Instructions:

Do two E problems and two problems in the area C, M, or S in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Prove that there is a computable group operation on ω whose center is not computable. So, you need a computable function $*$ from ω^2 into ω which makes ω into a group such that the center:

$$\{x \in \omega : \forall y \in \omega\, [x * y = y * x]\}$$

is not computable.

E2. Suppose A and B are sets of positive reals which are well-ordered by the ordering on the reals. For each of the following show that it is a well-order or give an example showing it may not be.

(a) $A + B = \{a + b \ : \ a \in A \text{ and } b \in B\}$

(b) $AB = \{ab \ : \ a \in A \text{ and } b \in B\}$

(c) $A^B = \{a^b \ : \ a \in A \text{ and } b \in B\}$

(d) $A/B = \{a/b \ : \ a \in A \text{ and } b \in B\}$

E3. Let L be the language containing one binary relation symbol. A graph is a symmetric irreflexive binary relation. It is n-colorable iff there is a map from its universe into n such that no two elements in the relation are assigned the same value.

(a) Show that there is a first order L-theory T whose models are exactly the 3-colorable graphs.

(b) Prove that T is not finitely axiomatizable.

Computability Theory

C1. Prove or disprove: There is a $\emptyset'$-partial computable function f such that for any index e, if $\lim_s \varphi_e(-, s)$ is the characteristic function of a c.e. set S, then $S = W_{f(e)}$.

In this problem $\{\varphi_e\}_{e\in\omega}$ is the standard uniformly computable enumeration of all partial computable functions of two variables.

C2. Let X be a noncomputable c.e. set. Prove that there are disjoint computably inseparable c.e. sets A and B such that $X = A \cup B$.

C3. Prove:

1. If G is 1-generic, then G is hyperimmune.
2. Conclude that if G is 1-generic, then $\overline{G}$ is hyperimmune.
3. Construct a non-1-generic set G such that both G and $\overline{G}$ are hyperimmune.

An infinite $A \subseteq \omega$ is hyperimmune iff for any strong pairwise-disjoint array $D_{f(n)}$ for $n < \omega$ there exists an n with $D_{f(n)}$ disjoint from A. A set $G \in 2^\omega$ is 1-*generic* iff for any computably enumerable set $\mathcal{E} \subseteq 2^{<\omega}$ there exists τ an initial segment of G such that either $\tau \in \mathcal{E}$ or no extension of τ is in $\mathcal{E}$.

Model Theory

M1. Assume that Σ is a complete theory with infinite models in a countable language L . Assume further that $P \in L$ is a unary predicate symbol, and that for any model M of Σ, P_M is an infinite sub-structure of M (so, P_M is closed under the functions of M). Let Σ_P be the theory of these P_M. Consider the following statements.

a. If Σ is $\aleph_0$-categorical, then Σ_P is $\aleph_0$-categorical.

b. If Σ is $\aleph_1$-categorical, then Σ_P is $\aleph_1$-categorical.

c. If Σ is ω-stable, then Σ_P is ω-stable.

Prove (a) and (c) and give a counter-example for (b).

M2. An L-structure M is pseudo-finite if for every L-sentence ϕ which M satisfies, there exists a finite L-structure also satisfying ϕ. Let M be a pseudo-finite L-structure. Let f be a surjective L_M-definable function from M back to itself, i.e., definable by an L-formula possibly using parameters from M. Show that f is bijective.

M3. For a graph G and $x, z \in G$ we say that z is in the n-neighborhood of x if there is path of length $\leq n$ connecting x to z. We say a graph G is locally transitive if for every $n \in \omega$, x and y in G, the n-neighborhoods of x and y are finite and isomorphic by a map taking x to y. A graph G is transitive if for every x and y in G there is an automorphism of G taking x to y.

(a) Prove that a locally transitive graph is transitive.

(b) Prove that if G is a locally transitive graph, then any definable subset of G is finite or cofinite, i.e., G is strongly minimal. Definable means by a formula in one variable (possibly using parameters) in the language with a single binary relation symbol naming the edge relation.

