Jan 11 4pm

Instructions: Do two E problems and two M problems.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Suppose T_i for $i < n$ (with $n < \omega$) are L-theories such that every L-structure M satisfies exactly one of the T_i. Prove that each T_i is finitely axiomatizable. If $n = \omega$ must this still be true? Prove or give a counterexample.

E2. Let $(A, <)$ be a dense total order without endpoints, and assume that A is homogeneous in the sense that (a, b) is isomorphic to A whenever $a, b \in A$ with $a < b$ (examples: $\mathbb{R}$, $\mathbb{Q}$). Let $\alpha(A)$ be the least ordinal which is not isomorphic to any subset of A. Prove that $\alpha(A)$ is a regular uncountable cardinal. Then, give examples of such A, B with $|A| = |B| = \aleph_1$ and $\alpha(A) = \omega_1$ and $\alpha(B) = \omega_2$.

E3. Prove that the set of validities in the language with two unary operation symbols is undecidable. You may assume without proof that the set of validities in the language with one binary relation symbol is undecidable.

Recall that a binary relation on a set A is a set $\subseteq A \times A$. A unary operation on a set A is a function $: A \to A$.

Model Theory

M1. Let L be the language whose signature contains a single unary function symbol. Show that the empty theory T in L has a model companion.

Recall that a theory U in L is a model companion of T iff U is model-complete, every model of T has an extension which is a model of U, and every model of U has an extension which is a model of T. A theory U is model-complete iff every embedding between its models is an elementary embedding.

M2. Show T has the JEP iff whenever ϕ and ψ are universal formulae and $T \vdash \phi \vee \psi$, then $T \vdash \phi$ or $T \vdash \psi$.

Recall that T has the joint embedding property (JEP) iff whenever A and B are models of T, there is some C modeling T such that both A and B are embeddable into C.

M3. Let T be a complete theory in a countable language with infinite models. Show that T has a countable model A such that for every tuple $\bar{a}$ from A, there is a formula $\psi(\bar{x})$ with $A \models \psi(\bar{a})$ such that either (1) ψ generates a complete type over T or (2) no principal complete type over T contains ψ.

Recall that a type Φ is principal iff there is a formula ψ consistent with T which generates Φ, i.e., $T \vdash \psi \to \rho$ for every $\rho \in \Phi$.

Sketchy Answers or Hints

E1 ans. For $n < \omega$, use compactness. For $n = \omega$, any T_0 which is not finitely axiomatizable but has a countable axiomatization can be turned into a counterexample.

E2 ans. Note that if ξ is isomorphic to a subset of A, then by homogeneity, ξ is isomorphic to a bounded subset. It follows that $\alpha(A)$ is a limit ordinal. Now, suppose that $\mathrm{cf}(\alpha(A)) = \theta < \alpha(A)$. Choose $a_\xi \in A$ for $\xi < \theta$ such that $\xi < \eta \to a_\xi < a_\eta$. For each $\xi < \theta$, choose a well-ordered $E_\xi \subseteq (a_\xi, a_{\xi+1})$ such that $\sup\{\mathrm{type}(E_\xi) : \xi < \theta\} = \alpha(A)$. But then $\mathrm{type}(\bigcup_\xi E_\xi) = \alpha(A)$, a contradiction.

For the examples, note that $\alpha(\mathbb{R}) = \omega_1$. This is homogeneous, with the isomorphisms given by rational functions. Of course, $|\mathbb{R}|$ is $2^{\aleph_0}$, not $\aleph_1$, so let A be an elementary submodel of the ordered field of real numbers of size $\aleph_1$. Then, let B be an ordered field of size $\aleph_1$ which elementarily equivalent to $\mathbb{R}$ but contains an increasing ω_1–sequence. Then $\alpha(B) > \omega_1$, so $\alpha(B) = \omega_2$.

E3 ans. Given $R \subseteq A^2$, consider the structure (U, f_1, f_2), where U is the disjoint union of A and R and $f_1(\langle a, b\rangle) = a$, $f_2(\langle a, b\rangle) = b$, and both are the identity on A.

M1 ans. Let L be the language with the single unary function f. Let T be the theory that says that every element has infinitely many f-predecessors and that there are infinitely many f-loops of length n for every n.

