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Computable Numberings in Hierarchies

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CONTENTS

NOTES AND ABBREVIATION

ω denotes the set of natural numbers
$A_0, A_1, A_2, \ldots$ denotes subsets of the set of natural numbers
$\alpha, \beta, \ldots$ denotes numberings
P_n denotes the n-th program of a Turing machine (TM)
φ_n denotes the unary function, computable by a TM with the n-th program P_n.
$\alpha \leq \beta$ denotes that a numbering α is reducible to a numbering β
$\{A_s\}_{s\in\omega}$ denotes an approximating function to a set A
Σ_n^0 denotes finite levels of arithmetical hierarchy for $n \in \omega$
c.e. set denotes computably enumerable set
d-c.e. set denotes difference of two c.e. sets
Σ_n^{-1} (or n-c.e.) sets denotes finite levels of the Ershov hierarchy for $n \in \omega$
Π_n^{-1} denotes the set of the complements of the Σ_n^{-1}-sets
Δ_n^{-1} denotes the intersection of the classes Σ_n^{-1}, Π_n^{-1}
$\{W_x\}_{x\in\omega}$ denotes the standard numbering of c.e. sets

INTRODUCTION

Uniformly computable sequences of sets of natural numbers is a classical object of a study of the computability theory. A major part of Hartley Rogers's famous monograph [1], a handbook for computability theory specialists, is devoted to investigation of properties of such sequences.

A nonempty computably enumerable set is precisely a set whose elements can be represented as a sequence of numbers generated by some algorithm. For an arbitrary sequence of computably enumerable sets, each set of the sequence has its own enumeration algorithm, and all these algorithms can be mutually independent. In the case when it is possible to parameterize in some effective way this set of algorithms, i.e. to come up with an algorithm that, for each set of the sequence, reproduces an algorithm of generation of its elements, we are dealing with a uniformly computably enumerable sequence.

Formally, for a sequence of subsets of the natural numbers $A_0, A_1, A_2, \ldots$, it means that the set of pairs of numbers $\{\langle x, n\rangle : x \in A_n\}$ is computably enumerable. A family $\mathcal{A}$ of computably enumerable (c.e. for short) sets is called computable if its elements can be numbered (not necessarily without repetitions) as a uniformly computably enumerable sequence. In other words, there exists a surjective mapping $\alpha : \omega \to \mathcal{A}$ of the set of all natural numbers ω onto the family $\mathcal{A}$ such that the sequence of sets $\alpha(0), \alpha(1), \alpha(2), \ldots$ is uniformly computably enumerable, i.e. the set of pairs

$$\{\langle x, n\rangle : x \in \alpha(n)\} \qquad (*)$$

is a c.e. set. The set $(*)$ is called the universal set of a numbering α. A numbering α is called computable if its universal set is c.e. A classical example of a computable numbering is the standard numbering $W_0, W_1, W_2, \ldots$ of the family of all c.e. subsets of ω. In this numbering, W_n denotes the domain of the unary function φ_n, that is computable by a Turing machine with n-th program P_n in the Gödel encoding of all Turing machines.

At the end of the 20th century, S.S. Goncharov and A. Sorbi [2] offered a general approach to define notion of a computable numbering of families of sets of a class of constructive objects, which admits a Gödel numbering. For example, the class of arithmetical sets described by formulas of arithmetic, and the class of sets of the Ershov hierarchy is described by Boolean combinations of $\exists$-formulas of arithmetic.

The Goncharov-Sorbi approach allowed to introduce the notion of a computable numbering for families of sets of many natural classes of constructive objects and to apply to them methods developed in general numberings theory. Dozens of papers on computable numberings in arithmetic, hyperarithmetical, and analytical hierarchies and the Ershov difference hierarchy were published over the past 15 years.

Relevance of the research. The thesis is devoted to the investigation of computable numberings of families of sets in the Ershov hierarchy. The relevance of studying the computable numberings in the Ershov hierarchy is confirmed by many publications in highly rated international journals such as Algebra and Logic, Journal of Symbolic Logic, Mathematical Logic Quarterly, Archive for Mathematical Logic as well as by plenary and sectional talks at many international conferences (Maltsev Readings, Logic Colloquium, Computability in Europe, Asian Logic Conference, Theory and Applications of Models of Computations). Related research has been done, and is being done, in the world in several important centers in computability theory: Novosibirsk State University, Sobolev Institute of Mathematics, Kazan State University (Russia), University of Wisconsin (USA), University of Heidelberg (Germany), University of Siena (Italy), Singapore National University, University of Auckland (New Zealand), Kazakh National University (Kazakhstan), and others.

Along with the notion of a computable numbering, the notion of reducibility of numberings, and the derived notion of a Rogers semilattice are the basic concepts of work.

As is common for numbering theory, an index n in a numbering α is regarded as a description (calculation program) of the set $\alpha(n)$. Two numberings α, β can be compared in terms of their algorithmic complexity. A numbering α is called reducible to a numbering β (in symbols $\alpha \leq \beta$), if there exists a translation algorithm for each program in α to a program of the same object in β, i.e. $\alpha(n) = \beta \circ f(n)$ for some computable function f for all $n \in \omega$. The reducibility relation of numberings is a preorder. Two numberings α, β are called equivalent if they are reducible to each other. A partial order relation is naturally defined on the equivalence classes of computable numberings of a family, which is induced by the reducibility relation. The algebraic structure thus obtained is an upper semilattice and is called the Rogers semilattice of computable numberings of the family.

The Rogers semilattice, as is noted in the monograph of Ershov [3], represents the complexity of all computations of the considered familyas as is a whole, unlike the complexity of concrete computations investigated in complexity theory. The purpose of the research of Rogers semilattices is to find relationships between structural and other characteristics of a given family, on one hand, and algebraic and elementary properties of Rogers semilattices, descriptions of invariants required to classify their isomorphism types and types of elementary equivalence on the other hand.

The goal of the research. The thesis is devoted to study problems that concern the following two invariants related to the extreme elements of Rogers semilattices: the problem of the existence of universal computable numberings and the problem of the existence of minimal computable numberings. These issues are developed in [4–23] by researchers from Germany, Italy, Kazakhstan, Russia and USA.

General methodology of the research. The common tools for studying the computable numberings in the Ershov hierarchy comes from computability theory and the theory of numberings. In the thesis, an important role is played by the fixed point theorem and its generalizations such as w_n-subobjects of a numbered set. Another important tool coming from computability theory is the so-called “priority method” for the construction of computable numberings. Indeed, the priority method is only a common scheme to construct computably enumerable sets and relations, but every concrete construction built by the priority method is based on very original, and sometimes very sophisticated specific ideas.

Scientific novelty. Scientific novelty of this research is that it is devoted to the solution of well known problems that are formulated, say, in [4]. Investigations in the dissertation throw light to some of these questions in the Ershov hierarchy. Interest in these problems comes from the facts that some properties of Rogers semilattices for families of sets in the Ershov hierarchy are significantly different from the properties of Rogers semilattices of families of c.e. sets and families of arithmetic sets (see, for example, [9–11]).

Theoretical and practical significance. The theoretical significance of this research is based on a very new and unexpected computational phenomena that we encountered during the study of non-monotonic computations like computations in the Ershov hierarchy. Therefore, the problem of studying further specific features of non-monotonic computations is very important for non-classical models of computation. Moreover, there are many long-standing problems of the theory of numberings that might be resolved by means of computable numberings in the Ershov hierarchy. Among such problems, one can mention the problem of Ershov on the cardinality of Rogers semilattices: is it true that Rogers semilattice of a computable family is either infinite or one-element. The practical significance of this research in the frame of the thesis is its usefulness for specialists in this field as well as for creation of up-to-date courses on computable numberings. Besides, non-monotonic computations seem to be a suitable mathematical model for algorithmic processes that occur during neurocomputing and for expert systems.

Publications. The results of the thesis were published in 11 [25–34] works. Two papers were published in ranking journals [30–31], 3 articles were published in journals recommended by the Committee for Control of Education and Science of RK, 4 abstracts were published in the proceedings of international conferences.

The main results of this thesis were presented at the following international conferences: 5th Conference on Computability in Europe, CiE-2009 (2009, Heidelberg, Germany); Logic Colloquium 2009, (2009, Sofia, Bulgaria); Association for Symbolic Logic 2012 North American Annual Meeting (2012, Madison, WI, USA); Mal’tsev Meeting - 2012, (2012, Novosibirsk, Russia).

Besides, the results were reported on the seminar "Spectral theory of linear operators and its applications" at the Department of Fundamental Mathematics (Almaty, 2014); and the seminar "Modern scientific problems of mathematics, mechanics and information technology" (Almaty, 2014).

The structure and scope of the thesis. The thesis of 51 pages consists of an introduction, three chapters, a conclusion and the list of references.

Statements are numbered by pairs of indices. The first index indicates the number of the section; the second one shows the number of the statement within the section.

The main content of the thesis.

Section 1 contains common concepts of computability theory and numberings theory and consists of two subsections. In subsection 1 we define basic notions and properties of the Ershov Hierarchy. In subsection 2 we define notions of computable numberings in hierarchies and describes properties of extreme elements in the Rogers semilattice in various hierarchies.

In section 2 we investigate properties of universal numberings of finite families of d.c.e. sets. We show different cases of finite families of d.c.e. sets for which there is a universal numbering and for which there is not.

The main results of this section are the following theorems. The first theorem is an answer for the question: "Do there exist finite families in Ershov's hierarchy without universal numberings?"

Theorem 2.4 There are nonempty, disjoint, d.c.e. sets A, B such that the finite family $\mathcal{F} = \{A, B\}$ has no universal numbering.

And the second theorem covers some cases of finite families of d.c.e. sets for which there is a universal numbering. The conditions below may appear rather complicated but encompass all the obstacles to building a universal numbering of which we are aware.

Theorem 2.3 If there are c.e sets A_0, A_1, B_0, B_1 and $A = A_0 - A_1$ and $B = B_0 - B_1$ and $A \nsubseteq B$ and $B \nsubseteq A$ such that

$$\forall x\,(x \in A_0 \Rightarrow x \notin A_1 \text{ or } x \notin B),$$
$$\forall x\,(x \in B_0 \Rightarrow x \notin B_1 \text{ or } x \notin A),$$

and partial computable functions ϕ and ψ such that

$$\forall s \forall x \in A_s(x \notin B \text{ or } (\phi_s(x,s) \downarrow > s \,\&\, x \in B_{\phi_s(x,s)})),$$
$$\forall s \forall x \in B_s(x \notin A \text{ or } (\psi_s(x,s) \downarrow > s \,\&\, x \in A_{\psi_s(x,s)})),$$

then there is a universal numbering π for $\mathcal{F} = \{A, B\}$**.**

The main result of section 3 is the following theorem on the existence of families of sets without minimal computable numberings in each level, whether finite or infinite, of the Ershov hierarchy.

Theorem 3.2 For every nonzero computable ordinal and any ordinal notation a of it, there exists a Σ^{-1}_a–computable family $\mathcal{A}$ of Σ^{-1}_a sets that has no Σ^{-1}_a–computable minimal numbering.

The results of sections 2 and 3 were published in journals in the Thomson list with non-zero impact factor.

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1 BACKGROUND

1.1 Ershov Hierarchy

The notion of a computable enumerable set, i.e. a set of integers whose members can be effectively listed, is a fundamental one. Another way of approaching this definition is via an approximating function $\{A_s\}_{s\in\omega}$ to the set A in the following sense: we begin by guessing $x \notin A$ at stage 0 (i.e. $A_0(x) = 0$); when later x enters A at stage $s+1$, we can change our approximation from $A_s(x) = 0$ to $A_{s+1}(x) = 1$. Note that this approximation (for fixed) x may change at most once as s increases, namely when x enters A. An obvious variation of this definition to allow us more than one change: a set A is 2-c.e. (or d-c.e.) if for each x, $A_s(x)$ change at most twice as s increases. This is equivalent to requiring the set A to be the difference of two c.e. sets $A_1 - A_2$. Similarly, one can define n-c.e. sets by allowing n changes for each x. A direct generalization of this reasoning leads to sets which are computably approximable in the following sense: for a set A there is a set of uniformly computable sequence $\{f(x,0), f(x,1), \ldots, f(x,s), \ldots | x \in \omega\}$ consisting of 0 and 1 such that for any x the limit of the sequence $f(x,0), f(x,1), \ldots$ exists and is equal to the value of the characteristic function $A(x)$ of A. The well-known Shoenfield Lemma states that the class of such sets coincides with the class off all Δ^0_2-sets. Thus, for a set A, $A \leq_T \emptyset'$ if and only if there is a computable function $f(x,s)$ such that $A(x) = \lim_s f(x,s)$.

The notion of d-c.e. and n-c.e. sets goes back to Putnam [35] and was first investigated and generalized by Ershov [36–38]. The arising hierarchy of sets is now known as the Ershov difference hierarchy. The position of set A in this hierarchy is determined by the number of changes in the approximation of A described above, i.e. by the number of different pairs of neighboring elements of the sequence.

The Ershov hierarchy consists of the finite and infinite levels. The finite levels of the hierarchy consists of the n-c.e. or Σ^{-1}_n sets for $n \in \omega$. Otherwise a set belongs to one of the infinite levels of the hierarchy. The infinite levels of the hierarchy are defined using infinite constructive ordinals. As it turned out, the resulting hierarchy of sets exhaused the whole class of Δ^0_2-sets. Each subsequent level of the hierarchy contains all previous ones but does not coincide with any of them.

Our notation and terminology are standard and generally follows Soare [39]. In particular, the standard enumeration of the c.e. sets and partial computable functions are denoted by $\{W_x\}_{x\in\omega}$ and $\{\varphi_x\}_{x\in\omega}$, respectively. As usual, we append $[s]$ to various functionals such as $\varphi^A_e(x)[s]$ to indicate the state of affairs at stage s. We mean by this notation the result of running the e th Turing machine for s steps on input x. For a set $A \subseteq \omega$, its complement $\omega - A$ is denoted by $\bar{A}$. The cardinality of a set A is denoted by $|A|$.

The pairing function $\langle x, y\rangle$ is defined as $\langle x, y\rangle = \frac{(x+y)^2+3x+y}{2}$ and bijectively maps ω^2 onto ω. We denote by l and r the uniquely defined functions such that for all x, y, $l(\langle x, y\rangle) = x$, $r(\langle x, y\rangle) = y$ and $\langle l(x), r(x)\rangle = x$; the n-place function $\langle x_1, \ldots x_n\rangle$ for

$n > 2$ is defined by $\langle x_1, \ldots, x_n \rangle = \langle\langle \ldots \langle x_1, x_2 \rangle, x_3 \rangle, \ldots, x_n \rangle$. In this case the s-th component of $\langle x_1, \ldots, x_n \rangle$ is denoted as $c_{n,s}$. Thus, $\langle c_{n,1}(x), \ldots, c_{n,n}(x) \rangle = x$ and $c_{n,s}(\langle x_1, \ldots, x_n \rangle) = x_s$. If a function f is defined at x, then we write $f(x) \downarrow$, otherwise $f(x) \uparrow$. The characteristic function of a set A is denoted by the same letter: $A(x) = 1$, if $x \in A$, and otherwise $A(x) = 0$.

The finite levels of the Ershov hierarchy. We begin with following characterization of the Δ^0_2-sets (i.e. sets $A \leq_T \emptyset'$).

Lemma 1.1 (Shoenfield Limit Lemma [40]) A set A is a Δ^0_2-set if and only if there is a computable function of two variables f such that $f(x, s) \in \{0, 1\}$ for all s, x, $f(x, 0) = 0$ and $\lim_s f(x, s)$ exists for each x (i.e. $|\{s : f(x, s) \neq f(x, s + 1)\}| < \infty$), and $\lim_s f(x, s) = A(x)$.

