 4.5, $r \sqcap q \trianglelefteq [\![\varphi(\mathbf{b}, \mathbf{a})]\!]$. Further, as $r \sqcap q \trianglelefteq q$ and $q \trianglelefteq [\![\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{a})]\!]$, there exist elements $\mathbf{c}$ such that $\mathfrak{A} \models [\![\mathrm{E}(\mathbf{c})]\!] = (r \sqcap q) \wedge [\![\mathrm{E}(\mathbf{c})]\!] \trianglelefteq [\![\psi(\mathbf{c}, \mathbf{a})]\!]$.

As $\Gamma^{\mathrm{ABA}} \vdash_c 1 \trianglelefteq [\![\forall \mathbf{y}\mathbf{x}_1\mathbf{x}_2\neg(\varphi(\mathbf{x}_1, \mathbf{y}) \wedge \psi(\mathbf{x}_2, \mathbf{y}))]\!]$, we get that

$$\mathfrak{A} \models [\![\mathrm{E}(\mathbf{a})]\!] \trianglelefteq [\![\forall \mathbf{x}_1\mathbf{x}_2\neg(\varphi(\mathbf{x}_1, \mathbf{a}) \wedge \psi(\mathbf{x}_2, \mathbf{a}))]\!].$$

An exercise on the translation schema shows that this is equivalent to $\forall \mathbf{x}_1\mathbf{x}_2(([\![\mathrm{E}(\mathbf{x}_1)]\!] \trianglelefteq [\![\mathrm{E}(\mathbf{a})]\!] \wedge [\![\mathrm{E}(\mathbf{x}_2)]\!] \trianglelefteq [\![\mathrm{E}(\mathbf{a})]\!] \wedge [\![\mathrm{E}(\mathbf{x}_1)]\!] \trianglelefteq [\![\varphi(\mathbf{x}_1, \mathbf{a})]\!] \wedge [\![\mathrm{E}(\mathbf{x}_2)]\!] \trianglelefteq [\![\psi(\mathbf{x}_2, \mathbf{a})]\!]) \rightarrow [\![\mathrm{E}(\mathbf{x}_1)]\!] \sqcap [\![\mathrm{E}(\mathbf{x}_2)]\!] = 0)$. We replace $\mathbf{x}_1$ with $\mathbf{b}$ and $\mathbf{x}_2$ with $\mathbf{c}$. The conclusion then is $[\![\mathrm{E}(\mathbf{b})]\!] \sqcap [\![\mathrm{E}(\mathbf{c})]\!] = r \sqcap q \sqcap r = q \sqcap r = 0$. Since $r \trianglelefteq (p \sqcup q)$ and $r \sqcap q = 0$, we have that $r \trianglelefteq p$. Thus $p = [\![\mathrm{E}(\mathbf{a}) \wedge \exists \mathbf{x}\varphi(\mathbf{x}, \mathbf{a})]\!]$. Since φ was arbitrary, we are done. $\dashv$

We get an immediate corollary from this result.

Corollary 5.16 *Let Γ be a Π^0_2 axiomatization of a model complete $\mathcal{L}$-theory, and let* $\mathfrak{A} \models$

Γ^{ABA}. *Then if* φ *is an existential* $\mathcal{L}(\mathrm{ST}(A))$*-sentence, then there exists an existential* $\mathcal{L}(\mathrm{ST}(A))$*-sentence* ψ *such that* $(\neg\varphi] = (\psi]$*, and both are principal ideals.*

Proof. Immediate from the proof of Lemma 5.15. ⊣

In the proof that Γ^{ABA} is model complete, this result proves very useful. The important fact is that for model complete theories, every existential sentence is equivalent to a universal sentence. Knowing that existential formulas are discrete is vital for finding the proper elements in $\mathfrak{A}$ to replace those from $\mathfrak{B}$.

With this, we now prove that Γ^{ABA} is model complete.

Theorem 5.17 *Let* Γ *be a* Π^0_2 *axiomatization of a model complete* $\mathcal{L}$*-theory. Then* Γ^{ABA} *is model complete.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be models of Γ^{ABA}, and suppose $\mathfrak{B} \models \varphi(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$, where φ is quantifier-free, $\mathbf{a} \in \mathrm{ST}(A)$, $\mathbf{p} \in \mathrm{BA}(A)$, $\mathbf{b} \in \mathrm{ST}(B)$, $\mathbf{q} \in \mathrm{BA}(B)$. Now, by Lemma 5.10, we may replace $\varphi(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{q})$ with $\psi(\mathbf{c}, \mathbf{r}, \mathbf{d}, \mathbf{e}, \mathbf{s})$, the decomposition of φ . Thus, we must find elements $\mathbf{d}', \mathbf{e}', \mathbf{s}' \in A$ such that $\mathfrak{A} \models \psi(\mathbf{c}, \mathbf{r}, \mathbf{d}', \mathbf{e}', \mathbf{s}')$. Note that the $\mathcal{L}$-part of φ is the same as the $\mathcal{L}$-part of ψ. We let t be this $\mathcal{L}$-part.

For $s \in \mathbf{s}$, we let $\mathbf{d}_s$ be the tuple $\{d \in \mathbf{d} \;:\; \mathfrak{B} \models s = [\![\mathrm{E}(d)]\!]\}$, and similarly for $\mathbf{e}_s$. For elements $\mathbf{d}, \mathbf{e}, s$, we let $u(\mathbf{d}, \mathbf{e}, s) \subseteq \mathcal{A}\mathrm{t}^{\pm}_t$ be the largest type such that $\mathfrak{B} \models s \trianglelefteq [\![\pi_{u(\mathbf{d},\mathbf{e},s)}(\mathbf{d}_s, \mathbf{e}_s)]\!]$ from Lemma 5.13. We choose an element r from $\mathbf{r}$. Without loss of generality, we may assume that r is such that $\mathfrak{B} \models s_0 \sqcup s_1 \sqcup \ldots \sqcup s_{n-1} = r$. Now, for each $i < n$, we have that $\mathfrak{B} \models s_i \trianglelefteq [\![\pi_{u(\mathbf{d},\mathbf{e},s_i)}(\mathbf{d}_{s_i}, \mathbf{e}_{s_i})]\!]$. Thus, for each i, $\mathfrak{B} \models s_i \trianglelefteq [\![\exists \mathbf{y} \pi_{u(\mathbf{d},\mathbf{e},s_i)}(\mathbf{d}_{s_i}, \mathbf{y})]\!]$. Let $\mathbf{c}_{s_i}$ be tuple such that, for each j, $(c_{s_i})_j \restriction s_i = (d_{s_i})_j$, that is, $(c_{s_i})_j$ is the element of $\mathbf{c}$ that, when restricted to s_i, is the element $(d_{s_i})_j$. By

Corollary 4.4, for each i, $\mathfrak{B} \models s_i \trianglelefteq [\![\exists \mathbf{y} \pi_{u(\mathbf{d},\mathbf{e},s_i)}(\mathbf{c}_{s_i}, \mathbf{y})]\!]$. Now, Γ is model complete, so that by Lemma 5.15, $\exists \mathbf{y} \pi_{u(\mathbf{d},\mathbf{e},s_i)}(\mathbf{c}_{s_i}, \mathbf{y})$ is discrete for each i.

Now, by the previous paragraph, we get that $\mathfrak{B} \models \exists \mathbf{z}(\bigwedge_{i<j<n} z_i \sqcap z_j = 0 \wedge \bigwedge_{i<n} z_i \neq 0 \wedge \bigsqcup_{i<n} z_i = r \wedge \bigwedge_{i<n} z_i \trianglelefteq [\![\exists \mathbf{y} \pi_{u(\mathbf{d},\mathbf{e},s_i)}(\mathbf{c}_{s_i}, \mathbf{y})]\!])$. Note that $[\![\exists \mathbf{y} \pi_{u(\mathbf{d},\mathbf{e},s_i)}(\mathbf{c}_{s_i}, \mathbf{y})]\!]$ is a Boolean element in $\mathfrak{A}$. Now, as $\mathrm{BA}(\mathfrak{A})$ and $\mathrm{BA}(\mathfrak{B})$ are atomless Boolean algebras, we have that $\mathrm{BA}(\mathfrak{A})$ is existentially closed in $\mathrm{BA}(\mathfrak{B})$. Thus, $\mathfrak{A} \models \exists \mathbf{z}(\bigwedge_{i<j<n} z_i \sqcap z_j = 0 \wedge \bigwedge_{i<n} z_i \neq 0 \wedge \bigsqcup_{i<n} z_i = r \wedge \bigwedge_{i<n} z_i \trianglelefteq [\![\exists \mathbf{y} \pi_{u(\mathbf{d},\mathbf{e},s_i)}(\mathbf{c}_{s_i}, \mathbf{y})]\!])$. Let $s'_0, \ldots, s'_{n-1}$ witness this in $\mathrm{BA}(A)$. We repeat this process for all other elements in $\mathbf{r}$, and set $\mathbf{s}'$ to be the collection of elements derived in this manner. Note that $(\mathrm{BA}(\mathfrak{B}), \mathbf{r}, \mathbf{s}) \equiv (\mathrm{BA}(\mathfrak{A}), \mathbf{r}, \mathbf{s}')$, so that in particular, $\mathbf{s}'$ forms a partition of $\mathrm{BA}(A)$ and the elements $\mathbf{s}'$ generate the elements $\mathbf{r}$ in the same way that the elements $\mathbf{s}$ generate $\mathbf{r}$. We have thus replaced the elements $\mathbf{s}$.

Next, for each $c \in \mathbf{c}$, there exists $\mathbf{d}_c \subseteq \mathbf{d}$ such that $\mathfrak{B} \models c = \bigoplus\{d : d \in \mathbf{d}_c\}$. That is, for each $c \in \mathbf{c}$ and $s_i \in \mathbf{s}$ with $\mathfrak{B} \models s_i \trianglelefteq [\![\mathrm{E}(c)]\!]$, there is a d_j such that $\mathfrak{B} \models c \restriction s_i = d_j$. We replace d_j with $d'_j := c \restriction s'_i$. As a result, for each i, we have that $\mathfrak{B} \models s_i = [\![\mathrm{E}(d_j)]\!]$ if and only if $\mathfrak{A} \models s'_i = [\![\mathrm{E}(d'_j)]\!]$ and $\mathfrak{B} \models s_i \sqcap [\![\mathrm{E}(d_j)]\!] = 0$ if and only if $\mathfrak{A} \models s'_i \sqcap [\![\mathrm{E}(d'_j)]\!] = 0$. Let $\delta(\mathbf{x})$ be an element of $\mathcal{A}\mathrm{t}^{\pm}_t$. Now, as s_i is an atom of $\langle S \rangle$, for any tuple $\hat{\mathbf{d}} \subseteq \mathbf{d}_{s_i}$, we have that $\mathfrak{B} \models (s_i \trianglelefteq [\![\delta(\hat{\mathbf{d}})]\!]) \vee (s_i \sqcap [\![\delta(\hat{\mathbf{d}})]\!] = 0)$. Now, for a set $\hat{\mathbf{d}} \subseteq \mathbf{d}_{s_i}$, we have a corresponding set $\hat{\mathbf{d}}' \subseteq \mathbf{d}'_{s_i}$. Suppose we have that $\mathfrak{B} \models s_i \trianglelefteq [\![\delta(\hat{\mathbf{d}})]\!]$. We claim that $\mathfrak{A} \models s'_i \trianglelefteq [\![\delta(\hat{\mathbf{d}}')]\!]$.

Note that for our given s_i, there is a unique $r \in \mathbf{r}$ such that $\mathfrak{B} \models s_i \trianglelefteq r$. Further, each $\hat{d}_j$ is equal to some $c \restriction s_i$. Thus, let $\hat{\mathbf{c}}$ be the tuple such that $\hat{c}_j \restriction s_i = d_j$ for each j. By Corollary 4.4, $\mathfrak{B} \models s_i \trianglelefteq [\![\delta(\hat{\mathbf{d}}_{s_i})]\!]$ implies that $\mathfrak{B} \models s_i \trianglelefteq [\![\delta(\hat{\mathbf{c}})]\!]$. However, $\mathfrak{A} \models (r \trianglelefteq [\![\delta(\hat{\mathbf{c}})]\!]) \vee (r \sqcap [\![\delta(\hat{\mathbf{c}})]\!] = 0)$. Since $s_i \trianglelefteq r$ and s_i is nonzero, we must have that

$\mathfrak{A} \models r \trianglelefteq [\![\delta(\hat{\mathbf{c}})]\!]$. Now, $\mathfrak{A} \models s_i' \trianglelefteq r$, and note that $\hat{\mathbf{c}} \restriction s_i' = \hat{\mathbf{d}}'$. By Lemma 4.5 and Corollary 4.3, we have that $\mathfrak{A} \models s_i' \trianglelefteq [\![\delta(\hat{\mathbf{d}}')]\!]$.

Now, for each $s \in \mathbf{s}$, there is a largest reduced-term type $u(\mathbf{d}, s) \subseteq \mathcal{A}\mathrm{t}_t^{\pm}$ such that $\mathfrak{B} \models s \trianglelefteq [\![\pi_{u(\mathbf{d},s)}(\mathbf{d}_s)]\!]$. Thus, for every type $u \subseteq \mathcal{A}\mathrm{t}_t^{\pm}$ over $\mathcal{L}$, $\mathfrak{B} \models s \trianglelefteq [\![\pi_u(\mathbf{d}_s)]\!]$ if and only if $\mathfrak{A} \models s' \trianglelefteq [\![\pi_u(\mathbf{d}_s')]\!]$. We have thus replaced the elements $\mathbf{d}$ with $\mathbf{d}'$.

Finally, we have to find elements in $\mathrm{ST}(A)$ that correspond to the set of elements $\mathbf{e}$. By Lemma 5.14, we may suppose that the elements $\mathbf{e}$ appear only in functions of the form $s \trianglelefteq [\![\pi_u(\mathbf{d_s}, \mathbf{e_s})]\!]$. Thus, suppose that, for a given $s \in \mathbf{s}$, $\mathfrak{B} \models s \trianglelefteq [\![\pi_u(\mathbf{d}_s, \mathbf{e}_s)]\!]$. Note that we have already replaced the elements $\mathbf{d}_s$ with elements $\mathbf{d}_s'$ from $\mathrm{ST}(A)$. For $\mathbf{d}_s, \mathbf{e}_s$, recall the definition of $u(\mathbf{d}, \mathbf{e}, s) \subseteq \mathcal{A}\mathrm{t}_t^{\pm}$. As $u \subseteq u(\mathbf{d}, \mathbf{e}, s)$, we may suppose that $u = u(\mathbf{d}, \mathbf{e}, s)$. Now, since $\mathfrak{B} \models s \trianglelefteq [\![\pi_u(\mathbf{d}_s, \mathbf{e}_s)]\!]$, we have that $\mathfrak{B} \models s \trianglelefteq [\![\exists \mathbf{y} \pi_u(\mathbf{d}_s, \mathbf{y})]\!]$. By Lemma 4.2, $\mathfrak{B} \models s \trianglelefteq [\![\exists \mathbf{y} \pi_u(\mathbf{c}_s, \mathbf{y})]\!]$. By the above discussion, $\mathfrak{A} \models s' \trianglelefteq [\![\exists \mathbf{y} \pi_u(\mathbf{c}_s, \mathbf{y})]\!]$, so again by Lemma 4.2, $\mathfrak{A} \models s' \trianglelefteq [\![\exists \mathbf{y} \pi_u(\mathbf{d}_s', \mathbf{y})]\!]$. Let $\mathbf{e}_s'$ witness this, so that $\mathfrak{A} \models s' \trianglelefteq [\![\pi_u(\mathbf{d}_{s'}', \mathbf{e}_{s'}')]\!]$. We have thus replaced $\mathbf{e}_s$ with $\mathbf{e}_{s'}'$. We continue this process for all $s \in \mathbf{s}$, and we get a set $\mathbf{e}' \in \mathrm{ST}(A)$. Then $\mathfrak{A} \models \psi(\mathbf{c}, \mathbf{r}, \mathbf{d}', \mathbf{e}', \mathbf{s}')$, so that $\mathfrak{A}$ is existentially-closed. As $\mathfrak{A}$ and $\mathfrak{B}$ are arbitrary, we get that Γ^{ABA} is model complete. $\dashv$

We now provide a brief description of the proof that Γ^{ABA} is model complete. We need to replace the elements $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{s}$ with elements in A. Over each $s \in \mathbf{s}$ there is an existential formula which describes how the elements $\mathbf{e}_s$ interact with the elements $\mathbf{d}_s$. As Γ is model complete, every existential sentence over $\mathcal{L}$ is discrete. We then use the model completeness of atomless Boolean algebras to find a set of elements $\mathbf{s}'$ in $\mathrm{BA}(A)$ that partitions $\mathrm{BA}(\mathfrak{B})$. Once we have $\mathbf{s}'$, we construct the elements $\mathbf{d}'$. Above $s' \in \mathbf{s}$ there is an existential sentence with elements $\mathbf{d}_{s'}'$ that matches the existential sentence above the element s. We then find elements $\mathbf{e}'$ which satisfy the existential sentences over each element of $\mathbf{s}'$.

