Instructions:

Do two E problems and two problems in the area C, M, or S in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. If α, β are non-zero ordinals, then there is a largest ordinal which divides both of them. Here, δ divides α iff $\alpha = \delta\xi$ for some ξ.

E2. Let $L = \{E\}$, where E is a binary relation symbol. Let Σ in L be the axioms that say that E is an equivalence relation. Let ϕ be a sentence of L which is consistent with Σ. Prove that ϕ is true in some finite model of Σ.

E3. Let L be the language with two non-logical symbols f and g. Let T declare that f and g are bijective functions which commute with each other, as well as the axiom scheme stating for all m and n integers (including negative ones) which are not both zero that

$$\forall x(f^m g^n(x) \neq x).$$

Show that T is complete, decidable, and not finitely axiomatizable.

Computability Theory

C1. Recall that $G \in 2^\omega$ is 1-*generic* iff for any computably enumerable set $\mathcal{D} \subseteq 2^{<\omega}$ there exists τ an initial segment of G such that either $\tau \in \mathcal{D}$ or no extension of τ is in $\mathcal{D}$. G is *weakly* 1-*generic* iff for any computable set $\mathcal{D} \subseteq 2^{<\omega}$ which is dense there exists $\tau \in \mathcal{D}$ which is an initial segment of G. Prove there exists a weakly 1-generic which is not 1-generic.

C2. Prove that there is a computable tree $T \subseteq 2^{<\omega}$ with the property that $\{\deg(x) : x \in [T]\}$ is exactly the set of c.e. degrees.

C3. Show that there is a computable linear order isomorphic to $\mathbb{Z} \times \mathbb{Q}$ such that there is no computable embedding of $\mathbb{Q}$ into it.

Model Theory

M1. Let T be a complete countable theory with an infinite model. Show that there are countable models M_q of T for each $q \in \mathbb{Q}$ so that $q < r \to M_q \prec M_r \wedge M_q \neq M_r$.

M2. Let T be a complete countable theory. Show that the following are equivalent:

a) T has a prime model N such that there is an $M \prec N$ with $M \neq N$.

b) T has an uncountable atomic model.

M3. Suppose that T is a first order theory in the language L. Show the following are equivalent.

a) T is equivalent to a set of sentences of the form $\forall x \exists \bar{y} \phi(x, \bar{y})$ where ϕ is quantifier-free,

b) If A is an L-structure and for every $a \in A$, there is a substructure of A which contains a and is a model of T, then A is a model of T.

Sketchy Answers or Hints

E1. Following (*roughly*) Euclid, Proposition VII.2:

Induct on $\max(\alpha, \beta)$. The result is trivial if $\alpha = \beta$, so WLOG $0 < \alpha < \beta$. Dividing, let $\beta = \alpha\mu + \rho$, where $\rho < \alpha$. Assume that $\rho > 0$ (otherwise the result is trivial). Observe that for all δ:

$$\delta \mid \alpha \wedge \delta \mid \beta \iff \delta \mid \rho \wedge \delta \mid \alpha \quad .$$

So, the result for α, β follows from the result for ρ, α, using induction.

E2. Let M be an equivalence relation. Show that $\mathrm{Th}(M)$ can be axiomitized by a set of sentences Γ for which it is easy to see that every finite subset of Γ has a finite model.

E3. Proceed as in Enderton (section 3.1) to show that T allows effective quantifier elimination to get completeness and decidability. Use the Compactness Theorem to show that if T were finitely axiomatizable, then finitely many of our axioms would already axiomatize T, which is clearly false since for any finite fragment of our axioms, we can build a "torus" with both perimeters larger than any iterate of f or g mentioned in the fragment.

C1. Use a priority argument to construct a computable sequence $(\sigma_n \in 2^{<\omega} : n < \omega)$ and a c.e. set $A \subseteq 2^{<\omega}$ such that $G = \lim_n \sigma_n$ is weakly 1-generic and A witnesses that G is not 1-generic. For example, we could make sure that:

1. $G{\restriction}n \notin A$ all n
2. $G{\restriction}n \,\hat{}\, 00 \in A$ for infinitely many n
3. for any $W_e \subseteq 2^{<\omega}$ for some n either $G{\restriction}n \in W_e$ or $G{\restriction}n \,\hat{}\, 01$ has no extension in W_e.

C2. Show that for any W_e there is a computable tree $T_e \subseteq \omega^{<\omega}$ such that T_e has only one infinite branch and that branch is Turing equivalent to W_e. For example, $[T_e] = \{f\}$ where $f(n) = s$ where s is the stage where n enters W_e or $f(n) = 0$ if $n \notin W_e$. Embedding $\omega^{<\omega}$ into $2^{<\omega}$ in the natural way takes T_e to a tree with a unique nonisolated branch.

C3. We can construct the linear order to have the stronger property:

R_e: If W_e infinite, then there are distinct $n, m \in W_e$ which are in the same $\mathbb{Z}$-chain.

Build the order by constructing $\mathbb{Z}$-chains $\mathbb{Z}_q$ for $q \in \mathbb{Q}$. If Z_q^s is the finite approximation at stage s, then at most finitely many are nonempty. Satisfy R_e by clumping together contiguous chains, i.e, put

$$Z_q^{s+1} = \bigcup_{q \leq r \leq q'} Z_r^s$$

and reseting all Z_r^{s+1} to the empty set for $q < r \leq q'$. Use priority to guarantee that no Z_q is reset to the empty set infinitely many times.

M1. Add predicates U_q for each rational and create T' which says that $U_q \subset U_r$ if $q < r$, U_q models T and $U_q \prec U_r$ if $q < r$.

M2. M and N are the prime model of T and isomorphic. Build a continuous elementary chain N_α for $\alpha < \omega_1$ with $(N_\alpha, N_{\alpha+1})$ isomorphic to (M, N).

M3. Let Σ be the set of $\forall\exists$ sentences that T proves. Show that for any $M \models \Sigma$ and $a \in M$ there exists M' with $M \prec M'$ and M' has an substructure containing a which is a model of T. Hint: Let $D_a^\forall(M)$ be all universal sentences with parameter a which are true in M. Show that $T \cup D_a^\forall(M)$ is consistent.