It follows easily from the Limit Lemma that

Theorem 1.1 [37] A set A is Turing reducible (T-reducible) to $\emptyset'$ if and only if there is a uniformly computably enumerable sequence of c.e. sets $\{R_x\}_{x\in\omega}$ such that

$$R_0 \supseteq R_1 \supseteq \ldots, \bigcap_{x=0}^{\infty} R_x = \emptyset, \text{ and } A = \bigcup_{x=0}^{\infty} (R_{2x} - R_{2x+1}) \tag{1}$$

Proof. ($\Rightarrow$) Let $A \leq \emptyset'$. By the Limit Lemma there is a computable numbering f such that $A = \lim_s f(x, s)$, and for all x, $f_s(x, s) = 0$. Define c.e. sets R_n, $n \in \omega$, as follows:

$R_0 = \{y : \exists(f(y, s) = 1)\}$,

$R_1 = \{y : \exists s_0, s_1(s_0 < s_1, f(y, s_0) = 1, f(y, s_1) = 0)\}$, and in general for $n > 0$,

$$R_n = \{y : \exists s_0 < s_1 < \ldots < s_n(f(y, s_0) = 1, f(y, s_1) = 0, \ldots,$$
$$f(y, s_n) = n + 1 \mod 2\}.$$

Obviously, all sets R_n are c.e., the sequence $\{R_x\}_{x\in\omega}$ is uniformly c.e. and $R_0 \supseteq R_1 \supseteq \ldots$. It is also easy to check that $\bigcap_{x=0}^{\infty} R_x = \emptyset$ and $A = \bigcup_{\infty}^{x=0} (R_{2x} - R_{2x+1})$.

($\Leftarrow$) For this direction the proof is straightforward.

Note that if A is an arbitrary Σ^0_2-set then it is easy to show that $A = \bigcup_{\infty}^{x=0} (R_{2x} - R_{2x+1})$ such that $R_0 \supseteq R_1 \supseteq \ldots$. Therefore, in theorem 1.1 the condition $\bigcap_{x=0}^{\infty} R_x = \emptyset$ is necessary.

If in (1) starting from some n all elements of the sequence $\{R_x\}_{x\in\omega}$ are empty, then we obtain sets from the finite levels of the Ershov Hierarchy.

Definition 1.1 [36] A set A is Σ^{-1}_n set, if either $n = 0$ and $A = \emptyset$, or $n > 0$ and there are c.e. sets $R_0 \supseteq R_1 \supseteq \ldots \supseteq R_{n-1}$ such that

$$A = \bigcup_{i=0}^{[\frac{n-1}{2}]} (R_{2i} - R_{2i+1}) \text{ (Here if } n \text{ is odd number then } R_n = \emptyset.)$$

It follows from this definition that if $n > 1$ and n is an even number then $(n = 2m)$

$$A = \bigcup_{x=0}^{m-1} (R_{2x} - R_{2x+1}),$$

and if $n > 1$ and n is an odd number $(n = 2m + 1)$ then

$$A = \{\bigcup_{x=0}^{m-1} (R_{2x} - R_{2x+1})\} \cup R_{2m}.$$

Therefore, the class of Σ_1^{-1} sets coincide with the class c.e. sets, Σ_2^{-1} sets can be written as $R_1 - R_2$, where $R_1 \supseteq R_2$ c.e. sets, therefore they also called d-c.e. (difference-c.e.) sets, Σ_3^{-1} sets can be written as $(R_1 - R_2) \cup R_3$ etc.

The n-c.e. sets are exactly those sets constitute the level Σ_n^{-1} of the Ershov hierarchy. The complement of the Σ_n^{-1}-sets constitute the level Π_n^{-1} of the hierarchy (Π_n^{-1}-sets). The intersection of these two classes is denoted by Δ_n^{-1}:

$$\Delta_n^{-1} = \Sigma_n^{-1} \cap \Pi_n^{-1}$$

The proof of the following statement is straightforward.

Theorem 1.2 A set A is an Σ_n^{-1} set for some $n \geq 0$ if and only if there is a computable function g of two variables s and x such that $A(x) = \lim_s g(x, s)$ for every x, $g(x, 0) = 0$ and

$$|\{s \mid g(x, s + 1) \neq g(x, s)\}| \leq n.$$

Comment. Addison, in [41], considered a general method of constructing "difference" hierarchies. In particular, his hierarchy, generated by c.e. sets, defines the same classes of n- and ω-c.e. sets. The notations Δ_n^{-1}, Π_n^{-1} and Σ_n^{-1} for the finite levels of Ershov hierarchy, as well as analogous notations for further levels (see Theorem 1.6) were introdused by Ershov [36–37].

The class of ω-c.e. sets. The n-c.e. sets for $n < \omega$ does not exhaust the collection of Δ_2^0-sets ([37], see Theorem 1.4). Therefore, to obtain in this way a description of all Δ_2^0-sets we need to consider infinite levels of the hierarchy.

In the definition of Σ_n^{-1} sets $n < \omega$ we have used non-increasing sequences $R_0 \supseteq R_1 \supseteq \ldots \supseteq R_{n-1}$ of c.e. sets. The infinite levels of the Ershov hierarvhy are defined using uniformly c.e. sequences of c.e. sets such that the c.e. sets in these sequences satisfy the same $\subseteq$-relation which are consistent with the order type of the original which defines the level of this set in the hierarchy.

Definition 1.2 Let $P(x, y)$ be a computable binary relation which partially orders the set of natural numbers (for convenience instead of $P(x, y)$ we will write $x \leq_P y$.) By definition, a uniformly c.e sequence $\{R_x\}$ of c.e. sets is a P-(or $\leq_P$-) sequence if for all pairs x, y,$x \leq_P y$ implies that $R_x \subseteq R_y$.

Note that we can easily redefine the Σ_n^{-1} sets for $n < \omega$ according to this definition. Indeed, if, for instance, for some c.e. sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n$ we have $A = (A_1 - A_2) \cup \ldots \cup A_{n-1} - A_n)$ (where n is an even number), then let $R_0 = A_n, R_1 = A_{n-1}, \ldots, R_{n-1} = A_1$. We have thus obtained an n-sequence

$(n = \{0 < 1 < \ldots < n-1\})$ $R_0 \subseteq R_1 \subseteq \ldots \subseteq R_{n-1}$ such that $A = \bigcup_{i=0}^{\frac{n-1}{2}} (R_{2i+1} - R_{2i})$.

The sets from the first infinite level of the Ershov hierarchy are the ω-c.e. sets. They are defined using ω-sequences of c.e. sets, i.e. sequences $\{R_x\}_{x\in\omega}$, in which the relation $R_x \subseteq R_y$ is consistent with the order type of $\omega = \{0 < 1 < \ldots\}$: $R_0 \subseteq R_1 \subseteq \ldots \subseteq R_{n-1} \subseteq \ldots$.

Definition 1.3 A set $A \subseteq \omega$ belongs to level Σ^{-1}_ω of the Ershov hierarchy (or A is a Σ^{-1}_ω-set) if there is an ω-sequence $\{R_x\}_{x\in\omega}$ such that $A = \bigcup_{n=0}^{\infty} (R_{2n+1} - R_{2n})$. A belongs to level Π^{-1}_ω of the Ershov hierarchy (or A is a Π^{-1}_ω), if $\bar{A} \in \Sigma^{-1}_\omega$. Finally, A belongs to level Δ^{-1}_ω of the Ershov hierarchy (A is a Δ^{-1}_ω-set), if A and $\bar{A}$ both are Σ^{-1}_ω -sets, i.e. $\Delta^{-1}_\omega = \Sigma^{-1}_\omega \cap \Pi^{-1}_\omega$. Δ^{-1}_ω-sets are also called ω-c.e. sets.

Theorem 1.3 ([42], or see [43]) A set $A \subseteq \omega$ belong to level Σ^{-1}_ω of the Ershov hierarchy iff there is a partial computable function ψ such that for every x,

$x \in A$ implies $\exists s(\psi(x,s)\downarrow)$ and $A(x) = \psi(\mu s(\psi(x,s)\downarrow), x)$;

$x \notin A$ implies $\forall s(\psi(x,s)\uparrow)$, or $\exists s(\psi(x,s)\downarrow)\,\&\, A(x) = \psi(\mu s(\psi(x,s)\downarrow), x)$.

In other words, $A \subseteq \mathrm{dom}(\psi(\mu s(\psi(x,s)\downarrow), x))$, and for every x,

$x \in \mathrm{dom}(\psi(\mu s(\psi(x,s)\downarrow), x))$ implies $A(x) = \psi(\mu s(\psi(x,s)\downarrow), x)$.

Definition 1.4 Let f be a total unary function. A set A is called f-computably enumerable (an f-c.e. set), if there is a computable function g such that for all s and x, $A(x) = \lim_s g(x,s)$, and

$$|\{s : g(x,s) \neq g(x,s+1)\}| \leq f(x).$$

Theorem 1.4

a) There is an id-c.e. set (where id is the identity function) which is not n-c.e. for any $n \in \omega$.

b) Let f and g be computable functions such that $\exists^\infty x(f(x) < g(x))$.

c) There is a Δ^0_2-set which is not f-c.e. for any computable function f.

d) Let A be an f-c.e. set for some computable function f, $A \neq \emptyset$, and let g be a computable function such that $\forall y \exists x(g(x) \geq y)$. Then there exists a g-c.e. set B such that $A \equiv_m B$.

Theorem 1.5 Let $A \subseteq \omega$. The following are equivalent:

a) A is ω-c.e.

b) There is an ω -sequence $\{R_x\}_{x\in\omega}$ such that $\bigcup_{x\in\omega}$; and $A = \bigcup_{n=0}^{\infty} (R_{2n+1} - R_{2n})$.

c) A is f-c.e. for some computable function f.

d) There is a partial computable function ψ such that for all x,

$$A(x) = \psi(x,k), \text{ where } k = \mu t(\psi(x,t)\downarrow).$$

e) A is tt-reducible to $\emptyset'$.

The infinite levels of Ershov hierarchy. The ω-c.e. sets are the first examples of sets from infinite levels of the Ershov hierarchy.

Let us recall some necessary notions and notations. For infinite levels of the Ershov we use Kleene's system of ordinal notations $(\mathcal{O}, <_{\mathcal{O}})$ (Kleene [44], see also Rogers [1] for details). For $a \in \mathcal{O}$, $|a|_{\mathcal{O}}$ denotes the ordinal of which a is a notation; Kleene's partial ordering on the set $\mathcal{O}$ is denoted by $<_{\mathcal{O}}$. Ordinal notations were used by Ershov in his article [37] to represent and study the sets lying in the infinite levels of his difference hierarchy. In the notation $\mathcal{O}$ a parity function $e(x)$ is defined as follows: for any $a \in \mathcal{O}$, $e(a) = 1$ if $|a|_{\mathcal{O}}$ is odd and $e(a) = 0$, if $|a|_{\mathcal{O}}$ is even.

Since $|a|_o$ has order-type $\langle \{x : x <_o a\}, <_o \rangle$, the sentence "$a$-sequence of c.e. sets $\{R_x\}$" for $a \in \mathcal{O}$ has to be understood in the sense of Definition 1.2. Define for $a \in \mathcal{O}$ the operators S_a and P_a, which map a-sequences $\{R_x\}_{x<_o a}$ to subset of ω as follows:

$$S_a(R) = \{z \mid \exists x <_o a(z \in R_x \& e(x) \neq e(a) \& \forall y <_o x(z \notin R_y))\}.$$

$$P_a(R) = \{z \mid \exists x <_o a(z \in R_x \& e(x) = e(a) \& \forall y <_o x(z \notin R_y))\} \cup \{\omega - \bigcup_{x<_o a} R_x\}.$$

It follows from these definition that $P_a(R) = \overline{S_a(R)}$ for all $a \in \mathcal{O}$ and all a-sequences R.

Definition 1.5 [38] The class $\Sigma_a^{-1}(\Pi_a^{-1})$ for $a \in \mathcal{O}$ is the class of sets $S_a(R)$ ($P_a(R)$, respectively), where $R = \{R_x\}_{x<_o a}$ runs thought all a-sequence of c.e. sets. Let $\Delta_a^{-1} = \Sigma_a^{-1} \cap \Pi_a^{-1}$.

It is easy to see that for natural numbers $n > 0$ and for $a \in \mathcal{O}$ such that $|a|_o = \omega$ these definition coincide with the previous ones. (The finite levels of the Ershov hierarchy are denoted by ordinals, not by their $\mathcal{O}$-notations.)

We will use another characterization of the sets in the Ershov hierarchy, as described in Ash and Knight's handbook [45].

Definition 1.6 Let a be a notation of a nonzero computable ordinal. We say that a set of numbers A belongs to the Σ_a^{-1} class of the Ershov hierarchy if there exist a pair of computable functions $f(z, s)$ and $h(z, s)$ such that, for all z, s:

1. $A(z) = \lim_s f(z, s)$ and $f(z, 0) = 0$ (here and hereinafter, we denote the value of the characteristic function of a set X on z by $X(z)$);

2. (a) $h(z, 0) = a \,\&\, h(z, s+1) \leq_{\mathcal{O}} h(z, s)$;
   (b) $f(z, s+1) \neq f(z, s) \Rightarrow h(z, s+1) \neq h(z, s)$.

The function h is called a change function for A with respect to f. A pair of functions $\langle f, h \rangle$ is called a Σ_a^{-1}–approximation of the Σ_a^{-1} set A.

There are different versions of this definition (for example [46] or [47]), however, the next one appears in the paper of R.L. Epstein, R.L. Haas and R.L. Kramer [42] and V.L. Selivanov [48-49].

Definition 1.7 For any $a \in \mathcal{O}$, a set A is a Σ_a^{-1}-set, if there are a computable function $f(x,s)$ and a partially computable function $g(x,s)$ such that for all $x \in \omega$ the following conditions are performed:

1. $A(x) = \lim_s f(x,s)$ and $f(x,0) = 0$.
2. $g(x,s) \downarrow \rightarrow g(x,s+1) \downarrow \leq_{\mathcal{O}} g(x,s) <_{\mathcal{O}} a$.
3. $f(x,s) \neq f(x,s+1) \rightarrow g(x,s+1) \downarrow \neq g(x,s)$

A pair of functions $\langle f, h\rangle$ is called a Σ_a^{-1}–approximation of the Σ_a^{-1} set A.

Theorem 1.6 (Hierarchy Theorem [37]) Let $a, b \in \mathcal{O}$ and $a <_o b$. Then $\Sigma_a^{-1} \cup \Pi_a^{-1} \subsetneq \Sigma_b^{-1} \cap \Pi_b^{-1}$.

Corollary 1.6.1 For every $a \in \mathcal{O}$, $\Sigma_a^{-1} \subsetneq \Sigma_2^0 \cap \Pi_2^0$.

Theorem 1.7 Let $|a|_o$ be a limit ordinal. The set A belongs to the class Δ_a^{-1} if and only if there is an a-sequence R such that $A = S_a(R)$ and $\cup_{b<_o a} R_b = \omega$.

Theorem 1.8 $\Delta_2^0 = \bigcup\limits_{a \in \mathcal{O}} \Sigma_a^{-1}$.

Generalizing Definition 1.3 of ω-c.e. sets to infinite ordinals we introduce the following definition:

Definition 1.8 Let $|a|_o$ be a limit ordinal. If $A \in \Delta_a^{-1}$, then the set A is called an $|a|_o$-c.e. set (or an α-c.e. set, if $|a|_o = \alpha$).

It is clear that if $A \in \Sigma_a^{-1}$ for some $a \in \mathcal{O}$, and $B \leq_m A$, then $B \in \Sigma_a^{-1}$, and if A is $|a|_o$-c.e. for some limit ordinal $|a|_o$, $a \in \mathcal{O}$, and $B \leq_m A$, then B is also $|a|_o$-c.e. set.

Obviously, definitions 1.6 and 1.7 are equivalent. The equivalence of definitions 1.5 and 1.7 was established in the paper [48].

The next three theorems show that we really have hierarchy of sets for the class Δ_2^0 of the arithmetical hierarchy. Our research is devoted to the problems of computable numbering of sets from both finite and infinite levels of the Ershov hierarchy.

1.2 Computable numberings in hierarchies

The theory of numberings was developed for investigating the algorithmic properties of classes of abstract objects by methods of classical computability theory, by coding the information about them and their relations through the properties of their numbers (names). For the first time the effectiveness of this approach has been demonstrated in the classical work of K. Gödel's on the incompleteness of arithmetic.

Later S.K. Kleene [50] constructed a universal partial computable function (in other words, a computable enumeration of all partially computable functions). Kleene's result has great importance for computability theory.

The concept of a numbering as a mathematical object was introduced by A.N. Kolmogorov [51] and his student V.A. Uspensky, they studied computable numberings of partial computable functions [52-53].