Chapter 6

An Alternative to Wheeler's Conjecture

In chapter 4 we show that the universal Horn fragment of $\Gamma^{\mathrm{BA}\mathbf{2}}$ is Γ^{BA}. In chapter 5 we show that if Γ is a set of Π^0_2 axiomatization of a model complete $\mathcal{L}$-theory, then the extension Γ^{ABA} of Γ^{BA} is model complete. In this chapter we show that if a set Γ of universal $\mathcal{L}$-sentences has a model companion axiomatized by a set Γ' of Π^0_2 sentences, then $(\Gamma')^{\mathrm{ABA}}$ axiomatizes the model companion of Γ^{BA}. We begin by describing models of $\emptyset^{\mathrm{BA}}$ which are obtained by taking a model of $\emptyset^{\mathrm{BA}}$ and a nonzero Boolean element and restricting all elements to that Boolean element. We apply category theory results to these restricted models in order to embed a model of $\emptyset^{\mathrm{BA}}$ into a model of $\emptyset^{\mathrm{ABA}}$. We then proceed to show that models of Γ^{BA} can be embedded into models of Γ^{ABA}. We conclude by drawing some deeper connections between our results and Wheeler's conjecture.

6.1 Restricted models

Definition 6.1 *For a model $\mathfrak{A}$ of $\emptyset^{\mathrm{BA}}$ with $p \in \mathrm{BA}(A)$ and $p \neq 0$, we define the* **restricted model** $\mathfrak{A}\restriction p$ *to be the model described as follows:*

- *The domain $A\restriction p$ consists of the structure $\mathrm{ST}(A\restriction p)$ which is $\{a \ : \ \mathfrak{A} \models [\![\mathrm{E}(a)]\!] \trianglelefteq p\}$*

and the Boolean algebra $\mathrm{BA}(A\restriction p)$ *which is* $\{q : \mathfrak{A} \models q \trianglelefteq p\}$*. Further, any element which is chaff is included in* $A\restriction p$*.*

- *We label the interpretation of functions* f *of* $\mathcal{L}^{\mathrm{BA}}$ *in* $\mathfrak{A}\restriction p$ *as* f^p*. Now,* $0^p = 0^{\mathfrak{A}}$*,* $1^p = p^{\mathfrak{A}}$*,* $\sqcap^p = \sqcap^{\mathfrak{A}}$ *and* $\sqcup^p = \sqcup^{\mathfrak{A}}$*, while* $-^p x = ((-x) \sqcap p)^{\mathfrak{A}}$*. For elements* $a \in \mathrm{ST}(A\restriction p)$ *and* $q \in \mathrm{BA}(A\restriction p)$*, define* $(a\restriction^p q) = (a\restriction q)^{\mathfrak{A}}$*.*

- *For an* $\mathcal{L}$*-predicate* P*,* $p \in \mathrm{BA}(A\restriction p)$*, and elements* $\mathbf{a} \in \mathrm{ST}(A\restriction p)$*, we set* $[\![P(\mathbf{a})]\!]^p = [\![P(\mathbf{a})]\!]^{\mathfrak{A}}$*.*

- *For an* $\mathcal{L}$*-function and elements* $\mathbf{a} \in \mathrm{ST}(A\restriction p)$*,* $f^p(\mathbf{a}) = f(\mathbf{a})^{\mathfrak{A}}$*.*

Note that we could have also defined the structure as $\{a\restriction p : a \in \mathrm{ST}(A)\}$, and the Boolean algebra to be $\{q \sqcap p : q \in \mathrm{BA}(A)\}$. We further note that for any model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, $\mathfrak{A}\restriction 1 = \mathfrak{A}$. We now show that these restricted models preserve the extent of $\mathcal{L}$-sentences.

Lemma 6.2 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$*, let* $\mathfrak{A} \models p \trianglelefteq q$ *with* $q \neq 0$*, and let* γ *be an* $\mathcal{L}$*-sentence. Then* $\mathfrak{A} \models p \trianglelefteq [\![\gamma]\!]$ *if and only if* $\mathfrak{A}\restriction q \models p \trianglelefteq [\![\gamma]\!]$*.*

Proof. This follows from a straightforward induction on the complexity of γ. ⊣

We now show a slight generalization of the previous lemma.

Lemma 6.3 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$*,* $\mathfrak{A} \models p \trianglelefteq r$*,* $\mathfrak{A} \models q \trianglelefteq -r$ *with* $r \neq 0, 1$*, and let* γ *be an* $\mathcal{L}$*-sentence. Then the following are equivalent:*

1. $\mathfrak{A} \models p \sqcup q \trianglelefteq [\![\gamma]\!]$*.*
2. $\mathfrak{A}\restriction r \models p \trianglelefteq [\![\gamma]\!]$ *and* $\mathfrak{A}\restriction(-r) \models q \trianglelefteq [\![\gamma]\!]$*.*

Proof. This follows from Lemma 6.2. $\dashv$

We now define a map from a model to a restricted model.

Definition 6.4 *For a model* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$*, and* $p \in \mathrm{BA}(A)$ *with* $p \neq 0$*, we define the map* $\pi_p^{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A} \restriction p$ *by* $\pi_p^{\mathfrak{A}}(a) = a \restriction p$ *for* $a \in \mathrm{ST}(A)$*,* $\pi_p^{\mathfrak{A}}(a) = a \sqcap p$ *for* $a \in \mathrm{BA}(A)$*, and* $\pi_p(a) = a$ *for* $a \in \mathrm{CH}(A)$*.*

We now show this map is a morphism.

Lemma 6.5 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$*, and* $p \in \mathrm{BA}(A)$ *with* $p \neq 0$*. Then* $\pi_p^{\mathfrak{A}}$ *is an onto morphism. Further, for any function* $f \in \mathcal{L}^{\mathrm{BA}}$*,* $\pi_p^{\mathfrak{A}}$ *maps the domain of* f *in* $\mathfrak{A}$ *onto the domain of* f *in* $\mathfrak{A} \restriction p$*.*

Proof. $\pi_p^{\mathfrak{A}}$ is obviously onto with respect to Boolean elements, structural elements, and chaff. Thus, we need only show this is a morphism. For predicates, if $\mathrm{E}(x)$, then $\mathrm{E}(\pi_p^{\mathfrak{A}}(x))$, and if $x = y$, then $\pi_p^{\mathfrak{A}}(x) = \pi_p^{\mathfrak{A}}(y)$. For functions, if f is an $\mathcal{L}$-function, then $f^p(\pi_p^{\mathfrak{A}}(\mathbf{x})) = \pi_p^{\mathfrak{A}}(f(\mathbf{x}))$ by axiom Rs6. For functions of the form $[\![P(\mathbf{x})]\!]$, $[\![P(\pi_p^{\mathfrak{A}}(\mathbf{x}))]\!]^p = \pi_p^{\mathfrak{A}}([\![P(\mathbf{x})]\!])$ by axiom Rs5. For $x \restriction y$, we have that $\pi_p^{\mathfrak{A}}(x) \restriction^p \pi_p^{\mathfrak{A}}(y) = (x \restriction p) \restriction (y \sqcap p)$. By axiom Rs4, $(x \restriction p) \restriction (y \sqcap p) = x \restriction (p \sqcap y \sqcap p) = x \restriction (y \sqcap p)$. Again, by axiom Rs4, $x \restriction (y \sqcap p) = (x \restriction y) \restriction p = \pi_p^{\mathfrak{A}}(x \restriction y)$. For $\sqcap$, $\pi_p^{\mathfrak{A}}(x) \sqcap^p \pi_p^{\mathfrak{A}}(y) = x \sqcap p \sqcap y \sqcap p = x \sqcap y \sqcap p = \pi_p^{\mathfrak{A}}(x \sqcap y)$. For $\sqcup$, $\pi_p^{\mathfrak{A}}(x) \sqcup^p \pi_p^{\mathfrak{A}}(y) = (x \sqcap p) \sqcup (y \sqcap p) = (x \sqcup y) \sqcap p = \pi_p^{\mathfrak{A}}(x \sqcap y)$, where the second equality holds by axiom Ba15. For $-$, note that since $\pi_p^{\mathfrak{A}}(x)$ is an element of $A \restriction p$, we have that $-^p \pi_p^{\mathfrak{A}}(x) = (-\pi_p^{\mathfrak{A}}(x)) \sqcap p$. Thus, we get that $\pi_p^{\mathfrak{A}}(-x) = (-x) \sqcap p = ((-x) \sqcap p) \sqcup ((-p) \sqcap p) = ((-x) \sqcup (-p)) \sqcap p = -(x \sqcap p) \sqcap p = -^p \pi_p^{\mathfrak{A}}(x)$. Next, $\pi_p^{\mathfrak{A}}(0) = 0 = 0^p$ while $\pi_p^{\mathfrak{A}}(1) = p = 1^p$. Finally, it is obvious that if $\mathfrak{A} \models [\![P(\mathbf{a})]\!]$, then

$\mathfrak{A}\restriction p \models \mathrm{E}([\![\mathbf{a}\restriction p]\!])$, and if $\mathfrak{A} \models \mathrm{E}(f(\mathbf{a}))$, then $\mathfrak{A}\restriction p \models \mathrm{E}(f(\mathbf{a}\restriction p))$. The same holds for the predicates BA and $=$, so the last claim follows. $\dashv$

We call these maps $\pi_p^{\mathfrak{A}}$ **restriction morphisms**. These morphisms have the extra property that for any function in $\mathcal{L}^{\mathrm{BA}}$, its domain in $\mathfrak{A}$ maps onto its domain in $\mathfrak{A}\restriction p$. We give this property the following definition:

Definition 6.6 *For models $\mathfrak{A}$ and $\mathfrak{B}$ of $\emptyset^{\mathrm{BA}}$ and a map $\pi : \mathfrak{A} \to \mathfrak{B}$, we say that π* **maps domains onto domains** *if, for every function g in $\mathcal{L}^{\mathrm{ABA}}$ and $\mathbf{a} \in A$, π maps the domain of g in $\mathfrak{A}$ onto the domain of g in $\mathfrak{B}$.*

We note a simple corollary of Lemma 6.5:

Corollary 6.7 *Suppose $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and p is a nonzero element of* $\mathrm{BA}(A)$. *Then $\mathfrak{A}\restriction p \models \emptyset^{\mathrm{BA}}$. Further, if $\mathfrak{A} \models \emptyset^{\mathrm{ABA}}$ and $p \neq 0$, then $\mathfrak{A}\restriction p \models \emptyset^{\mathrm{ABA}}$.*

Proof. This result holds because $\pi_p^{\mathfrak{A}}$ maps domains onto domains, because $A\restriction p \subseteq A$, and because these domains are disjoint by Proposition 3.2. We show that $\mathfrak{A}\restriction p \models \mathrm{Ba2}$ as an example. Suppose $\mathfrak{A}\restriction p \models \mathrm{BA}(q) \wedge \mathrm{BA}(r)$. Then $\mathfrak{A} \models \mathrm{BA}(q) \wedge \mathrm{BA}(r)$, so that $\mathfrak{A} \models \mathrm{BA}(q \sqcap r)$. Since $\pi_p^{\mathfrak{A}}$ is a morphism, $\mathfrak{A}\restriction p \models \mathrm{BA}(q \sqcap r)$. $\dashv$

We immediately get the following corollary:

Corollary 6.8 *Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and $p, q \in \mathrm{BA}(A)$ with $p \sqcap q$ nonzero. Then $\pi_{p\sqcap q}^{\mathfrak{A}\restriction p}$ is a morphism which maps domains onto domains.*

Proof. First, $\mathfrak{A}\restriction p \models \emptyset^{\mathrm{BA}}$ by Corollary 6.7. As $\mathfrak{A}\restriction(p \sqcap q) = (\mathfrak{A}\restriction p)\restriction q$ by Definition 6.1, the result holds by Lemma 6.5. $\dashv$

We now show that restriction morphisms can be used to decompose models of $\emptyset^{\mathrm{BA}}$.

Lemma 6.9 *Suppose* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$*, and let* $p, q, r \in \mathrm{BA}(A)$ *such that* $q \neq 0$*,* $r \neq 0$*, and* $p = q \sqcup r$*. Then the map* $\langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle : \mathfrak{A}\restriction p \to \mathfrak{A}\restriction q \times \mathfrak{A}\restriction r$ *is an embedding. Additionally, if* $q \sqcap r = 0$ *and* $\mathfrak{A}$ *is clean, then* $\langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle$ *is an isomorphism.*

Proof. As each of $\pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r}$ are morphisms by Lemma 6.5, we need only show that, for every atom $\delta(\mathbf{x})$ of $\mathcal{L}^{\mathrm{ABA}}$, if $\mathfrak{A}\restriction q \times \mathfrak{A}\restriction r \models \delta(\langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle(\mathbf{a}))$ then $\mathfrak{A}\restriction p \models \delta(\mathbf{a})$. Now, the only predicates of $\mathcal{L}^{\mathrm{ABA}}$ are E, =, and BA. We show the case for equality, with the cases of E and BA being similar. Suppose $\mathfrak{A}\restriction q \times \mathfrak{A}\restriction r \models \langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle(a) = \langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle(b)$. Now, since $\pi^{\mathfrak{A}\restriction p}_{p\sqcap q}$ and $\pi^{\mathfrak{A}\restriction p}_{p\sqcap r}$ map domains to domains, then either a and b are both structural, both Boolean, or both chaff. First, suppose a, b are both structural elements. Then $\mathfrak{A}\restriction q \models a\restriction q = b\restriction q$ and $\mathfrak{A}\restriction r \models a\restriction r = b\restriction r$. Then $\mathfrak{A}\restriction p \models (a\restriction q = b\restriction q) \wedge (a\restriction r = b\restriction r)$. By Lemma 3.9.10, we get $\mathfrak{A}\restriction p \models a\restriction(q \sqcup r) = b\restriction(q \sqcup r)$. Since $[\![\mathrm{E}(a)]\!] \trianglelefteq p$ and $[\![\mathrm{E}(b)]\!] \trianglelefteq p$, we have $\mathfrak{A} \models a = a\restriction p = b\restriction p = b$, so that $\mathfrak{A}\restriction p \models a = b$. Thus, $\langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle$ preserves equality, so that the function is injective. Next, suppose s, t are Boolean elements. Then $\mathfrak{A}\restriction q \times \mathfrak{A}\restriction r \models \langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle(s) = \langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle(t)$ means that $\mathfrak{A}\restriction q \models s \sqcap q = t \sqcap q$ and $\mathfrak{A}\restriction r \models s \sqcap r = t \sqcap r$. Since $s \trianglelefteq p$ and $t \trianglelefteq p$, we have that $\mathfrak{A}\restriction p \models s = s \sqcap (q \sqcup r) = (s \sqcap q) \sqcup (s \sqcap r) = (t \sqcap q) \sqcup (t \sqcap r) = t \sqcap (q \sqcup r) = t$. Finally, suppose a, b are chaff. Then $\mathfrak{A}\restriction q \times \mathfrak{A}\restriction r \models \langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle(a) = \langle \pi^{\mathfrak{A}\restriction p}_{p\sqcap q}, \pi^{\mathfrak{A}\restriction p}_{p\sqcap r} \rangle(b)$ means that $\mathfrak{A}\restriction q \models a = b$ and $\mathfrak{A}\restriction r \models a = b$. Since $\mathrm{CH}(A) = \mathrm{CH}(A\restriction s)$ for any $s \in \mathrm{BA}(A)$, we have that $\mathfrak{A}\restriction p \models a = b$. Finally, if $\mathfrak{A}$ is clean and $q \sqcap r = 0$, the embedding is clearly onto, so it is an isomorphism. $\dashv$

6.2 Extending Boolean indexed models to atomless Boolean indexed models

In this section we show how to embed models of $\emptyset^{\mathrm{BA}}$ into models of $\emptyset^{\mathrm{ABA}}$. Recall that a model $\mathfrak{A}$ of $\emptyset^{\mathrm{BA}}$ is a model of $\emptyset^{\mathrm{ABA}}$ if $\mathfrak{A}$ has infinitely much chaff, has a global element, and is atomless. One easily embeds $\mathfrak{A}$ into a model with an infinite amount of chaff by simply adding an infinite set to the domain A. Embedding $\mathfrak{A}$ into a model with an atomless Boolean algebra requires more work. We show that this last embedding preserves the existence of a global element.

We first show how to create a model of $\emptyset^{\mathrm{BA}} \cup \{\mathrm{Ba19}\}$ from a model of $\emptyset^{\mathrm{BA}}$. Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and let $p \in \mathrm{BA}(A)$ with $p \neq 0$. Recall the structure $\mathfrak{A}\upharpoonright -p$ and the morphism $\pi^{\mathfrak{A}}_{-p}$ from Lemma 6.5. Further, we let $\pi_1 : \mathfrak{A} \times \mathfrak{B} \to \mathfrak{A}$ be the projection onto the first coordinate, and we similarly define π_2. We form the pullback depicted in Figure 1.

