Instructions:

Do two E problems and two problems in the area C or M in which you signed up.

Write your letter code on all of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $L$ be a language which includes a unary relation symbol $R$. Let $\phi$ be an $L$-sentence and $\Gamma$ a set of $L$-sentences neither of which contains the symbol $R$. If $\Gamma$ proves $\phi$ in the language $L$, must there be a deduction of $\phi$ from $\Gamma$ in which $R$ does not occur (i.e., in the language $L - \{R\}$)? If so, prove that there is always such a deduction; and if not, describe $\Gamma$ and $\phi$ which provide a counterexample.

E2. Show that there exists an $\mathcal{N} \models \text{PA}$ and an $a \in \mathcal{N} \setminus \mathbb{N}$ so that $a$ is definable in $\mathcal{N}$.

E3. Let $\alpha$, $\beta$ and $\gamma$ be ordinals. Prove that the six sums,

\[
\begin{align*}
\alpha + \beta + \gamma, & \quad \alpha + \gamma + \beta, \\
\beta + \alpha + \gamma, & \quad \beta + \gamma + \alpha, \\
\gamma + \alpha + \beta, & \quad \gamma + \beta + \alpha,
\end{align*}
\]

cannot all be different.
Computability Theory

**C1.** Say that a computable function $f$ has a limit if for all $x$, $\lim_s f(x, s)$ exists. Show that the index set $\{e \mid \phi_e \text{ has a limit}\}$ is $\Pi_3$-complete.

**C2.** Show that no 1-generic set computes a diagonally noncomputable function. (Recall that a function $f$ is diagonally noncomputable if for all $e$, $f(e) \neq \phi_e(e)$.)

**C3.** Show that $a$ is a hyperimmune degree if and only if $a$ computes a function $f$ that agrees with every total computable function infinitely often. (Recall that a Turing degree $a$ is hyperimmune if it computes a function $g$ that is not dominated by any total computable function.)
Model Theory

M1. Let $T$ be a theory in the language of a single unary function $f$ stating that $f$ has no loops (i.e., for every $n > 0$ and every $x$, $f^n(x) \neq x$) and for every $x$, there are infinitely many $y$ with $f(y) = x$. Show that $T$ has quantifier elimination, is complete and not $\kappa$-categorical for any infinite cardinal $\kappa$.

M2. Find a complete theory $T$ in a countable first-order language such that the space $S_1(T)$ of 1-types is uncountable but $T$ is atomic. (Recall that $T$ is atomic if every formula $\phi(x_1, \ldots, x_n)$ is contained in a principal $n$-type.)

M3. Show that a complete countable first-order theory with infinite models is $\aleph_0$-categorical if and only if all of its models are pairwise back-and-forth equivalent.

Recall $A$ and $B$ are back-and-forth equivalent if there is a set $I$ comprised of pairs $(\bar{a}, \bar{b})$ where $\bar{a} \subset A$ and $\bar{b} \subset B$ such that the following hold:

- $(\emptyset, \emptyset) \in I$,

- If $(\bar{a}, \bar{b}) \in I$, then $|\bar{a}| = |\bar{b}| < \omega$ and $\text{tp}^{A}_{q.f.}(\bar{a}) = \text{tp}^{B}_{q.f.}(\bar{b})$ (i.e., their quantifier-free types coincide),

- If $(\bar{a}, \bar{b}) \in I$ and $c \in A$, then there exists a $d \in B$ so that $(\bar{ac}, \bar{bd}) \in I$, and

- If $(\bar{a}, \bar{b}) \in I$ and $d \in B$, then there exists a $c \in A$ so that $(\bar{ac}, \bar{bd}) \in I$. 
Sketchy Answers or Hints

E1 ans. Straightforward application of the Completeness theorem: If Γ proves φ, then any model M of Γ is a model of φ. The same then also holds for any model M of Γ in the language L − {R}, so again by Completeness, there is a deduction of φ from Γ in the language L − {R}.

E2 ans. By the Incompleteness Theorem, we can find a ∆₀-sentence φ(x) such that N |= ∀x ¬φ(x) but PA + ∃x φ(x) is consistent. Then any model N |= PA + ∃x φ(x) contains, by induction, a least witness a for φ, which must be both nonstandard and definable.

E3 ans. Write α, β and γ in Cantor normal form as

\[ \omega^{\alpha_n} \cdot a_n + \cdots + \omega^{\alpha_0} \cdot a_0, \ \omega^{\beta_n} \cdot b_n + \cdots + \omega^{\beta_0} \cdot b_0, \ \omega^{\gamma_n} \cdot c_n + \cdots + \omega^{\gamma_0} \cdot c_0, \]

respectively, where \( a_n, \ldots, a_0, b_n, \ldots, b_0, c_n, \ldots, c_0 \) are non-negative integers. Now use the fact that for \( \delta < \epsilon \), \( \omega^{\delta} \cdot d + \omega^{\epsilon} = \omega^{\epsilon} \).