H. Rogers [54-55] investigated the computable numberings of the family of all partial computable functions and computably enumerable sets. He introduced the concept of $\mathrm{Com}(A)$ of all computable numberings of the family A. Rogers considered so-call acceptable numbering, i.e. computable numberings of the family of all unary partial computable functions and the family of all c.e. subsets of ω such that every computable numbering of the family is reducible to it. Numberings that are minimal with respect to reducibility were studied by R.F. Friedberg [56], A.I. Malt'sev [57], M.B. Pour-El [58].

The results of the classical theory of computable numberings are used most frequently in recursive mathematics [59-60]. Thus, the method of constructing families of computably enumerable sets with a finite number of computable Friedberg numberings proposed by Goncharov in [14], served as a starting point for the study of algorithmic dimensions of recursive models [61-63].

The theory of numberings found applications in classical recursive theory. For example, using the theorem of Goncharov [14] on the numbers of computable Friedberg numberings of families of computably enumerable sets, Kummer [64] found a solution to the problem of the known types of recursive isomorphism of partial computable functions ([55], Chapter 4). More precisely, he proved that for any $k \in \omega$ there is a computable function with exactly k recursive isomorphism types.

In [2], S.S. Goncharov and A.Sorbi offered a general approach for studying classes of objects which admit a constructive description in a formal language via a Gödel numbering for formulas of the language. Within the approach of Goncharov - Sorbi, has been possible to approach from unified position the notion of computability of families of such constructive objects as computably enumerable sets, constructive models, families of computable morphisms, and so on. This approach also allows us to introduce the notion of a computable family of sets for the Ershov and Kleene - Mostowski (arithmetic) hierarchies, as well as the concept of the Rogers semilattice for such families. According to their approach, a numbering is computable if there exists a computable function which, for every object and each index of this object in the numbering, produces some Gödel index of its constructive description.

Now, we will give precise definition of the main notions used in the thesis. We follow the monograph [3] of Yu.L. Ershov in Russian and the survey papers [4], [65] for terminology and notations that are commonly used in the theory of numberings. The notion of a computable numbering for a family $\mathcal{A}$ of sets in the class Σ^i_n, with $i \in \{-1, 0\}$, may be deduced from the Goncharov–Sorbi approach [2] as follows.

Definition 1.9 A numbering α of a family $\mathcal{A} \subseteq \Sigma^i_n$ is Σ^i_n*-computable* if $\{\langle x, m\rangle : x \in \alpha(m)\} \in \Sigma^i_n$, i.e. the sequence $\alpha(0), \alpha(1), \ldots$ of the members of $\mathcal{A}$ is uniformly Σ^i_n.

We will denote the set of all Σ^i_n-computable numberings of a family $\mathcal{A} \subseteq \Sigma^i_n$ by $\mathrm{Com}^i_n(\mathcal{A})$. For the families of sets from infinite levels of the Ershov hierarchy we use following definition:

Definition 1.10 Let a be a notation of a nonzero computable ordinal. A numbering $\alpha : \omega \to \mathcal{A}$ of a family of Σ^{-1}_a sets is Σ^{-1}_a–computable, if

$$\{\langle n, x\rangle : x \in \alpha(n)\} \in \Sigma^{-1}_a.$$

Hence, α is a Σ^{-1}_a–computable numbering of a family $\mathcal{A}$ if there exists a Σ^{-1}_a–approximation of the universal set $\{\langle n, x\rangle : x \in \alpha(n)\}$ of α, i.e. there are computable functions $f(n, x, s)$ and $h(n, x, s)$ such that $\alpha(n)(x) = \lim_s f(n, x, s)$, and $f(n, x, 0) = 0$ for all n, x, and $h(n, x, s)$ is a change function of the set $\{\langle n, x\rangle : x \in \alpha(n)\}$ with respect to f.

The precise meaning of the phase "a uniform Σ^i_n sequence $\alpha(0), \alpha(1), \dots$ of the members of $\mathcal{A}$" can be explained as follows. Let $A(n, x, t)$ denote a function satisfying the following conditions:

1. ran$(A) \subseteq \{0, 1\}$;
2. $A(e, x, 0) = 0$, for all e and x.

We can treat this function as a uniform procedure for computing the sets $\alpha(e)$. Given e and x, $A(e, x, 0) = 0$ means that initially the number x is not enumerated into $\alpha(e)$. The number x stays outside of $\alpha(e)$ until the function $\lambda t A(e, x, t)$ changes its value from 0 to 1. When this happens, the number x is enumerated into $\alpha(e)$. Now, x remains in $\alpha(e)$ until $\lambda t A(e, x, t)$ changes the value from 1 to 0. In this case, the number x is taken out of the set $\alpha(e)$. And again we wait for the value of $\lambda t A(e, x, t)$ to change from 0 to 1, to put x into $\alpha(e)$ for the second time, and so on.

It is easy to check that, for $\mathcal{A} \subseteq \Sigma^0_1$, a numbering α is Σ^0_1-computable if and only if there exists a computable function A such that, for all e, x, $\lambda t A(e, x, t)$ is a monotonic function, and

$$x \in \alpha(e) \iff \lim_t A(e, x, t) = 1.$$

If $\mathcal{A} \subseteq \Delta^0_2$ then a numbering α is Δ^0_2-computable if and only if there exists a computable function A such that, for all e, x,

$$\lim_t A(e, x, t) \text{ exists, and } x \in \alpha(e) \iff \lim_t A(e, x, t) = 1.$$

If $\mathcal{A} \subseteq \Sigma^0_2$ then a numbering α is Σ^0_2-computable if and only if there exists a computable function A such that, for all e, x,

$$x \in \alpha(e) \iff \lim_t A(e, x, t) \text{ exists and } \lim_t A(e, x, t) = 1.$$

If $\mathcal{A} \subseteq \Sigma^{-1}_{n+1}$ then a numbering α is Σ^{-1}_{n+1}-computable if and only if there exists a computable function A such that, for all e, x,

$$\mid \{t : A(e, x, t + 1) \neq A(e, x, t)\} \mid \leq n + 1.$$

For a Σ_n^i-computable numbering α, we say that such a computable function A represents a Σ_n^i computation of $\alpha(e)$.

Finally, for families $\mathcal{A} \subseteq \Sigma_{n+3}^0$ we can use criteria of computability similar to the ones given above, but with the relevant function A computable relatively to the appropriate iteration of the jump of the empty set.

Note that the computable function $A(e, x, t)$ above is monotonic in t only in the classical case of c.e. sets (i.e. $\mathcal{A} \subseteq \Sigma_1^0$).

It is very important question on which one of the infinite collection of computable numberings of a family is the most natural. The notion of universal numbering seems to be the most adequate for this purpose.

Roughly speaking, a universal numbering for a class of numberings is a numbering in the class which can simulate any numbering in the class. For instance, if we consider the computable numberings of the unary partial computable functions , i.e. the uniformly computable sequences $\psi_0, \psi_1, \ldots$ of the unary partial computable functions, then the standard Gödel numbering $\varphi_0, \varphi_1, \ldots$ is a classical example of a universal numbering, since for any such sequence, $\psi_e = \varphi_{f(e)}$ for some computable function f and all $e \in \omega$. Analogously, the standard Gödel numbering $\{W_e\}_{e\in\omega}$ of the computably enumerable sets is another example of a universal numbering for the class of c.e. sets. Let us give the precise definition, [3] of a family of c.e. sets.

Definition 1.11 A numbering $\alpha : \omega \rightarrow \mathcal{A}$ is called universal (principal) if $\alpha \in \mathrm{Com}(\mathcal{A})$ and $\beta \leqslant \alpha$ for each numbering $\beta \in \mathrm{Com}(\mathcal{A})$.

It is easy to see that α is a computable numbering of a family of c.e. sets if and only if $\alpha \leqslant W$, i.e. $\alpha(e) = W_{f(e)}$ for some computable function f and all $e \in \omega$. Thus, the numbering W is a universal numbering of the class Σ_1^0 of all c.e. sets. For arbitrary numberings α and β with $\alpha = \beta \circ f$, we can think of α as being computable relatively to β (f allows to simulate α from β). Therefore, a universal numbering of a computable family $\mathcal{A}$ of c.e. sets is just one by means of which we can simulate all possible uniform computations of the sets from $\mathcal{A}$. We can generalize definition 1.11 in the following way.

Definition 1.12 Let a be a notation of a nonzero computable ordinal. A numbering $\alpha : \omega \rightarrow \mathcal{A}$ of a family of Σ_a^{-1} sets is called universal if $\alpha \in \mathrm{Com}_a^{-1}(\mathcal{A})$ and $\beta \leqslant \alpha$ for each numbering $\beta \in \mathrm{Com}_a^{-1}(\mathcal{A})$.

And the notion of reducibility of computable numberings defines preorder on the class of all computable numberings. Factorization by the equivalence relation $\equiv$, defined according to the preorder, allows us to construct partially ordered set $\mathcal{R}_a^i(A) = \langle \mathrm{Com}_a^i(A)/_{\equiv}, \leq \rangle$, forming an upper semilattice of computable numberings of the family A. Partial ordered set $\mathcal{R}_a^i(A)$ is upper semilattice [3] and is called the Rogers semilattice of family A.

There are three main research directions to study Rogers semilattices. The first direction is connected with the global characteristics of the Rogers semilattices: the cardinality, being a lattice, elementary theory and so on. The second is devoted to the characterization type of isomorphism of numberings, generators in the Rogers semilattices elements with special study algebraic properties: maximal, minimal, irreducible (atomic) elements, and so on. The third is directed at the study of local algebraic properties of the Rogers semilattices: structure of segments, ideals, and so on. Our research is related to latter two directions.

Definition 1.13 [3] A numbering α of a family $\mathcal{A}$ is called minimal if $\beta \leq \alpha$ implies $\alpha \leq \beta$ for every numbering β.

Definition 1.14 [3] A numbering $\alpha : \omega \to A$ induces a numbering equivalence η_α on ω:

$$\eta_\alpha \leftrightharpoons \{\langle x, y\rangle \mid x, y \in \omega, \alpha(x) = \alpha(y)\}.$$

Definition 1.15 [3] A numbering α is called decidable (positive, negative), if η_α is a computable (computably enumerable, co-computably enumerable) set.

Definition 1.16 [3] A numbering α is called single-valued (Friedberg), if α is one-one ($\eta_\alpha = \{\langle x, x\rangle) \mid x \in \omega\}$).

Note that every Friedberg numbering is decidable, and hence a positive numbering, and each positive numbering is minimal numbering, but not vice versa.

The beginning of the study of minimal numberings was the well-known theorem of Friedberg [56] on the existence of one-to-one numbering of family of all computable enumerable sets. Pour-El and Putnam [66] constructed an example of a family of computably enumerable sets without computable one-to-one (Friedberg) numbering:

$$\{\{2x, 2x+1\} \mid x \in K\} \cup \{\{2x\}, \{2x+1\} \mid x \notin K\},$$

where K is a creative set.

Khutoretsky proved that the family of all c.e. sets has a computable minimal numbering that is not positive [67]. The main objectives of the study of minimal numberings can be roughly divided for two problems:

1. Search for conditions under which a family $\mathcal{A} \in \Sigma^i_n$ has minimal Σ^i_n-computable minimal numberings.

2. Determine the number of minimal elements of the Rogers semilattice of a given family of sets.

In the classical case, some necessary conditions on the existence of a Friedberg numbering of families of computably enumerable sets were found by Lachlan [68]. A member of necessary or sufficient conditions for a family of c.e. sets have been proposed in the works of Ershov [69], Kummer [70-71], Mal'tcev [57] and others. For a family of sets of the arithmetical hierarchy we have the following results give solution for both two problems above.

Theorem 1.9 (Badaev, Goncharov [13]) The Rogers semilatice $\mathcal{R}^0_{n+2}(\mathcal{A})$ of every infinite Σ^0_{n+2}-computable family $\mathcal{A}$ contains infinitely many minimal elements.

This result is contained in the hyperarithmetical hierarchy (N. Baklanova, 72).

Theorem 1.10 (Badaev, Goncharov [13]) A family $\mathcal{A} \subseteq \Sigma^0_{n+2}$ has a positive Σ^0_{n+2} -computable numbering if and only if there exists a Σ^0_{n+2}-computable numbering α of the family $\mathcal{A}$ such that $\{\langle x, y\rangle \mid \alpha(x) = \alpha(y)\} \in \Delta^0_2$.

Theorem 1.11 (Goncharov, Sorbi [73]) If the infinite family $\mathcal{A} \subseteq \Sigma^0_{n+2}$ has a positive Σ^0_{n+2}-computable numbering then $\mathcal{A}$ has a Σ^0_{n+2}-computable Friedberg numbering.

In the Ershov hierarchy, we note the following three results obtained in [15] by Goncharov, Lempp, Solomon

Theorem 1.12 For any $n > 0$ there exists a Σ^{-1}_{n+2}-computable Friedberg numbering of the family of all Σ^{-1}_{n+2}-computably enumerable sets.

Theorem 1.13 There exists an infinite Σ^{-1}_n-computable family without a Σ^{-1}_n-computable Friedberg numbering.

Theorem 1.14 There exists an infinite family of computably enumerable sets with a unique computable numbering regarded as a Σ^{-1}_2 -computable numbering of Σ^{-1}_2-sets.

S.S. Ospichev generalized these results above for every finite level in the Ershov hierarchy, [18]:

Theorem 1.15 For any $k > 1$ there exists a Σ^{-1}_{2k}-computable Friedberg numbering of the family of all Σ^{-1}_k-computably enumerable sets and a computable function m-reducing the Friedberg numbering of the family of all Σ^{-1}_{k-1}-computably enumerable sets to the Friedberg numbering of the family of all Σ^{-1}_k-computably enumerable sets. The Friedberg numbering and reducing function are constructed uniformly with respect to k.

Theorem 1.16 There exists a minimal ω-computable numbering of the family of all sets in $\bigcup\limits_{k\in\omega} \Sigma^{-1}_k$.

Theorem 1.17 For any n there exists an Σ^{-1}_n-computable Friedberg numberings of the family of all Σ^{-1}_n-computably enumerable sets.

In [22], Talasbaeva showed that, for any finite level n of the Ershov hierarchy, and every infinite computable family containing $\emptyset$ if n is even, or ω for n odd, has infinitely many computable positive undecidable numberings pairwise incomparable with respect to reducibility of numberings (for $n = 1$ it was first proved by Badaev [12]). Later, this result was generalized by Andrea Sorbi and Manat Mustafa [16] for all levels of Σ^{-1}_a of the Ershov hierarchy, where a is a notation for any nonzero computable ordinal. All these results in whole implies the the following theorem.

Theorem 1.18 Let an infinite family $\mathcal{A}$ is a Σ_a^{-1}-computable family and $\emptyset \in \mathcal{A}$, if $e(a) = 0$ ($\omega \in \mathcal{A}$, if $e(a) = 1$). Then there are infinitely many positive and undecidable Σ_a^{-1}-computable numberings, pairwise incomparable with respect to reducibility of numberings.

Using this result, Manat and Sorbi showed in [16] that for every $a \in \mathcal{O}$ there is an infinite Σ_a^{-1}-computable family without Σ_a^{-1}-computable Friedberg numberings, but there are infinitely many positive Σ_a^{-1}-computable numberings of a given family.

We proceed to study the local algebraic properties of the Rogers semilattices, more precisely, the questions on the existence ideals, with and without minimal elements. Every ideal of the Rogers semilattice $\mathcal{R}(\mathcal{A})$ contains a minimal element if $\mathcal{A}$ is a family of computable functions or $\mathcal{A}$ is a finite family [3]. If $\mathcal{C}$ is the family of all computably enumerable sets, then $\mathcal{R}(\mathcal{C})$ contains an ideal with minimal elements, and the ideal without minimal elements [69], [67]. Also, there is a family $\mathcal{A}$ of c.e. sets such that $\mathcal{R}(\mathcal{A})$ does not contain the minimal elements [8]. For arithmetic numberings similar research has been initiated in [13].

In [13], it was shown that, for $n \geq 2$, there exists infinite families, the Rogers semilattice of which contain ideals without minimal elements.