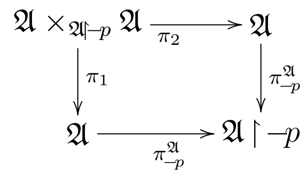

Figure 1: Pullback over $-p$

Recall that the model $\mathfrak{A} \times_{\mathfrak{A}\upharpoonright -p} \mathfrak{A}$ is defined as the submodel of $\mathfrak{A} \times \mathfrak{A}$ where the domain is the set $\{(x, y) : \pi_{-p}(x) = \pi_{-p}(y)\}$. Note that if we replace $\mathfrak{A} \times_{\mathfrak{A}\upharpoonright -p} \mathfrak{A}$ with $\mathfrak{A}$ in the above diagram and π_i with $\mathrm{id}_{\mathfrak{A}}$, the diagram obviously commutes. Because of this, by a well-known category theory result, see [7, page 71], there is a unique embedding from $\mathfrak{A}$ to $\mathfrak{A} \times_{\mathfrak{A}\upharpoonright -p} \mathfrak{A}$ which makes Figure 2 commute.

We now discuss the form of this embedding. Let $\mathfrak{A}'$ be the pullback $\mathfrak{A} \times_{\mathfrak{A}\upharpoonright -p} \mathfrak{A}$, and

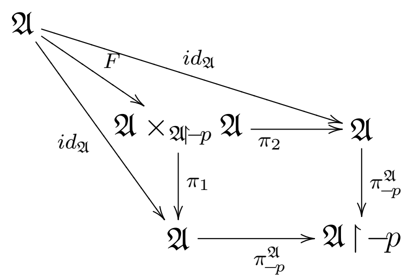

Figure 2: Factoring $\mathfrak{A}$ through the pullback

define $F : \mathfrak{A} \to \mathfrak{A}'$ by $x \mapsto (x, x)$. We now get the following result:

Lemma 6.10 *F is the unique embedding from $\mathfrak{A}$ to $\mathfrak{A}'$ that makes Figure 2 commute.*

Proof. It suffices to show that $\pi_1 \circ F = \mathrm{id}_{\mathfrak{A}}$ and $\pi_2 \circ F = \mathrm{id}_{\mathfrak{A}}$. That both hold is obvious. $\dashv$

Thus, $\mathfrak{A}$ embeds into $\mathfrak{A}'$, so we may assume that $\mathfrak{A} \subseteq \mathfrak{A}'$. We also get the following:

Lemma 6.11 *Suppose $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, let $p \in \mathrm{BA}(A)$ be nonzero, and set $\mathfrak{A}'$ as the pullback model $\mathfrak{A} \times_{\mathfrak{A}\restriction -p} \mathfrak{A}$. Then $\mathfrak{A}' \models \emptyset^{\mathrm{BA}}$.*

Proof. As $\emptyset^{\mathrm{BA}}$ is universal Horn, and $\mathfrak{A}'$ is a submodel of a product of models of $\emptyset^{\mathrm{BA}}$, we have that $\mathfrak{A}' \models \emptyset^{\mathrm{BA}}$. $\dashv$

We now suppose that p is an atom, so we have that for all Boolean q either $p \sqcap q = 0$ or $p \sqcap q = p$. Then, for each $q \in \mathrm{BA}(A)$, we set $q_0 = (-p) \sqcap q$. Further, if $q \sqcap p = 0$, then set $q_1 = 0$, and if $q \sqcap p = p$, then set $q_1 = 1$. Thus, $q = q_0 \sqcup (q_1 \sqcap p)$. We now prove that the elements q_0 and q_1 are unique.

Lemma 6.12 *Suppose $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and $p \in \mathrm{BA}(A)$ is an atom. Recall that $\mathbf{2}$ is the nondegenerate Boolean algebra consisting of two elements, 0 and 1. Then for each $q \in \mathrm{BA}(A)$, there is a unique $(q_0, q_1) \in (\mathrm{BA}(A \restriction (-p))) \times \mathbf{2}$ such that $q = q_0 \sqcup (q_1 \sqcap p)$.*

Proof. Suppose $q = q_0 \sqcup (q_1 \sqcap p) = r_0 \sqcup (r_1 \sqcap p)$. As mentioned above, since p is an atom, we have either $q \sqcap p = 0$ or $q \sqcap p = p$. Further, note that $q_0 \sqcap p = 0 = r_0 \sqcap p$, since $q_0 \trianglelefteq (-p)$ and $r_0 \trianglelefteq (-p)$. Thus, $q_0 = (q_0 \sqcup (q_1 \sqcap p)) \sqcap (-p) = (r_0 \sqcup (r_1 \sqcap p)) \sqcap (-p) = r_0$. Further, $q \sqcap p = q_1 \sqcap p = r_1 \sqcap p$. Since $q_1, r_1 \in \mathbf{2}$, we have that $q_1 = r_1$. $\dashv$

With this, we get the following:

Lemma 6.13 $\mathrm{BA}(A) \cong (\mathrm{BA}(A \restriction (-p))) \times \mathbf{2}$, *by the map* $G(q) = (q_0, q_1)$.

Proof. By Lemma 6.12, this function is surjective. Thus, we need only show it is a morphism. For $\sqcup$, note that $q \sqcup r = q_0 \sqcup (q_1 \sqcap p) \sqcup r_0 \sqcup (r_1 \sqcap p) = q_0 \sqcup r_0 \sqcup (q_1 \sqcap p) \sqcup (r_1 \sqcap p) = q_0 \sqcup r_0 \sqcup ((q_1 \sqcup r_1) \sqcap p)$, so that $G(q \sqcup r) = G(q) \sqcup G(r)$. Further, $q \sqcap r = (q_0 \sqcup (q_1 \sqcap p)) \sqcap (r_0 \sqcup (r_1 \sqcap p)) = (q_0 \sqcap r_0) \sqcup (q_0 \sqcap r_1 \sqcap p) \sqcup (q_1 \sqcap p \sqcap r_0) \sqcup (q_1 \sqcap r_1 \sqcap p)$. Now, since $q_0 \sqcap p = 0 = r_0 \sqcap p$, this reduces to $(q_0 \sqcap r_0) \sqcup (q_1 \sqcap r_1 \sqcap p)$, so we have that $G(q \sqcap r) = G(q) \sqcap G(r)$. Further, $G(0) = (0, 0) = 0^{(\mathrm{BA}(A \restriction (-p))) \times \mathbf{2}}$, while $G(1) = ((-p), 1) = 1^{(\mathrm{BA}(A \restriction (-p))) \times \mathbf{2}}$. $\dashv$

For an element $a \in \mathrm{ST}(A)$, we have that $\mathfrak{A} \models [\![\mathrm{E}(a)]\!]$ maps to $([\![\mathrm{E}(a \restriction (-p))]\!], 0)$ if $p \not\trianglelefteq [\![\mathrm{E}(a)]\!]$, and it maps to $([\![\mathrm{E}(a \restriction (-p))]\!], 1)$ if $p \trianglelefteq [\![\mathrm{E}(a)]\!]$.

We now discuss the Boolean algebra of $\mathfrak{A}'$ derived from a model $\mathfrak{A}$ and an atom $p \in \mathrm{BA}(A)$. By the pullback construction, $\mathrm{BA}(A')$ is the set $(x, y) \in \mathrm{BA}(A) \times \mathrm{BA}(A)$ such that $(x_0 \sqcup (x_1 \sqcap p) \sqcap (-p)) = (y_0 \sqcup (y_1 \sqcap p) \sqcap (-p))$, that is, $x_0 = y_0$. As above, each element in $\mathrm{BA}(A')$ can be uniquely written as an element in $(\mathrm{BA}(A \restriction (-p))) \times \mathbf{2} \times \mathbf{2}$. We map the element (x, y) to the element (x_0, x_1, y_1). Note that this is the same as (y_0, x_1, y_1). By this map, we have that $\mathrm{BA}(A') \cong (\mathrm{BA}(A \restriction (-p))) \times \mathbf{2} \times \mathbf{2}$. For an element $(x, y) \in \mathrm{ST}(A')$, we let $x_0 = x \restriction (-p)$, $x_1 = x \restriction p$, $y_0 = y \restriction (-p)$ and $y_1 = y \restriction p$. Note that $x_0 = y_0$. Thus $[\![\mathrm{E}((x, y))]\!] = ([\![\mathrm{E}(x_0)]\!], [\![\mathrm{E}(x_1)]\!], [\![\mathrm{E}(y_1)]\!])$.

Now, as $\mathfrak{A} \subseteq \mathfrak{A}'$, we have that $\mathrm{BA}(A) \subseteq \mathrm{BA}(A')$, with $x \in \mathrm{BA}(A)$ identified with (x, x). Since $\mathrm{BA}(A') \cong (\mathrm{BA}(A \restriction (-p))) \times \mathbf{2} \times \mathbf{2}$, the element (x, x) maps to (x_0, x_1, x_1). Further, for an element $a \in \mathrm{ST}(A)$, $[\![\mathrm{E}(a)]\!]^{\mathfrak{A}'} = ([\![\mathrm{E}(a \restriction (-p))]\!], 0, 0)$ if $p \not\trianglelefteq [\![\mathrm{E}(a)]\!]$ and $[\![\mathrm{E}(a)]\!] = ([\![\mathrm{E}(a \restriction (-p))]\!], 1, 1)$ if $p \trianglelefteq [\![\mathrm{E}(a)]\!]$. We note that the image of p in $(\mathrm{BA}(A \restriction (-p))) \times \mathbf{2} \times \mathbf{2}$ is the element $(0, 1, 1)$. Let $p_1 = (0, 1, 0)$ and $p_2 = (0, 0, 1)$. Thus, in $(\mathrm{BA}(A \restriction (-p))) \times \mathbf{2} \times \mathbf{2}$, the element p is no longer an atom, as p_1 and p_2 are nonzero elements that are below p. Further, each element $q \in \mathrm{BA}(A')$ can be uniquely written as $q_0 \sqcup (q_1 \sqcap p_1) \sqcup (q_2 \sqcap p_2)$, where $q_0 \in \mathrm{BA}(A \restriction (-p))$, and $q_1, q_2 \in \mathbf{2}$.

We now discuss the structural elements in the pullback.

Lemma 6.14 *Suppose $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and suppose $p \in \mathrm{BA}(A)$ is an atom. Let $\mathfrak{A}' = \mathfrak{A} \times_{\mathfrak{A} \restriction -p} \mathfrak{A}$, and let $q \in \mathrm{BA}(A')$. We write q as (q_0, q_1, q_2).*

1. *If $q = (q_0, 0, 0)$, then $\mathfrak{A}'_q = \mathfrak{A}_q$.*

2. *If $q = (q_0, 1, 0)$, then $\mathfrak{A}'_q \cong \mathfrak{A}_{q_0 \sqcup p}$.*

3. *If $q = (q_0, 0, 1)$, then $\mathfrak{A}'_q \cong \mathfrak{A}_{q_0 \sqcup p}$.*

4. *If $q = (q_0, 1, 1)$, then $\mathfrak{A}'_q$ is the submodel of $\mathfrak{A}_{q_0 \sqcup p} \times \mathfrak{A}_{q_0 \sqcup p}$ with domain $\{(x, y) \in (A_{q_0 \sqcup p})^2 : x \restriction (-p) = y \restriction (-p)\}$. Equivalently, $\mathfrak{A}'_q$ is the submodel of $\mathfrak{A}_{q_0 \sqcup p} \times \mathfrak{A}_{q_0 \sqcup p}$ with domain $\{(x, y) \in (A_{q_0 \sqcup p})^2 \ : \ x \restriction q_0 = y \restriction q_0\}$.*

Proof. For part 1, clearly $A_q \subseteq A'_q$. For the other direction, let $(a, b) \in A'_q$. Then $\mathfrak{A} \models (a = a \restriction (-p)) \wedge (b = b \restriction (-p))$. Further, since $\pi^{\mathfrak{A}}_{(-p)}(a) = \pi^{\mathfrak{A}}_{(-p)}(b)$, we have that $a \restriction (-p) = b \restriction (-p)$. Thus $a = b$, so that $A'_q \subseteq A_q$.

For part 2, we have that

$$A'_q = \{(x, y) \ : \ x \restriction (-p) = y \restriction (-p), [\![\mathrm{E}(x)]\!] = q_0 \sqcup p, \text{ and } [\![\mathrm{E}(y)]\!] = q_0\}.$$

In other words, for an element to be in the domain of A'_q, it must be of the form $(x, x \restriction p)$, where $[\![\mathrm{E}(x)]\!] = q_0 \sqcup p$. Let $F : \mathfrak{A}_{q_0 \sqcup p} \to \mathfrak{A}'_q$ be defined as $F(x) = (x, x \restriction p)$. By the previous comments, this function is well defined. Further, it is clearly onto and, by the previous comments, it is one-to-one. Now, $\mathfrak{A}_{q_0 \sqcup p} \models P(\mathbf{a})$ holds if and only if $\mathfrak{A} \models [\![P(\mathbf{a})]\!] = q_0 \sqcup p$. This holds if and only if $\mathfrak{A}' \models [\![P((\mathbf{a}, \mathbf{a}))]\!] = (q_0, 1, 1)$, which is equivalent to $\mathfrak{A}' \models [\![P(\mathbf{a}, \mathbf{a} \restriction (-p))]\!] = (q_0, 1, 0) = q$. Thus, $\mathfrak{A}'_q \models P(F(\mathbf{a}))$ if and only if $\mathfrak{A}_{q_0 \sqcup p} \models P(\mathbf{a})$. Further, suppose $\mathfrak{A}_{q_0 \sqcup p} \models f(\mathbf{a}) = b$. Then $\mathfrak{A} \models f(\mathbf{a}) = b$. Thus, $\mathfrak{A}' \models f((\mathbf{a}, \mathbf{a})) = (b, b)$, so that $\mathfrak{A}' \models f((\mathbf{a}, \mathbf{a} \restriction (-p))) = (b, b \restriction (-p))$. Thus, $F(f(\mathbf{a})) = f(F(\mathbf{a}))$.

The proof of part 3 is nearly identical to the proof of part 2.

The proof of part 4, suppose $(x, y) \in A'_q$. Then $\mathfrak{A}' \models [\![\mathrm{E}((x, y))]\!] = (q_0, 1, 1)$. Thus, $x \restriction (-p) = y \restriction (-p)$. The last statement follows from the fact that $q_0 = q \restriction (-p)$. $\dashv$

For a model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, recall that $\mathfrak{A}^\circ$ is the largest clean submodel of $\mathfrak{A}$. We now describe the largest clean submodel of the pullback $\mathfrak{A}'$.

Lemma 6.15 *Let $\mathfrak{A}'$ be the pullback described above. Then $(\mathfrak{A}')^\circ \cong \mathfrak{A}^\circ \times (\mathfrak{A}^\circ \restriction p)$.*

Proof. Note that Figure 3 obviously commutes.

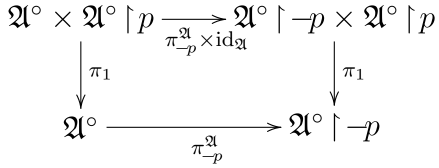

Figure 3: Commutative square

It suffices to show that $\mathfrak{A}^\circ \times \mathfrak{A}^\circ \restriction p$ is also a pullback. Thus, suppose we have a model $\mathfrak{B}$ and that Figure 4 commutes.

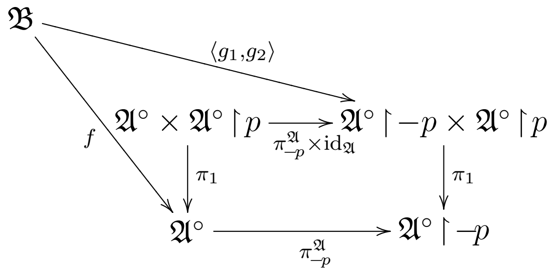

Figure 4: Factoring $\mathfrak{B}$ through $\mathfrak{A}^\circ \times \mathfrak{A}^\circ \restriction p$

Then a straightforward diagram chase shows that $\langle f, g_2 \rangle : \mathfrak{B} \to \mathfrak{A}^\circ \times \mathfrak{A}^\circ \restriction p$ is the unique factorization. So $\mathfrak{A}^\circ \times \mathfrak{A}^\circ \restriction p$ is a pullback, and as pullbacks are unique up to isomorphism, the result holds. $\dashv$

We now show that this pullback $\mathfrak{A}'$ preserves additional axioms from $\emptyset^{\mathrm{ABA}}$.

Lemma 6.16 *Suppose* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$ *and* $p \in \mathrm{BA}(A)$ *is an atom. Let* $\mathfrak{A}'$ *be the pullback formed as above.*

1. *If* $\mathfrak{A}$ *is clean, then* $\mathfrak{A}'$ *is clean;*

2. *If* $\mathfrak{A}$ *has a global element, then* $\mathfrak{A}'$ *has a global element.*

3. *Suppose* $\mathfrak{A} \models 1 \trianglelefteq [\![\gamma]\!]$ *for an* $\mathcal{L}$*-sentence* γ*. Then* $\mathfrak{A}' \models 1 \trianglelefteq [\![\gamma]\!]$*.*

Proof. Parts 1 and 2 follow immediately from Lemma 6.15. For 3, combine Lemma 6.15 with Lemma 6.3. $\dashv$

The next step in the procedure is to take the union of a chain of models derived through this pullback procedure so as to remove all atoms to get an atomless Boolean indexed model. It is well-known that Π^0_2-sentences are preserved under unions of chains. Therefore, we show that translations of Π^0_2-sentences are also Π^0_2.

