Instructions:

Do two E problems and two problems in the area C, M, or S in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $\mathcal{L} = \{<, F\}$ where $<$ is a binary relation symbol and F a unary operation symbol. Given any real-valued function $f : \mathbb{R} \to \mathbb{R}$ consider the $\mathcal{L}$-structure

$$M_f = (\mathbb{R}, <, f).$$

(a) Show that there exists an $\mathcal{L}$-sentence θ such that for any f

$$f \text{ is continuous iff } M_f \models \theta.$$

(b) Prove that there is no $\mathcal{L}$-sentence θ such that for any f

$$f \text{ is differentiable iff } M_f \models \theta.$$

E2. Prove that the class of simple groups is not axiomatizable, i.e., there is no set of first-order sentences Σ such that the class of simple groups is exactly the class of models of Σ.

Recall that a group is simple iff its only normal subgroups are itself and the trivial subgroup. A subgroup H of G is normal iff $gHg^{-1} = H$ for every $g \in G$.

E3. Let X be any set and $f : \mathcal{P}(X) \to \mathcal{P}(X)$ be order preserving, i.e., for any $A, B \in \mathcal{P}(X)$ if $A \subseteq B$, then $f(A) \subseteq f(B)$. Prove there exists $Y \subseteq X$ such that $f(Y) = Y$.

Computability Theory

C1. Let $g\colon \omega \to \omega$ be Δ^0_2. Prove that there is an $e \in \omega$ such that W_e is computable, but $\mu n\, [W_n = \overline{W_e}] > g(e)$.

Hint: ensure that W_e intersects every nonempty c.e. set W_i for $i \leq g(e)$.

C2. Prove there is a nonempty Δ_2 set A that is not the range of a limitwise monotonic function, i.e., an $f : \omega \to \omega$ for which there is a computable function $g(s,x)$ such that $g(s,x) \leq g(s+1,x)$ for all s, x and

$$f(x) = \lim_{s \to \infty} g(s,x).$$

C3. Prove there are hyperimmune A and B such that $A \cup B$ is immune but not hyperimmune.

Recall that a subset of ω is immune iff it is infinite but contains no infinite computable subset. An infinite $A \subseteq \omega$ is hyperimmune iff for any strong pairwise-disjoint array $D_{f(n)}$ for $n < \omega$ there exists an n with $D_{f(n)}$ disjoint from A.

Answers

E1. For (a), write out the usual definition of continuity in a crowded LOTS:

$$\forall x, y_1, y_2\, [f(x) \in (y_1, y_1) \to \exists x_1, x_2\, [x \in (x_1, x_2) \wedge f((x_1, x_2)) \subseteq (y_1, y_2)]]$$

Of course, you have to translate away the $\in$ and $\subseteq$.

For (b), the notion isn't even invariant under isomorphism:
Let $f(x) = -x$, and let $\Gamma(x)$ be x for $x \leq 0$ and $2x$ for $x \geq 0$. Then f is differentiable, but $g = \Gamma f \Gamma^{-1}$ is not.

E2. The only abelian simple groups are the cyclic groups of prime order. But any elementary class with arbitrarily large finite models must have an infinite model.

E3. Let $Y = \bigcup\{A : A \subseteq f(A)\}$. Another way to construct Y is by using a transfinite chain argument.

C1. Let $g(s, e)$ be computable with $g(e) = \lim_s g(s, e)$. Construct a computable $h(e)$ so that $W_{h(e)}$ is finite and meets any W_i for $i < g(s, e)$ which is nonempty. Since $g(s, e)$ can change at most finitely many times this is possible. By the recursion theorem we can find e with $W_e = W_{h(e)}$.

C2. Construct an infinite A computable in $0'$ as follows. Suppose at stage n we have $a_1 < \cdots < a_n$ and for every $e < n$ we have either declared e finished or we have (permanently) assigned a column x_e such that

$$\exists s\ \psi_e(x_e, s) \downarrow > a_n$$

First we consider the next e, ie., $e = n$. We ask $0'$ if it is possible to find x, s, t with $s < t$ such that $\psi_e(x, s) \downarrow > \psi_e(x, t) \downarrow$. If yes, we declare e finished. Next we ask $0'$ if it is possible to find x, s with $\psi_e(x, s) \downarrow > a_n$. If no, we declare e finished, if yes, we find such an x and make it x_e.

Second we pick $a_{n+1} > a_n$. For each $e \leq n$ which is unfinished, we ask the oracle if there exists an s such that $\psi_e(x_e, s) \downarrow > a_n + n + 2$. If the answer is no, then we may use the oracle to find the permanent value p_e such that $\exists s\ \psi_e(x_e, s) \downarrow = p_e$ and for any $t > s$ if $\psi_e(x_e, t)$ converges, then it converges to p_e also. Since there at most $n + 1$ of these p_e we may choose a_{n+1} with $a_n < a_{n+1} < a_n + n + 2$ unequal to any of the p_e.

Finally for any unfinished $e \leq n$ for which $p_e < a_{n+1}$ we declare e finished.

C3. Any infinite set can be split into two hyperimmune sets. Let f_k list all computable functions coding a strong pairwise disjoint array. Given C construct a strictly increasing sequence $n_k < n_{k+1}$ such that the interval $[n_k, n_{k+1})$ meets C and contains some $D_{f_k(m)}$ and some $D_{f_{k+1}(l)}$. Decompose C using every other interval.