Set Theory

S1. Prove that there is no order preserving map from a Suslin tree into the real numbers.

Here, a Suslin tree is a tree T of size $\aleph_1$ in which every chain and every antichain is countable. Let $\sqsubset$ be the tree order. We call $f : T \to \mathbb{R}$ *order preserving* iff $x \sqsubset y \to f(x) < f(y)$ for all $x, y \in T$.

S2. Assume $V = L$. A *nice theory* is a complete theory T in the language of set theory such that $\{\alpha < \omega_1 : L_\alpha \models T\}$ is uncountable. Prove that there are $\aleph_1$ nice theories.

You may use Tarski's Theorem on the undefinability of truth without proof.

S3. Assume MA. Let E be any subset of $\mathbb{R}$ with $|E| < 2^{\aleph_0}$. Prove that there is a Cantor set $K \subset \mathbb{R}$ and real numbers r_n for $n \in \omega$ such that $E \subseteq \bigcup_n (K + r_n)$.

A Cantor set is a homeomorphic copy of the Cantor space 2^ω.

Hints or Answers

E1. For each prime p, let $B_p = \{p, p^2, p^3, p^4 \ldots\}$, and choose a (possibly empty) subset $A_p \subseteq B_p \backslash \{p, p^2\}$. Let G_p be the group of permutations on B_p generated by (p, p^2) plus all (p, p^n) with $p_n \in A_p$, and let G be the group of permutations on ω generated by $\bigcup_p G_p$. Then $(p, p^2) \in Z(G_p)$ iff $(p, p^2) \in Z(G)$ iff $A_p = \emptyset$. Now, assume that $\{(p, p^n) : p^n \in A_p\}$ is decidable and $\{p : A_p = \emptyset\}$ is undecidable; so G is a decidable set of permutations and $Z(G)$ is undecidable. Then, $*$ is obtained via a computable bijection from ω onto G.

E2. yes, yes, no, no. Show that any sequence in a well-ordered set has a subsequence which is either constant or strictly increasing.

E3. (a) For each $n \geq 3$ there is a first-order sentence which says that every subset of size n can be partitioned into three subsets none of which contains adjacent vertices. (b) For any odd $n > 1$ an n-cycle is not 2-colorable. Adding another point adjacent to all vertices in the n-cycle gives a graph which is not 3-colorable but every proper subgraph is.

C1. Suppose there is such an f. Let $\{f_s\}_s$ be uniformly computable such that $\lim_s f_s(e) = f(e)$ whenever e in the domain of f. Construct $F_{e,s}$ as follows:

1. $F_{e,0} = \{0\}$
2. if $f_s(e) \neq f_{s+1}(e)$, then $F_{e,s+1} = \{s+1\}$
3. if $f_s(e) = f_{s+1}(e)$, $F_{e,s} = \{x\}$, and $x \in W_{f_s(e),s}$, then $F_{e,s+1} = \{\}$
4. otherwise $F_{e,s+1} = F_{e,s}$.

By the recursion theorem there is an e such that $\varphi_e(-, s)$ is the characteristic function of $F_{e,s}$ all s. But $W_{f(e)}$ is not the limit of $F_{e,s}$.

C2. If φ_e is total, show that there must be infinitely many s such that $\varphi_{e,s}(x_s) \downarrow$.

C3. (a) Given a disjoint strong array $D_{f(n)}$ for $n < \omega$ consider

$$\{\sigma \in 2^{<\omega} \ : \ \exists n \ \ D_{f(n)} \subseteq \sigma^{-1}(1)\}$$

(c) Construct G such that for any $n \in G$ either $n+1 \in G$ or $n-1 \in G$.

M1. (a) Observe that every $\mathfrak{B} \models \Sigma_P$ has an elementary extension which is a P_M for some $M \models \Sigma$. When $\kappa = \aleph_0$, it follows that Σ_P has finitely many n–types for each n, and is hence $\aleph_0$–categorical.