Lemma 6.17 *Let φ be a Π^0_2 $\mathcal{L}$-formula. Then, over $\emptyset^{\mathrm{BA}}$, $1 \trianglelefteq [\![\varphi]\!]$ is equivalent to a Π^0_2 Horn formula.*

Proof. By Theorem 4.35, we may suppose that φ has the form $\forall\mathbf{x}\exists\mathbf{y}(\bigwedge_{r<s}(\bigwedge_{j<m}\delta_{rj} \to \bigvee_{k<l}\epsilon_{rk}))$, where each δ_{rj} and ϵ_{rk} is atomic. By Lemma 4.33, $1 \trianglelefteq [\![\varphi]\!]$ translates to

$$\forall\mathbf{x}(\bigwedge_{i<n}\mathrm{ST}(x_i) \to [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\exists\mathbf{y}(\bigwedge_{r<s}(\bigwedge_{j<m}\delta_{rj} \to \bigvee_{k<l}\epsilon_{rk}))]\!]).$$

This translates to

$$\forall\mathbf{x}(\bigwedge_{i<n}\mathrm{ST}(x_i) \to \exists\mathbf{y}(\bigwedge_{p<q}[\![\mathrm{E}(y_p)]\!] = [\![\mathrm{E}(\mathbf{x})]\!] \wedge [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\bigwedge_{r<s}(\bigwedge_{j<m}\delta_{rj} \to \bigvee_{k<l}\epsilon_{rk})]\!])).$$

This is equivalent to

$$\forall\mathbf{x}\exists\mathbf{y}(\bigwedge_{i<n}\mathrm{ST}(x_i) \to (\bigwedge_{p<q}[\![\mathrm{E}(y_p)]\!] = [\![\mathrm{E}(\mathbf{x})]\!] \wedge [\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\bigwedge_{r<s}(\bigwedge_{j<m}\delta_{rj} \to \bigvee_{k<l}\epsilon_{rk})]\!])).$$

As in Lemma 4.37, we have that $[\![\bigwedge_{r<s}(\bigwedge_{j<m}\delta_{rj} \to \bigvee_{k<l}\epsilon_{rk})]\!]$ is the same as $\bigsqcap_{r<s}(\bigsqcup_{j<m} -[\![\delta_{rj}]\!] \sqcup \bigsqcup_{k<l}[\![\epsilon_{rk}]\!])$. Thus, our formula is equivalent to

$$\forall\mathbf{x}\exists\mathbf{y}(\bigwedge_{i<n}\mathrm{ST}(x_i) \to (\bigwedge_{p<q}[\![\mathrm{E}(y_p)]\!] = [\![\mathrm{E}(\mathbf{x})]\!] \wedge ([\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq \bigsqcap_{r<s}(\bigsqcup_{j<m} -[\![\delta_{rj}]\!] \sqcup \bigsqcup_{k<l}[\![\epsilon_{rk}]\!])))).$$

This is Π^0_2 Horn, so we are done. ⊣

We now proceed with our construction. For a model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, we enumerate the set of atoms as $\{p_i : i < \kappa\}$ for some cardinal κ. Note that as they are atoms, we have that $p_i \sqcap p_j = 0$ for $i \neq j$. We let $\mathfrak{A}'$ be the pullback formed by $\mathfrak{A} \times_{\mathfrak{A}\upharpoonright p_0} \mathfrak{A}$. Further, $p_1 \in \mathrm{BA}(A')$ is the element $(p_1, 0, 0)$, so that p_1 is an atom in $\mathfrak{A}'$. In this manner, we create a chain of models: let $\mathfrak{A}^0 = \mathfrak{A}$, for a successor cardinal $\lambda = \alpha + 1$, set $\mathfrak{A}^\lambda = \mathfrak{A}^\alpha \times_{\mathfrak{A}^\alpha\upharpoonright p_\alpha} \mathfrak{A}^\alpha$, and for a limit ordinals λ, set $\mathfrak{A}^\lambda = \bigcup_{i<\lambda}\mathfrak{A}^i$. We set $\mathfrak{A}^{(1)} = \bigcup_{i<\kappa}\mathfrak{A}^i$. With this, we get the following:

Lemma 6.18 *Let $\mathfrak{A}^{(1)}$ be the model described above. Then $\mathfrak{A}^{(1)} \models \emptyset^{\mathrm{BA}}$. Additionally, if $\mathfrak{A} \models \Gamma^{\mathrm{BA}}$ for some set Γ of Π^0_2-sentences, then $\mathfrak{A}^{(1)} \models \Gamma^{\mathrm{BA}}$.*

Proof. By Lemma 6.11, each pullback $\mathfrak{A}^i \models \emptyset^{\mathrm{BA}}$. Since $\emptyset^{\mathrm{BA}}$ is Π^0_2, it is preserved under unions of chains of models. Thus $\mathfrak{A}^{(1)} \models \emptyset^{\mathrm{BA}}$. The last claim follows from Lemma 6.17. $\dashv$

All atoms in $\mathfrak{A}$ are no longer atoms in $\mathfrak{A}^{(1)}$. However, $\mathfrak{A}^{(1)}$ may contain new atoms. So we repeat this procedure for the model $\mathfrak{A}^{(1)}$, and construct a model $\mathfrak{A}^{(2)}$. In this way, we create a chain of models $\mathfrak{A}^{(1)} \subseteq \mathfrak{A}^{(2)} \subseteq \ldots$. Set $\mathfrak{A}^{(\omega)} = \bigcup_{n\in\omega} \mathfrak{A}^{(n)}$. Further, as per the discussion at the beginning of this section, we may suppose that $\mathfrak{A}^{(\omega)}$ has infinitely much chaff. We are now ready to prove our main result:

Theorem 6.19 *Suppose $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$ and $\mathfrak{A}$ has a global element. Let $\mathfrak{A}^{(\omega)}$ be constructed as above. Then $\mathfrak{A}^{(\omega)} \models \emptyset^{\mathrm{ABA}}$. Further, if $\mathfrak{A} \models \Gamma^{\mathrm{BA}}$ for some set Γ of Π^0_2 $\mathcal{L}$-sentences, then $\mathfrak{A}^{(\omega)} \models \Gamma^{\mathrm{BA}}$.*

Proof. Each $\mathfrak{A}^{(n)} \models \emptyset^{\mathrm{BA}}$, so that $\mathfrak{A}^{(\omega)} \models \emptyset^{\mathrm{BA}}$. To see it is atomless, let $p \in \mathrm{BA}(A^{(\omega)})$. Then $p \in \mathrm{BA}(A^{(n)})$ for some $n \in \omega$. If p is not an atom in $\mathfrak{A}^{(n)}$, we are done. Otherwise, p is not an atom in $\mathfrak{A}^{(n+1)}$. The final claim follow from Lemma 6.17. $\dashv$

6.3 A modified Wheeler's conjecture

We are now ready to give a positive alternative to Wheeler's conjecture.

Theorem 6.20 *Let Γ be a set of $\mathcal{L}$-sentences that has a nonempty model. If Γ has a model companion axiomatized by a set of Π^0_2 $\mathcal{L}$-sentences Γ', then the universal Horn fragment of $\Gamma^{\mathrm{BA}\mathbf{2}}$ has as its model companion $(\Gamma')^{\mathrm{ABA}}$.*

Proof. We may suppose that Γ is a set of universal sentences. By Theorem 4.38, $(\Gamma^{\mathrm{BA}\mathbf{2}})_{\mathrm{UH}} = \Gamma^{\mathrm{BA}}$ is universal Horn. Every model of $(\Gamma')^{\mathrm{ABA}}$ is a model of Γ^{BA} by Corollary 4.36. By Theorem 5.17, $(\Gamma')^{\mathrm{ABA}}$ is model complete. Thus, it suffices to show that every model of Γ^{BA} embeds into a model of $(\Gamma')^{\mathrm{ABA}}$.

Let $\mathfrak{A} \models \Gamma^{\mathrm{BA}}$. By Theorem 4.34, $\mathfrak{A}_U \models \Gamma$ for all $U \in \mathcal{U}_{\mathfrak{A}}$. By Corollary 2.13 and Proposition 3.19, $(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}}$. As Γ^{BA} is universal Horn, $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}} \models \Gamma^{\mathrm{BA}}$. As $\mathfrak{A}$ embeds into $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}}$ by Lemma 4.32, it suffices to prove that $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}}$ embeds into a model of $(\Gamma')^{\mathrm{ABA}}$. Now, each $\mathfrak{A}_U \models \Gamma$ so by supposition $\mathfrak{A}_U$ embeds into a model $\mathfrak{A}'_U \models \Gamma'$ which must be nonempty. Then $(\mathfrak{A}'_U)^{\mathrm{BA}\mathbf{2}} \models (\Gamma')^{\mathrm{BA}}$ by Corollary 2.13 and Proposition 3.19. Further, $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}_U)^{\mathrm{BA}\mathbf{2}}$ embeds into $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}'_U)^{\mathrm{BA}\mathbf{2}}$. By Lemma 6.17, $(\Gamma')^{\mathrm{BA}}$ is a set of Horn sentences. Thus, $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}'_U)^{\mathrm{BA}\mathbf{2}} \models (\Gamma')^{\mathrm{BA}}$. As each $(\mathfrak{A}'_U)^{\mathrm{BA}\mathbf{2}}$ has a global element, $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}'_U)^{\mathrm{BA}\mathbf{2}}$ has a global element. By Theorem 6.19, $\Pi_{U\in\mathcal{U}_{\mathfrak{A}}}(\mathfrak{A}'_U)^{\mathrm{BA}\mathbf{2}}$ embeds in a model of $(\Gamma')^{\mathrm{ABA}}$. $\dashv$

In the presence of the existence operator $\mathrm{E}(x)$, consistent theories may contain the empty model. This is not true for the usual predicate logics without $\mathrm{E}(x)$. For a proof, see Proposition A.1. Thus, we include the seemingly trivial condition that Γ has a nonempty model in the statement of Theorem 6.20.

Our modified version of Wheeler's conjecture can be described as follows: We start with a universal theory Γ that has a model companion Γ'. Then $\Gamma^{\mathrm{BA}\mathbf{2}}$ is essentially the same theory as Γ; in particular, $\Gamma^{\mathrm{BA}\mathbf{2}}$ has as its model companion $(\Gamma')^{\mathrm{BA}\mathbf{2}}$. We take the universal Horn fragment Γ^{BA} of $\Gamma^{\mathrm{BA}\mathbf{2}}$. Out of Γ' we are able to construct a model companion for Γ^{BA}. By encoding a Boolean algebra into the language, we have sufficiently enriched its expressive power to be able to axiomatize the theory of the existentially closed models.

The following draws an even closer connection between Theorem 6.20 and Wheeler's conjecture.

Theorem 6.21 *Let Γ be a set of Π^0_2 $\mathcal{L}$-sentences. Then Γ^{ABA} is axiomatizable by a set of Π^0_2 Horn sentences. In particular, if Γ has a model companion, then the model companion of Γ^{BA} has a Π^0_2 Horn axiomatization.*

Proof. The set $\emptyset^{\mathrm{BA}}$ is Horn. By Lemma 6.17, Γ^{BA} is Π^0_2 Horn. Thus, it suffices to show that Ba19 and So3 are equivalent to Π^0_2 Horn sentences. The former is equivalent to $\forall x \exists y((x \sqcap y = 0 \rightarrow x = 0) \wedge (x \sqcap y = x \rightarrow x = 0))$, while the latter schema is equivalent to the set of sentences $\exists \mathbf{x}((\bigwedge_{i<n} \mathrm{BA}(x_i) \rightarrow \bot) \wedge (\bigwedge_{i<n} \mathrm{ST}(x_i) \rightarrow \bot) \wedge (\bigwedge_{i<j<n} x_i = x_j \rightarrow \bot))$ for $n \in \mathbb{N}$. These are both conjunctions of Π^0_2 Horn sentences, so we are done. $\dashv$

We end with two corollaries to Theorem 6.20.

Corollary 6.22 *Let Γ be a set of universal $\mathcal{L}$-sentences. If Γ has a model companion, then $(\Gamma_{\mathrm{UH}})^{\mathrm{BA}}$ has a model companion.*

Proof. This follows from Theorem 4.40 and Theorem 6.20. $\dashv$

We note that the connection between Theorem 6.20 and Wheeler's conjecture is drawn even closer by Corollary 6.22.

Recall that the empty theory has a model companion, see [2]. Let $\emptyset^*$ be a Π^0_2 axiomatization of that model companion. Then the following is a simple corollary of Theorem 6.20.

Corollary 6.23 *The model companion of $\emptyset^{\mathrm{BA}}$ is $(\emptyset^*)^{\mathrm{ABA}}$.*

Chapter 7

Intuitionistic Derivability and Non-Discrete Models

In this chapter we discuss the logical strength of $y \trianglelefteq [\![\varphi]\!]$. We demonstrate that its deductive power includes intuitionistic derivability plus the discreteness schema for atomic formulas. We also discuss the logical strength of $y \trianglelefteq_K [\![\varphi]\!]$. Due to its connection with forcing on Kripke models, we easily show its deductive power includes intuitionistic derivability. We then show that if Γ and Δ are intuitionistically equivalent theories, then Γ^{BA} and Δ^{BA} are equivalent. We also show that neither $y \trianglelefteq [\![\varphi]\!]$ nor $y \trianglelefteq_K [\![\varphi]\!]$ contain full classical derivability. We demonstrate this by creating models of $\emptyset^{\mathrm{BA}}$ and sentences φ such that the ideal associated with φ is not principal. We first show that the Boolean translation is weaker than classical derivability. Our particular φ does not contain any disjunctions, so that the result then follows from Proposition 3.21.

7.1 Kripke forcing

We show that $p \trianglelefteq_K [\![\varphi]\!]$ obeys the rules of intuitionistic logic.

Theorem 7.1 *Let $\Gamma \cup \{\varphi\}$ be a set of $\mathcal{L}$-sentences. If $\Gamma \vdash_i \varphi$, then there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\emptyset^{\mathrm{BA}} \vdash y \trianglelefteq_K [\![\bigwedge_{\gamma\in\Gamma_0} \gamma]\!] \to y \trianglelefteq_K [\![\varphi]\!]$.*

Proof. By compactness, we have that $\Gamma \vdash_i \varphi$ if and only if $\Gamma_0 \vdash_i \varphi$ for some finite subset Γ_0. By Kripke soundness, $\Gamma_0 \vdash_i \varphi$ implies that for all models $\mathfrak{A}$ and Boolean elements p, if $(K(\mathfrak{A}), p) \Vdash \Gamma_0$ then $(K(\mathfrak{A}), p) \Vdash \varphi$. The result then follows from Proposition 3.23. $\dashv$

7.2 Boolean forcing

We now discuss the power of $y \trianglelefteq [\![\varphi]\!]$. As there is no clear connection between the Boolean translation and Kripke forcing, we show that each step of the intuitionistic sequent calculus is preserved. In this case, it is important to keep track of the set of free variables which occur in the $\mathcal{L}$-formulas.

Lemma 7.2 *Suppose $\varphi(\mathbf{x})$ is an $\mathcal{L}$-formula, with $\mathbf{x}$ the set of all free variables in φ. Then the free variables in the $\mathcal{L}^{\mathrm{BA}}$-formula $y \trianglelefteq^0 [\![\varphi(\mathbf{x})]\!]$ are $\mathbf{x} \cup y$.*

Proof. We proceed by induction on the complexity of $\varphi(\mathbf{x})$. If φ is atomic of the form $P(t_0(\mathbf{x}), \ldots, t_{n-1}(\mathbf{x}))$, then $y \trianglelefteq^0 [\![P(t_0(\mathbf{x}), \ldots, t_{n-1}(\mathbf{x}))]\!]$ is translated as $y \sqcap [\![P(t_0(\mathbf{x}), \ldots, t_{n-1}(\mathbf{x}))]\!] = y$, so that the free variables are y and $\mathbf{x}$.

Suppose $\varphi(\mathbf{x})$ is $\psi(\mathbf{x}) \wedge \theta(\mathbf{x})$. Then $y \trianglelefteq^0 [\![\varphi(\mathbf{x})]\!]$ is translated as $y \trianglelefteq^0 [\![\psi(\mathbf{x})]\!] \wedge y \trianglelefteq^0 [\![\theta(\mathbf{x})]\!]$. By induction, the free variables in this latter formula are y and $\mathbf{x}$.

Suppose $\varphi(\mathbf{x})$ is $\psi(\mathbf{x}) \vee \theta(\mathbf{x})$. Then $y \trianglelefteq^0 [\![\varphi(\mathbf{x})]\!]$ is translated as $\exists y_1, y_2 (y_1 \trianglelefteq^0 [\![\psi(\mathbf{x})]\!] \wedge y_2 \trianglelefteq^0 [\![\theta(\mathbf{x})]\!] \wedge y_1 \sqcup y_2 = y)$. By induction, the free variables in this latter formula are y and $\mathbf{x}$.

