Instructions: Do all six problems.¹

If you think that a problem has been stated incorrectly, mention this to
the proctor and indicate your interpretation in your solution. In such cases,
do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial
credit by weakening a conclusion or strengthening a hypothesis. In this case,
include such information in your solution, so the graders know that you know
that your solution is not complete.

If you want to ask a grader a question during the exam, write out your
question on an 8\(\frac{1}{2}\) by 11 sheet of paper. Give it to the proctor. The proctor
will contact one of the logic graders who will retrieve your written question,
write a response, copy the sheet of paper, and return it to the proctor.

E1. Let \(\kappa\) be an infinite cardinal and let \(A \subseteq [\kappa]^{<\kappa}\) have cardinality \(\kappa\). Prove
that there is a 1-1 enumeration \(\{A_\alpha : \alpha < \kappa\} = A\) such that \(|\bigcup_{\xi < \alpha} A_\alpha| < \kappa\)
for all \(\alpha < \kappa\).

As usual, \([\kappa]^{<\kappa} = \{x \subseteq \kappa : |x| < \kappa\}\). Caution. This problem is trivial
when \(\kappa\) is regular.

E2. Prove that any decidable consistent \(L\)-theory \(T\) which is decidable
complete consistent \(L\)-theory \(T' \supseteq T\). Recall that \(T\) is decidable iff there is
an algorithm which will decide for any \(L\)-sentence \(\theta\) whether or not \(T \vdash \theta\).

E3. Prove or disprove: there exists a partial computable function \(f\) such
that the domain of \(f\) and range of \(f\) are not computable but the graph of \(f\)
is computable.

C1. Let \(f_0, f_1, f_2, \ldots\) be a sequence of functions on \(\omega\) such that each \(f_i\)
is hyper-immune relative to the join of the others (in other words, \(f_i\) is not
dominated by any \((\bigoplus_{j \neq i} f_j)\)-computable function). Show that there is a
1-generic computable from \(\bigoplus_{j \in \omega} f_j\).

C2. Prove there exists a partial computable \(\psi : \omega \rightarrow \omega\) such that the
domain of \(\psi\), \(\text{dom}(\psi)\), is co-infinite but for every partial computable \(\rho\) which
extends \(\psi\) we have that \(\text{dom}(\rho) \setminus \text{dom}(\psi)\) is finite. Can we have such a \(\psi\)
with range \(\{0, 1\}\)?

C3. Let \(x\) be an incomputable real, find two Turing incomparable reals \(a\) and \(b\) such that \(a + b = x\).

¹Note that this is different from past exams.
Sketchy Answers or Hints

**E1 answer.** Assume that $\kappa$ is a limit cardinal. There are two cases.

Case 1. For some cardinal $\gamma < \kappa$ the set $A \cap [\kappa]^{< \gamma}$ has cardinality $\kappa$.

Case 2. Not case 1. In case 2 list the family so that if $\alpha < \beta$ then $|A_\alpha| \leq |A_\beta|$. In case 1 construct a listing with $|A_\alpha| \leq |\alpha| + |\gamma|$ by “filling in” using the elements of $A \cap [\kappa]^{< \gamma}$.

**E2 answer.** Use that $T \cup \{\theta\}$ is inconsistent iff $T \vdash \neg \theta$.

**E3 answer.** There is such an $f$. Let $a_n$ be an effective 1-1 enumeration of a c.e. set which is not computable. Let $f$ be the function whose graph is $\{(2n, 2a_n), (2a_n + 1, 2n + 1) : n < \omega\}$.

**C1 answer.** J Miller - result due to Damir and Adam.

We construct an infinite binary string $A$ by finite initial segments $\sigma_n$ for $n \in \omega$. The requirement is to force the jump (1-generic). At $\sigma_n$, for each $e < n$, in order to force $\varphi^A_e(e)$ to converge, we search for extensions of $\sigma_n$ up to length $f_e(n)$ for convergence of the oracle computation $\varphi^A_e(e)$, and pick the highest priority $e$ for which we find such extension and let it be $\sigma_n+1$.

So the construction is recursive in the join of all $f_i$'s. Now if some jump-forcing requirement is not satisfied, say the least such is $e_0$, then it is easy to see that the construction is recursive in the join of all others since the jump-forcing requirement for $e_0$ has never acted. Therefore at each $\sigma_n$, the first extension which forces $\varphi^{A_{e_0}}_{e_0}(e_0)$ to converge is bounded by some length recursive in the join of all other $f_i$'s. Then $f_{e_0}$ being hyperimmune relative to the join of all others gives the desired contradiction.

**C2 answer.** Let $a_n$ be a 1-1 effective enumeration of a simple set and define $\psi(a_n) = n$. To get such a $\psi$ with range $\{0, 1\}$. Apply two theorems of Friedberg. Let $A$ be a maximal set and let $A = A_0 \sqcup A_1$ be a splitting into c.e. non-computable sets. Define $\psi$ by $\psi^{-1}(i) = A_i$.

**C3 answer.** Construct $a$ and $b$ by finite initial segments. Say we have $\alpha$ and $\beta$, in order to force $\varphi^a_e \neq b$. There are three subcases here:

1. if there is an $n$ such that $\varphi^a_e(n)$ is always divergent, then we have forced $\varphi^a_e$ to be partial.
2, if there are extensions of $\alpha$ and $\beta$ which satisfy $\varphi_e^a(n) \downarrow \neq b(n)$ and $a + b = x$ is still possibly true, then we can take these extensions.

3, otherwise, then we can compute $x$ by searching for long enough extensions of $\alpha$ which converge on long enough bits, which now agree with all possible $b$’s which satisfy $a + b = x$. This allows us to limit $x$ into smaller and smaller intervals and so compute longer and longer initial segments of $x$. Note that this algorithm will fail if $x$ is a dyadic rational (say we are in base 2), but of course $x$ is incomputable here.