For the Ershov hierarchy Badaev and Talasbaeva in [9] presented some sufficient conditions for the Rogers semilattice $\mathcal{R}_2^{-1}(\mathcal{A})$ contain to a family $\mathcal{A} \subseteq \Sigma_2^{-1}$ the principal ideal that is isomorphic to the semilattice of computably enumerable m-degrees. In [20], Ospichev showed, that for every infinite Σ_a^{-1}-computable family $\mathcal{A}$, the Rogers semilattice $\mathcal{R}_c^{-1}(\mathcal{A})$ contains infinitely many disjoint principal ideals of the Rogers semilattice $\mathcal{R}_c^{-1}(\mathcal{A})$, isomorphic to the upper semilattice L_m^0, where $c = a +_o a$, if $e(a) = 0$ and $c = 2^{a+_o a}$, if $e(a) = 0$.

For today the question on the cardinality and the type of algebraic structure of the Rogers semilattices of families of sets in the Ershov hierarchy attracted the attention of researchers. This is due to the fact that the algebraic properties of the Rogers semilattices of families of sets in the Ershov hierarchy is very different from the corresponding properties of the Rogers semilattices of families of computably enumerable sets and families of sets of the arithmetical hierarchy. Consider one of the most important results in classical computability theory.

Theorem 1.19 (A.B. Khutoretskii [74]) Let $\mathcal{A}$ be a family of computable enumerable sets.

1. If $\nu \nleq \mu$ are computable numberings of $\mathcal{A}$ then there is a computable numbering π of $\mathcal{A}$ with $\pi \nleq \nu$ and $\mu \nleq \pi \oplus \nu$.
2. If the Rogers semilattice $\mathcal{R}_1^0(\mathcal{A})$ of $\mathcal{A}$ contains more than one element, then it is infinite.

This theorem is also true in the arithmetical hierarchy. Its statement follows from the proof of theorem S.S. Goncharov and A. Sorbi, in [73], that Rogers semilattice any non-trivial Σ^0_{n+2}-computable family of arithmetical sets is infinite and is not a lattice. However, in [10] S.A. Badaev and S. Lempp showed:

Theorem 1.20 There is a family $\mathcal{F}$ of d.c.e. sets, and there are computable numberings μ and ν of the family $\mathcal{F}$ such that $\mu \nleq \nu$ and such that for any computable numbering π of $\mathcal{F}$, either $\mu \leq \pi$ or $\pi \leq \nu$. In addition, we can ensure the following:

- $\mathcal{F}$ is a family of c.e. sets and ν is a computable numbering of $\mathcal{F}$ as a family of c.e. sets;
- both μ and ν can be made Friedberg and thus minimal numberings; and so
- any computable numbering π of $\mathcal{F}$ satisfies $\pi \equiv \nu$ or $\mu \leq \pi$.

In other words, the first part of Khutoretskii's Theorem, does not hold for the second level of the Ershov hierarchy. It remains an open question whether the second part of the Khutoretskii's theorem holds.

The following theorem (which follows along the lines of a similar theorem proved by Badaev and Talasbaeva in [9] for all finite levels of the Ershov hierarchy) shows that there is no problem when we consider families without any structural restrictions: it is easy to construct a family consisting of any given number of elements whose Rogers semilattice consists of one element.

Theorem 1.21 (Badaev, Manat, Sorbi [11]) For every nonzero $n \in \omega \cup \{\omega\}$, and for every ordinal notation a of a nonzero ordinal, there exists a Σ^{-1}_a-computable family $\mathcal{A}$ of exactly n sets, such that $|\mathcal{R}^{-1}_a(A)| = 1$.

We proceed to the study of universal numberings, that include in the Rogers semilattices the greatest element.

For a given computable family $\mathcal{A}$ of c.e. sets, two main ways of constructing universal numberings are known. The first way is based on the idea of considering uniform computations of all computable numberings, or at least of witnesses from each equivalence class of numberings, lying in $\mathrm{Com}(\mathcal{A})$. Essentially, this way is epitomized in Rice's description of the classes of c.e. sets whose index sets in W are c.e.

The second way originated from the notion of a standard class, introduced by A.Lachlan in [75]. Generalizations of the notion of standard class by A.I.Mal'tsev [76] and Yu.L. Ershov [3] provided very fruitful tools for constructing universal numberings. We will discuss them later. One of the finest results on universal numberings for the classical case are the following two theorems of A. Lachlan [75].

Theorem 1.22 Every finite family of c.e. sets with a least set under inclusion has a universal numbering.

Theorem 1.23 If a family of c.e. sets has a universal numberings then it is closed under union of increasing computable chains of sets.

Later, Y.L. Ershov extended Theorem 1.22 as follows

Theorem 1.24 Every finite family of c.e. sets has a principal numbering.

For the case of the arithmetical hierarchy, we recall the following known result.

Theorem 1.25 (Badaev, Goncharov, Sorbi, [6]) Let $\mathcal{A}$ be any finite family of Σ^0_{n+2} sets. Then $\mathcal{A}$ has a universal numbering in $\mathrm{Com}^0_{n+2}(\mathcal{A})$ if and only if $\mathcal{A}$ contains the least set under inclusion.

Note the following results for the cases of the Ershov hierarchy:

Theorem 1.26 (Abeshev, Badaev, [24]) For every n, the class Σ^{-1}_{n+2} of the Ershov hierarchy has a universal numbering in $\mathrm{Com}^{-1}_{n+2}(\Sigma^{-1}_{n+2})$.

The proof is straightforward since it is easy to construct uniformly all Σ^{-1}_{n+2}-computable numberings for all Σ^{-1}_{n+2}-computable families. We will denote this universal numbering by $W^{(-1,n+2)}$.

2 UNIVERSAL NUMBERINGS

2.1 Universal numberings for finite families in the Ershov hierarchy

We try to find universal numberings for finite families in the Ershov hierarchy.

The next theorem originated as an attempt to adapt the idea of a wn-subset to the class Σ^{-1}_{n+2} equipped with the numbering $W^{(-1,n+2)}$. We will use the notion of a wn-subset in a form which is slightly different from the original one of Ershov [3].

Definition 2.1 A family $\mathcal{A} \subseteq \Sigma^{-1}_k$ is called a wn-subset of Σ^{-1}_k if there exist a c.e. set I and a sequence $\{V_e\}_{e\in\omega}$ such that

1. I contains the index set of the family $\mathcal{A}$ with respect to the numbering $W^{(-1,k)}$
2. V is a Σ^{-1}_k-computable numbering;
3. for every $e \in I$, $V_e \in \mathcal{A}$, and
4. for every $e \in I$, if $W^{(-1,k)}_e \in \mathcal{A}$ then $V_e = W^{(-1,k)}_e$.

Lemma 2.1 If a family $\mathcal{A} \subseteq \Sigma^{-1}_k$ is a wn-subset of Σ^{-1}_k then $\mathcal{A}$ has a universal numbering in $\mathrm{Com}^{-1}_k(\mathcal{A})$.

Proof. The proof is a straightforward modification of the original proof of Ershov [3].

Using Lemma 2.1, Badaev and the author have obtained the following

Theorem 2.1 (Abeshev, Badaev, [24]) Let $k > 1$ and $m > 0$ be any numbers. If $F_0, F_1, \ldots, F_m$ is a sequence of finite sets and $B \in \Sigma^{-1}_k$ is a set such that no F_i in the sequence intersects B, then the family $\mathcal{A} = \{B \cup F_i : i \leq m\}$ is a wn-subset of Σ^{-1}_k.

Proof. We build by a construction in stages a computable function $V(e, x, s)$ which will represent a numbering V. The set I will be defined at the end of the construction.

Let $B(x, s)$ be a computable function which represents a Σ^{-1}_k computation of B. We can assume that $B(x, s) = 0$ for all pairs (x, s) with $x \in F_0 \cup F_1 \cup \cdots \cup F_m$. We let $V(e, x, s) = B(x, s)$ for all triples (e, x, s) with $x \notin F_0 \cup F_1 \cup \cdots \cup F_m$. So, we can fix e and describe, uniformly in e, how to construct the values needed to define $V(e, x, s)$ for all pairs (x, s) with $x \in F_0 \cup F_1 \cup \cdots \cup F_m$.

We let $P(e, x, s)$ be a computable function which represents the numbering $W^{(-1,k)}$, and denote $F_0 \cup F_1 \cup \cdots \cup F_m$ by F.

Construction.

For $s = 0$ we let $V(e, x, 0) = 0$ for all $x \in F$. For the definition of $V(e, x, s+1)$ we distinguish the following cases.

Case 1: There exists $i \leq m$ such that

$$P(e, x, s) = 1 \iff x \in F_i$$

for all $x \in F$.

Then let

$$V(e,x,s+1)=\begin{cases}1, & \text{if } x\in F_i;\\ 0, & \text{if } x\in F\setminus F_i.\end{cases}$$

Case 2: Otherwise, let $V(e,x,s+1)=V(e,x,s)$ for all $x\in F$.

Now, we define the set I. If the sequence $F_0,F_1,\dots,F_m$ contains the empty set then let $I=\omega$. Otherwise, let

$$I=\{e:\exists x\exists s(x\in F\wedge V(e,x,s)=1)\}.$$

It remains only to check that the requirements of definition 2.1 are satisfied by this sequence $\{V_e\}_{e\in\omega}$ and the set I.

Remark 2.2 Theorem 2.1 can't be extended to infinite number of the sets F_i even for a strong array $F_0,F_1,F_2,\dots$.

For instance, if $B=\emptyset$, and $D_0,D_1,D_2,\dots$ is the canonical numbering of all finite sets then the family $\mathcal{A}=\{D_0,D_1,D_2,\dots\}$ has no universal numbering in $\mathrm{Com}^{-1}_{n+1}(\mathcal{A})$ for every n. [18]

The main question we address here is: "For which finite families of d.c.e. sets is there a universal numbering?" Now we can formulate the two main results of this section.

The first theorem covers some cases of finite families of d.c.e. sets for which there is a universal numbering for finite family of Ershov's hierarchy. The conditions below may appear rather complicated but encompass all the obstacles to building a universal numbering of which we are aware; we do not know if this theorem is sharp.

Theorem 2.3 If there are c.e sets A_0,A_1,B_0,B_1 and $A=A_0-A_1$ and $B=B_0-B_1$ and $A\not\subseteq B$ and $B\not\subseteq A$ such that

$$\forall x\,(x\in A_0\Rightarrow x\notin A_1 \text{ or } x\notin B),$$
$$\forall x\,(x\in B_0\Rightarrow x\notin B_1 \text{ or } x\notin A),$$

and partial computable functions ϕ and ψ such that

$$\forall s\forall x\in A_s(x\notin B \text{ or } (\phi_s(x,s)\downarrow>s\,\&\,x\in B_{\phi_s(x,s)})),$$
$$\forall s\forall x\in B_s(x\notin A \text{ or } (\psi_s(x,s)\downarrow>s\,\&\,x\in A_{\psi_s(x,s)})),$$

then there is a universal numbering π for $\mathcal{F}=\{A,B\}$.

Now we will proof Theorem 2.3 by an infinite-injury priority argument using a tree of strategies ([77]).

Proof of Theorem 2.3:

We need to build a universal numbering π of the family $\mathcal{F}=\{A,B\}$. We fix a number $a\in A-B$ and fix another number $b\in B-A$. We now need to meet, for all computable numberings α of all families of d.c.e. sets, and for all computable functions f, the following

Requirements:

$$\mathcal{R}_\alpha:\mathcal{F}_\alpha=\{A,B\}\Rightarrow\exists f\,\forall e\,(\alpha(e)=\pi\circ f(e))$$

If all requirements $\mathcal{R}_\alpha$ hold then π is a universal numbering.

Background Action:

1. We set $\pi(0) = A, \pi(1) = B$ and $\pi(i) = A$ for $i > 1$ at first initially for $i \leq s$.

2. For all $e \leq s$ if a is first enumerated into $\alpha(e)$ (or, b into $\alpha(e)$) then define $f(e)$ large and copy the set A (or B, respectively) into $\pi f(e)$.

Procedure "Switch":

For all $e \leq s$, if a is removed from $\alpha(e)$ and b is enumerated into $\alpha(e)$ for all $e \leq s$ then check all elements $x \in A_s$ using the partial computable function ϕ such that $\forall s \forall x \in A_s\ (x \notin B$ or $(\phi_s(x,s)\downarrow \& \, x \in B_{\phi(x,s)}))$. If for some $x \in A_s$, $\phi_s(x,s)\uparrow$ then this $x \notin B$. If for some $x \in A_s$, $\phi_s(x,s)\downarrow$ and $x \in B_{\phi_s(x,s)}$ then preserve this x in $\pi f(e)$ until step $\phi_s(x,s)$. For all $x \notin A_s$, put x into $\pi f(e)$ if $x \in B_s$. Note that x cannot have entered and left $\pi f(e)$ before by condition (1). (Proceed symmetrically, if b is removed and a is enumerated).

Strategy for $\mathcal{R}_\alpha$:

For each α we build the computable function f as follows:

1. Wait for $a \in \alpha(e)$ (or $b \in \alpha(e)$).

2. Define $f(e)$ large.

3. Copy A into $\pi f(e)$ (or B).

4. Wait for b (or a, respectively) to be enumerated into $\alpha(e)$ and a (or b, respectively) to be extracted from $\alpha(e)$.

5. Run procedure "Switch", copy B (or A) into $\pi f(e)$ and stop.

Construction:

Every strategy acts independently.

We also ensure $\mathrm{Ran}(f_\alpha) \cap \mathrm{Ran}(f_\beta) = \emptyset$ for all $\alpha \neq \beta$.

Verification:

Suppose $\mathcal{F}_\alpha = \{A, B\}$, then $\forall e\, (a \in \alpha(e) \Leftrightarrow b \notin \pi f(e))$. Fix e.

Case 1.

If $a \in \alpha(e)$ then:

Case 1a.

There is $\exists s\, \exists t > s(b \in \alpha_s(e) - \alpha_t(e))$, say, that at stage s, b is first enumerated into $\alpha(e)$ and at stage $t > s$, b is extracted from $\alpha(e)$. Then $\alpha(e) = A = \pi f(e)$, since by (4) and there are three possible cases for each $x \in B$ at stage t, when we switch from copying B to copying A:

Case 1a.1.

At stage t, if $(x \in A \Leftrightarrow x \in \pi f(e))$ then $\alpha(e)(x) = A(x) = \pi f(e)(x)$.

Case 1a.2.

At stage t, if $(x \notin A \,\&\, x \in \pi f(e))$ then check using the partial computable function ψ such that $\forall t \forall x \in B_t (x \notin A$ or $(\psi_t(x,t)\downarrow \& \, x \in A_{\psi_t(x,t)})$. If $\psi_t(x,t)\uparrow$ then this $x \notin A$. If $\psi_t(x,t)\downarrow$ and $x \in B_{\psi_t(x,t)}$ then we preserved this x in $\pi f(e)$ such that x will be enumerated into $A_{\psi_t(x,t)}$ (and possibly later extracted if x later leaves A).

Case 1a.3.

At stage t, if $(x \in A \,\&\, x \notin \pi f(e))$ then there are two possible cases:
1. In the limit x will be extracted from A then $\alpha(e)(x) = A(x) = \pi f(e)(x)$.
2. $x \in A$, then x cannot have entered and left $\pi f(e)$ before stage t by condition (2).

Case 1b.

a is first enumerated into $\alpha(e)$ and $b \notin \alpha(e)$ at stage s. Then $\alpha(e) = A = \pi f(e)$ directly.

Case 2.

If $b \in \alpha(e)$ then apply a symmetric argument.

Finally, if $i \notin \mathrm{Ran}(f_\alpha)$ for any α, then $\pi(i) = A$. This proves Theorem 2.3.

2.2 Families without universal numberings

And the second theorem is the answer for the question: "Do there exist finite families in Ershov's hierarchy without universal numberings?"

Theorem 2.4 There are nonempty, disjoint, d.c.e. sets A, B such that the finite family $\mathcal{F} = \{A, B\}$ has no universal numbering.

The rest of this section is devoted to the proof of our Theorem 2.4, which is an infinite-injury priority argument using a tree of strategies (see, [77]).