Suppose $\varphi(\mathbf{x})$ is $\psi(\mathbf{x}) \rightarrow \theta(\mathbf{x})$. Then $y \trianglelefteq^0 [\![\varphi(\mathbf{x})]\!]$ is translated as $\forall z((z \trianglelefteq^0 y \wedge z \trianglelefteq^0 [\![\psi(\mathbf{x})]\!]) \rightarrow z \trianglelefteq^0 [\![\theta(\mathbf{x})]\!])$. By induction, the free variables in this latter formula are y and $\mathbf{x}$.

Suppose $\varphi(\mathbf{x})$ is $\forall x\psi(x, \mathbf{x})$. Then $y \trianglelefteq^0 [\![\varphi(\mathbf{x})]\!]$ translates to $\forall x([\![\mathrm{E}(x)]\!] \trianglelefteq^0 y \rightarrow [\![\mathrm{E}(x)]\!] \trianglelefteq^0 [\![\psi(x, \mathbf{x})]\!])$. By induction, the free variables in this latter formula are y and $\mathbf{x}$.

Finally, suppose $\varphi(\mathbf{x})$ is $\exists x\psi(x, \mathbf{x})$. Then $y \trianglelefteq^0 [\![\varphi(\mathbf{x})]\!]$ translates to $\exists x([\![\mathrm{E}(x)]\!] = y \wedge y \trianglelefteq^0 [\![\psi(x, \mathbf{x})]\!])$. By induction, the free variables in this latter formula are y and $\mathbf{x}$.

$\dashv$

Before we prove our main result, we need one last result.

Lemma 7.3 *Let φ be an $\mathcal{L}$-formula, and t an $\mathcal{L}$-term where no free variables of t occur as bound variables in φ or in $y \trianglelefteq [\![\varphi]\!]$. Then $(y \trianglelefteq [\![\varphi(x)]\!])(x/t)$ is the same string as $y \trianglelefteq [\![\varphi(x)(x/t)]\!]$.*

Proof. We proceed by induction on the complexity of φ. If φ is an atomic formula of the form $P(t_0(x), \ldots, t_{n-1}(x))$, then $y \trianglelefteq [\![P(t_0(x), \ldots, t_{n-1}(x))]\!](x/t)$ means $(y \sqcap [\![P(t_0(x), \ldots, t_{n-1}(x))]\!] = y)(x/t)$. This becomes $y \sqcap [\![P(t_0(t), \ldots, t_{n-1}(t))]\!] = y$. But this is the same string as $y \sqcap [\![P(t_0(x), \ldots, t_{n-1}(x))(x/t)]\!] = y$, which is the same as $y \trianglelefteq [\![P(t_0(x), \ldots, t_{n-1}(x))(x/t)]\!]$.

Suppose φ is $\psi \wedge \theta$. Then $(y \trianglelefteq [\![\varphi(x)]\!])(x/t)$ means $((y \trianglelefteq [\![\psi(x)]\!]) \wedge (y \trianglelefteq [\![\theta(x)]\!]))(x/t)$. By induction, this is the same as $y \trianglelefteq [\![\psi(x)(x/t)]\!]$ and $y \trianglelefteq [\![\theta(x)(x/t)]\!]$. This is the same as $y \trianglelefteq [\![(\psi(x) \wedge \theta(x))(x/t)]\!]$.

Suppose φ is $\psi \vee \theta$. Then $(y \trianglelefteq [\![\varphi(x)]\!])(x/t)$ is

$$(\exists y_1, y_2((y_1 \trianglelefteq [\![\psi(x)]\!]) \wedge (y_2 \trianglelefteq [\![\theta(x)]\!]) \wedge y_1 \sqcup y_2 = y))(x/t).$$

By induction, this is the same as

$$\exists y_1, y_2(y_1 \trianglelefteq [\![\psi(x)(x/t)]\!] \wedge y_1 \trianglelefteq [\![\theta(x)(x/t)]\!] \wedge y_1 \sqcup y_2 = y).$$

This is the same as $y \trianglelefteq [\![(\psi(x) \vee \theta(x))(x/t)]\!]$.

Suppose φ is $\psi \to \theta$. Then $(y \trianglelefteq [\![\varphi(x)]\!])(x/t)$ is the same as

$$\forall z((z \trianglelefteq y \wedge z \trianglelefteq [\![\psi(x)]\!]) \to (z \trianglelefteq [\![\theta(x)]\!]))(x/t).$$

By induction, this is the same as

$$\forall z(z \trianglelefteq y \wedge z \trianglelefteq [\![\psi(x)(x/t)]\!] \to z \trianglelefteq [\![\theta(x)(x/t)]\!]).$$

This is the same as $y \trianglelefteq [\![(\psi(x) \to \theta(x))(x/t)]\!]$.

Suppose φ is $\forall x'(\psi(x, x'))$. Then $(y \trianglelefteq [\![\varphi(x)]\!])(x/t)$ is

$$(\forall x'([\![\mathrm{E}(x')]\!] \trianglelefteq y \to [\![\mathrm{E}(x')]\!] \trianglelefteq [\![\varphi(x, x')]\!]))(x/t),$$

which, by induction, is the same as

$$\forall x'([\![\mathrm{E}(x')]\!] \trianglelefteq y \to [\![\mathrm{E}(x')]\!] \trianglelefteq [\![\varphi(x, x')(x/t)]\!]).$$

This is the same as $y \trianglelefteq [\![(\forall x'(\varphi(x, x')))(x/t)]\!]$.

Finally, suppose φ is $\exists x'(\psi(x, x'))$. Then $(y \trianglelefteq [\![\varphi(x)]\!])(x/t)$ is

$$(\exists x'([\![\mathrm{E}(x')]\!] = y \wedge y = [\![\varphi(x, x')]\!]))(x/t).$$

By induction, this is the same as $\exists x'([\![\mathrm{E}(x')]\!] = y \wedge y = [\![\varphi(x, x')(x/t)]\!])$. This is the same as $y \trianglelefteq [\![(\exists x'(\varphi(t, x')))(x/t)]\!]$. $\dashv$

We now prove the main result for this section.

Lemma 7.4 *Suppose* $\vdash_i \gamma(\mathbf{x}) \Rightarrow \varphi(\mathbf{x})$, *where* $\mathbf{x}$ *is the set of all free variables that occur in* γ *and* φ. *Then* $\emptyset^{\mathrm{BA}} \vdash_i 1 \trianglelefteq [\![\forall \mathbf{x}(\gamma \to \varphi)]\!]$.

Proof. We proceed by induction on the length of the proof of $\gamma \Rightarrow \varphi$. Thus, we suppose that the result holds for shorter lengths, and show the result holds for each step in the intuitionistic sequent calculus presented in Appendix A. Recall that $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$ means that $z \trianglelefteq \bigsqcap_{i<n}[\![\mathrm{E}(x_i)]\!]$, so that the extent of every element of $\mathbf{x}$ is at least as big as z.

We start by showing that $1 \trianglelefteq [\![\forall\mathbf{x}(\bot \to \varphi)]\!]$. This translates to

$$\forall\mathbf{x}(\bigwedge \mathrm{ST}(x_i) \to \forall z((z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge z \trianglelefteq [\![\bot]\!]) \to z \trianglelefteq [\![\varphi]\!])).$$

But if $z \trianglelefteq [\![\bot]\!]$, then $z = 0$, so that $z \trianglelefteq [\![\varphi]\!]$ by Proposition 4.8.

We now show that $1 \trianglelefteq [\![\forall\mathbf{x}(\varphi \to \top)]\!]$. This translates to

$$\forall\mathbf{x}(\bigwedge \mathrm{ST}(x_i) \to \forall z((z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge z \trianglelefteq [\![\varphi]\!]) \to z \trianglelefteq [\![\top]\!])).$$

But $[\![\top]\!] = 1$, so that if $z \trianglelefteq [\![\varphi]\!]$, then clearly $z \trianglelefteq 1$.

Next, we show that $1 \trianglelefteq [\![\forall\mathbf{x}(\varphi \to \varphi)]\!]$. This case translates to

$$\forall\mathbf{x}(\bigwedge \mathrm{ST}(x_i) \to \forall z((z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge z \trianglelefteq [\![\varphi]\!]) \to z \trianglelefteq [\![\varphi]\!])).$$

This is obviously true.

Next, we show that $1 \trianglelefteq [\![\forall\mathbf{x}(\top \to x = x)]\!]$. This translates to

$$\forall\mathbf{x}(\bigwedge \mathrm{ST}(x_i) \to \forall z((z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge z \trianglelefteq [\![\top]\!]) \to z \trianglelefteq [\![x = x]\!])).$$

But if $z \trianglelefteq [\![\mathrm{E}(x)]\!]$, then $z \trianglelefteq [\![x = x]\!]$.

Next, we show $1 \trianglelefteq [\![\forall\mathbf{x}((x = y \wedge \varphi(x)) \to \varphi(y))]\!]$. This translates to

$$\forall\mathbf{x}(\bigwedge \mathrm{ST}(x_i) \to \forall z(z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge z \trianglelefteq [\![x = y]\!] \wedge z \trianglelefteq [\![\varphi(x)]\!] \to z \trianglelefteq [\![\varphi(y)]\!])).$$

But if $z \trianglelefteq [\![x = y]\!]$ and $z \trianglelefteq [\![\varphi(x)]\!]$, then by Corollary 4.4, we have that $z \trianglelefteq [\![\varphi(y)]\!]$.

We now show that

$$1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \to \psi) \wedge (\psi \to \theta))]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}(\varphi \to \theta)]\!].$$

Then, for any $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$, $z \trianglelefteq [\![\varphi]\!]$ implies that $z \trianglelefteq [\![\psi]\!]$, and for any $w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$, $w \trianglelefteq [\![\psi]\!]$ implies that $w \trianglelefteq [\![\theta]\!]$. Thus, if some $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$ is such that $z \trianglelefteq [\![\varphi]\!]$, then $z \trianglelefteq [\![\psi]\!]$. But then $z \trianglelefteq [\![\theta]\!]$ so that $z \trianglelefteq [\![\varphi \to \theta]\!]$.

Next, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \to \psi) \wedge (\theta \to \psi))]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \vee \theta) \to \psi)]\!].$$

Then we have that if $\bigwedge_{i<n} \mathrm{ST}(x_i)$, then $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\varphi \to \psi]\!]$ and $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\theta \to \psi]\!]$. Now suppose $w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$ with $w \trianglelefteq [\![\varphi \vee \theta]\!]$. Then there exist $w_1, w_2 \trianglelefteq w$ with $w_1 \trianglelefteq [\![\varphi]\!]$ and $w_2 \trianglelefteq [\![\theta]\!]$ and $w_1 \sqcup w_2 = w$. By our suppositions, $w_1 \trianglelefteq [\![\psi]\!]$ and $w_2 \trianglelefteq [\![\psi]\!]$. Then $w_1 \sqcup w_2 = w \trianglelefteq [\![\psi]\!]$ by Lemma 4.6. Thus $z \trianglelefteq [\![(\varphi \vee \theta) \to \psi]\!]$.

Conversely, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \vee \theta) \to \psi)]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \to \psi) \wedge (\theta \to \psi))]\!].$$

Suppose we have that $\bigwedge_{i<n} \mathrm{ST}(x_i)$ and $w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$. We need to show that $w \trianglelefteq [\![\varphi \to \psi]\!]$ and $w \trianglelefteq [\![\theta \to \psi]\!]$. Let $v \trianglelefteq w$ and suppose $v \trianglelefteq [\![\varphi]\!]$. Then by our supposition, $v \trianglelefteq [\![\psi]\!]$. The case for $w \trianglelefteq [\![\theta \to \psi]\!]$ is similar.

Next, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \to \psi) \wedge (\varphi \to \theta))]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}(\varphi \to (\psi \wedge \theta))]\!].$$

Then we have that if $\bigwedge_{i<n} \mathrm{ST}(x_i)$, then $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\varphi \to \psi]\!]$ and $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\varphi \to \theta]\!]$. Suppose $\bigwedge_{i<n} \mathrm{ST}(x_i)$ and $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$. We need to show that $z \trianglelefteq [\![\varphi \to \psi \wedge \theta]\!]$. Thus, suppose $w \trianglelefteq z$ and $w \trianglelefteq [\![\varphi]\!]$. By our supposition, $w \trianglelefteq [\![\psi]\!]$ and $w \trianglelefteq [\![\theta]\!]$, so $w \trianglelefteq [\![\psi \wedge \theta]\!]$.

Conversely, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}(\varphi \to (\psi \wedge \theta))]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \to \psi) \wedge (\varphi \to \theta))]\!].$$

Suppose $\bigwedge_{i<n} \mathrm{ST}(x_i)$ and $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$. We need to show that $z \trianglelefteq [\![(\varphi \to \psi) \wedge (\varphi \to \theta)]\!]$. Thus, suppose $w \trianglelefteq z$ and $w \trianglelefteq [\![\varphi]\!]$. Then by our supposition, $w \trianglelefteq [\![\psi \wedge \theta]\!]$, so certainly $w \trianglelefteq [\![\psi]\!]$. Thus, $z \trianglelefteq [\![\varphi \to \psi]\!]$. The case for $z \trianglelefteq [\![\varphi \to \theta]\!]$ is similar.

Next we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}(\varphi(x) \to \psi(x))]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}'(\varphi(t) \to \psi(t))]\!]$$

where t is a term where no variable in t becomes bound, and $\mathbf{x}'$ is the tuple obtained from $\mathbf{x}$ by removing x and adding the free variables of t. Then, the antecedent is equivalent to: if $\bigwedge_{i<n} \mathrm{ST}(x_i)$, then $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![\varphi(x) \to \psi(x)]\!]$, or, equivalently, if $w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge w \trianglelefteq [\![\varphi(x)]\!]$, then $w \trianglelefteq [\![\psi(x)]\!]$. We wish to show that $1 \trianglelefteq [\![\forall \mathbf{x}'(\varphi(t) \to \psi(t))]\!]$. Let $\mathbf{x}'$ be any set of structural elements, and suppose $w \trianglelefteq [\![\mathrm{E}(\mathbf{x}')]\!]$ and $w \trianglelefteq [\![\varphi(t)]\!]$. Then $w \trianglelefteq [\![\varphi(t)]\!]$ if and only if $w \trianglelefteq [\![\varphi(t \restriction w)]\!]$ by Corollary 4.3. Then, by Lemma 7.3 and applying the antecedent to the elements $\mathbf{x}'$ and $t \restriction w$, we have that $w \trianglelefteq [\![\psi(t)]\!]$.

Next, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}((\mathrm{E}(x) \wedge \varphi) \to \psi)]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}'(\exists x \varphi \to \psi)]\!],$$

where x is a variable which is not free in ψ. Then our supposition is

$$\forall \mathbf{x}(\bigwedge_{i<n} \mathrm{ST}(x_i) \to \forall z((z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge z \trianglelefteq [\![\mathrm{E}(x) \wedge \varphi(x)]\!]) \to z \trianglelefteq [\![\psi]\!])).$$

We need to show that $1 \trianglelefteq [\![\forall \mathbf{x}'(\exists x \varphi \to \psi)]\!]$. This translates to

$$\forall \mathbf{x}'(\bigwedge_{x' \in \mathbf{x}'} \mathrm{ST}(x') \to \forall w((w \trianglelefteq [\![\mathrm{E}(\mathbf{x}')]\!] \wedge \exists x([\![\mathrm{E}(x)]\!] = w \wedge w \trianglelefteq [\![\varphi(x)]\!])) \to w \trianglelefteq [\![\psi]\!])).$$

Thus, suppose we have a set of structural elements $\mathbf{x}'$ and an element x such that $[\![\mathrm{E}(x)]\!] = w$ and $w \trianglelefteq [\![\varphi(x)]\!]$. Then we have $w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge w \trianglelefteq [\![\varphi(x)]\!]$, so by our supposition, $w \trianglelefteq [\![\psi]\!]$.

Conversely, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}'(\exists x\varphi \to \psi)]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}((\mathrm{E}(x) \wedge \varphi) \to \psi)]\!],$$

again with x not free in ψ. As above, the antecedent translates to

$$\forall \mathbf{x}'(\bigwedge_{x' \in \mathbf{x}'} \mathrm{ST}(x') \to \forall z((z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge \exists x([\![\mathrm{E}(x)]\!] = z \wedge z \trianglelefteq [\![\varphi]\!])) \to z \trianglelefteq [\![\psi]\!])).$$

We wish to show that

$$\forall \mathbf{x}(\bigwedge_{i<n} \mathrm{ST}(x_i) \to \forall w((w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge w \trianglelefteq [\![\mathrm{E}(x) \wedge \varphi(x)]\!]) \to w \trianglelefteq [\![\psi]\!])).$$

Thus, suppose $\mathbf{x}$ are structural elements, $w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$ and $w \trianglelefteq [\![\mathrm{E}(x) \wedge \varphi(x)]\!]$. Then, by replacing x with $x \upharpoonright w$, we have that $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq z, w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$, $[\![\mathrm{E}(x)]\!] = w$, and $w \trianglelefteq [\![\varphi(x)]\!]$. Thus $w \trianglelefteq [\![\psi]\!]$.

