When any three solutions are independent

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I want to think about solutions of systems of algebraic differential equations:

$$P(x, \delta x, \delta^2 x, \delta^3 x) = 0$$

over a differential field (k, δ).

- e.g. $k = \mathbb{C}(t)$ and $\delta = \frac{d}{dt}$ and all of the solutions are meromorphic on some domain in $\mathbb{C}$.

The issue we want to consider involves one of the main problems in algebraic differential equations:

- Understanding the algebraic relations between solutions of a differential equation. Main issues for this lecture:
  1. Why should you care about this problem?
  2. What does this problem have to do with model theory?
  3. What are some interesting examples and what can we hope to prove?

Example: 1

- Starting about a decade ago, Nagloo and Pillay wrote a series of influential papers on Painlevé equations.
- Consider a Painlevé equation from the families P_{II} through P_V with generic coefficients:

$$y'' = 2y^3 + ty + \pi$$

  over the differential field $(\mathbb{C}(t), \frac{d}{dt})$.
- Nagloo and Pillay (2017) show that if $f_1, \ldots, f_n$ are solutions to the equation, then $f_1, f_1', f_2, f_2', \ldots, f_n, f_n'$ are algebraically independent over $\mathbb{C}(t)$.
- There are no algebraic relations between solutions of generic Painlevé equations.

The general setup

$$P(y, \delta y, \ldots, \delta^{(n)} y) = 0,$$

where $P \in k[X_0, \ldots, X_n]$ is irreducible. For each $m \geq 1$, consider the following condition:

(C_m) For any m distinct solutions $a_1, \ldots, a_m \notin k^{alg}$, the sequence

$$(\delta^{(i)} a_j : i = 0, \ldots, n-1, j = 1, \ldots, m)$$

is algebraically independent over k.

- (C_2) is implied by (C_3) is implied by (C_4) ...
- The Nagloo-Pillay results say that (C_n) holds for all $n \in \mathbb{N}$ for generic Painlevé equations.
- Can we say anything interesting in general?
- What kinds of interesting applications does this sort of result have?

Motivation from diophantine geometry

- About 15 years ago, Pila-Wilkie proved an influential theorem connecting diophantine and o-minimal geometry:

Theorem

Let $X \subset \mathbb{R}^n$ be definable in an o-minimal expansion of $\mathbb{R}$. Then for all $\epsilon > 0$, there is c such that for all T,

$$N(X^{tr}, T) \leq cT^{\epsilon}.$$

- $X^{alg} =$ union of all connected infinite semi-algebraic subsets of X.
- $X^{tr} = X \setminus X^{alg}$.
- $N(X^{tr}, T)$ is the number of rational points on X^{tr} of height $\leq T$.
- $H(\frac{a}{b}) = \max\{|a|, |b|\}$ and height of a tuple is the max of the heights of the elements of the tuple.

Motivation from number theory: bi-algebraicity problems

- X, Y algebraic varieties[1] over $\mathbb{C}$ and let

$$\phi : X \to Y$$

  be a complex analytic map **which is not algebraic.**
- For **most** algebraic subvarieties $X_0 \subset X$, $\phi(X_0)$ is not algebraic.
- Pairs of algebraic subvarieties (X_0, Y_0) with $X_0 \subset X$ and $Y_0 \subset Y$ such that $\phi(X_0) = Y_0$ are called **bi-algebraic** for ϕ.
- For many applications of Pila-Wilkie, X^{alg} is closely related to identifying bi-algebraic subvarieties of some analytic function.

[1] More generally an open subset of an algebraic variety.

Specific problem 1: Torsion on abelian varieties

- X an Abelian variety of dimension g over $\mathbb{C}$. $\pi : \mathbb{C}^g \to X(\mathbb{C})$.
- By scaling coordinates appropriately, we can make the torsion of X the image of $\mathbb{Q}^{2g}$.
- Take $Y \subset X$ an algebraic subvariety.
- Can we describe the torsion points on Y?
- Pila-Wilkie: After removing the **algebraic part** of $\pi^{-1}(Y)$, there are $\leq cT^{\epsilon}$ many rational points - for any $\epsilon > 0$,

$$N(\pi^{-1}(Y) \setminus (\pi^{-1}(Y))^{alg}, t) = \mathcal{O}(t^{\epsilon}).$$

- $(\pi^{-1}(Y))^{alg} =$ union of all bi-algebraic subvarieties of $(\pi^{-1}(Y))^{alg}$
- bi-algebraic $=$ coset of an abelian subvariety