Proof of Theorem 2.4:

We need to build a family $\mathcal{F}$ of d.c.e. sets. For an arbitrary numbering π of a family of d.c.e. sets, we denote by $\mathcal{F}_\pi$ the family of d.c.e. sets enumerated by π. We put a number a into A but not into B and put another number b into B but not into A. We now need to meet, for all partial computable numberings π of a family of d.c.e. sets, and for all computable functions f, the following

Requirements:

$$\mathcal{R}_\pi : \mathcal{F}_\pi = \{A, B\} \Rightarrow \exists \alpha_\pi\, (\mathcal{F}_{\alpha_\pi} = \{A, B\}), \text{ and}$$
$$\mathcal{R}_{\pi,f} : \mathcal{F}_\pi = \{A, B\} \Rightarrow \exists e\, (\alpha(e) \neq \pi \circ f(e))$$

If π is a numbering of $\mathcal{F}$ and requirements $\mathcal{R}_\pi$ and $\mathcal{R}_{\pi,f}$ hold for all total f, then there is no universal numbering π. Here the α_π computable numberings built by us.

Background Action:

1. We set $\alpha_\pi(0) = A, \alpha_\pi(1) = B$ and $\alpha_\pi(i) = A$ for $i > 1$ at first initially for $i \leq s$.

2. Ensure that a is enumerated into $\alpha_\pi(e)$ (or, without loss of generality, b into $\alpha_\pi(e)$) for all $e \leq s$.

3. If x is enumerated into (or removed from) A then enumerate x into (or remove x from) $\alpha_\pi(e)$ for all $e \leq s$ with $a \in \alpha_\pi(e)$, except when removed for the sake of (4) in the $\mathcal{R}_{\pi,f}$ strategy until itm:stream2 of the $\mathcal{R}_\pi$ strategy is complete.

4. If x is enumerated into (or removed from) B then x is enumerated into (or removed from) $\alpha_\pi(e)$ for all $e \leq s$ with $b \in \alpha_\pi(e)$.

Strategy for $\mathcal{R}_\pi$:

1. Wait for

(a) an $\mathcal{R}_{\pi,f}$ strategy below to have enumerated x into A and $\alpha_\pi(e)$,

(b) an $\mathcal{R}_{\pi,f}$ strategy below to have extracted x from A, or

(c) an $\mathcal{R}_{\pi,f}$ strategy below to have enumerated x into B.

2. If (a), wait for x to be enumerated into $\pi f(e)$ (or a to be extracted from $\pi f(e)$). Go to (1).

3. If (b), wait for x to be extracted from $\pi f(e)$ (or a to be extracted from $\pi f(e)$). Put x into B, extract a from $\alpha_\pi(e)$ and enumerate b into $\alpha_\pi(e)$. Go to (1).

4. If (c), wait for a to be extracted from $\pi f(e)$. Stop.

Outcomes of the $\mathcal{R}_\pi$ strategy:

1 (finite) Wait at step (2) or (3) forever or stop at step (4): Then π is not a numbering of the family $\mathcal{F}$.

0 (∞) Wait at step (1) forever or go from (3) to (1) infinitely often: Then $\mathcal{F}_{\alpha_\pi} = \mathcal{F}$, since we ensure $\alpha_\pi(e) = A$ or B for all e.

An $\mathcal{R}_\pi$ strategy is a strategy with many $\mathcal{R}_{\pi,f}$ substrategies below the (∞) outcome. On the true path (*TP*) of the tree T we need to ensure all $\mathcal{R}_{\pi,f}$.

Strategy for $\mathcal{R}_{\pi,f}$:

1. Pick a fresh number $e > s$, enumerate a into $\alpha_\pi(e)$.

2. Wait until $f(e)\downarrow$ and a is in $\pi f(e)$.

3. Pick x fresh, enumerate x into A and $\alpha_\pi(e)$. End the stage. (Go to (1a) of the $\mathcal{R}_\pi$ strategy.)

4. Extract x from A. Do not extract x from $\alpha_\pi(e)$ for now. (Go to (1b) of the $\mathcal{R}_\pi$ strategy.) End the stage.

5. Stop.

Outcomes of the $\mathcal{R}_{\pi,f}$ strategy:

1 (wait) Wait at step (2) forever: Then $f(e)$ is partial or $a \in \alpha_\pi(e)$ but $a \notin \pi f(e)$.

0 (stop) Reach (5): If $\mathcal{F}_\pi = \mathcal{F}$, then $\alpha_\pi(e) \neq \pi \circ f(e)$.

We need to restrict the number of these changes to ensure $\alpha_\pi(e)$ to be d.c.e.; therefore eventually $\alpha_\pi(e)$ will be equal either to A or to B. For requirements $\mathcal{R}_{\pi,f}$ we will succeed easily if π is a numbering of $\mathcal{F}$ and f is total.

Tree of the strategies:

Effectively order the requirements (of order type ω). Inductively define a tree $T \subseteq 2^{<\omega}$ such that for any path $TP \in [T]$

$$\forall \pi \, \exists \sigma \subset TP(\sigma \text{ works for } \mathcal{R}_\pi) \text{ and}$$
$$\forall \pi \, \exists \sigma \, (\sigma \text{ works for } \mathcal{R}_\pi \,\&\, \sigma^\frown\langle\infty\rangle \subseteq \mathrm{TP})$$
$$\Rightarrow \forall f \, \exists \tau \subset TP \, (\tau \text{ works for } \mathcal{R}_{\pi,f})$$

Construction:

At stage 0, the sets A and B are empty, all functions are undefined.

We will inductively show that the following properties of the construction hold at the end of each stage s:

At stage $s > 0$, at substages $t \leq s+1$ let some $\sigma \in T$ act at substage $t = |\sigma|$, initialize $\tau > \sigma_s$, the approximation to the true path *TP* at stages.

There are two cases for s:

Case 1. σ is an $\mathcal{R}_\pi$ strategy:

1. Last time σ waited for x to be enumerated into $\pi f(e)$ without success.

(a) If $x \in \pi f(e)$ now, then the outcome is **(0)**. End the substage.

(b) If $x \notin \pi f(e)$ now, then the outcome is **(1)**. End the substage.

2. Last time σ waited for x to be extracted from $\pi f(e)$ without success.

(a) If $x \notin \pi f(e)$ now, then put x into B, extract a from $\alpha_\pi(e)$ and enumerate b into $\alpha_\pi(e)$. The outcome is **(0)**. End the substage.

(b) If $x \in \pi f(e)$ now, then the outcome is **(1)**. End the substage.

3. Last time σ waited for a to be extracted from $\pi f(e)$.

(a) If $a \notin \pi f(e)$ now, then the outcome is **(1)**. End the stage.

(b) If $a \in \pi f(e)$ now, then the outcome is **(0)**. End the substage.

4. Otherwise: Check if there is a new $\tau \supset \sigma$ waiting for x to be enumerated into, or removed from, $\pi f(e)$ at a previous stage s' when σ acted.

(a) Yes: Then proceed as in (1) or (2) or (3).

(b) No: Do nothing. The outcome is **(1)**. Then go to the next substage.

Case 2. σ is an $\mathcal{R}_{\pi,f}$ strategy:

1. Check if the strategy has stopped.

(a) No: Go to (2).

(b) Yes: The outcome is **(0)**. End the substage.

2. Check if e, x are defined.

(a) No: Define two fresh numbers e and x, enumerate a, x into A and $\alpha_\pi(e)$. End the stage.

(b) Yes: Go to (3).

3. Check if $f(e)$ is defined and $a, x \in \pi f(e)$ and $x \in A$.

(a) No: The outcome is **(1)**. End the substage.

(b) Yes: Go to (4).

4. Extract x from A and end the stage.

Each stage $s > 0$ consists of substages $t \leq s$ (where stage s may end before reaching substage s). All parameters will remain defined the same way as at the previous stage unless explicitly redefined. At substage t of stage s, a strategy $\sigma \in T$ of length t (determined at substage $t-1$ if $t > 0$) will be eligible to act and proceed as described above (unless we end stage s at substage $t-1$). (If σ has already stopped at a previous stage and not been initialized since then, then σ immediately ends the substage, taking outcome finite.)

Verification:

We now verify that the above construction satisfies the requirements for our Theorem in a sequence of lemmas. We prove some technical lemmas on important properties of our construction.

Lemma 2.2 A, B are d.c.e sets.

Proof. By the construction, for all $\sigma \in$ T the set $\alpha_\pi(e)$ behaves like A or B, and each element can be enumerated into and removed from α_π and π only once. So by definition they are d.c.e. sets.

Lemma 2.3 If an $\mathcal{R}_\pi$ strategy along the true path, then

1. if the outcome is finite then $\mathcal{F}_\pi \neq \mathcal{F}$, and
2. if the outcome is (∞) then $\mathcal{F}_{\alpha_\pi} = \mathcal{F}$.

Proof. Indeed, if an $\mathcal{R}_\pi$ strategy works on the true path and

Case 1. The the outcome is (finite), then it this means that the $\mathcal{R}_\pi$ strategy waits at step (2) or (3) or stops at (4), i.e., π is not a numbering of the family $\mathcal{F}$. So $\mathcal{F}_\pi \neq \mathcal{F}$.

Case 2. The outcome is (∞), then this means that the $\mathcal{R}_\pi$ strategy waits at step (1) forever or goes from (3) to (1) infinitely often, i.e., α_π is a numbering of the family $\mathcal{F}$ by the $\mathcal{R}_\pi$ strategy and all the $\mathcal{R}_{\pi,f}$ strategies below. So $\mathcal{F}_{\alpha_\pi} = \mathcal{F}$.

Lemma 2.4 If the $\mathcal{R}_\pi$ strategy on the true path has infinite outcome, then every $\mathcal{R}_{\pi,f}$ strategy on TP works successfully.

Proof. If σ is an $\mathcal{R}_\pi$ strategy and has infinite outcome this means that $\mathcal{R}_\pi$ waits at step (1) forever or goes from (3) to (1) infinitely often, i.e., if the $\mathcal{R}_{\pi,f}$ strategy waits at step (3) or (4), this means that the $\mathcal{R}_{\pi,f}$ strategy will eventually stop waiting. In other words, for every $\mathcal{R}_{\pi,f}$ strategy when the outcome is (**1**), then $\mathcal{F}_\pi \neq \mathcal{F}$, and when the outcome is (**0**), then $\exists e\, \alpha_\pi(e) \neq \pi \circ f(e)$.

These lemmas establish our Theorem 2.4.

3 MINIMAL NUMBERINGS

Families without minimal computable numberings:Families without minimal computable numberings

Families without minimal computable numberings can be viewed as analogues, in the theory of numberings, of Blum's speedup theorem. Two examples of these families in the class of c.e. sets were constructed by V'yugin [23] and Badaev [8], and are based on different ideas. In the Rogers semilattice of the family built by V'yugin, every element is the least upper bound of two incomparable elements. The families of c.e. sets without minimal computable numberings built by Badaev are based on the following criterion of minimality of a numbering (not necessarily a computable one).

Theorem 3.1 (Badaev [8]) A numbering $\nu : \omega \to S$ is minimal if and only if, for every c.e. set W, if $\nu(W) = S$, then there exists a positive equivalence ε such that

$$\forall x \forall y\, ((x, y) \in \varepsilon \to \nu(x) = \nu(y))\, \&\, \forall x \exists y\, (y \in W\, \&\, (x, y) \in \varepsilon).$$

The minimality criterion allows to construct a computable family of c.e. sets without computable minimal numberings by using simple diagonal considerations: if a computable numbering indexes the family then it is not a minimal numbering of this family.

Theorem 3.2 For every nonzero computable ordinal and any ordinal notation a of it, there exists a Σ^{-1}_a–computable family $\mathcal{A}$ of Σ^{-1}_a sets that has no Σ^{-1}_a–computable minimal numbering.

If $\langle f, h\rangle$ is a Σ^{-1}_a–approximation of a numbering α then by α^s we denote the numbering whose universal set is

$$\{\langle n, x\rangle : f(n, x, s) = 1\}.$$

We denote by π a computable numbering of the family of all possible Σ^{-1}_a–computable numberings $\{\pi_k\}_{k\in\omega}$. There is a pair of computable functions $f_\pi(k, n, x, s), h_\pi(k, n, x, s)$ that uniformly in k gives Σ^{-1}_a–approximations of the numberings in the sequence $\{\pi_k\}_{k\in\omega}$.

By $p_\pi(k, n, x, s)$ we denote a computable function defined as follows. We define $p_\pi(k, n, x, s) = a$ if $h_\pi(k, n, x, s) = a$; otherwise, we let $p_\pi(k, n, x, s) = h_\pi(k, n, x, t)$, where $t < s$ is the greatest number such that $h_\pi(k, n, x, s) <_{\mathcal{O}} h_\pi(k, n, x, t)$.

Evidently, $h_\pi(k, n, x, s) <_{\mathcal{O}} p_\pi(k, n, x, s)$, if $h_\pi(k, n, x, s) \neq a$, and, if $h_\pi(k, n, x, s) \neq a$ and $h_\pi(k, n, x, s+1) \neq h_\pi(k, n, x, s)$ then $p_\pi(k, n, x, s+1) <_{\mathcal{O}} p_\pi(k, n, x, s)$.

By $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots,$ we denote a computable numbering of all c.e. equivalence relations on ω. $\{\varepsilon^s_m\}$ stands for a double-indexed strong array of these equivalences on the initial segments of ω such that for all m, s

$$\varepsilon_m^s \subseteq \varepsilon_m^{s+1}, \varepsilon_m = \bigcup_{s\in\omega} \varepsilon_m^s.$$

Proof of the Theorem 3.2:

We will build a family $\mathcal{A}$ satisfying the statement of Theorem 3.2 by constructing a Σ_a^{-1}–computable numbering α of $\mathcal{A}$, in which each set of $\mathcal{A}$ will have either one or two indices.

Requirements:

We build a numbering α and a sequence of c.e. sets $\Pi_k, k \in \omega$, that meet the following requirements:

– $\mathcal{C}$: α is Σ_a^{-1}–computable;

– $\mathcal{P}_k$: if π_k is a numbering of $\mathcal{A}$ then $\pi_k(\Pi_k) = \mathcal{A}$;

– $\mathcal{N}_{k,m}$: if π_k is a numbering of $\mathcal{A}$ then

$$\exists x \exists y((x, y) \in \varepsilon_m \ \&\ \pi_k(x) \neq \pi_k(y)) \vee \exists x \forall y \in \Pi_k((x, y) \notin \varepsilon_m).$$

Let us describe the strategies to meet these requirements.

Strategy for $\mathcal{C}$:

We define a Σ_a^{-1}–approximation $\langle f, g\rangle$ to a numbering α by a stage by stage construction. Indeed, we could describe the construction without even mentioning α, since α is defined uniquely by the pair of functions f, g. But it is more convenient to give an informal explanation of the ideas of our construction by means of the numbering α.

In the numbering α, each pair of consecutive indices $2x, 2x + 1$ is targeted to meet exactly one $\mathcal{N}_{k,m}$ requirement. For this reason, we will consider the number x as the standard index $\langle k, m\rangle$ of some pair (k, m). The sets $\alpha(2x)$ and $\alpha(2x + 1)$ will have some common static part, but each of them will also have its own dynamically changeable part. At each stage of the construction below, the dynamic part of a set consists of exactly two numbers. At each stage, one element of the dynamic part of both sets $\alpha(2x)$ and $\alpha(2x + 1)$ can be moved into the common static part of these sets, but no number of the static part of a set can be moved into its dynamic part. Besides, two numbers of sets $\alpha(2y)$ and $\alpha(2y + 1)$ with $y \neq x$ can be enumerated simultaneously into both sets $\alpha(2x)$ and $\alpha(2x)$, and later they can be extracted simultaneously again from both $\alpha(2x)$ and $\alpha(2x + 1)$. In the limit, the two sets $\alpha(2x)$ and $\alpha(2x + 1)$ can coincide or be distinct; this depends on whether the process of movement of elements from the dynamic parts into the static part is infinite or finite. At all stages since stage 1, for all $z \notin \{2x, 2x + 1\}$, the static part of the sets $\alpha(2x)$, $\alpha(2x + 1)$ is not included into the static part of $\alpha(z)$, and at stage 1 the static parts of $\alpha(2x)$ and $\alpha(z)$ are nonempty and disjoint. Thus, in the numbering α each set of $\mathcal{A}$ will have either one or two indices: in the latter case, these two indices are consecutive natural numbers of the form $2x, 2x + 1$. Let us describe in precise terms the process of increase of both sets $\alpha(2x)$ and $\alpha(2x + 1)$ due to their dynamic parts.