Next we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}(\varphi \to (\mathrm{E}(x) \to \psi))]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}'(\varphi \to \forall x\psi)]\!],$$

where x is not free in φ. Then we have that if $\mathbf{x}$ are structural and for all z, if $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!] \wedge z \trianglelefteq [\![\varphi]\!]$, then $z \trianglelefteq [\![\mathrm{E}(x) \to \psi]\!]$. We need to show that $1 \trianglelefteq [\![\forall \mathbf{x}'(\varphi \to \forall x\psi)]\!]$, or equivalently,

$$\forall \mathbf{x}'(\bigwedge_{x' \in \mathbf{x}'} \mathrm{ST}(x') \to \forall z(z \trianglelefteq [\![\mathrm{E}(\mathbf{x}')]\!] \wedge z \trianglelefteq [\![\varphi]\!] \to \forall x([\![\mathrm{E}(x)]\!] \trianglelefteq z \to [\![\mathrm{E}(x)]\!] \trianglelefteq [\![\psi]\!]))).$$

Let $\mathbf{x}'$ be structural and suppose $z \trianglelefteq [\![\varphi]\!]$, $z \trianglelefteq [\![\mathrm{E}(\mathbf{x}')]\!]$, and x is such that $[\![\mathrm{E}(x)]\!] \trianglelefteq z$. Let $w = [\![\mathrm{E}(x)]\!]$. Then $w \trianglelefteq [\![\mathrm{E}(\mathbf{x}')]\!]$, $w \trianglelefteq [\![\varphi]\!]$, and $w \trianglelefteq [\![\mathrm{E}(x)]\!]$, so that $w = [\![\mathrm{E}(x)]\!] \trianglelefteq [\![\psi]\!]$.

Conversely, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}'(\varphi \to \forall x\psi)]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}(\varphi \to (\mathrm{E}(x) \to \psi))]\!],$$

where again x is not free in φ. Then our supposition is that

$$\forall \mathbf{x}'(\bigwedge_{x' \in \mathbf{x}'} \mathrm{ST}(x') \to \forall z(z \trianglelefteq [\![\mathrm{E}(\mathbf{x}')]\!] \wedge z \trianglelefteq [\![\varphi]\!] \to \forall x([\![\mathrm{E}(x)]\!] \trianglelefteq z \to [\![\mathrm{E}(x)]\!] \trianglelefteq [\![\psi]\!]))).$$

We need to show that $1 \trianglelefteq [\![\forall \mathbf{x}(\varphi \to (\mathrm{E}(x) \to \psi))]\!]$. Suppose we have structural elements $\mathbf{x}$ and a Boolean element z where $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$ and $z \trianglelefteq [\![\varphi]\!]$. Then we have that $\mathbf{x}'$ are structural elements, $z \trianglelefteq [\![\mathrm{E}(\mathbf{x}')]\!]$, and $z \trianglelefteq [\![\varphi]\!]$. By replacing x with $x \restriction z$, we have that $[\![\mathrm{E}(x)]\!] \trianglelefteq z$, so we can apply our supposition to get $z = [\![\mathrm{E}(x)]\!] \trianglelefteq [\![\varphi]\!]$.

Next, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \wedge \psi) \to \theta)]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}(\varphi \to (\psi \to \theta))]\!].$$

Then we have that if $\mathbf{x}$ are structural, $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$, and $z \trianglelefteq [\![\varphi \wedge \psi]\!]$, then $z \trianglelefteq [\![\theta]\!]$. We wish to show that if $\mathbf{x}$ are structural, $w \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$, and $w \trianglelefteq [\![\varphi]\!]$, then $w \trianglelefteq [\![\psi \to \theta]\!]$. To this end, suppose that $v \trianglelefteq w$ and $v \trianglelefteq [\![\varphi]\!]$. We need to show that $v \trianglelefteq [\![\psi \to \theta]\!]$. So let $u \trianglelefteq v$ and $u \trianglelefteq [\![\psi]\!]$. Then $u \trianglelefteq [\![\varphi]\!]$, so $u \trianglelefteq [\![\varphi \wedge \psi]\!]$. By our supposition, $u \trianglelefteq [\![\theta]\!]$.

Finally, we show that

$$1 \trianglelefteq [\![\forall \mathbf{x}(\varphi \to (\psi \to \theta))]\!] \to 1 \trianglelefteq [\![\forall \mathbf{x}((\varphi \wedge \psi) \to \theta)]\!].$$

Suppose $\mathbf{x}$ are structural. We need to show that $[\![\mathrm{E}(\mathbf{x})]\!] \trianglelefteq [\![(\varphi \wedge \psi) \to \theta]\!]$. Suppose $z \trianglelefteq [\![\mathrm{E}(\mathbf{x})]\!]$ and $z \trianglelefteq [\![\varphi \wedge \psi]\!]$. Then $z \trianglelefteq [\![\varphi]\!]$, so by our supposition, $z \trianglelefteq [\![\psi \to \theta]\!]$. But since $z \trianglelefteq [\![\psi]\!]$, then $z \trianglelefteq [\![\theta]\!]$. $\dashv$

We get the following theorem as an immediate result.

Theorem 7.5 *Let* Γ *be a set of* $\mathcal{L}$*-formulas, let* $\Delta = \{\forall \mathbf{x}(\delta \vee \neg\delta) : \delta$ *a quantifier-free* $\mathcal{L}$*-formula*$\}$. *If* $\Gamma \cup \Delta \vdash_i \varphi$*, then* $\Gamma^{\mathrm{BA}} \vdash_i 1 \trianglelefteq [\![\varphi]\!]$.

Proof. This follows from Lemma 7.4 and the fact that quantifier-free formulas are discrete. ⊣

Thus, by Lemma 7.4, intuitionistically equivalent axiomatizations of a theory Γ have the same translated theory Γ^{BA}: if we include $1 \trianglelefteq [\![\varphi]\!]$ and $\Gamma \vdash_i \varphi \to \psi$, then $\Gamma^{\mathrm{BA}} \models 1 \trianglelefteq [\![\psi]\!]$. Theorem 7.5 shows that the deductive strength of $y \trianglelefteq [\![\varphi]\!]$ is strictly stronger than intuitionistic logic.

We end with an obvious corollary to Lemma 7.4.

Corollary 7.6 *Let* Γ *and* Δ *be intuitionistically equivalent theories. Then* Γ^{BA} *and* Δ^{BA} *axiomatize the same theory.*

7.3 Non-discrete models

In this section, we present models of $\emptyset^{\mathrm{BA}}$ with $\mathcal{L}(A)$-sentences φ such that there is no $p \in \mathrm{BA}(A)$ where $\mathfrak{A} \models [\![\varphi]\!] = p$. This shows that the Boolean translation is not as powerful as full classical derivability. As our example of φ contains no disjunction, by Proposition 3.21 the result also holds for the Kripke translation.

Recall that an $\mathcal{L}(\mathrm{ST}(A))$-sentence is discrete if its ideal is principal.

Definition 7.7 *Let* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$. *We say that* $\mathfrak{A}$ *is* **discrete** *if every* $\mathcal{L}(\mathrm{ST}(A))$*-sentence is discrete. We say* $\mathfrak{A}$ *is* **non-discrete** *otherwise.*

In other words, $\mathfrak{A}$ is non-discrete if there is an $\mathcal{L}(\mathrm{ST}(A))$-sentence φ such that $(\varphi]$ is not principal.

We present some simple examples of non-discrete models. For the examples we present, it is easy to describe the structure of the model so that it does not satisfy the axiom Pt5. By a pullback construction like we made in Chapter 6, we can embed this into a model of Pt5. We present this construction in Appendix B

7.3.1 A two element non-discrete model

We begin by discussing the Boolean algebra for our model $\mathfrak{A}$. Let $\mathcal{C}$ be Cantor space 2^ω. It is well known to be a compact metric 0-dimensional space. It has a countable clopen basis, which we describe as follows: let $\alpha \in 2^{<\omega}$ be of length n. We define $\hat{\alpha} = \{f \in \mathcal{C} : \forall i < n(f(i) = \alpha(i))\}$. Thus each $\hat{\alpha}$ is a clopen subset of $\mathcal{C}$, and all clopen subsets of $\mathcal{C}$ are constructed by taking finite unions of elements $\hat{\alpha}$. We let $\mathfrak{C}$ represent the collection of clopen subsets of $\mathcal{C}$. The Boolean algebra $\mathrm{BA}(A)$ for our model is $\mathfrak{C}$.

The only functions symbols in our language $\mathcal{L}^{\mathrm{BA}}$ are $x \restriction y$, $[\![\mathrm{E}(x)]\!]$, and $[\![x = y]\!]$. We have one global element, which we call a. Now, the rest of our structure is generated as follows: let p_0 be the clopen subset of $\mathcal{C}$ which corresponds to the cone of elements above $\hat{0}$. Let p_1 be the clopen subset which corresponds to the cone of elements above $\widehat{10}$. Let p_2 be the clopen subset which corresponds to the cone of elements above $\widehat{110}$. Let p_3 be the clopen subset which corresponds to the cone of elements above $\widehat{1110}$. Continue in this fashion to get a sequence of Boolean elements p_i. Over each p_i, we introduce a new structural element which we call a_i. Thus for each p_i, $|A_{p_i}| = 2$. By Lemma B.8, this embeds into a model of $\emptyset^{\mathrm{BA}}$, which we call $\mathfrak{A}'$.

We now investigate the extent of the sentence $\exists xy(\neg(x = y))$:

Lemma 7.8 *Let* $\mathbf{1}$ *represent the function in* 2^ω *which is the constant* 1*. Then*

$p \trianglelefteq [\![\exists xy(\neg(x = y))]\!]$ *if and only if* $\mathbf{1} \notin p$.

Proof. For $q \in \mathfrak{C}$, $q \trianglelefteq [\![\exists xy(\neg(x = y))]\!]$ if and only if

$$\exists x([\![\mathrm{E}(x)]\!] = p \wedge p \trianglelefteq [\![\exists y(\neg(x = y))]\!]).$$

This holds if and only if

$$\exists x(([\![\mathrm{E}(x)]\!] = p) \wedge \exists y([\![\mathrm{E}(y)]\!] = p \wedge p \trianglelefteq (\neg(x = y)))),$$

which holds if and only if $\exists xy([\![\mathrm{E}(x)]\!] = [\![\mathrm{E}(y)]\!] = p \wedge \forall q(q \trianglelefteq p \wedge q \trianglelefteq [\![x = y]\!] \rightarrow q = 0))$.

Suppose $\mathbf{1} \notin p$. Then p is covered by some finite set of p_i. Call this finite set I. Now, each $p_i \trianglelefteq [\![\exists xy(\neg(x = y))]\!]$, as witnessed by $a \restriction p$ and a_i. Thus, by Lemma 4.6, we have that $\bigsqcup_{i \in I} p_i \trianglelefteq [\![\exists xy(\neg(x = y))]\!]$. Since $p \trianglelefteq \bigsqcup_{i \in I} p_i$, then by Lemma 4.5, $p \trianglelefteq [\![\exists xy(\neg(x = y))]\!]$.

Conversely, suppose $\mathbf{1} \in p$, and assume $p \trianglelefteq [\![\exists xy(\neg(x = y))]\!]$. Let x and y witness that $p \trianglelefteq [\![\exists xy(\neg(x = y))]\!]$. Since $\mathbf{1} \in p$, and each of x and y is constructed by piecing together finitely many pieces of a and elements a_i, then there exists a nonzero element $q \in \mathrm{BA}(A')$ with $\mathbf{1} \in q$ such that $q \trianglelefteq [\![x = a]\!]$ and $q \trianglelefteq [\![y = a]\!]$, so that $q \trianglelefteq [\![x = y]\!]$. Then $q \sqcap p \neq 0$ since $\mathbf{1} \in p$ and $\mathbf{1} \in q$. But $q \sqcap p \trianglelefteq 0$ and $q \sqcap p \trianglelefteq [\![x = y]\!]$, but $q \sqcap p \neq 0$, a contradiction. Thus we must have that $p \not\trianglelefteq [\![\exists xy(\neg(x = y))]\!]$. ⊣

From this we get an immediate corollary:

Corollary 7.9 $\mathfrak{A}'$ *is non-discrete.*

Proof. Since there is no element of $\mathfrak{C}$ which consists of all of $\mathcal{C}$ except $\mathbf{1}$, the extent of $\exists xy(\neg(x = y))$ is not in $\mathfrak{C}$. ⊣

Note that $(\neg\exists xy\neg(x=y)] = 0$. To see this, let $p \in (\neg\exists x\exists y\neg(x=y)]$. Then for all q, if $q \trianglelefteq p$ and $q \trianglelefteq [\![\exists x\exists y\neg(x=y)]\!]$, then $q = 0$. Assume $p \neq 0$. As $\mathcal{C}$ is atomless, there exists $q \trianglelefteq p$ such that $q \neq 0$ and $\mathbf{1} \notin q$. Then $q \trianglelefteq p$ and $q \trianglelefteq [\![\exists x\exists y\neg(x=y))]\!]$, so that $q = 0$, a contradiction. Thus $p = 0$. So we get the following.

Theorem 7.10 *Let φ be $\exists x\exists y\neg(x=y)$. Then $[\![\varphi \vee \neg\varphi]\!] \neq 1$.*

As $\vdash_c \varphi \vee \neg\varphi$, we see that the Boolean translation does not contain the full power of classical predicate logic. Since φ does not have any disjunctions, with Proposition 3.21 the Kripke translation is also weaker than classical predicate logic.

7.3.2 An infinite non-discrete model

We again start with a model $\mathfrak{A}$ with Boolean algebra $\mathfrak{C}$. The language $\mathcal{L}$ consists of only one binary predicate, $\leq$. We take a properly descending sequence of elements of $\mathfrak{C}$, $\{p_n : n \in \omega\}$, with $p_0 = 1^{\mathfrak{C}}$, whose intersection is a single point, which we name $\mathbf{c}$. Over each element p_n, we introduce an element $\mathbf{n}$ with extent p_n, i.e. $[\![\mathrm{E}(\mathbf{n})]\!] = p_n$. If $m, n \in \mathbb{N}$ with $m \leq n$, then $[\![\mathbf{m} \leq \mathbf{n}]\!] = p_n$. The structure of this model consists of finite piecing together of these elements $\mathbf{n}$. Again, by Lemma B.8, this embeds into a model $\mathfrak{A}' \models \emptyset^{\mathrm{BA}}$.

We look at the extent of $\exists x\forall y(y \leq x)$, i.e. there is a largest element.

Theorem 7.11 *Let $q \in \mathfrak{C}$. Then $q \trianglelefteq [\![\exists x\forall y(y \leq x)]\!]$ holds if and only if $\mathbf{c} \notin q$.*

Proof. We start by noting that $q \trianglelefteq [\![\exists x\forall y(y \leq x)]\!]$ holds if and only if $\exists x([\![\mathrm{E}(x)]\!] = q \wedge q \trianglelefteq [\![\forall y(y \leq x)]\!])$. This holds if and only if $\exists x([\![\mathrm{E}(x)]\!] = q \wedge \forall y([\![\mathrm{E}(y)]\!] \trianglelefteq q \rightarrow [\![\mathrm{E}(y)]\!] \trianglelefteq [\![y \leq x]\!]))$. Further, for every $n \in \mathbb{N}$, the Boolean element $p_n \sqcap (-p_{n+1}) \trianglelefteq$

$[\![\exists x \forall y(y \leq x)]\!]$. In fact, we have that $p_n \sqcap (-p_{n+1}) \trianglelefteq [\![\forall y(y \leq \mathbf{n})]\!]$. Also, $-p_i = ((-p_i) \sqcap p_{i-1}) \sqcup ((-p_{i-1}) \sqcap p_{i-2}) \sqcup \ldots \sqcup ((-p_1) \sqcap p_0)$. Since each $p_n \sqcap (-p_{n+1}) \trianglelefteq [\![\exists x \forall y(y \leq x)]\!]$, by Lemma 4.6, we have $-p_i \trianglelefteq [\![\exists x \forall y(y \leq x)]\!]$ for each i.

Suppose $\mathbf{c} \notin c$. Since $\mathbf{c} \notin c$, q is covered by some finite set of $-p_i$. As each $-p_i \trianglelefteq [\![\exists x \forall y(y \leq x)]\!]$, we have that $q \trianglelefteq [\![\exists x \forall y(y \leq x)]\!]$ by Lemma 4.5.