We must verify that every L-structure embeds into a model of T and that T is model complete. Fix M any L-structure. Let M' be the disjoint union of M and infinitely many n-loops for every n. Then let $N_0 = M'$, let N_{i+1} be formed from N_i by adding infinitely many predecessors to every element of N_i. Then $N_\omega = \bigcup_{i\in\omega} N_i$ is a model of T and M embeds in N_ω.

Finally, we have to show that T is model complete. In fact, T even has quantifier elimination. Denoting by $\langle \bar{c}\rangle$ the substructure generated by $\bar{c}$, we will use the following criterion to verify quantifier elimination: T has QE if and only if $\forall \bar{x}, \bar{y}, a \in M \models T$ if $\langle \bar{x}\rangle \cong \langle \bar{y}\rangle$, then there is an element

$b \in N \succeq M$ so that $\langle \bar{x}, a \rangle \cong \langle \bar{y}, b \rangle$.

Let $\bar{x}, \bar{y}, a \in M \models T$, and suppose that $\langle \bar{x} \rangle \cong \langle \bar{y} \rangle$. Consider the cases: If there is some n so that $f^{(n)}(a) \in \langle \bar{x} \rangle$, then take the minimal n as such and say $f^{(n)}(a) = x_0$. Let y_0 be the image of x_0 in $\langle \bar{y} \rangle$. Choosing b so that $f^{(n)}(b) = y_0$ and $f^{(n-1)}(b) \notin \langle \bar{y} \rangle$ will work. The other case is that for every n, $f^{(n)}(a) \notin \langle \bar{x} \rangle$. If a is part of an f-loop, then we can choose b to be a part of an f-loop of the same length (T guarantees this exists). If a is not part of an f-loop, then since there are f-loops of arbitrary length, an easy compactness argument gives us that there are elements b which are not part of an f-loop and not in $\bar{y}$ in an elementary extension N of M. Such a b works. So, T has QE (which is stronger than model completeness).

M2 ans. ($\Rightarrow$) Recall that if $A \subseteq B$ and ϕ is universal, then $B \models \phi \implies A \models \phi$. Let ϕ, ψ be universal such that $T \models \phi \vee \psi$, but $T \not\models \phi$, i.e., there is $A \models T \cup \{\neg\phi\}$. Then for any $B \models T$, by JEP, there exists $C \models T$ such that (WLOG) $A, B \subseteq C$. Because $A \models \neg\phi$, it must be that $C \models \neg\phi$, hence $C \models \psi$, and therefore $B \models \psi$. So $T \models \psi$.

($\Leftarrow$) Let $A, B \models T$. To get JEP, it is enough to show that $T_{AB} = Diag(A) \cup Diag(B) \cup T$ is finitely satisfiable, where $Diag(A)$ is the atomic diagram of A (with the elements of A as constants). If T_{AB} is not satisfiable, then $T \models \neg(\phi(\bar{a}) \wedge \psi(\bar{b}))$ for ϕ, ψ quantifier free where $A \models \phi(\bar{a})$ and $B \models \psi(\bar{b})$. Therefore, modulo some syntactical massaging, $T \models \forall\bar{x}\neg\phi(\bar{x}) \vee \forall\bar{y}\neg\psi(\bar{y})$. By assumption, without loss of generality, $T \models \forall\bar{x}\neg\phi(\bar{x})$, which by $A \models T$ contradicts $A \models \varphi(\bar{a})$.

M3 ans. Proceed as in the omitting types theorem, but instead of trying to omit any given type, you are trying to make types principal. That is, we have requirements of the form

$$S_{\bar{c}} : \text{The type of } \bar{c} \text{ is principal}$$

We proceed as in the usual Henkin construction to build a model where every element is named by a constant c_i. When we get to an $S_{\bar{c}}$ requirement, we try to find some formula $\psi(\bar{x})$ that generates a principal type, so that the addition of $\psi(\bar{c})$ to what we already constructed is consistent. If we cannot do this consistently, then from the single formula $\Gamma(\bar{d}, \bar{c})$ we have committed to so far, we can extract the formula $\psi(\bar{x}) := \exists\bar{y}\Gamma(\bar{y}, \bar{x})$ which is not contained in any principal type.