Instructions: Do all six problems.[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Prove that a consistent finitely axiomatizable (possibly incomplete) theory T with less than continuum many completions must have a finitely axiomatizable completion.

E2. Let A be a non-empty set and let $<$ be a strict total order on A with no largest element. Let C be the set of all ordinals that are isomorphic to an unbounded subset of A. Prove that C is non-empty and that its least element is a regular cardinal.

E3. Let $\mathcal{L}$ be a countable language and T an $\mathcal{L}$-theory with infinite models. Show that there is an ω_1-sequence of models $\{\mathcal{M}_\alpha\}_{\alpha<\omega_1}$ of T of size $\aleph_1$ so that $\mathcal{M}_\beta \prec \mathcal{M}_\alpha$ (i.e., $\mathcal{M}_\beta$ is a proper elementary submodel of $\mathcal{M}_\alpha$) whenever $\alpha < \beta$.

[1] Note that this is different from exams before January 2014.

Computability Theory

C1. Show that a maximal c.e. set has minimal m-degree. (Here, a c.e. set M is *maximal* if M is coinfinite but any c.e. superset of M is either cofinite or differs from M at only finitely many elements. An m-degree $\mathbf{a}$ is *minimal* if it is nonzero but there is no m-degree strictly between $\mathbf{0}$ and $\mathbf{a}$.)

C2. Suppose S is a uniformly c.e. family of sets containing all finite sets. Show that there is a c.e. enumeration of S which lists each set in S exactly once. (Here, a family S of sets is *uniformly c.e.* if $S = \{A_n \mid n \in \omega\}$ where $A_n = \{x \mid \langle n, x\rangle \in A\}$ for some c.e. set A; this sequence $\{A_n\}_{n\in\omega}$ is then called a *c.e. enumeration* of S.)

C3. If A is 1-generic and B is 1-generic relative to A, then $A \cap B$ is 1-generic.

Sketchy Answers or Hints

E1 ans. Suppose T has no finitely axiomatizable completion. Then for any finite set of sentences F such that $T \cup F$ is consistent, there is a sentence φ such that both $T \cup F \cup \{\varphi\}$ and $T \cup F \cup \{\neg\varphi\}$ are consistent. This easily allows one to build a tree of formulas $\{\varphi_\sigma \mid \sigma \in 2^{<\omega}\}$ such that for any path $p \in 2^\omega$, $T \cup \{\varphi_\sigma \mid \sigma \subset p\}$ is consistent and $\varphi_{\sigma 1}$ is $\neg\varphi_{\sigma 0}$ for all σ.

E2 ans. C is non-empty since we can define, using Choice, a $<$-increasing sequence of elements $\{a_\alpha\}_\alpha \in A$ for ordinals α. By Replacement, this process must stop at some ordinal α_0, say, and the resulting sequence will be unbounded in A. α_0 is a limit ordinal since A has no largest element, and the least such ordinal is regular since any sequence of ordinals contains a cofinal subsequence of length a regular ordinal.

E3 ans. Let $\mathcal{L}'$ be the language generated from $\mathcal{L}$ by adding new unary predicates P_α for $\alpha < \omega_1$. Let T' be the theory which says that $P_0 \models T$ and that P_β defines an elementary submodel of P_α whenever $\alpha < \beta$. Check by compactness that T' is consistent: For finitely many axioms, these are witnessed by a finite elementary chain. This exists by an upward Skolem argument starting with an infinite model of T, so T' is consistent. By downward Skolem, we can take a model $\mathcal{N}$ of T' of size $\leq \aleph_1$ (the size of $\mathcal{L}'$). We now have our ω_1-length elementary descending sequence with $\mathcal{M}_\alpha$ defined by P_α in $\mathcal{N}$. It remains to see that each of the models has size $\aleph_1$. For any $\gamma < \omega_1$, we argue that M_γ has size $\aleph_1$ as follows: For every $\delta \in (\gamma, \omega_1)$, choose an element $x_\delta \in M_\delta \smallsetminus M_{\delta+1}$. Then $\{x_\delta \mid \delta \in (\gamma, \omega_1)\}$ is a set of $\aleph_1$ many distinct elements, all of which are in M_γ, so M_γ is not countable. Of course, since $M_\gamma \subseteq N$, its size cannot be larger than $\aleph_1$.

C1 ans. Clearly, no maximal set M can be computable since otherwise its computable complement can be split into two infinite computable subsets. Suppose $A \leq_m M$ for some set A via some computable function f. Then the range of f forms a c.e. set B. If $M \cup B$ differs from M only finitely, say, $B - M = \{b_1, \ldots, b_n\}$, then $A = f^{-1}(\omega - \{b_1, \ldots, b_n\})$ is computable. Otherwise,

$M \cup B$ is cofinite, and so by changing f at finitely many arguments, we may assume that $M \cup B = \omega$. But then $M \leq_m A$ via g, where $g(n)$ is defined as follows: Start enumerating M and look for some m with $f(m) = n$. If we see $n \in M$ first then set $g(n) = a$, where $a \in A$ is some fixed number. If we first find some m with $f(m) = n$, then set $g(n) = m$.

C2 ans. this that S Without loss of generality, we may assume that S contains ω, since we can simply remove a single set from the enumeration produced, so let $\{A_n\}_{n\in\omega}$ be any c.e. enumeration of S with $A_0 = \omega$. We now build a c.e. enumeration $\{B_n\}_{n\in\omega}$ of S and a $\mathbf{0}'$-partial computable function f (approximated by uniformly partial computable functions f_s in the sense that $f(n)\downarrow = m$ if $f_s(n) = m$ for cofinitely many s, and $f(n)$ is undefined otherwise). We meet the following requirements:

(1) If $A_n = A_{n'}$ for some $n' < n$ then $f(n)$ is undefined.