Specific problem 1: Torsion on abelian varieties

- Pila-Wilkie: After removing the algebraic part of $\pi^{-1}(Y)$, there are only subpolynomially many rational points - for any $\epsilon > 0$,

$$N(\pi^{-1}(Y) \setminus (\pi^{-1}(Y))^{alg}, t) = \mathcal{O}(t^{\epsilon}).$$

- The bi-algebraic subvarieties for $\pi : \mathbb{C}^g \to X(\mathbb{C})$, are cosets of abelian subvarieties.
- When Y is defined over k, we can find $\delta > 0$ and C such that for a torsion point of degree d, y, $[k(y) : k] > Cd^{\delta}$.
- Every Galois conjugate of y lies in Y.
- Manin-Mumford: There are only finitely many torsion points on Y unless Y contains a torsion coset of an abelian subvariety.

- $\bar{j} : \mathbb{H}^n \to (\mathbb{C})^n$ given by:

$$\bar{j}(x_1, \ldots, x_n) \mapsto (j(x_1), \ldots, j(x_n)).$$

- Replace torsion points in the last example by CM-points.
- Pila proved the Andre-Oort conjecture for $\mathbb{C}^n$ (2011). A large portion of the proof is dedicated to solving the bi-algebraicity problem for the j-function.

Bi-algebraicity:

- Which algebraic X_0 have $\bar{j}(X_0)$ algebraic?

Ax-Lindemann-Weierstrass:

- Let $t_1, \ldots t_n \in \mathbb{C}(X)$. X an algebraic variety.
- Classify when $(j_\Gamma(t_1), \ldots, j_\Gamma(t_n))$ alg. dependent.

- Understanding the bi-algebraic varieties is equivalent to understanding algebraic relations between the solutions of the equation (sometimes also some closely related equations).
- But what does model theory have to offer?

Example: 2

- But what does model theory have to offer?

We call a differential equation X of order n over F, strongly minimal if whenever y is a solution of X then over any differential field K extending F,

$$tr.deg_K(y, y', y'', \dots, y^{(n-1)}) = 0 \textbf{ or } n.$$

- (F. and Nagloo) If strongly minimal differential equation X has constant coefficients and order > 1, then X is geometrically trivial - if $a_1, \dots, a_n$ of X are algebraically dependent over K then some **pair of solutions** is algebraically dependent.
- **Good news:** F.-Scanlon showed this property for the j-function.

$$S(y) + \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2}(y')^2 = 0,$$

where S is the Schwarzian derivative: $S(y) = \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2$.

- We call a differential equation X of order n over $\mathbb{F}$, strongly minimal if whenever y is a solution of X then over any differential field K extending F,

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- **Good news:** F.-Scanlon showed this property for the j-function.

$$S(y) + \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2}(y')^2 = 0.$$

- The algebraic relations between solutions are precisely given by modular polynomials. (C_2) fails.
- Casale-F.-Nagloo showed this for the uniformizing functions of genus zero Fuchsian groups, covering maps of Shimura varieties,...
- **Bad news:** There is no general purpose method of proving this condition. Usually it is very difficult to establish. e.g. for uniformizing functions of genus zero Fuchsian groups it was a famous conjecture of Painlevé (1895).
- **Other news:** (F.-Devilbiss) The property holds for "generic" nonlinear differential equations with nonconstant coefficients.

Example: 3

- There are many equations for which (C_2) holds, but (C_3) fails.
- An construction of Jin and Moosa:

$$X' = AX$$

where A is a generic enough $n \times n$ matrix $(n > 3)$. Then if one takes the projective quotient of the solution space $V \setminus \{0\}$, any two such solutions are independent, but

- (C_3) can be shown to fail.
- There are other examples coming from Liénard type equations.

Example: 4

- There are equations for which (C_3) holds, but (C_4) fails.
- Shown by Nagloo (also F.-Moosa):

$$y' = ay^2 + by + c$$

for $a, b, c \in \mathbb{C}$ so that the equation has no solutions in $\mathbb{C}(t)^{alg}$. After a linear change of variables, the equation can be rewritten:

$$y' = y^2 + r$$

- Any three solutions y_1, y_2, y_3 are algebraically independent over $\mathbb{C}(t)$.
- Given any fourth solution, $y(t)$, there is $\alpha \in \mathbb{C}$ so that:

$$y(t) = \frac{y_2(y_3 - y_1) + \alpha y_1(y_2 - y_3)}{(y_3 - y_1) + \alpha(y_3 - y_1)}.$$

- So, (C_4) fails for this equation (over any field containing $\mathbb{Q}(r, \alpha)$).