Conversely, suppose $\mathbf{c} \in c$ and assume there exists an element $a \in A_q$ such that $\forall y [\![\mathrm{E}(y)]\!] \trianglelefteq q \to [\![y \leq a]\!] = [\![\mathrm{E}(y)]\!]$. Since $\mathbf{c} \in c$, there exists an $n \in \mathbb{N}$ such that $p_n \trianglelefteq c$. Further, since a is made by piecing together a finite number of elements, then there is a largest $m \in \mathbb{N}$ such that $\mathbf{m}$ is used in constructing a. Now $p_{m+1} \trianglelefteq p_n$, so that $p_{m+1} \trianglelefteq c$. Further, $[\![\mathrm{E}(\mathbf{m+1})]\!] = p_{m+1}$ and $[\![\mathbf{m+1} \leq a]\!] = 0$, a contradiction. Thus, we must have that $q \ntrianglelefteq [\![\exists x \forall y(y \leq x)]\!]$. $\dashv$

We get an immediate corollary:

Corollary 7.12 *The sentence $\exists x \forall y(y \leq x)$ is not discrete.*

Proof. By the previous theorem, $(\exists x \forall y(y \leq x)] = \{c : \mathbf{c} \notin c\}$. However, there is no element in $\mathfrak{C}$ that consists of all subsets that do not contain $\mathbf{c}$. Thus, $(\exists x \forall y(y \leq x)]$ is not a principal ideal. $\dashv$

Note that $(\neg \exists x \forall y(y \leq x)] = 0$. To see this, let $p \in (\neg \exists x \forall y(y \leq x)]$. Then for all q, if $q \trianglelefteq p$ and $q \trianglelefteq [\![\exists x \forall y(y \leq x)]\!]$, then $q = 0$. Assume $p \neq 0$. As $\mathcal{C}$ is atomless, there exists $q \trianglelefteq p$ such that $q \neq 0$ and $\mathbf{c} \notin q$. Then $q \trianglelefteq p$ and $q \trianglelefteq [\![\exists x \forall y(y \leq x)]\!]$, so that $q = 0$, a contradiction. Thus $p = 0$. So we get the following.

Theorem 7.13 *Let φ be $\exists x \forall y(y \leq x)$. Then $[\![\varphi \vee \neg \varphi]\!] \neq 1$.*

Again we see that the Boolean translation does not contain the full power of classical predicate logic.

7.3.3 A non-discrete model with a complete Boolean algebra

Recall that a Boolean algebra is **complete** if it is closed under arbitrary joins and meets. We show that there exists a model of $\emptyset^{\mathrm{BA}}$ with a complete Boolean algebra. Note in a complete Boolean algebra, an ideal is principal exactly when it is closed under arbitrary joins of elements in the ideal. We construct a model and a sentence φ such that the ideal generated by φ is not closed under arbitrary joins of elements, so that the ideal is not principal, and thus φ is not discrete.

The language $\mathcal{L}$ contains no function symbols and no predicates besides the usual $\mathrm{E}(x)$ and $x = y$. We construct a model $\mathfrak{A}$ over $\mathcal{L}^{\mathrm{BA}}$ as follows: We let $\mathrm{BA}(A)$ be the power set Boolean algebra on $\mathbb{N}$. For each singleton $\{n\} \in \mathcal{P}(\mathbb{N})$, we place above it the one element structure. We call its element a_n. We set $\mathrm{ST}(A) = \bigcup_{n\in\mathbb{N}}\{a_n\}$. For each n, we set $[\![\mathrm{E}(a_n)]\!] = \{n\}$, and we set $[\![a_n = a_m]\!] = 0$ for all $n \neq m$. Further, for $N \subseteq \mathbb{N}$, we set $a_n \restriction N$ equal to a_n if $n \in N$ and ϖ otherwise. One easily shows that $\mathfrak{A} \models \emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$. By Lemma B.8, this embeds into a model $\mathfrak{A}'$. We note that $\mathfrak{A}'$ does not have a global element, as we are only permitted to piece together finitely many elements.

Proposition 7.14 *Let $\mathfrak{A}'$ be as above, and let φ be the $\mathcal{L}$-sentence $\exists x(x = x)$. Then $(\varphi]$ is the set of all finite subsets of $\mathcal{P}(\mathbb{N})$. Additionally φ is non-discrete.*

Proof. The first claim follows from the fact that we are able to glue together finitely many a_i. Note that $\mathbb{N}$ is not in $(\varphi]$. Thus, the ideal $(\varphi]$ is contains every finite subset of $\mathbb{N}$, but does not contain the join of these elements. $\dashv$

Note that it was vital to the proof that there is no global element. This example shows that having a complete Boolean algebra is not sufficient to showing the model is discrete.

7.4 Sufficient conditions for discrete models

We now present two sufficient conditions for a model to be discrete.

7.4.1 Internally finite models

Many of our models have structures with infinitely many elements. For example, if a model has an atomless Boolean algebra and a structural element not equal to ϖ, then the structure of the model must be infinite. However, some models will have a finitely many structural elements such that every structural element can be partitioned into those places where it equals those finite structural elements.

Definition 7.15 *A model* $\mathfrak{A}$ *of* $\emptyset^{\mathrm{BA}}$ *is* **internally finite** *if there exist finitely many elements* $\mathbf{a} \in \mathrm{ST}(A)$ *such that* $[\![\mathrm{E}(a_i)]\!] = 1$ *for all* i *and for every* $a' \in \mathrm{ST}(A)$*,* $[\![\mathrm{E}(a)]\!] = \bigsqcup_{i<n} [\![a_i = a]\!]$*.*

In some sense, this is an approximation of a finite model. More precisely, $\mathfrak{A}$ is internally finite if and only if $\mathfrak{A} \models 1 \trianglelefteq [\![\exists \mathbf{x} \forall y (\bigvee_{i<n} y = x_i)]\!]$ for some n. We now show that being internally finite is a sufficient condition for being a discrete model.

Proposition 7.16 *Suppose* $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$*. If* $\mathfrak{A}$ *is internally finite, then* $\mathfrak{A}$ *is a discrete model.*

Proof. Let $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$, and let $\mathbf{a} \in \mathrm{ST}(A)$ witness that $\mathfrak{A}$ is internally finite. Let φ be a $\mathcal{L}(\mathrm{ST}(A))$-sentence. We show by induction that $[\![\varphi]\!]$ is Boolean. By Corollary 4.17 and Corollary 4.18, we only need to show the inductive case for the universal and existential cases.

Suppose φ is $\forall x \psi(x)$. Now, for each element a of $\mathbf{a}$, $[\![\psi(a)]\!]$ is discrete. Let $q = \bigsqcap_{i<n} [\![\psi(a_i)]\!]$. We show that $q = [\![\forall x \psi(x)]\!]$. First, $q \trianglelefteq [\![\forall x \psi(x)]\!]$: let a be such that $[\![\mathrm{E}(a)]\!] \trianglelefteq q$. Now, $\bigsqcup_{i<n} [\![a_i = a]\!] = [\![\mathrm{E}(a)]\!]$. Now, by Corollary 4.4, $[\![a_i = a]\!] \trianglelefteq [\![\psi(a)]\!]$. By Lemma 4.6, $[\![\mathrm{E}(a)]\!] = \bigsqcup_{i<n} [\![a_i = a]\!] \trianglelefteq [\![\psi(a)]\!]$.

We now show $[\![\forall x \psi(x)]\!] \trianglelefteq q$. Suppose $p \trianglelefteq [\![\forall x \psi(x)]\!]$. Now, $[\![\mathrm{E}(a_i \restriction p)]\!] = p$, since $[\![\mathrm{E}(a_i)]\!] = 1$ for all i. Thus, $p \trianglelefteq [\![\psi(a_i \restriction p)]\!]$, so by Corollary 4.3, $p \trianglelefteq [\![\psi(a_i)]\!]$ for all i. Thus, $p \trianglelefteq \bigsqcap_{i<n} [\![\psi(a_i)]\!] = q$.

Suppose φ is $\exists x \psi(x)$. Let $q = \bigsqcup_{i<n} [\![\psi(a_i)]\!]$. We show that $q = [\![\exists x \psi(x)]\!]$. First, $q \trianglelefteq [\![\exists x \psi(x)]\!]$: let $a_i' = a_i \restriction [\![\psi(a_i)]\!]$, so that $[\![\psi(a_i)]\!] = [\![\psi(a_i')]\!]$. By Corollary 4.3, we have that $[\![\mathrm{E}(a_i')]\!] \trianglelefteq [\![\psi(a_i')]\!]$. Thus, $[\![\psi(a_i')]\!] \trianglelefteq [\![\exists x \psi(x)]\!]$. By Lemma 4.6, $\bigsqcup_{i<n} [\![\psi(a_i)]\!] = q \trianglelefteq [\![\exists x \psi(x)]\!]$.

Conversely, suppose $p \trianglelefteq [\![\exists x \psi(x)]\!]$. Let a be such that $[\![\mathrm{E}(a)]\!] = p$ and $p \trianglelefteq [\![\psi(a)]\!]$. Now, $\bigsqcup_{i<n} [\![a_i = a]\!] = p$. By Lemma 4.5, $[\![a_i = a]\!] \trianglelefteq [\![\psi(a)]\!]$, and by Corollary 4.4, $[\![a_i = a]\!] \trianglelefteq [\![\psi(a_i)]\!]$. Thus $p = \bigsqcup_{i<n} [\![a_i = a]\!] \trianglelefteq \bigsqcup_{i<n} [\![\psi(a_i)]\!] = q$ $\dashv$

7.4.2 Model completeness

We show that if $\mathfrak{A}$ is a model of a translated model complete theory, then $\mathfrak{A}$ is discrete. This encompasses Lemma 5.15.

Proposition 7.17 *Let Γ be a Π_2^0 axiomatization of a model complete $\mathcal{L}$-theory, and let*

$\mathfrak{A} \models \Gamma^{\mathrm{ABA}}$. *Then* $\mathfrak{A}$ *is discrete.*

Proof. By Lemma 5.15, it suffices to show that all $\mathcal{L}(\mathrm{ST}(A))$-sentences have the same extent as an existential $\mathcal{L}(\mathrm{ST}(A))$-sentence. We proceed by induction on the complexity of φ. The result obviously holds for atoms, and is preserved under conjunction, disjunction, and existential closure.

Suppose φ is $\psi \to \gamma$. By induction, ψ has the same extent as $\exists \mathbf{x}\psi'$ and γ has the same extent as $\exists \mathbf{y}\gamma'$. As ψ and γ are discrete, $[\![\varphi]\!]$ is the same as $[\![\neg\exists\mathbf{x}\psi']\!] \sqcup [\![\exists\mathbf{y}\gamma']\!]$. By Corollary 5.16, there is a sentence $\exists\mathbf{z}\psi''$ such that $[\![\neg\exists\mathbf{x}\psi']\!] = [\![\exists\mathbf{z}\psi'']\!]$. Then φ has the same extent as $\exists\mathbf{z}\psi'' \vee \forall\mathbf{y}\gamma'$.

Suppose φ is $\forall x\psi$. By induction, ψ has the same extent as $\exists\mathbf{x}\psi'$. By Corollary 5.16, there is a sentence $\exists\mathbf{y}\gamma$ such that $[\![\psi]\!] = [\![\neg\exists\mathbf{y}\gamma]\!]$. So $(\varphi] = (\forall x\neg\exists\mathbf{y}\gamma]$. By Theorem 7.5, $(\forall x\neg\exists\mathbf{y}\gamma] = (\neg\exists x\exists\mathbf{y}\gamma]$. Apply Corollary 5.16 to get $\exists\mathbf{z}\theta$ such that $(\neg\exists x\exists\mathbf{y}\gamma] = (\exists\mathbf{z}\theta]$. $\dashv$

Appendix A

Intuitionistic Sequent Calculus

We include as a reference a version of the intuitionistic sequent calculus. Here, $\varphi \Rightarrow \psi$ means that ψ is derivable from φ. Our particular choice of predicate logic allows us to have empty models. We discuss this at the end of the appendix. Here φ and ψ are formulas, a single line means that the suppositions on top imply the bottom, and a double line means that the implication goes both ways.

$\bot \Rightarrow \varphi$

$\varphi \Rightarrow \top$

$\varphi \Rightarrow \varphi$

$\top \Rightarrow x = x$

$x = y \wedge \varphi(x) \Rightarrow \varphi(y)$ where x and y are not bound after substitution

$$\frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \theta}{\varphi \Rightarrow \theta}$$

$$\frac{\varphi \Rightarrow \psi \quad \theta \Rightarrow \psi}{\overline{\varphi \vee \theta \Rightarrow \psi}}$$

$$\frac{\varphi \Rightarrow \psi \quad \varphi \Rightarrow \theta}{\overline{\varphi \Rightarrow \psi \wedge \theta}}$$

$$\frac{\varphi(x) \Rightarrow \psi(x)}{\varphi(t) \Rightarrow \psi(t)}$$ where t is any term, and no free variable in t becomes bound.

$$\frac{(\mathrm{E}(x) \wedge \varphi) \Rightarrow \psi}{\exists x\varphi \Rightarrow \psi}$$ where x is not free in ψ.

$$\frac{\varphi \Rightarrow (\mathrm{E}(x) \rightarrow \psi)}{\varphi \Rightarrow \forall x\psi}$$ where x is not free in φ.

$$\frac{\varphi \wedge \psi \Rightarrow \theta}{\varphi \Rightarrow \psi \rightarrow \theta}$$

This is a well-known version of the intuitionistic sequent calculus. We refer the reader to [12], [11], and [1] for reference.

This system is known to contain the empty model. We show that the usual predicate calculus does not permit this.

Proposition A.1 *In the usual predicate calculus with* $\mathrm{E}(x)$ *replaced by* $\top$*, the system proves* $\exists x(x = x)$.

Proof. We have that $\top \Rightarrow x = x$. We also have that $\exists x(x = x) \Rightarrow \exists x(x = x)$. Further, from $\exists x(x = x) \Rightarrow \exists x(x = x)$ we derive $x = x \Rightarrow \exists x(x = x)$. From this we get $\top \Rightarrow \exists x(x = x)$. $\dashv$

Appendix B

Piecing Together

We now show how embed a clean, nondegenerate model of $\emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$ into a model of $\emptyset^{\mathrm{BA}}$. Often when constructing models of $\emptyset^{\mathrm{BA}}$, it is easier to introduce the structure of the model without piecing together. We show that this is sufficient, that is, such a model embeds into a model with piecing together. Further, our construction will preserve axioms Ex5, Ba19 and So2. Throughout this appendix, we fix a clean, nondegenerate model $\mathfrak{A} \models \emptyset^{\mathrm{BA}}$.

Let p, q be elements of $\mathrm{BA}(A)$ such that $p \sqcup q = 1$. We form the following pullback:

$$\begin{array}{ccc}
(\mathfrak{A}\upharpoonright p) \times_{\mathfrak{A}\upharpoonright(p\sqcap q)} (\mathfrak{A}\upharpoonright q) & \xrightarrow{\pi_2} & \mathfrak{A}\upharpoonright q \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi^{\mathfrak{A}\upharpoonright q}_{p\sqcap q}} \\
\mathfrak{A}\upharpoonright p & \xrightarrow[\pi^{\mathfrak{A}\upharpoonright p}_{p\sqcap q}]{} & \mathfrak{A}\upharpoonright (p \sqcap q)
\end{array}$$

Figure 5: Pullback over p and q

Recall that the model $\mathfrak{A} \upharpoonright p \times_{\mathfrak{A}\upharpoonright(p\sqcap q)} \mathfrak{A} \upharpoonright q$ is the submodel of the product model $\mathfrak{A}\upharpoonright p \times \mathfrak{A}\upharpoonright q$ with domain $\{(x, y) \in (A\upharpoonright p \times A\upharpoonright q) : \pi^{\mathfrak{A}\upharpoonright p}_{p\sqcap q}(x) = \pi^{\mathfrak{A}\upharpoonright q}_{p\sqcap q}(y)\}$. We label this pullback model $\mathfrak{A}'$. We wish to show that $\mathfrak{A}$ embeds into $\mathfrak{A}'$. By [7, page 71], if Figure 6 commutes, then there is a unique embedding from $\mathfrak{A}$ to $\mathfrak{A}'$.

Lemma B.1 $\pi^{\mathfrak{A}\upharpoonright p}_{p\sqcap q} \circ \pi^{\mathfrak{A}}_{p} = \pi^{\mathfrak{A}\upharpoonright q}_{p\sqcap q} \circ \pi^{\mathfrak{A}}_{q}$, *so that* $\mathfrak{A}$ *factors through* $\mathfrak{A}'$.

Proof. This holds because $\pi^{\mathfrak{A}\upharpoonright p}_{p\sqcap q} \circ \pi^{\mathfrak{A}}_{p} = \pi^{\mathfrak{A}}_{p\sqcap q} = \pi^{\mathfrak{A}\upharpoonright q}_{p\sqcap q} \circ \pi^{\mathfrak{A}}_{q}$. $\dashv$

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow[\pi_q^{\mathfrak{A}}]{} & \mathfrak{A}\upharpoonright q \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_{p\sqcap q}^{\mathfrak{A}\upharpoonright q}} \\ \mathfrak{A}\upharpoonright p & \xrightarrow[\pi_{p\sqcap q}^{\mathfrak{A}\upharpoonright p}]{} & \mathfrak{A}\upharpoonright (p\sqcap q) \end{array}$$

Figure 6: Commutative square over p and q

Thus, the diagram commutes, and we get a unique embedding from $\mathfrak{A}$ to $\mathfrak{A}'$. We now discuss the form of this embedding.