Fix three arbitrary injective computable functions $d_0 : \omega^2 \mapsto \omega$, $d_1 : \omega^2 \mapsto \omega$, and $\sigma : \omega^3 \mapsto \omega$ with pairwise disjoint ranges. For every x, the values of the function $\lambda e \lambda s\, \sigma(x, e, s)$ form the static part of the sets $\alpha(2x)$ and $\alpha(2x + 1)$, while some, or all the values of the function $\lambda s\, d_j(x, s)$ are enumerated stage by stage into the dynamic part of $\alpha(2x + j), j \leq 1$. At the beginning of the construction, for each x, we assume that

– $\alpha^0(2x) = \alpha^0(2x + 1) = \emptyset$,
– $\alpha^1(2x) = \{\sigma(x, e, i) : e, i \in \omega,\ e \neq \langle x\rangle_0\} \cup \{d_0(x, 0), d_0(x, 1)\}$,
– $\alpha^1(2x + 1) = \{\sigma(x, e, i) : e, i \in \omega, e \neq \langle x\rangle_0\} \cup \{d_1(x, 0), d_1(x, 1)\}$

and we declare the numbers $d_0(x, 0), d_1(x, 0)$ *active*, and declare the numbers $d_0(x, 1)$, $d_1(x, 1)$ *semi-active*.

The beginning of the process of changing the dynamic parts of $\alpha(2x)$ and $\alpha(2x + 1)$ (under certain conditions) is carried out as follows.

– The number $d_0(x, 0)$ is enumerated into $\alpha(2x + 1)$ and thereafter ceases to be active,

– the number $d_1(x, 0)$ is enumerated into $\alpha(2x)$ and thereafter ceases to be active,

– the semi-active numbers $d_0(x, 1), d_1(x, 1)$ are declared to be active,

– the number $d_0(x, 2)$ is enumerated into $\alpha(2x)$ and is declared to be semi-active,

– the number $d_1(x, 2)$ is enumerated into $\alpha(2x + 1)$ and is declared to be semi-active.

Thus, as a result of these actions, the pair of active numbers is moved from the dynamic part of the sets $\alpha(2x)$ and $\alpha(2x + 1)$ into their static part and ceases to be active; the pair of semi-active numbers is activated; the next two unused numbers are declared semi-active, one of them is enumerated into the dynamic part of $\alpha(2x)$, and the other one is enumerated into the dynamic part of $\alpha(2x)$.

The process of changing the dynamic parts of the sets $\alpha(2x)$ and $\alpha(2x + 1)$ is carried as an iterative process. Let us describe an iteration step in the following procedure for a pair of active numbers $d_0(x, i)$, $d_1(x, i)$.

Procedure $\mathcal{D}(x, i)$:

– The number $d_0(x, i)$ is enumerated into $\alpha(2x + 1)$ and thereafter ceases to be active,

– the number $d_1(x, i)$ is enumerated into $\alpha(2x)$ and thereafter ceases to be active,

– the semi-active numbers $d_0(x, i + 1), d_1(x, i + 1)$ are declared to be active,

– the number $d_0(x, i + 2)$ is enumerated into $\alpha(2x)$ and is declared to be semi-active,

– the number $d_1(x, i + 2)$ is enumerated into $\alpha(2x + 1)$ and is declared to be semi-active.

Without loss of generality, we can assume that $\pi_0^s(y) = \emptyset$ for all s, y. According to the construction, for every x, i, the number $\sigma(x, 0, i)$ is not extracted from $\alpha(2x), \alpha(2x+1)$ and is never enumerated into $\alpha(2y), \alpha(2y+1)$ for all $y \neq x$. Therefore,

$$\{\alpha(2x), \alpha(2x+1)\} \cap \{\alpha(2y), \alpha(2y+1)\} = \emptyset$$

for any distinct numbers x, y.

Strategy for $\mathcal{P}_k$ in isolation:

For every k, we split the set Π_k of π_k-indices into three parts: L_k, R_k, O_k. If π_k is a numbering of $\mathcal{A}$ then we need

– for L_k to contain exactly one π_k-index for every set $\alpha(2x)$, with $\langle x\rangle_0 = k$;

– for R_k to contain exactly one π_k-index for every set $\alpha(2x+1)$, with $\langle x\rangle_0 = k$ if $\alpha(2x) \neq \alpha(2x+1)$;

– for O_k to consist of all π_k-indices of the remaining sets of $\mathcal{A}$, i.e. all the sets $\alpha(2z)$, $\alpha(2z+1)$, with $\langle z\rangle_0 \neq k$.

Note that if so then $\Pi_k = L_k \cup R_k \cup O_k$, and Π_k contains exactly one π_k-index for each set of the subfamily

$$\{\alpha(2x), \alpha(2x+1) : \langle x\rangle_0 = k\}.$$

The important point is that the equality $\alpha(2x) = \alpha(2x+1)$ can occur only in the limit, and if $\alpha(2x) \neq \alpha(2x+1)$ then this inequality is easily recognized at a finite stage.

It is clear that achievement of the above needs ensures the equality $\alpha(\Pi_k) = \mathcal{A}$. And in order for the set Π_k to be c.e. we have to carry out some effective way to meet these needs. The sets L_k, R_k, O_k will consist of some values of the partial computable functions $l(x)$, $r(x)$, $o(x, e, i)$, which we define as follows. Let $x = \langle k, m\rangle_0$. In the approximation π_k^s, choose two distinct π_k-indices y_0 and y_1 such that $d_0(x, 0) \in \pi_k^s(y_0)$, $d_1(x, 0) \in \pi_k^s(y_1)$ and define $l(x) = y_0, r(x) = y_1$. For every pair of numbers (e, i) with $e \neq k$, define $o(x, e, i) = i$ if $\sigma(x, e, i) \in \pi_e(i)$. Note that

$$o(x, e, i)\downarrow \Rightarrow \sigma(x, e, i) \in \pi_e(i) \,\&\, o(x, e, i) = i.$$

We define

$$L_k = \{l(x) : \langle x\rangle_0 = k\},$$
$$R_k = \{r(x) : \langle x\rangle_0 = k \,\&\, \alpha(2x) \neq \alpha(2x+1)\},$$
$$O_k = \{o(x, k, i) : (x, k, i) \in \mathrm{dom}(o) \,\&\, \langle x\rangle_0 \neq k\}.$$

When the values $l(x)$, $r(x)$, $o(x, e, i)$ are defined, we should take care that these values be correct. For a numbering π_k of the family $\mathcal{A}$, we will show below how to achieve that $\pi_k(o(x, k, i)) \in \{\alpha(2x), \alpha(2x+1)\}$, if $\langle x\rangle_0 \neq k$, and the equalities $\alpha(2x) = \pi_k(l(x))$ and $\alpha(2x+1) = \pi_k(r(x))$, if $\langle x\rangle_0 = k$.

Procedure $\mathcal{O}(x, e, i)$:

It can be performed only after that stage when the value $o(x, e, i)$ had been defined, it can start at those stages s only (not all) when the inactive number $h(2x + 1, y, 1) = 1$ is not in the set $\pi_e(i)$ and proceeds as follows:

1. enumerate $\sigma(x, e, i)$ into all sets $\alpha(z)$, $z \notin \{2x, 2x + 1\}$;

2. wait for $\sigma(x, e, i)$ to appear in $\pi_e(i)$ at some stage $s' > s$;

3. at that stage s', remove the number $\sigma(x, e, i)$ from all the sets $\alpha(z)$, $z \notin \{2x, 2x + 1\}$, and thereby complete the procedure.

If π_k is a numbering of $\mathcal{A}$ and $o(x, k, i)$ is defined then Procedure $\mathcal{O}(x, k, i)$ guarantees that $\pi_k(i)$ is one of the sets $\alpha(2x)$ or $\alpha(2x + 1)$. Really, at stage 0 the number $f(2x + 1, y, 1) = 1$, is enumerated into the sets $\alpha(2x)$ and $\alpha(2x + 1)$ and never leaves them. Until the moment when the value $o(x, k, i)$ is defined, the number $\sigma(x, k, i)$ cannot enter the sets $\alpha(2z)$ and $\alpha(2z + 1)$ for all $z \neq x$. At that stage s when the value $o(x, k, i)$ is defined in the construction, the number $\sigma(x, k, i)$ is in the set $\pi^s_k(i)$. If $\sigma(x, k, i)$ is not extracted from $\pi_k(i)$ at later stages then $\pi_k(i) \in \{\alpha(2x), \alpha(2x + 1)\}$, since $\sigma(x, k, i)$ is not contained in the sets $\alpha(2z)$, $\alpha(2z + 1)$ with $z \neq x$. On the other hand, if the number $\sigma(x, k, i)$ leaves $\pi_k(i)$ after stage s then later, either it will be enumerated in $\pi_k(i)$, or Procedure $\mathcal{O}(x, e, i)$ will be performed. In the latter case, due to item (1) of Procedure $\mathcal{O}(x, e, i)$, all the sets of the family $\mathcal{A}$ will contain the number $\sigma(x, k, i)$, and, because of $\pi_k(i) \in \mathcal{A}$, in item (2) of the procedure we successfully stop waiting, and, hence, the number $\sigma(x, k, i)$ will be removed from all sets $\alpha(2z)$, $\alpha(2z + 1)$ for $z \neq x$ by item (3) of Procedure $\mathcal{O}(x, e, i)$.

It is clear that Procedure $\mathcal{O}(x, e, i)$ cannot be performed at infinitely many stages. This implies that $\pi_k(i) \in \{\alpha(2x), \alpha(2x + 1)\}$.

Note that, indeed, we have shown that if π_k is a numbering of $\mathcal{A}$ then $o(x, k, i)$ is defined and

$$\sigma(x, k, i) \in \pi_k(i) \iff \pi_k(i) \in \{\alpha(2x), \alpha(2x + 1)\}.$$

Procedure $\mathcal{L}(x, i)$:

This procedure is performed for the active number $d_0(x, i)$ and the set $\pi_k(l(x))$ with $k = \langle x \rangle_0$. Recall that at stage s the active number $d_0(x, i)$ is contained in $\alpha^s(2x)$ and is not contained in $\alpha^s(2x + 1)$.

So, if the active number $d_0(x, i)$ is not in $\pi_k(l(x))$ at stage s then

1. enumerate $d_0(x, i)$ into all sets $\alpha(z)$, $z \neq 2x$;

2. wait until $d_0(x, i)$ is enumerated in $\pi_k(l(x))$ at some stage $s' > s$;

3. at stage s', remove the number $d_0(x, i)$ from all sets $\alpha(z)$, $z \neq 2x$.

It is not difficult to show that if the number $d_0(x, i)$ is declared to be active at some stage, and remains active afterwards, then $\alpha(2x) = \pi_k(l(x))$.

Procedure $\mathcal{R}(x,i)$:

Procedure $\mathcal{R}(x,i)$ can be described similar to Procedure $\mathcal{L}(x,i)$, but replacing the number $d_0(x,i)$ and the set $\pi_k(l(x))$ with the number $d_1(x,i)$ and the set $\pi_k(r(x))$, respectively.

A priori, it may appear that we may need to carry out simultaneously two or more of the above described procedures. To avoid such conflicts, we can easily order, in a suitable priority list, the executions of the procedures on the elements of $\alpha(2x)$ and $\alpha(2x+1)$ so that, at every stage, no more than one procedure can be carried out, and item (1) of any procedure begins to start only after item (3) of the previous procedure has been completed. Note that, in the case when π_k is a numbering of $\mathcal{A}$, if a procedure starts then it is eventually completed.

Thus, our strategy to meet Requirement $\mathcal{P}_k$ is the following:

– do not extract inactive numbers from those sets of $\mathcal{A}$ in which these numbers are enumerated for the first time;

– for every $i \le 1$, do not extract the active numbers of the sequence $d_i(x,j), j \in \omega$, from the set $\alpha(2x+i)$;

– do perform the procedure $\mathcal{O}(x,e,i)$, as soon as it needs and does not lead to a conflict;

– do perform the procedure $\mathcal{L}(x,i)$ as soon as it needs and does not lead to a conflict;

– do perform the procedure $\mathcal{R}(x,i)$ as soon as it needs and does not lead to a conflict.

It remains only to describe a way to recognize the moment in constructing the set R_k when we could insure that $\alpha(2x) \ne \alpha(2x+1)$. We are able to do this already now if we note that

$$R_k = \{r(x) : \langle x\rangle_0 = k \,\&\, (l(x), r(x)) \in \varepsilon_{\langle x\rangle_1}\}.$$

We provide an explanation of why this equality holds in the description of the following strategy.

Strategy for $\mathcal{N}_{k,m}$ in isolation:

To meet Requirement $\mathcal{N}_{k,m}$ we use π_k-indices $l(x)$ and $r(x)$ of the sets $\alpha(2x)$ and $\alpha(2x+1)$, where $x = \langle k, m\rangle$. Under the assumption that π_k is a numbering of $\mathcal{A}$, we wait for the values $l(x)$ and $r(x)$ to have been defined in the strategy for $\mathcal{P}_k$, after that begin to enumerate more and more new numbers into the sets $\alpha(2x)$ and $\alpha(2x+1)$ through their dynamic parts, so that they could become equal in the limit (remaining distinct at each finite stage) and simultaneously check whether the pair $(l(x), r(x))$ appears in the enumeration of $\varepsilon(m)$. If and only if the latter occurs, we stop adding numbers to the sets $\alpha(2x)$ and $\alpha(2x+1)$ from their dynamic parts. It remains only to ensure that

$$\alpha(2x) = \pi_k(l(x)), \quad \alpha(2x+1) = \pi_k(r(x)).$$

For every $x = \langle k, m \rangle$, we execute the following Procedure $\mathcal{N}(k, m)$ to meet Requirement $\mathcal{N}_{k,m}$. At each step s, for the active numbers $d_0(x, i)$ and $d_1(x, i)$,

– check whether the pair of numbers $(l(x), r(x))$ belongs to the set $\varepsilon^s(m)$;

– check whether the following two inclusions are valid:

$$\{d_0(x, i), d_0(x, i + 1)\} \subseteq \pi^s_{\langle x \rangle_0}(l(x)),$$
$$\{d_1(x, i), d_1(x, i + 1)\} \subseteq \pi^s_{\langle x \rangle_0}(r(x));$$

– execute Procedure $\mathcal{D}(x, i)$ if and only if the answer to the first question is negative while the answer to the second one is affirmative.

Conflicts between strategies:

The strategies of the same type, obviously, cannot conflict with each other, since their instructions are executed for distinct pairs of sets. By the same reason, there are no conflicts between the strategies $\mathcal{P}_{k'}$ and $\mathcal{N}_{k,m}$ if $k' \neq k$.

Formally, a conflict between the strategies $\mathcal{P}_k$ and $\mathcal{N}_{k,m}$ arises when, for the same pair of sets $\alpha(2x), \alpha(2x + 1)$ with $x = \langle k, m \rangle$, the conditions to start execution of Procedure $\mathcal{N}_{k,m}$ on one hand, and the procedures $\mathcal{L}(x, i)$ or $\mathcal{R}(x, i)$ on the other hand, may occur at the same stage. In such a situation, we give the following priority list of execution to the procedures, from higher to lower: $\mathcal{L}(x, i)$, $\mathcal{R}(x, i)$, $\mathcal{D}(x, i)$.

Construction:

We build by stages computable functions $f(x, y, s)$, $h(x, y, s)$ and partial computable functions $o(x, e, i)$, $l(x)$, and $r(x)$.

Stage $\mathbf{s = 0}$:

Let $f(x, y, 0) = 0$, $h(x, y, 0) = a$ for all x, y. The values $o(x, e, i)$, $l(x)$, and $r(x)$ are undefined at stage 0 for all x, e, i.

Later on, we let by default $f(x, y, s + 1) = f(x, y, s)$, $h(x, y, s + 1) = h(x, y, s)$ for those x, y such that the values of $f(x, y, s + 1)$, $h(x, y, s + 1)$ are not explicitly defined at stage $s + 1$.

Stage $\mathbf{s = 1}$:

For all $y \in \{\sigma(x, e, i) : e, i \in \omega\} \cup \{d_0(x, 0), d_0(x, 1)\}$ and every x, define $f(2x, y, 1) = 1, h(2x, y, 1) = 1$.