Are there more examples? How bad does this get?

- Hrushovski (1986): for any equation which is order one in one variable:

$$(C_4) \implies (C_n)$$

  for all $n \in \mathbb{N}$.
- Can one prove a higher order version of Hrushovski's result?
- In early 2020 Moosa and I did this, eventually showing that for an equation of order $n > 1$ in one variable,

$$(C_{n+2}) \implies (C_m)$$

  for all $m \in \mathbb{N}$.
- We did this by introducing a new invariant, called the degree of nonminimality (or forking degree) and relating the problem of the bounding this degree to an interesting conjecture of Borovik and Cherlin, which we had to verify for algebraic groups in characteristic zero.

In purely geometric terms...

Algebraic differential equations with coefficients in $\mathbb{C}$	$\Longleftrightarrow$	(V, s) where $s : V \to TV$ is a rational section.

- We say that $W \subset V$ is s-invariant if $s|_W : W \to TW$.
- Most subvarieties are not s-invariant.
- Let $s^{(n)}$ denote the section of $V^n \to T(V^n)$ given by s on each copy of V

Algebraic relations between solutions of (V, s)	$\Longleftrightarrow$	$s^{(n)}$-invariant subvarieties of V^n not contained in a "diagonal".

- Is there some d such that if there are no $s^{(d)}$-invariant subvarieties of V^d, then there are no $s^{(n)}$ subvarieties of V^n for any n?
- (F.-Moosa) Yes and $d \leq \dim(V) + 2$.

A stronger result

- Surprisingly, last Fall, we were able to show a remarkable improvement: for an equation of order $n > 1$,

$$(C_3) \implies (C_m)$$

  for all m.
- With Jaoui, we also showed that when the equation is additionally assumed to have constant coefficients,

$$(C_2) \implies (C_m)$$

  for all m.
- In the remaining time, I want to talk about some of the ideas of the proof...

In purely geometric terms...

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Algebraic relations between solutions of (V, s) $\Longleftrightarrow$ $s^{(n)}$-invariant subvarieties of V^n not contained in a "diagonal".

- Is there some d such that if there are no $s^{(d)}$-invariant subvarieties of V^d, then there are no $s^{(n)}$ subvarieties of V^n for any n?
- (F.-Jaoui-Moosa) Yes and $d = 2$.

What are the ideas in the proof?

Theorem

For an equation of order $n > 1$, for all m,

$$(C_3) \implies (C_m).$$

- We develop a "theory of fibrations" of differential equations.
- This part relies on a technical part of model theory called <u>geometric stability theory</u>.
- Basically, we reduce the problem to pieces in the fibration which are "close" to being strongly minimal.
- On these minimal parts of our fibration, we have to use a sort of nonlinear differential galois theory, and the properties (C_m) are related to certain notions of <u>higher transitivity</u> of the galois groups.

Transitive group actions

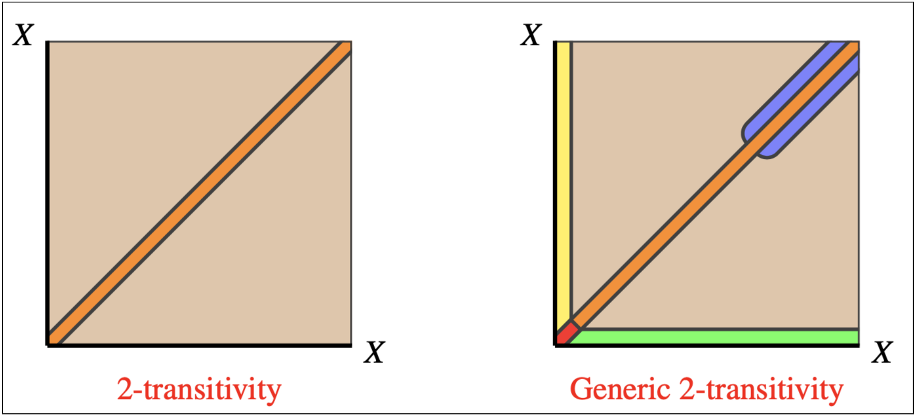
Definition

An action of G on S is <u>k-transitive</u> if for any two k-tuples of <u>distinct</u> elements $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$, there is some $g \in G$ such that $(gx_1, \ldots, gx_k) = (y_1, \ldots, y_k)$.