Let $F : \mathfrak{A} \to \mathfrak{A}\upharpoonright p \times \mathfrak{A}\upharpoonright q$ by $F(x) = \begin{cases} (x\upharpoonright p, x\upharpoonright q) & \text{if } \mathfrak{A} \models \mathrm{E}([\![\mathrm{E}(x)]\!]) \\ (x \sqcap p, x \sqcap q) & \text{if } \mathfrak{A} \models \mathrm{BA}(x) \end{cases}$

Thus, we get the following diagram:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow[\pi_2\circ F]{} & \mathfrak{A}\upharpoonright q \\ \downarrow{\scriptstyle \pi_1\circ F} & & \downarrow{\scriptstyle \pi_{p\sqcap q}^{\mathfrak{A}\upharpoonright q}} \\ \mathfrak{A}\upharpoonright p & \xrightarrow[\pi_{p\sqcap q}^{\mathfrak{A}\upharpoonright p}]{} & \mathfrak{A}\upharpoonright (p\sqcap q) \end{array}$$

Figure 7: Mapping through F

We now show the following:

Lemma B.2 *F is the unique embedding from $\mathfrak{A}$ to $\mathfrak{A}'$ which makes Figure 7 commute.*

Proof. It suffices to show that the above diagram commutes. Now, $\pi_1 \circ F = \pi_p^{\mathfrak{A}}$, and $\pi_2 \circ F = \pi_q^{\mathfrak{A}}$. The result then follows by Lemma B.1. $\dashv$

With this Lemma, we get the following result:

Lemma B.3 *Let $p, q \in \mathrm{BA}(A)$ be such that $\mathfrak{A} \models p \sqcup q = 1$ and $\mathfrak{A}' = (\mathfrak{A}\upharpoonright p) \times_{\mathfrak{A}\upharpoonright(p\sqcap q)} (\mathfrak{A}\upharpoonright q)$. Then $\mathrm{BA}(A) \cong \mathrm{BA}(A')$.*

Proof. For the Boolean algebra, all that needs to be shown is that the function F from Lemma B.2 is onto. Let $(r, s) \in \mathrm{BA}(A')$. Then $r \sqcap q = s \sqcap p$. Let $t \in \mathrm{BA}(A)$ be such that $t = r \sqcup s$. Then $F(t) = (t \sqcap p, t \sqcap q)$. But $t \sqcap p = (r \sqcup s) \sqcap p = (r \sqcap p) \sqcup (s \sqcap p) = (r \sqcap p) \sqcup (r \sqcap q) = r$. Similarly, $t \sqcap q = s$. Thus F is onto $\mathrm{BA}(A')$, so that $\mathrm{BA}(A) \cong \mathrm{BA}(A')$. $\dashv$

By Lemma B.2, we have that $\mathfrak{A}$ embeds into $\mathfrak{A}'$. Thus, we may assume that $\mathfrak{A} \subseteq \mathfrak{A}'$, and by Lemma B.3, we have that $\mathrm{BA}(A) = \mathrm{BA}(A')$. Similarly, we identify the elements $a \in A$ with the image $(a \restriction p, a \restriction q) \in A'$. We now discuss the theory of $\mathfrak{A}'$.

Lemma B.4 *Let $p, q \in \mathrm{BA}(A)$ be such that $p \sqcup q = 1$, and let $\mathfrak{A}'$ be $(\mathfrak{A} \restriction p) \times_{\mathfrak{A} \restriction (p \sqcap q)} (\mathfrak{A} \restriction q)$. Then $\mathfrak{A}'$ is a clean model of $\emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$. Further,*

1. *$\mathfrak{A}$ is atomless if and only if $\mathfrak{A}'$ is atomless;*
2. *If $\mathfrak{A}$ has a global element, then $\mathfrak{A}'$ has a global element.*
3. *If $\mathfrak{A} \models \mathrm{Pt5}$ and $p \sqcap q = 0$, then $\mathfrak{A} = \mathfrak{A}'$.*

Proof. Since $\emptyset^{\mathrm{BA}}$ is universal Horn, and $\mathfrak{A}'$ is a submodel of a product of models of $\emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$, we have that $\mathfrak{A}' \models \emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$. Further, the domain of $\mathfrak{A}$ is the set of elements of $\mathfrak{A} \restriction p \times \mathfrak{A} \restriction q$ such that $\{(x, y) \in (A \restriction p \times A \restriction q) : \pi^{\mathfrak{A} \restriction p}_{p \sqcap q}(x) = \pi^{\mathfrak{A} \restriction q}_{p}(y)\}$. Since $\pi^{\mathfrak{A} \restriction p}_{p \sqcap q}(x) = \pi^{\mathfrak{A} \restriction q}_{p}(y)$ implies that both x and y are Boolean or both x and y are structural, we do not create any chaff. Thus, $\mathfrak{A}'$ is clean.

Claim 1 also follows from the fact that $\mathrm{BA}(A) = \mathrm{BA}(A')$. For part 2, let $a \in A$ such that $\mathfrak{A} \models [\![\mathrm{E}(a)]\!] = 1$. Then $(a \restriction p, a \restriction q) \in A'$, and $\mathfrak{A}' \models [\![\mathrm{E}((a \restriction p, a \restriction q))]\!] = (p, q) = 1^{\mathfrak{A}'}$. For part 3, we show that F from Lemma B.2 is onto. Let $(a, b) \in A'$. Then $\mathfrak{A} \models [\![\mathrm{E}(a)]\!] \sqcap [\![\mathrm{E}(b)]\!] = 0$. Since $\mathfrak{A}$ satisfies the piecing together axiom, there is an element $a \oplus b$ with $[\![\mathrm{E}(a \oplus b)]\!] = [\![\mathrm{E}(a)]\!] \sqcup [\![\mathrm{E}(b)]\!]$, $(a \oplus b) \restriction [\![\mathrm{E}(a)]\!] = a$, and $(a \oplus b) \restriction [\![\mathrm{E}(b)]\!] = b$. Then $F(a \oplus b) = (a, b)$. $\dashv$

We now prove the following Lemma:

Lemma B.5 *Let $p, q, r, s \in \mathrm{BA}(A)$ be such that $p \sqcap q = 0$, $p \sqcup q = 1$, $r \trianglelefteq p$, and $s \trianglelefteq q$, and let $\mathfrak{A}' = (\mathfrak{A}\restriction p) \times_{\mathfrak{A}\restriction(p\sqcap q)} (\mathfrak{A}\restriction q)$. Then if $a \in A_r$ and $b \in A_s$, then $a \oplus b \in A'_{r\sqcup s}$.*

Proof. We have that $(a, b) \in A'$. Then $[\![\mathrm{E}((a, b))]\!] = (r, s)$, which is the image of $r \sqcup s$ in $\mathfrak{A}'$. Further, $(a, b)\restriction[\![\mathrm{E}((a, 0))]\!] = (a, 0)$, the image of a in $\mathfrak{A}'$, and $(a, b)\restriction[\![\mathrm{E}((0, b))]\!] = (0, b)$, the image of b in $\mathfrak{A}'$. Thus, $(a, b) = a \oplus b$. ⊣

Lemma B.6 *Let $p, q \in \mathrm{BA}(A)$ be such that $p \sqcap q = 0$ and $p \sqcup q = 1$. Set $\mathfrak{A}'$ equal to the pullback model $\mathfrak{A}\restriction p \times_{\mathfrak{A}\restriction(p\sqcap q)} \mathfrak{A}\restriction q$. Let $r \trianglelefteq p$ and $s \trianglelefteq q$. Then $\mathfrak{A}'_{r\sqcup s} \cong \mathfrak{A}'_r \times \mathfrak{A}'_s$.*

Proof. We need only show that the function in Lemma 3.13 is onto. To see it is surjective, let $(a, b) \in \mathfrak{A}'_r \times \mathfrak{A}'_s$. Then $a \in A\restriction p$ and $b \in A\restriction q$, and $\pi^{\mathfrak{A}\restriction p}_{p\sqcap q}(a) = \pi^{\mathfrak{A}\restriction q}_{p\sqcap q}(b)$. Thus, $(a, b) \in \mathfrak{A}'$. We show that $[\![\mathrm{E}((a, b))]\!] = r\sqcup s$. Let $t = [\![\mathrm{E}((a, b))]\!]$. Then $t\sqcap p = [\![\mathrm{E}(a)]\!] = r$ and $t \sqcap q = [\![\mathrm{E}(b)]\!] = s$. Thus, $t = t \sqcap (p \sqcup q) = r \sqcup s$. Thus, $(a, b) \in \mathfrak{A}'_{r\sqcup s}$, and maps to (a, b). ⊣

We now show that a clean model $\mathfrak{A}$ of $\emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$ embeds into a clean model of $\emptyset^{\mathrm{BA}}$ with the same underlying Boolean algebra. We enumerate all elements $p \in \mathrm{BA}(A)$ as p_i with $i < \kappa$ for a cardinal κ. We set $\mathfrak{A}_0 = \mathfrak{A}$, and set $\mathfrak{A}_1$ equal to the pullback model $\mathfrak{A}_0\restriction p_0 \times_{\mathfrak{A}\restriction(p_0\sqcap -p_0)} \mathfrak{A}_0\restriction -p_0$.

By Lemma B.3, we have that $\mathrm{BA}(A) = \mathrm{BA}(A_1)$. We create a chain of models in the following fashion: if λ is a successor ordinal, with $\lambda = \alpha + 1$, we set $\mathfrak{A}_\lambda = \mathfrak{A}_\alpha\restriction p_\alpha \times_{\mathfrak{A}_\alpha\restriction(p_\alpha\sqcap -p_\alpha)} \mathfrak{A}_\alpha\restriction -p_\alpha$, and if λ is a limit ordinals, we set $\mathfrak{A}_\lambda = \bigcup_{i<\lambda} \mathfrak{A}_i$. Thus, we get a chain of models $\mathfrak{A}_i$ for $i < \kappa$. We call $\mathfrak{A}^{(1)}$ the union of this chain of models.

Lemma B.7 *$\mathfrak{A}^{(1)}$ is a clean model of $\emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$ with $\mathrm{BA}(A^{(1)}) = \mathrm{BA}(A)$. Further,*

1. *$\mathfrak{A}$ is atomless if and only if $\mathfrak{A}^{(1)}$ is atomless;*

2. *If $\mathfrak{A}$ has a global element, then $\mathfrak{A}^{(1)}$ has a global element.*

3. *If $\mathfrak{A} \models \mathrm{Pt5}$, then $\mathfrak{A}^{(1)} = \mathfrak{A}$.*

Finally, if r, s are such that $r \sqcap s = 0$, then for all $a \in A_r$, $b \in A_s$, there exists $c \in \mathfrak{A}^{(1)}_{r \sqcup s}$ such that $c \restriction r = a$, $c \restriction s = b$, that is, $c = a \oplus b$.

Proof. As the axioms of $\emptyset^{\mathrm{BA}}$ are all Π^0_2, they are preserved under chains of models. Further, since $\mathfrak{A}^{(1)}$ is a union of clean models, it itself is clean. Finally, since each pullback has the same Boolean algebra, $\mathrm{BA}(A^{(1)}) = \mathrm{BA}(A)$.

Part 1 follows from the fact that $\mathrm{BA}(A^{(1)}) = \mathrm{BA}(A)$. For part 2, note that Ex5 is a Π^0_2 sentence. Now, by Lemma B.4, if $\mathfrak{A}$ satisfies Ex5 then each pullback satisfies Ex5. Since Π^0_2 sentences are preserved under unions of chains, we get that $\mathfrak{A}^{(1)}$ satisfies Ex5 as well. Part 3 follows from Lemma B.4.

For the last claim, let $a \in A_r$ and $b \in A_s$, and let p_λ be such that $r \trianglelefteq p_\lambda$ and $s \trianglelefteq -p_\lambda$. Then, $a \oplus b$ is in the pullback model $\mathfrak{A}_{p_{\lambda+1}}$ by Lemma B.5. Thus, $a \oplus b$ is in $\mathfrak{A}^{(1)}$. $\dashv$

By construction, we have that $\mathfrak{A} \subseteq \mathfrak{A}^{(1)}$. We repeat this process with the same enumeration of Boolean elements to get a model $\mathfrak{A}^{(2)}$ with $\mathfrak{A}^{(1)} \subseteq \mathfrak{A}^{(2)}$. We continue in this manner to get a chain of models $\mathfrak{A}^{(1)} \subseteq \mathfrak{A}^{(2)} \subseteq \mathfrak{A}^{(3)} \subseteq \ldots$. Thus, for every $n \in \mathbb{N}$ and any $r, s \in \mathrm{BA}(A)$ with $r \sqcap s = 0$, we have the picture in Figure 8. In Figure 8, $\oplus$ is the map taking (a, b) to $a \oplus b$, h is the isomorphism from Lemma B.6, and g is the inclusion map.

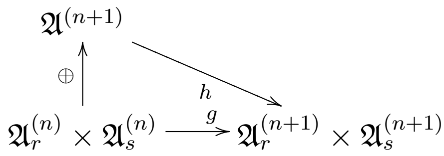

Figure 8: Comparing $\mathfrak{A}^{(n+1)}$ to $\mathfrak{A}^{(n)}$

We set $\mathfrak{A}^{(\omega)}$ equal to the union of the chain of models $\bigcup_{n\in\omega} \mathfrak{A}^{(n)}$. We are now ready to prove our major result:

Theorem B.8 $\mathfrak{A}^{(\omega)}$ *is a clean model of* $\emptyset^{\mathrm{BA}}$ *with* $\mathrm{BA}(A^{(\omega)}) = \mathrm{BA}(A)$. *Further,*

1. $\mathfrak{A}$ *is atomless if and only if* $\mathfrak{A}^{(\omega)}$ *is atomless;*

2. *If* $\mathfrak{A}$ *has a global element, then* $\mathfrak{A}^{(\omega)}$ *has a global element.*

3. *If* $\mathfrak{A} \models \mathrm{Pt5}$*, then* $\mathfrak{A}^{(\omega)} = \mathfrak{A}$.

Proof. As each $\mathfrak{A}^{(n)} \models \emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$, and $\mathfrak{A}^{(\omega)}$ is a union of a chain of models, then $\mathfrak{A}^{(\omega)} \models \emptyset^{\mathrm{BA}}$. Now, let $a, b \in \mathrm{ST}(A^{(\omega)})$ such that $\mathfrak{A}^{(\omega)} \models [\![\mathrm{E}(a)]\!] \sqcap [\![\mathrm{E}(b)]\!] = 0$. If $[\![\mathrm{E}(a)]\!] = 0$, then b satisfies the piecing together axiom. Thus, we may suppose $[\![\mathrm{E}(a)]\!] \neq 0$ and $[\![\mathrm{E}(b)]\!] \neq 0$. Let p and q be any elements such that $[\![\mathrm{E}(a)]\!] \trianglelefteq p$, $[\![\mathrm{E}(b)]\!] \trianglelefteq q$, $p \sqcap q = 0$, and $p \sqcup q = 1$. Now, let $N \in \mathbb{N}$ such that $a, b \in A^{(N)}$. Now, by Lemma B.7, the model $\mathfrak{A}_r^{(N)} \times \mathfrak{A}_s^{(N)}$ maps into $\mathfrak{A}_{r\sqcup s}^{(N+1)}$. Thus, there is an element in $\mathfrak{A}^{(N+1)}$ corresponding to (a, b). Call this element $a \oplus b$. Then $[\![\mathrm{E}(a \oplus b)]\!] = [\![\mathrm{E}(a)]\!] \sqcup [\![\mathrm{E}(b)]\!]$, $a \oplus b \restriction [\![\mathrm{E}(a)]\!] = a$ and $a \oplus b \restriction [\![\mathrm{E}(b)]\!] = b$.

Again, part 1 follows from the fact that $\mathrm{BA}(A^{(\omega)}) = \mathrm{BA}(A)$. For part 2, if $\mathfrak{A}$ satisfies Ex5, then each model $\mathfrak{A}^{(n)}$ does as well. As Ex5 is a Π_2^0 sentence, so that $\mathfrak{A}^{(\omega)} \models \mathrm{Ex5}$. Finally, part 3 follows from Lemma B.7. $\dashv$

We now show a simple corollary of this theorem.

Corollary B.9 *Suppose* $\mathfrak{A} \models \emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$*. Then* $\mathfrak{A}$ *embeds into a model* $\mathfrak{B}$ *of* $\emptyset^{\mathrm{BA}}$ *with* $\mathrm{BA}(B) = \mathrm{BA}(A)$.

Proof. Let $\mathfrak{A}'$ be largest clean submodel of $\mathfrak{A}$ given by Proposition 3.4, and let $\mathfrak{A}^{(\omega)}$ be the model of $\emptyset^{\mathrm{BA}} \cup \{\mathrm{Pt5}\}$ given by Theorem B.8. We set $B = A^{(\omega)} \cup \mathrm{CH}(A)$, with predicates and functions interpreted as in $\mathfrak{A}^{(\omega)}$. Then $\mathfrak{B}$ clearly models $\emptyset^{\mathrm{BA}} \cup \{\mathrm{Pt5}\}$, with $\mathrm{BA}(B) = \mathrm{BA}(A^{(\omega)})$. $\dashv$

We combine the results form Chapter 6 and this appendix into one result.

Theorem B.10 *Suppose* $\mathfrak{A} \models \emptyset^{\mathrm{BA}} \setminus \{\mathrm{Pt5}\}$*. Then* $\mathfrak{A}$ *embeds into a model* $\mathfrak{B}$ *of* $\emptyset^{\mathrm{ABA}}$.

Proof. By Theorem B.8, $\mathfrak{A}$ embeds into a model of $\emptyset^{\mathrm{BA}}$. By Theorem 6.19, this model embeds into a model of $\emptyset^{\mathrm{ABA}}$. $\dashv$

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