C1 ans. It is easy to see that it is Π₀³. Let \( R(n, x, m, t) \) be a total computable predicate. Let \( f_n(x, s) = \) the least \( m \) such that (\( \forall t \leq s \)) \( R(n, x, m, t) \), or \( s \), if no such \( m \) exists. Then \( f_n \) has a limit iff (\( \forall x \)(\( \exists m \))(\( \forall t \)) \( R(n, x, m, t) \).

C2 ans. Let \( G \) be 1-generic. Let Γ be a Turing functional. We want to prove that \( \Gamma^G \) is not a DNC function. If \( \Gamma^G \) is partial, then there is nothing to show, so assume that it is total. Consider the \( \Sigma^0 \) set of strings

\[ W = \{ \sigma \in 2^{<\omega} : (\exists e, s) \ \Gamma^\sigma_{e,s}(e) = \phi_{e,s}(e) \ (\text{and both converge}) \} \]

If there is a \( \tau \prec X \) such that \( \tau \in W \), then \( \Gamma^G \) is not DNC. The only case that remains (thanks to the 1-genericity of \( G \)) is that there is a \( \tau \prec X \) that has no extension in \( W \). We will show that this is impossible. Define a computable function \( f : \omega \to \omega \) as follows. To find \( f(e) \), search for a \( \sigma \geq \tau \) and an \( s \in \omega \) such that \( \Gamma^\sigma_{s,e}(e) \downarrow \) and let \( f(e) = \Gamma^\sigma(e) \). The totality of \( \Gamma^G \) implies that some extension of \( \tau \) makes \( \Gamma \) converge, so \( f \) is total. The fact that \( \tau \) has no
extension in $W$ implies that $f$ is DNC. But no computable function can be DNC, so we have the necessary contradiction.

**C3 ans.** A function $f$ that agrees with every total computable function infinitely often cannot be dominated by a total computable function. For the other direction, let $g$ be an $a$-computable function that is not dominated by any computable function. The $a$-computable function $f$ defined by

$$f(\langle e, n \rangle) = \begin{cases} \phi_e(\langle e, n \rangle) & \text{if } \phi_{e,g}(\langle e, n \rangle) \downarrow, \\ 0 & \text{otherwise} \end{cases}$$

agrees with every total computable function infinitely often. If it fails on the total computable function $\phi_e$, then

$$n \mapsto \text{least } s \text{ such that } \phi_{e,s}(\langle e, n \rangle) \downarrow$$

dominates $g$.

**M1 ans.** Proof of QE 1: Let’s consider a formula of the form $\exists y (\phi(\bar{x}, y))$ where $\phi$ is a conjunction of literals: $\bigwedge t_1(x, y) = t_2(x, y)$. Each term can take in only one parameter (as $f$ is unary), so this really is $\bigwedge t_1(x_i) = t_2(y)$. Whether or not this configuration can hold is determined only by the configuration of $\bar{x}$ - this can be verified in cases: The only hard-ish case is when two $x$’s are connected and $f(x_1) = y$ and $f(y) = x_2$, but $f^2(x_1) \neq x_2$ Proof of QE 2 (the better one): We show that every type $p \in S_1(A)$ is determined by its q.f.-type. Suppose we had a model $M$ with 2 element realizing the q.f.-type $p$. If the q.f.-type says it’s connected to an $a \in A$, then show that the two elements are automorphic in $M$ over $A$. If it’s not connected, then in a saturated elementary extension (which must be homogeneously splitting), it’s easy to automorph the two elements while fixing $A$. Completeness follows from QE-ness. Not $\aleph_0$-categorical: one model with 1 tree and one model with 2 trees. Not $\aleph_1$-categorical: One model with $\aleph_1$-splittings on a single tree, and one model with $\aleph_0$-splittings but $\aleph_1$-many trees.

**M2 ans.** Take a tree in $2^{<\omega}$ with infinitely many paths but a dense set of isolated paths. Let $T$ be the theory associated to this tree (ie. the 1-types
in $T$ are exactly the paths through this tree, and all 2-types are controlled by 1-types. This works

**M3 ans.** $\leftarrow$: Take any 2 countable models. The back-and-forth builds an isomorphism. $\rightarrow$: Using Ryll-Nardzewski, build the back-and-forth. Given $(\bar{a}, \bar{b}) \in I$ by stage $s$, and any element $c \in A$, let $\phi$ isolate the type of $c$ over $\bar{a}$. $\exists x \phi(x)$ is in the type of $\bar{a}$, thus also of $\bar{b}$. Let $d$ be a realization of this formula, and put $(\bar{a}c, \bar{b}d)$ into $I$ at stage $s + 1$. Do the back direction too.