(b) A counter-example: Let $L = \{P, Q, R, S, F, G\}$, where P, Q, R, S are unary predicate symbols and F, G are a binary predicate symbols. Let Σ say that Q, R, S partition the universe into infinite sets, $P = Q \cup R$, $F \subseteq Q \times S$ and F a bijection from Q onto S, and $G \subseteq R \times S$ and G a bijection from R onto S.

(c) as in (a).

M2. Suppose $f(x) = y$ is defined via the formula $\phi(x, y, \bar{a})$ where $\bar{a}$ is some set of parameters from M. Take the formula:

"$\exists \bar{z} \phi(x, y, \bar{z})$ defines a surjective function which is not injective".

This formula is first order, so by pseudo-finiteness of M has a finite model. This is a contradiction as all surjections where the domain and range has the same finite size must be injections.

M3. (taken from Constructive Models of Uncountably Categorical Theories, Herwig, Lempp, Ziegler Lemma on pg. 3)

a) Fix c, d to be any elements of the graph. Let (C_i, c) and (D_i, d) be the i-neighborhoods of c and d respectively, and let (C, c) and (D, d) be the connected components of c and d respectively. Look at the set of maps $\{p | \exists i \in \omega \text{ such that } p : (C_i, c) \cong (D_i, d)\}$ ordered by extension. This set forms a finitely branching tree with infinite height, which by König's lemma has an infinite branch. This infinite branch gives an isomorphism between (C, c) and (D, d).

b) Let H be a saturated model of the theory of G, and let $A \subset H$ be any finite set. We show that there is a unique non-algebraic type realized in H over A. It is clear that the type of any element within a connected component of an element of A is algebraic over A via the formula stating (for some n) $x \in Nbh_n(a)$ for some $a \in A$, as this set is given to be finite. It remains only to show that there is a unique type of an element outside of the connected components of elements of A. Let c and d be two elements outside of the connected components of the elements of A. Take an isomorphism between the connected components of c and of d which maps c to d. Extend this to a map on H by fixing every other point of H. Check that this gives an automorphism of H which fixes A and moves c to d. Thus there is a unique non-algebraic type over A realized in H.

S1. Suppose that we had such an f. In some ccc extension of the universe, $V[G]$, we have a path through T (first prune T, and then force with it). But then, f restricted to this path would, in $V[G]$, yield an order preserving map from ω_1 into $\mathbb{R}$, which is impossible.

S2. Let A be the set of all nice theories, and assume that A is countable. Then A is a countable family of subsets of HF $= L(\omega)$, so $A \in L(\omega_1)$, and A is first-order definable in $L(\omega_1)$. Then the L–first injection from A into ω is also first-order definable in $L(\omega_1)$, so every member of A is first-order definable in $L(\omega_1)$.

Let $T = Th(L(\omega_1))$, which is nice because $L(\omega_1)$ has a club of elementary submodels. But then $Th(L(\omega_1)$ is first-order definable in $L(\omega_1)$, contradicting Tarski's theorem on non-definability of truth.

S3. The r_n can enumerate any countable dense set A; so we'll get $E \subseteq K + A$. Let $\mathbb{P}$ be the set of all pairs $p = (U_p, e_p)$, where U_p is a finite union of rational open intervals and $e_p \in [U_p]^{<\omega}$. U_p is an outer approximation to K and e_p is a promise that the "generic" K will contain all points of e_p. So, define $q \leq p$ iff $e_p \subseteq U_q \subseteq U_p$ and $e_q \supseteq e_p$. Note that $\{q : \overline{U_q} \subseteq U_p\}$ is dense below p, so that $\bigcap\{U_p : p \in G\}$ will be a Cantor set if G is generic enough. For $x \in \mathbb{R}$, $\{p : x \in e_p\}$ is not dense, since once x gets kicked out, it stays out, but $\{p : (x + A) \cap e_p \neq \emptyset\}$ is dense.