For every x and all $y \in \{\sigma(x, e, i) : e, i \in \omega\} \cup \{d_1(x, 0), d_1(x, 1)\}$ let $f(2x + 1, y, 1) = 1$,$h(2x + 1, y, 1) = 1$. Declare the numbers $d_0(x, 0), d_1(x, 0)$ as active, and declare the numbers $d_0(x, 1)$, $d_1(x, 1)$ as semi-active.

Stage $\mathbf{s + 1, s > 0}$:

Let $\langle s \rangle_0 = \langle x, e, i \rangle$, $x = \langle k, m \rangle$. Follow the instructions of the following 10 steps, one by one in the natural order if there is no explicit instruction to change that order or go to a different step. If the conditions of a step do not hold then, by default, proceed to the next step.

1. If $k \neq e$, $o(x,e,i)$ is still undefined, and $f_\pi(e,i,\sigma(x,e,i),s+1) = 1$, then define $o(x,e,i) = i$.

2. If $k \neq e$, $o(x,e,i)$ is defined, $f_\pi(e,i,\sigma(x,e,i),s+1) = 0$, and $f(2x+2,\sigma(x,e,i),s) = 0$, then let

$$f(z,\sigma(x,e,i),s+1) = 1,\ h(z,\sigma(x,e,i),s+1) = h_\pi(e,i,\sigma(x,e,i),s+1)$$

for all $z \notin \{2x, 2x+1\}$.

3. If $k \neq e$, $o(x,e,i)$ is defined, $f_\pi(e,i,\sigma(x,e,i),s+1) = 1$, and $f(2x+2,\sigma(x,e,i),s) = 1$, then let

$$f(z,\sigma(x,e,i),s+1) = 0,\ h(z,\sigma(x,e,i),s+1) = h_\pi(e,i,\sigma(x,e,i),s+1)$$

for all $z \notin \{2x, 2x+1\}$.

4. If $k = e$, $l(x)$ and $r(x)$ are undefined, $f_\pi(k,y_0,d_0(x,0),s+1) = 1$, and $f_\pi(k,y_1,d_1(x,0),s+1) = 1$ for some distinct y_0, y_1, then choose the least such y_0, y_1, and define $l(x) = y_0, r(x) = y_1$.

5. If $k = e$, $l(x)$ and $r(x)$ are defined, the number $d_0(x,i)$ is active or has been active, and

$$f_\pi(k,l(x),d_0(x,i),s+1) = f(2x+2,d_0(x,i),s) = 0,$$

then define

$$f(z,d_0(x,i),s+1) = 1, h(z,d_0(x,i),s+1) = p_\pi(k,l(x),d_0(x,i),s+1)$$

for all $z \neq 2x$ if $d_0(x,i)$ is active, or for all $z \notin \{2x, 2x+1\}$ if $d_0(x,i)$ is not active. Go to step 10.

6. If $k = e$, $l(x)$ and $r(x)$ are defined, the number $d_0(x,i)$ is active or has been active, and

$$f_\pi(k,l(x),d_0(x,i),s+1) = f(2x+2,d_0(x,i),s) = 1$$

then define

$$f(z,d_0(x,i),s+1) = 0, h(z,d_0(x,i),s+1) = p_\pi(k,l(x),d_0(x,i),s+1)$$

for all $z \neq 2x$ if $d_0(x,i)$ is active, or for all $z \notin \{2x, 2x+1\}$ if $d_0(x,i)$ is not active, and go to step 7. Otherwise, go to step 10.

7. If $k = e$, $l(x)$ and $r(x)$ are defined, the number $d_1(x,i)$ is active or has been active, and

$$f_\pi(k,r(x),d_1(x,i),s+1) = f(2x+2,d_1(x,i),s) = 0$$

then define

$$f(z,d_1(x,i),s+1) = 1, h(z,d_1(x,i),s+1) = p_\pi(k,r(x),d_1(x,i),s+1)$$

for all $z \neq 2x+1$ if $d_1(x,i)$ is active, or for all $z \notin \{2x, 2x+1\}$ if $d_1(x,i)$ is not active. Go to step 10.

8. If $k = e$, $l(x)$ and $r(x)$ are defined, the number $d_1(x,i)$ is active or has been active, and

$$f_\pi(k,r(x),d_1(x,i),s+1) = f(2x+2,d_1(x,i),s) = 1$$

then let

$$f(z, d_1(x,i), s+1) = 0, h(z, d_1(x,i), s+1) = p_\pi(k, r(x), d_1(x,i), s+1)$$

for all $z \neq 2x+1$ if $d_1(x,i)$ is active, or for all $z \notin \{2x, 2x+1\}$ if $d_1(x,i)$ is not active, and go to step 9. Otherwise, go to step 10.

9. If $k = e$, $l(x)$ and $r(x)$ are defined, $(l(x), r(x)) \notin \varepsilon_m^{s+1}$, the numbers $d_0(x,i), d_1(x,i)$ are active, and

$$f_\pi(k, l(x), d_0(x,i), s+1) = f_\pi(k, l(x), d_0(x,i+1), s+1) = 1,$$
$$f_\pi(k, r(x), d_1(x,i), s+1) = f_\pi(k, r(x), d_1(x,i+1), s+1) = 1,$$
$$f(2x, d_1(x,i), s) = f(2x+1, d_0(x,i), s) = 0$$

then let

$$f(2x, d_1(x,i), s+1) = 1,\ f(2x+1, d_0(x,i), s+1) = 1,$$
$$h(2x, d_1(x,i), s+1) = 1,\ h(2x+1, d_0(x,i), s+1) = 1,$$

declare the numbers $d_0(x,i), d_1(x,i)$ to be inactive. Declare the semi-active numbers $d_0(x,i+1), d_1(x,i+1)$ to be active. Define

$$f(2x, d_0(x,i+2), s+1) = 1, f(2x+1, d_1(x,i+2), s+1) = 1,$$
$$h(2x, d_0(x,i+2), s+1) = 1, h(2x+1, d_1(x,i+2), s+1) = 1,$$

and declare the numbers $d_0(x,i+2), d_1(x,i+2)$ to be semi-active.

10. Go to the next stage.

The construction is now completely described. It is clear, that steps 2, 3 of the construction correspond to Procedure $\mathcal{O}(x,e,i)$, steps 5, 6 correspond to Procedure $\mathcal{L}(x,i)$, steps 7, 8 correspond to Procedure $\mathcal{R}(x,i)$, and step 9 aims to meet Requirement $\mathcal{N}_{k,m}$.

Obviously, the functions f, h are computable, and the functions l, r, o are partial computable. Let us define the numbering α by the equalities $\alpha(n)(y) = \lim_s f(n,y,s)$ for all n, y, and define $\mathcal{A}$ as the family of sets $\{\alpha(n) : n \in \omega\}$. Denote by $S(x,e,i)$ the set of all stages $s+1$ such that $\langle s \rangle_0 = \langle x, e, i \rangle$.

The theorem follows from the following lemmas.

Lemma 3.1 The pair of functions $\langle f, h \rangle$ is a Σ_a^{-1}–approximation to α and, hence, the numbering α is Σ_a^{-1}–computable.

Proof. It is sufficient to verify that the function h is a change function of the numbering α. By the construction, $h(n,y,0) = a$ for every n, y. We have to verify trueness of the following two properties:

i. $h(n,y,s+1) \leq_{\mathcal{O}} h(n,y,s)$;

ii. $f(n,y,s+1) \neq f(n,y,s) \Rightarrow h(n,y,s+1) \neq h(n,y,s)$,

for every n, y, s.

These properties are evident for every s and all pairs (n,y) of the following forms:

– $n = 2x, y \in \{\sigma(x,e,i) : e, i \in \omega\} \cup \{d_0(x,i) : i \in \omega\}$;

– $n = 2x+1, y \in \{\sigma(x,e,i) : e,i \in \omega\} \cup \{d_1(x,i) : i \in \omega\}$;
– $n \in \omega, y \notin \mathrm{range}(d_0) \cup \mathrm{range}(d_1) \cup \mathrm{range}(\sigma)$.

Let us verify these properties for the other pairs.

For every x, y, n, if $y \in \{\sigma(x,e,i) : e,i \in \omega\}$ and $n \notin \{2x, 2x+1\}$, then changing the values of the function $\lambda s\, h(n,y,s)$ can occur only at stages $s+1 \in S(x,e,i)$ with $x = \langle k,m\rangle$ and $k \neq e$, as a result of executing steps 2, 3 of the construction. At such stages, $h(n,y,s+1) = h_\pi(e,i,y,s)$ and, therefore, properties (i),(ii) for the function $\lambda sh(n,y,s)$ follow from the appropriate properties of the function $\lambda sh_\pi(e,i,y,s)$ because of transitivity of the relation $\leq_\mathcal{O}$.

Now let us consider the case when (n,y) is a pair with $n \neq 2x$, $y = d_0(x,i)$ for arbitrary x, i. If the number $d_0(x,i)$ never becomes active, or $x \notin \mathrm{dom}(l)$, then $f(n,y,s) = 0$, $h(n,y,s) = a$ for all s, and, hence, there is nothing to prove.

Let $x = \langle k,m\rangle$ for some m and suppose that at stage $s_0+1 \in S(x,k,i)$ the active number $d_0(x,i)$ is enumerated for the first time into $\alpha(n)$ for every $n \neq 2x$ due to step 5. Then $f(n,d_0(x,i),s_o) = 0$, $h(n,d_0(x,i),s_o) = a$, and $f(n,d_0(x,i),s_o+1) = 1$, $h(n,d_0(x,i),s_o+1) = p_\pi(k,l(x),d_0(x,i),s_0+1)$. Let us show that $h(n,d_0(x,i),s_o+1) <_\mathcal{O} h(n,d_0(x,i),s_o)$. Indeed, at stage s_0+1 the value $l(x)$ is already defined. Let s_1+1 be the stage at which this value was defined due to the construction. Then $s_1 < s_0$ and at stage s_1+1 step 4 holds and $f_\pi(k,l(x),d_0(x,0),s_1+1) = 1$. Therefore, until stage s_0+1 the function $\lambda sp_\pi(k,l(x),d_0(x,i),s)$ has changed its value at least twice. This implies that

$$p_\pi(k,l(x),d_0(x,i),s_0+1) <_\mathcal{O} p_\pi(k,l(x),d_0(x,i),s_1+1) \leq_\mathcal{O} a.$$

Hence, $h(n,d_0(x,i),s_o+1) <_\mathcal{O} h(n,d_0(x,i),s_o)$.

Thus, properties (i), (ii) are true for the desired pairs if $s \leq s_0$. For all pairs $\langle n, d_0(x,i)\rangle$, if $n \notin \{2x, 2x+1\}$, and $s \leq s_0$, then properties (i), (ii) for the function $\lambda sh(n,d_0(x,i),s)$ follows from the approppriate properties of the function $\lambda s\, p_\pi(k,l(x),d_0(x,i),s)$.

After stage s_0 up to stage s_2+1, when the number $d_0(x,i)$ is enumerated into the set $\alpha(2x+1)$ by Procedure $\mathcal{D}(x,i)$ (if this happens at all), trueness of properties (i),(ii) for the function $\lambda sh(2x+1,d_0(x,i),s)$ follows from the properties of the function $\lambda s\, p_\pi(k,l(x),d_0(x,i),s)$.

Thus, it is necessary to consider properties (i),(ii) for $s = s_2$. According to the instructions of step 9, $f(2x+1,d_0(x,i),s_2) = 0$, $f(2x+1,d_0(x,i),s_2+1) = 1$, $h(2x+1,d_0(x,i),s_2+1) = 1$. Besides,

$$p_\pi(k,l(x),d_0(x,0),s_2) \leq_\mathcal{O} h(2x+1,d_0(x,i),s_2),$$

if $d_0(x,i)$ has been enumerated up to stage s_2+1 at least once into $\alpha(2x+1)$ due to step 5 of the construction, and $h(2x+1,d_0(x,i),s_2) = a$ otherwise. From here, by the inequalities

$$1 \leq_{\mathcal{O}} h_\pi(k, l(x), d_0(x, 0), s_2) <_{\mathcal{O}} p_\pi(k, l(x), d_0(x, 0), s_2),$$

we obtain that $h(2x+1, d_0(x,i), s_2+1) <_{\mathcal{O}} h(2x+1, d_0(x,i), s_2)$. Hence, properties (i), (ii) are true also for $s = s_2$. Finally, we note that $f(2x+1, d_0(x,i), s) = h(2x+1, d_0(x,i), s) = 1$ for all $s \geq s_2 + 1$. Thus, properties (i), (ii) for the pair $\langle 2x+1, d_0(x,i)\rangle$ are true for every s.

In the symmetric case, when $n \neq 2x+1$, and $y = d_1(x,i)$, properties (i), (ii) can be proved in a similar way. Lemma 3.1 is proved.

Lemma 3.2 If π_k is a numbering of the family $\mathcal{A}$, then, for every x, i with $\langle x\rangle_0 \neq k$, the value $o(x,k,i)$ is defined if and only if $\pi_k(i) = \alpha(2x)$ or $\pi_k(i) = \alpha(2x+1)$.

Proof. (Necessity). Let x, i be any pair of numbers such that $\langle x\rangle_0 \neq k$ and the value $o(x,k,i)$ is defined. Let $s_0 + 1 \in S(x,k,i)$ be the stage at which the value $o(x,k,i)$ has been defined. It is possible only due to execution of step 1 at this stage. Then $f_\pi(k, i, \sigma(x,k,i), s_0+1) = 1$, i.e. $\sigma(x,k,i) \in \pi_k^{s_0+1}(i)$. By the construction, the number $\sigma(x,k,i)$ is enumerated due to the instructions of step 1 into the sets $\alpha(2x)$ and $\alpha(2x+1)$ only, and, later on, it can be enumerated into other sets of $\mathcal{A}$ only by step 2. If step 2 is not performed at all stages of $S(x,k,i)$ then, obviously, $\pi_k(i) = \alpha(2x)$ or $\pi_k(i) = \alpha(2x+1)$.

Let $s_1 + 1 \in S(x,k,i)$ be the least stage at which step 2 holds. Then $s_1 > s_0$ and

$$f_\pi(k, i, \sigma(x,k,i), s_1+1) = 0 \text{ and } f(z, \sigma(x,k,i), s_1+1) = 1$$

for every z. This means that at stage $s_1 + 1$ the number $\sigma(x,k,i)$ is contained in all sets of the family $\mathcal{A}$ except the set $\pi_k(i)$. Since π_k is a numbering of the family $\mathcal{A}$, it follows that at some later stage the number $\sigma(x,k,i)$ will be again enumerated in $\pi_k(i)$. Let $s_2 + 1 \in S(x,k,i)$ be the least stage such that $s_2 > s_1$ and $f_\pi(k, i, \sigma(x,k,i), s_1+1) = 1$. Then step 3 holds at stage $s_2 + 1$ and, therefore, $f(z, \sigma(x,k,i), s_1+1) = 0$ for all $z \notin \{2x, 2x+1\}$. Beginning from some stage s_3, the values of the function $\lambda s\, f_\pi(k, i, \sigma(x,k,i), s)$ will stabilize at 1, hence, $f(z, \sigma(x,k,i), s) = 0$ for every $z \notin \{2x, 2x+1\}$, and all $s \geq s_3$. Therefore, $\pi_k(i) = \alpha(2x)$ or $\pi_k(i) = \alpha(2x+1)$.

(Sufficiency). Let $\langle x\rangle_0 \neq k$ and $\pi_k(i) = \alpha(2x)$ or $\pi_k(i) = \alpha(2x+1)$. By the construction, the number $\sigma(x,k,i)$ is enumerated at some stage by step 1 into the sets $\alpha(2x)$ and $\alpha(2x+1)$, and is never extracted from them later. Then there is a stage s_0 such that $f_\pi(k, i, \sigma(x,k,i), s) = 1$ for all $s \geq s_0$. Let $s_1 + 1 \in S(x,k,i)$ be the least stage such that $s_1 \geq s_0$. If the value $o(x,k,i)$ has not yet been defined before stage $s_1 + 1$ then, at this stage, all the conditions of step 1 hold, and $o(x,k,i)$ will be defined equal to i. Lemma 3.2 is proved.