- A high degree of transitivity is known to impose strong structural conditions on the group.
- e.g. Jordan (1872): if a finite group G acts 4-transitively on a set S with the pointwise stabilizer of any four distinct elements being trivial (that is, G acts <u>sharply</u> 4-transitively) then G must be one of S_4, S_5, A_6 or the Mathieu group M_{11}.
- Later generalizations loosened the sharpness requirement (Tits, Hall,...)
- I don't know general power structural results for highly transitive infinite group actions without some sharpness condition
- <u>Our galois groups are closely related to algebraic groups acting regularly on algebraic varieties</u> and k-transitivity turns out to be not

Definition

Let (G, X) be an algebraic group acting regularly on an algebraic variety X. The action is generically k-transitive if the diagonal action of G on X^k has an open orbit $\mathcal{O}$.



Generic transitivity

- The action of GL_n on $\mathbb{A}^n$ is generically n-transitive.
- The action of PGL_n on $\mathbb{P}^{n-1}$ is generically $(n+1)$-transitive.

Conjecture

(Borovik and Cherlin 2008) If G acts generically k-transitively on X, then k is at most $\dim(X) + 2$.

- BC made their conjecture in a much more general setting as well - groups of finite Morley rank
- The general version of their conjecture seems very hard, but it is known in dimension 1 (Hrushovski's thesis) and dimension 2 (Altinel and Wiscons, JEMS 2018).
- (F.-Moosa) proved BC for algebraic groups in char 0.
- The conjecture seems to be open in characteristic p.

Generic transitivity

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Conjecture

(Borovik and Cherlin 2008) If G generically k-transitively on X, then k is at most $\dim(X) + 2$.

- Idea of the proof:
  - One can reduce to the "primitive" case and assume that the action is isomorphic to G acting on the left cosets of a point-stabilizer G_a and G_a is a maximal proper closed subgroup
  - In the late '90s, model theorists developed a version of the O'Nan-Scott theorem
  - In the O'Nan-Scott type theorem, one reduces to the group actions "very close" to the affine group acting a vector space <u>or</u> a simple algebraic group acting on cosets of parabolic subgroups

Higher transitivity

Conjecture

(Borovik and Cherlin 2008) If G generically k-transitively on X, then k is at most $\dim(X) + 2$.

- We show when $k = dim(X) + 2$ we must have $X = \mathbb{P}^k$ and $G = PGL_{k+1}$
- Ultimately to show our result about 3 solutions, we found a way to reduce to the case where the Galois group could be assumed to be 2-transitive.
- Knop classified 2-transitive regular actions of algebraic groups (G, X) showing they are either:
  - PGL_{n+1} acting on $\mathbb{P}^n$ or
  - Certain subgroups of the group of affine transformations.
- We ultimately prove our theorem by noticing that there is only one group action on the list that can be 3-transitive.
- In the autonomous case, we rule out the possibility of 2-transitivity altogether - every group in Knop's list is centerless.

Back to strong minimality 2

- I started developing the ideas explained here because Nagloo and I had this satisfactory picture of the algebraic relations between autonomous nonlinear equations that are strongly minimal, but
- We had no general method of establishing that a differential equations is strongly minimal.

We call a differential equation X of order n over F, strongly minimal if whenever y is a solution of X then over any differential field K extending F,

$$tr.deg_K(y, y', y'', \ldots, y^{(n-1)}) = 0 \textbf{ or } n.$$

- (F.-Jaoui-Moosa) To verify strong minimality, it is sufficient to consider $F = K\langle y_1, y_2\rangle$ where y_1, y_2 are solutions of X.