(2) If $A_n \neq A_{n'}$ for all $n' < n$ then either $f(n)$ is defined and $A_n = B_{f(n)}$; or A_n is of the form $[0, x]$ for some x, and there is $m \in \omega - \text{ran}(f)$ such that $A_n = B_m$.

(3) Any set B_m with $m \notin \text{ran}(f)$ is of the form $[0, x]$ for some x.

(4) For any set of the form $[0, x]$ for some x, there is a unique m with $B_m = [0, x]$.

Now, at stage $s = 0$, we define $B_0 = \omega$ and $f(0) = f_0(0) = 0$, while $f_0(n)$ is undefined for all $n > 0$. At a stage $s + 1$, we perform the following steps:

Step 1: If $f_s(n)$ is defined and for some $n' < n$,

$$A_{n',s} \upharpoonright (f_s(n) + 1) = A_{n,s} \upharpoonright (f_s(n) + 1)$$

(i.e., if n does not appear to be the least index for A_n), then let $f_{s+1}(n)$ be undefined (and keep $f_s(n)$ permanently out of the range of f from now on).

Step 2: If $f_s(n)$ is defined, $n > 0$, and, for some $s' < s$ and some $m \in \text{ran}(f_{s'}) - \text{ran}(f_s)$,

$$B_{m,s} \upharpoonright (f_s(n) + 1) = B_{f_s(n),s} \upharpoonright (f_s(n) + 1)$$

(i.e., if the set B_m seems to appear twice in the B-sequence of sets, including once as a set with index no longer in the range of f), then let $f_{s+1}(n)$ be undefined (and keep $f_s(n)$ permanently out of the range of f from now on).

Step 3: If $f_s(n)$ is defined but $f_{s+1}(n)$ is undefined (i.e., if $f(n)$ just became undefined via Step 1 or Step 2), then for each such n (in increasing order of n), set

$$B_{f_s(n)} = B_{f_s(n),s+1} = [0, x]$$

for some x larger than any number mentioned thus far in the construction.

Step 4: If $f_s(n)$ is undefined for $n \leq s$, then for each such n (in increasing order of n), let $f_{s+1}(n)$ be the least m not in $\bigcup_{s' \leq s} \operatorname{ran}(f_{s'})$ and not equal to $f_{s+1}(n')$ for some $n' < n$.

Step 5: If $f_{s+1}(n)$ is defined then let $B_{f_{s+1}(n),s+1} = A_{n,s+1}$.

To verify that this works, we first note that since for each m, there is at most one n such that $f_s(n) = m$ at some stage s, Step 5 can be carried out since no number has to be removed from $B_{f_{s+1}(n)}$ to carry out Step 5. Similarly, since x is chosen large in Step 3, this step can be carried out without removing numbers from $B_{f_s(n)}$.

We now verify the satisfaction of the above requirements:

(1) If $A_n = A_{n'}$ for some $n' < n$ then $f_s(n)$ is undefined for infinitely many s by Step 1.

(2) If $A_n \neq A_{n'}$ for all $n' < n$ then $f(n)$ becomes undefined via Step 1 at most finitely often. If $f(n)$ becomes undefined via Step 2 for the same m infinitely often, then $A_n = B_m$ as desired. Otherwise, since A_n is computably enumerable, $A_{n,s} = [0, x]$ at various stages s for larger and larger x; thus $A_n = \omega$, and so $n = 0$ and Step 2 never applies to n.

(3) This is immediate by Step 4.

(4) Fix x. Steps 2 and 4 ensure that there is at most one m such that $B_m = [0, x]$. Fix n least such that $A_n = [0, x]$. Then either $f(n)$ is defined and $B_{f(n)} = [0, x]$; or else we can argue as in (2) above that there is some m such that $B_m = [0, x]$.

C3 ans. Let $W \subseteq 2^{<\omega}$ be a c.e. set that contains no prefix of $A \cap B$.

Consider the A-c.e. set of strings

$$V = \{\sigma \in 2^{<\omega} \mid (\exists \tau \prec A)\ |\tau| = |\sigma| \text{ and } \tau \cap \sigma \in W\}.$$

Since B is 1-generic relative to A, it either meets or avoids V. If it meets V at σ as witnessed by $\tau \prec A$, then $\tau \cap \sigma$ is a prefix of $A \cap B$ in W. Therefore, B avoids V; say, that $\sigma \prec B$ has no extension in V. Now consider the c.e. set of strings

$$U = \{\tau \in 2^{<\omega} \mid (\exists \sigma' \succeq \sigma)\ |\sigma'| = |\tau| \text{ and } \tau \cap \sigma' \in W\}.$$

If A has a prefix $\tau \in U$, then the witnessing σ' would be an extension of σ in V, which does not exist. But A is 1-generic, so A avoids U: There is some $\tau \prec A$ such that no extension of τ is in U. Fix $\sigma' \prec B$ such that $|\sigma'| = |\tau|$. Then $\tau \cap \sigma'$ is a prefix of $A \cap B$ with no extension in W, otherwise τ would have an extension in U, hence $A \cap B$ avoids W. But $W \subseteq 2^{<\omega}$ was an arbitrary c.e. set that contains no prefix of $A \cap B$, so $A \cap B$ is 1-generic.