Lemma 3.3 For every x, z, if $x \neq z$ then $\alpha(2x)$ and $\alpha(2x+1)$ are distinct from all the sets $\alpha(z)$ with $z \notin \{2x, 2x+1\}$. Besides, $\alpha(2x) = \alpha(2x+1)$ if and only if Procedure $\mathcal{D}(x, i)$ carried out for all i.

Proof. Let $x = \langle k, m \rangle$. If $k \neq 0$, then at stage 1 the numbers $\sigma(x, 0, i), i \in \omega$, are enumerated into both $\alpha(2x)$ and $\alpha(2x+1)$. Since $f_\pi(0, i, \sigma(x, 0, i), s) = 0$ for all i, s, it follows that $o(x, 0, i)$ cannot be defined at any stage of the construction due to step 1. Therefore, the numbers $\sigma(x, 0, i)$ cannot be enumerated due to step 2 into any set $\alpha(z)$ for $z \notin \{2x, 2x+1\}$.

If $k = 0$ then at stage 1 the number $d_0(x, 0)$ is enumerated into $\alpha(2x)$ while $d_1(x, 0)$ is enumerated into $\alpha(2x+1)$. Since $f_\pi(0, y, d_0(x, 0), s) = 0$, $f_\pi(0, y, d_1(x, 0), s) = 0$ for all y, s, it follows that the values $l(x), r(x)$ cannot be defined at any stage of the construction via step 4. Hence, because of steps 5, 7, the numbers $d_0(x, 0), d_1(x, 0)$ cannot be enumerated into any set $\alpha(z)$ for $z \notin \{2x, 2x+1\}$.

Thus, each of sets $\alpha(2x)$ and $\alpha(2x+1)$ contains a number that is never enumerated into the sets $\alpha(z)$, for every $z \notin \{2x, 2x+1\}$.

To prove the second statement of lemma, note that $\{\sigma(x, e, i) : e, i \in \omega\} \subseteq \alpha(2x) \cap \alpha(2x+1)$ and that, for every $z \notin \{2x, 2x+1\}$, any number of $\alpha(z)$ is contained in $\alpha(2x)$ if and only if it is contained in $\alpha(2x+1)$. Hence, equality or inequality of the sets $\alpha(2x)$ and $\alpha(2x+1)$ is completely determined by the total number of stages at which Procedure $\mathcal{D}(x, i)$ holds. Lemma 3.3 is proved.

Lemma 3.4 If π_k is a numbering of $\mathcal{A}$ then $\lambda m\, l(\langle k, m \rangle)$, $\lambda m\, r(\langle k, m \rangle)$ are computable functions and

$$\alpha(2\langle k, m \rangle) = \pi_k(l(\langle k, m \rangle)), \alpha(2\langle k, m \rangle + 1) = \pi_k(r(\langle k, m \rangle)).$$

Proof. Let π_k be a numbering of the family $\mathcal{A}$. Firstly, we will show by contradiction that the functions $\lambda m\, l(\langle k, m \rangle)$, $\lambda m\, r(\langle k, m \rangle)$ are total. Choose an arbitrary number m and let $x = \langle k, m \rangle$. At stage 1, $d_0(x, 0)$ is enumerated into $\alpha(2x)$ while $d_1(x, 0)$ is enumerated into $\alpha(2x+1)$. Note that these numbers are never extracted respectively from $\alpha(2x)$ and $\alpha(2x+1)$. Besides, note that, at each stage of the construction the value $l(x)$ is defined if and only if the value $r(x)$ is defined.

Assume that the values $l(x)$ and $r(x)$ are never defined in the construction. Then at each stage $s+1 \in S(x, k, i)$ steps 1–9 do not hold and, therefore, $d_0(x, 0)$ is not enumerated into any set $\alpha(z)$ with $z \neq 2x$, and $d_1(x, 0)$ is not enumerated into any set $\alpha(z)$ with $z \neq 2x+1$. Let y_0 and y_1 be any π_k-indices of $\alpha(2x)$ and $\alpha(2x+1)$. It is obvious that $y_0 \neq y_1$. Then $f_\pi(k, y_0, d_0(x, 0), s) = 1$, $f_\pi(k, y_1, d_1(x, 0), s) = 1$ for some s_0 and all $s \geq s_0$. This implies that the conditions of step 4 hold at infinitely many stages of $S(x, k, i)$. A contradiction. Thus, the values $l(x)$ and $r(x)$ will be defined at some stage $s_1 + 1 \in S(x, k, i)$.

Let us fix a number i and show that $d_0(x,i) \notin \alpha(z)$ for all $z \notin \{2x, 2x+1\}$. If $d_0(x,i)$ is never declared semi-active then $d_0(x,i) \notin \alpha(z)$ for all z. If $d_0(x,i)$ is declared semi-active at some stage and remains semi-active at all later stages then $d_0(x,i) \in \alpha(2x)$ and $d_0(x,i) \notin \alpha(z)$ for all $z \neq 2x$.

Now we consider the case when the number $d_0(x,i)$ is declared active at some stage s_0 and remains active at all stages $s \geq s_0$. Then $f(2x, d_0(x,i), s) = 1$ for all $s \geq s_0$. If $f(2x+2, d_0(x,i), s) = 0$ for all $s \geq s_0$ then, evidently, $d_0(x,i) \notin \alpha(z)$ for all $z \notin \{2x, 2x+1\}$.

Let $s_1 + 1$ be the least stage such that $f(2x+2, d_0(x,i), s_1+1) = 1$. Then $s_1 + 1 > s_0$ and at stage $s_1 + 1$ step 5 holds and, hence, $f_\pi(k, l(x), d_0(x,i), s_1+1) = 0$ and $f(z, d_0(x,i), s_1+1) = 1$ for all z. If $f_\pi(k, l(x), d_0(x,i), s+1) = 0$ for all $s > s_1$, then step 6 does not hold at all further stages of $S(x,k,i)$, and $f(z, d_0(x,i), s+1) = 1$ for every z and all $s > s_1$. Then the number $d_0(x,i)$ is contained in all sets of the family $\mathcal{A}$ but is not contained in $\pi_k(l(x))$, and, therefore, π_k is not a numbering of $\mathcal{A}$: contradiction. Hence, there is the least stage $s_2 + 1 > s_1 + 1$ of $S(x,k,i)$ at which step 6 holds. Then $f(z, d_0(x,i), s+1) = 0$ for all $z \neq 2x$. Thus, at stages s such that $s_1 + 1 \leq s \leq s_2 + 1$, Procedure $\mathcal{L}(x,i)$ is performed completely, and the function $\lambda s\, h_\pi(k, l(x), d_0(x,i), s)$ has changed its value at least once. Therefore, Procedure $\mathcal{L}(x,i)$ can be carried out only at finitely many stages. Since π_k is a numbering of $\mathcal{A}$, it follows that every execution of this procedure will ended by carrying out step 6.

If the number $d_0(x,i)$ remains active until the end of the construction then step 9 does not hold at the stages in $S(x,k,i)$, and, therefore, $\alpha(2x)$ is the only set of $\mathcal{A}$ that contains the number $d_0(x,i)$. If at some stage $s_3 + 1 \in S(x,k,i)$ the number $d_0(x,i)$ became inactive then at this stage step 9 holds, i.e., Procedure $\mathcal{D}(x,i)$ is performed. The number $d_0(x,i)$ remains inactive also at the further stages. At stages $s > s_3 + 1$, the number $d_0(x,i)$ can be enumerated into all the sets $\alpha(z)$ with $z \notin \{2x, 2x+1\}$ by Procedure $\mathcal{L}(x,i)$ only. As was shown above, every execution of Procedure $\mathcal{L}(x,i)$ will be ended by carrying out step 6. Therefore, the number $d_0(x,i)$ is not contained in any set $\alpha(z)$ with $z \notin \{2x, 2x+1\}$.

We will show now that $\alpha(2x) = \pi_k(l(x))$. By Lemma 3, $\alpha(2x) \neq \alpha(z)$ for all $z \notin \{2x, 2x+1\}$. If, for some i and a stage s', the number $d_0(x,i)$ is active at all stages $s \geq s'$ then $d_0(x,i) \in \pi_k(l(x)) \cap \alpha(2x)$, and $d_0(x,i)$ is not contained in any set $\alpha(z)$ with $z \neq 2x$. Hence, $\alpha(2x) = \pi_k(l(x))$.

On the other hand, if every number $d_0(x,i)$, $i \in \omega$, is declared active then it means that Procedure $\mathcal{D}(x,i)$ is performed for all i. Then $\{d_0(x,i), d_1(x,i) : i \in \omega\} \subseteq \alpha(2x) \cap \alpha(2x+1)$ and every number $d_0(x,i)$ is not contained in any set $\alpha(z)$ with $z \notin \{2x, 2x+1\}$. Furthermore, for every such z, and each $y \in \alpha(z)$,

$$y \in \alpha(2x) \iff y \in \alpha(2x+1).$$

Hence, $\alpha(2x) = \alpha(2x+1) = \pi_k(l(x))$.

The equality $\alpha(2x+1) = \pi_k(r(x))$ is proved similarly. Lemma 3.4 is proved.

Let

$$L_k = \{l(x) : \langle x\rangle_0 = k\},\ R_k = \{r(x) : \langle x\rangle_0 = k \,\&\, \langle l(x), r(x)\rangle \in \varepsilon_{\langle x\rangle_1}\},$$
$$O_k = \{o(x,k,i) : (x,k,i) \in \mathrm{dom}(o) \,\&\, \langle x\rangle_0 \neq k\},\ \Pi_k = L_k \cup R_k \cup O_k.$$

Lemma 3.5 If π_k is a numbering of the family $\mathcal{A}$ then Π_k is a c.e. set that contains exactly one π_k-index for each set of $\{\alpha(2\langle k,m\rangle), \alpha(2\langle k,m\rangle+1) : m \in \omega\}$ and all π_k-indices of the other sets.

Proof. The set Π_k is c.e., since the functions l, r, o are partial computable, and ε_m is a c.e. relation for every m. The other statements of lemma follow from Lemmas 3.2–3.4. Lemma 3.5 is proved.

We now proceed to complete the proof of the theorem. By Lemma 3.1, the numbering α is Σ^{-1}_a-computable. By using the criterion of Theorem 3.1, we show that the family $\mathcal{A} = \alpha(\omega)$ has no Σ^{-1}_a-computable minimal numbering.

Let ν be an arbitrary Σ^{-1}_a-computable numbering of the family $\mathcal{A}$. Then $\nu = \pi_k$ for some k. Theorem 3.1 can be reformulated in a more convenient form as follows.

Theorem 3.3 A numbering π_k of the family $\mathcal{A}$ is not minimal if and only if there is a c.e. set W such that $\pi_k(W) = \mathcal{A}$ and, for every m,

$$\exists u \exists v((u,v) \in \varepsilon_m \,\&\, \pi_k(u) \neq \pi_k(v)), \qquad \text{(a)}$$

or

$$\exists u \forall w \in W((u,w) \notin \varepsilon_m) \qquad \text{(b)}$$

We take the set Π_k as W. By Lemma 3.5, $\pi_k(\Pi_k) = \mathcal{A}$. Let ε_m be an arbitrary c.e. equivalence. Denote $\langle k,m\rangle$ by x. By Lemma 3.4, the values $l(x)$, $r(x)$ are defined. Consider the following two cases.

Case 1. $\langle r(x), l(x)\rangle \in \varepsilon_m$.

Then $\langle r(x), l(x)\rangle \in \varepsilon^{s+1}_m$ for all stages $s+1 \in S(x,k,i), i \in \omega$, beginning with some stage s_0+1. Hence, at these stages, step 9 of the construction does not hold, and, therefore, Procedure $\mathcal{D}(x,i)$ is performed only for finitely many numbers i. By Lemmas 3.3, 3.4, $\pi_k(r(x)) \neq \pi(l(x))$. Hence, condition (a) of Theorem 3.3 holds with $u = r(x)$ and $v = l(x)$.

Case 2. $\langle r(x), l(x)\rangle \notin \varepsilon_m$.

We will show that in this case Procedure $\mathcal{D}(x,i)$ is performed for every i.

Let $i > 0$ be a number such that, at some stage $s_1+1 \in S(x,k,i-1)$, the numbers $d_0(x,i+1)$, $d_1(x,i+1)$ are declared semi-active. Then Procedure $\mathcal{D}(x,i-1)$ is performed at stage s_1+1, and, in accordance with it,

$$f_\pi(k, l(x), d_0(x,i), s_1+1) = 1,\ f_\pi(k, r(x), d_1(x,i), s_1+1) = 1,$$

and the numbers $d_0(x,i)$, $d_1(x,i)$ are declared active.

Let us show that, at some later stage of $S(x,k,i)$, Procedure $\mathcal{D}(x,i)$ will be performed. As was shown in the proof of Lemma 3.4, if at some stage of $S(x,k,i)$ step 5 (step 7) holds then at some later stage of $S(x,k,i)$ step 6 (step 8) is carried out. Therefore, for some $s_2+1 \in S(x,k,i)$

$$f_\pi(k,l(x),d_0(x,i),s+1)=1,\ f_\pi(k,r(x),d_1(x,i),s+1)=1,$$
$$f(2x,d_1(x,i),s)=0,\ f(2x+1,d_0(x,i),s)=0,$$

for all $s+1 \in S(x,k,i)$ with $s \geq s_2$. By lemma 3.4, $\alpha(2x)=\pi_k(l(x))$, $\alpha(2x+1)=\pi_k(r(x))$, and as long as the numbers $d_0(x,i+1)$, $d_1(x,i+1)$ remain semi-active, they cannot be enumerated into the sets that are distinct from respectively $\alpha(2x)$ and $\alpha(2x+1)$. Therefore, for some $s_3 \geq s_2$

$$f_\pi(k,l(x),d_0(x,i+1),s+1)=1,\ f_\pi(k,r(x),d_1(x,i+1),s+1)=1$$

for all $s+1 \in S(x,k,i)$ after stage s_3, as long as the numbers $d_0(x,i+1)$, $d_1(x,i+1)$ remains semi-active. Hence, at the least stage $s_4+1 \geq s_3$ of $S(x,k,i)$, step 9 is performed. Then at this stage Procedure $\mathcal{D}(x,i)$ is carried out.

Thus, $\langle r(x), l(x)\rangle \notin \varepsilon_m$, and Procedure $\mathcal{D}(x,i)$ is carried out for every i. Then $r(x) \notin \Pi_k$, and by Lemmas 3,4, $\pi_k(r(x))=\pi(l(x))$. If $\langle r(x), w\rangle \notin \varepsilon_m$ for all $w \in \Pi_k$, different from $l(x)$, then for $u=r(x)$ and all $w \in \Pi_k$, condition (b) of Theorem 3.3 holds. On the other hand, if $\langle u, w\rangle \notin \varepsilon_m$ for some $w \in \Pi_k$ different from $l(x)$, then condition (a) of Theorem 3.3 holds for $u=r(x)$ and $v=w$ by Lemma 3.5.

Thus, in both cases, at least one of the conditions (a) or (b) of Theorem 3.3 holds. Therefore, the numbering π_k is not minimal. The proof of Theorem 3.2 is completed.

CONCLUSION

The study of computable numberings of families of sets in the Ershov hierarchy revealed new phenomena compared to computable numberings of families of c.e. sets. For example, in [9] such a family consisting of two sets, one of which is included in the other, was constructed, that its Rogers semilattice consists of just one element. In [10], it was observed that a complete analogue of Khutoretskii's classical theorem does not hold in the Ershov hierarchy. All these results were obtained for d.c.e. sets, i.e. sets lying in the Σ_2^{-1} class of the Ershov hierarchy.

Along with the new phenomenon of computable numberings in the Ershov Hierarchy, it is natural to raise questions about which properties of computable numberings that hold in the classical case, are also valid for computable numberings in the Ershov hierarchy. In this regard, an interesting problem is a question of whether or not there exist families of sets without universal computable numberings. In [30], the family which consists of two disjoint sets and has no universal computable numbering was constructed. And the other interesting problem is whether or not there exist families of sets without minimal computable numberings (see [4], Question 11). The main result of this thesis is the following theorem, in [31], on the existence of such families in each level, whether finite or infinite, of the Ershov hierarchy.

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