Instructions:

Do two E problems and two problems in the area C, M, or S in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $\kappa < \gamma$ be regular cardinals. Partially order $\kappa \times \gamma$ by saying $(\alpha_1, \beta_1) \le (\alpha_2, \beta_2)$ iff $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$. Then $x < y$ means $x \le y$ and $x \ne y$. Prove that there is no ordinal α and function $f : \alpha \to \kappa \times \gamma$ such that f is both *cofinal* and *increasing*:

$$\forall x \in \kappa \times \gamma \, \exists \xi < \alpha \;\; [x < f(\xi)] \quad \text{and} \quad \forall \xi, \eta < \alpha \;\; [\xi < \eta \to f(\xi) < f(\eta)].$$

E2. Let $T = \mathrm{Th}(\mathbb{Z}, +)$. Prove that T has uncountably many pairwise nonisomorphic countable models.

E3. A linear order is scattered iff it does not contain a suborder isomorphic to the rationals. Prove that the class of scattered linear orders is not axiomatizable.

Model Theory

M1. Prove or disprove:

(a) Suppose T is a theory in a countable language, $(\Sigma_n : n < \omega)$ are partial types, and for every $N < \omega$ T has a model omitting the types $(\Sigma_n : n < N)$. Then T has a model omitting the types $(\Sigma_n : n < \omega)$.

(b) Suppose T is a complete theory in a countable language, $(\Sigma_n : n < \omega)$ are partial types, and for every $N < \omega$ T has a model omitting the types $(\Sigma_n : n < N)$. Then T has a model omitting the types $(\Sigma_n : n < \omega)$.

M2. Show the following are equivalent:

a) If A is a model of T, then the intersection of any 2 elementary substructures of A is also an elementary substructure of A.

b) If A is a model of T, then the intersection of any family of elementary substructures of A is also an elementary substructure of A.

c) If A is a model of T and B is an algebraically closed subset of A, then B is an elementary substructure of A.

M3. (a) Suppose T is complete first order theory in a countable language. Show that T has a countable saturated model implies T has a countable prime model.

(b) Suppose A is an infinite structure in a countable language and $\bar{a}$ a finite tuple from A. Show that $Th(A, \bar{a})$ is countably categorical if and only if $Th(A)$ is.

Set Theory

Notation. For a cardinal θ, $[A]^\theta = \{x \subseteq A : |x| = \theta\}$.

S1. Assume Martin's Axiom.

(a) If the continuum is ω_2, prove there exists $f : \omega_2 \times \omega_2 \to 2$ for which there does not exist $H \in [\omega_2]^\omega$ and $K \in [\omega_2]^{\omega_2}$ such that f restricted to $H \times K$ is constant.

(b) If the continuum is larger than ω_2, prove that for every $f : \omega_2 \times \omega_2 \to 2$ there exists $H \in [\omega_2]^\omega$ and $K \in [\omega_2]^{\omega_2}$ such that f restricted to $H \times K$ is constant.

Hint for (b): You can find $H \subseteq \omega$.

S2. Assume $\diamondsuit$, and prove that there is a Suslin tree $T \subseteq 2^{<\omega_1}$ such that $s \in T \leftrightarrow t \in T$ whenever $s =^* t$.

Here $s =^* t$ means that s, t are functions with the same domain and $\{x \in \mathrm{dom}(s) : s(x) \neq t(x)\}$ is finite. Your tree should be a sequence tree in the usual sense. So, in the tree order, $s \leq t$ iff t extends s; and, the root of the tree is the empty sequence.

S3. A γ–*tower* is a sequence $\vec{A} = \langle a_\alpha : \alpha < \gamma \rangle$, where each $a_\alpha \in [\omega]^\omega$, and γ is a limit ordinal, and $\alpha < \beta \to a_\beta \subseteq^* a_\alpha$, and $\neg \exists x \in [\omega]^\omega \, \forall \alpha \, [x \subseteq^* a_\alpha]$.

Let M be a countable transitive model for ZFC, and assume that in M, $\vec{A}$ is a γ–tower and $\mathbb{P}$ is a countable forcing poset.

Let G be $\mathbb{P}$–generic over M. Prove that $\vec{A}$ is a still a tower in $M[G]$.

Sketchy Answers or Hints

E1 ans. It would imply that there is a cofinal map from κ to α and another from γ to α.

E2 ans. For any set X of primes, it is possible to have a countable model with an element which is only divisible by the primes in X.

E3 ans. Any infinite linear order is elementarily equivalent to some non-scattered order.

M1 ans. (a) False. Let $T = Th(\omega, <, n)_{n\in\omega}$ in language with one additional constant symbol c. Let $\Sigma_0 = \{x > n : n < \omega\}$ and $\Sigma_k = \{(x = x \wedge c < k)\}$ for $k > 0$. (b) True, see Omitting Types Theorem.

M2 ans. See Thm 5.3.5 of Hodges short model theory or Exercise 8 p.293 of his long book. Hint for (a) implies (c): first show that for $X \subseteq A$ algebraically closed and $b \in A \setminus X$ that there exists C and B with A, B elementary substructures of C with $X \subseteq B$ and $b \notin B$.

M3 ans. (a) T has only countable many types, hence the atomic types are dense, so T has a prime model. (b) See the proof of Vaught's never two Theorem.

S1 ans. (a) Let H_α for $\alpha < \omega_2$ list $[\omega_2]^\omega$. Inductively define f so that $f \restriction H_\alpha \times \{\beta\}$ is not constant for $\alpha < \beta$. (b) Assume f's domain is $\omega \times \omega_2$. Let U be a nonprincipal ultrafilter on ω. Find $H_\alpha \in U$ such that $f \restriction H_\alpha \times \{\alpha\}$ is constant. By MA there exists an infinite H with $H \subseteq^* H_\alpha$ for all α. Hence for some n we have $(H \setminus n) \subseteq H_\alpha$ for ω_2 many α.

S2 ans. In the standard construction make sure that when you build a sequence going to the top that you also include all mod finite sequences and that they go thru the dense set the diamond sequence is giving you.

S3 ans. If the tower is filled by $x \in M[G]$, then for some $n < \omega$, $p \in G$, and unbounded $\Gamma \subseteq \gamma$ we have that for each $\alpha \in \Gamma$ that p forces $x \setminus n \subseteq a_\alpha$. But then $\bigcap_{\alpha \in \Gamma} a_\alpha$ fills the tower in M.