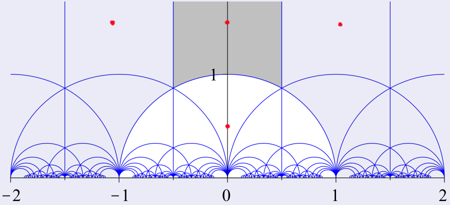
Some papers of mine appearing in the talk:

- Strong minimality and the j-function, with Scanlon. JEMS. 2017
- Ax-Lindemann-Weierstrass with derivatives and the genus 0 Fuchsian groups. with Casale and Nagloo. Annals. 2020.
- Bounding nonminimality and a conjecture of Borovik-Cherlin with Moosa. JEMS. $\geq$2022.
- When any three solutions are independent with Jaoui and Moosa. Inventiones. $\geq$2022.
- The degree of nonminimality is at most two with Jaoui and Moosa. Journal of Mathematical Logic. $\geq$2022.
- Generic differential equations are strongly minimal with DeVilbiss. Compositio $\geq$2022.
- On the equations of Poizat and Liénard with Jaoui, Marker, and Nagloo. IMRN. $\geq$2022.

Fuchsian Groups

Let Γ be a genus zero Fuchsian group of the first kind - a discrete, finitely generated subgroup $\Gamma \leq PSL_2(\mathbb{R})$. For instance:

- $SL_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



Fundamental domain for $SL_2(\mathbb{Z})$ on $\mathbb{H}$.

- For any Γ the fundamental domain for the action of Γ on $\mathbb{H}$ is semialgebraic.

- Another example - congruence groups:

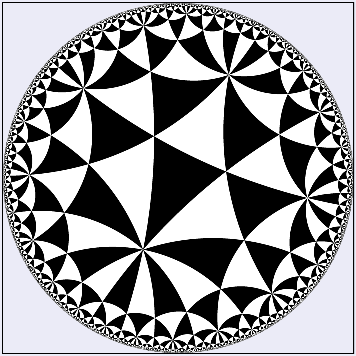
$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ a, d \equiv 1(\mathrm{mod} n), \quad b, c \equiv 0(\mathrm{mod} n) \right\}$$

Examples of Fuchsian groups

- A third example: triangle groups

$$\Delta(l,m,n) = \langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^l = (bc)^n = (ca)^m = 1\rangle.$$

with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

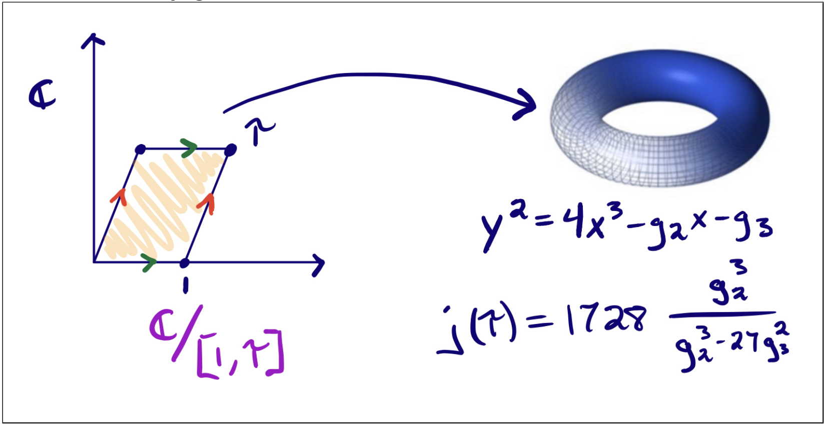


Here $l = m = 3, n = 6$.

- $\Delta(l,m,n)$ is the group of reflections of triangles tiling the hyperbolic plane with angles $(\pi/l, \pi/m, \pi/n)$.
- $SL_2(\mathbb{Z})$ is a triangle group with $(2,3,\infty)$.

Automorphic for Γ

- An analytic function j_Γ on $\mathbb{H}$ is Γ-**automorphic** if $j_\Gamma(g\tau) = j_\Gamma(\tau)$ for $\tau \in \mathbb{H}$ and any $g \in \Gamma$.



Automorphic functions

- The function $j_\Gamma(t)$ satisfies a nonlinear third order differential equation:
$$S_{\frac{d}{dt}}(y) + (y')^2 \cdot R_\Gamma(y) = 0 \qquad (\star)$$
where R_Γ is a rational function and
$$S_{\frac{d}{dt}}(x) = \left(\frac{x''}{x'}\right)' - \frac{1}{2}\left(\frac{x''}{x'}\right)^2$$
is the Schwarzian derivative.
- (Casale-F.-Nagloo) The algebraic relations between solutions come precisely from the action of the commensurator of Γ.
- Conclude by Margulis' theorem that there are only infinitely many algebraic relations between solutions if and only if Γ is arithmetic.
- Our work isolated an interesting connection between arithmetic groups and categoricity, a central topic in model theory.