Instructions:

Do two E problems and two problems in the area C, M, or S in which you signed up.
Write your letter code on all of your answer sheets.
If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. For an abelian group $G$ and prime $p$ we say that $G$ is divisible by $p$ iff for every $x \in G$ there is a $y \in G$ such that $py = x$. Prove that there is a computable $G \subseteq \mathbb{Q}$ which is a subgroup of $(\mathbb{Q}, +)$ but

$$\{ p : G \text{ divisible by } p \}$$

is not computable.

E2. For each prove or disprove:

(a) There exists a set $D$ of reals with the same order type as the rationals which is a closed subset of the real number line.

(b) There exists a set $D$ of reals with the same order type as the rationals which is discrete, i.e., no point of $D$ is a limit point of $D$.

(c) There exists a set $D$ of reals with the same order type as the rationals such that every point of $D$ is a limit point of $D$ but only from below and not above.

E3. Let $B = \bigcup_{i \in \omega} [p_i, q_i] \subset \mathbb{R}$, where each $p_i < q_i$, the intervals $[p_i, q_i]$ are pairwise disjoint, and all $p_i, q_i$ are rational. Let $A = B \cap \mathbb{Q}$. View $A, B$ as structures for $\mathcal{L} = \{ < \}$. Prove that $A$ is an elementary substructure of $B$. 
Computability Theory

C1. Show that the intersection of two hyper-simple sets is hyper-simple.

Recall that $A$ is hyper-simple\(^1\) iff it is c.e., co-finite and there is no computable function $f$ such that for all $n$, $f(n)$ is greater than the $n$th element of the complement of $A$, i.e., the complement is hyper-immune.

C2. Let $S$ be a class of c.e. sets closed under finite variation that contains the computable sets but not all the c.e. sets. Let $I = \{e : \text{W}_e \in S\}$ and let $A$ be any $\Pi_2^0$-set. Prove that $A \leq_m I$.

Comment: In fact, this can be shown for any $\Sigma_3^0$-set $A$ as well.

C3. Show that every non-computable c.e. set computes a 1-generic.

Recall that a set $G \in 2^\omega$ is 1-generic iff for any computably enumerable set $D \subseteq 2^{<\omega}$ there exists $\tau$ an initial segment of $G$ such that either $\tau \in D$ or no extension of $\tau$ is in $D$.

\(^1\)Of course, if you prefer, you may use the definition of hyper-simple in terms of disjoint strong arrays, as it is given in Soare’s book.
Set Theory

**S1.** Let $\lambda$ be an infinite cardinal, and assume $\text{MA}(\lambda)$. Let $\mathbb{P}$ be a ccc poset. Let $D_\alpha \subseteq \mathbb{P}$ be dense in $\mathbb{P}$ for $\alpha < \lambda$, and fix $E \subseteq \mathbb{P}$ with $|E| \leq \lambda$. Prove that there are filters $G_n \subseteq \mathbb{P}$ for $n < \omega$ such that $G_n \cap D_\alpha \neq \emptyset$ for each $n$ and $\alpha$, and such that $E \subseteq \bigcup_n G_n$.

Note that without the $E$, this is obvious by the definition of $\text{MA}(\lambda)$.

**S2.** Let $T$ be any special Aronszajn tree, and let $L_\alpha$ be the $\alpha^{\text{th}}$ level of $T$ for $\alpha < \omega_1$. Prove that there is a map $\varphi : T \to \mathbb{Q}$ such that $\varphi$ is order-preserving ($x < y \rightarrow \varphi(x) < \varphi(y)$) and such that $\varphi|L_\alpha$ is a 1-1 function for each $\alpha$.

A tree is a partial order $(T, \leq)$ such that $\{y \in T : y < x\}$ is well-ordered by $<$ for each $x \in T$. The level of $x$ is the corresponding ordinal. A tree $T$ is Aronszajn iff $T$ does not have an uncountable chain but $L_\alpha$ is nonempty and countable for every $\alpha < \omega_1$. An Aronszajn tree is special iff it is a countable union of antichains.

**S3.** In the ground model, let $\mathbb{P}$ be a ccc poset, and assume that $1$ forces that $\mathbb{P}$ adds a new subset of $\omega_2$. Prove that $1$ forces that $\mathbb{P}$ adds a new subset of $\omega_1$. 


E1. Take a set $P$ of primes which is computably enumerable but not computable with computable enumeration $P_s$ and look at the group generated by the set of $1/p$ where $p$ is a prime not in $P_s$.

E2. (a) This is impossible. $\mathbb{Q}$ is not Dedekind complete, so fix $A \subset D$ so that $A$ is bounded above but has no least upper bound in $D$. Then in $\mathbb{R}$, $\sup(A) \notin D$, so $D$ is not closed.

(b) Copy the proof that every countable order type embeds into $\mathbb{R}$. List $\mathbb{Q}$ as $\{q_n : n \in \omega\}$ and choose disjoint closed intervals $I_n = [a_n, b_n] \subset \mathbb{R}$ such that $q_n < q_m$ iff $I_n < I_m$. Then let $D = \{(a_n + b_n)/2 : n \in \omega\}$.

(c) Choose the $I_n$ as in (b), but make sure that $\bigcup_n I_n$ is dense in $\mathbb{R}$ (for example, make sure that $\mathbb{Q} \subset \bigcup_n I_n$). Then let $D = \{a_n : n \in \omega\}$.

E3. Observe that for all $a_1, \ldots, a_n \in A$ and $b \in B$, there is an automorphism $\alpha$ of $B$ such that $\alpha(b) \in A$ and each $\alpha(a_i) = a_i$, since you can always move an irrational to a nearby rational by a piecewise linear map. Now, apply the Tarski–Vaught criterion for elementary submodel.

C1. Assume $A$ and $B$ are c.e. and cofinite but $A \cap B$ is not hyper-simple. So there is a computable function $f$ such that, for all $n$, the $n$th element of the complement of $A \cap B$ is less than $f(n)$. Consider the c.e. set $S$ of all $n$ such that the $n$th element of the complement of $A$ is not less than $f(2n-1)$. If $S$ is finite, then a finite variation of $f(2n-1)$ witnesses that $A$ is not hyper-simple. So assume that $S$ is infinite. For $n \in S$, there are at least $n$ elements in the complement of $B$ less than $f(2n-1)$. Define a computable function $g$ as follows. To compute $g(m)$, find $n \in S$ with $n \geq m$ and let $g(m) = f(2n-1)$. Then $g$ witnesses that $B$ is not hyper-simple.

C2. Let $W$ be a c.e. set not in $S$ and let $A = \{m : (\forall n)(\exists k) R(m, n, k)\}$, where $R$ is computable. Let $f$ be a computable map such that $W_f(m) = W \cup \{q : (\forall n \leq q)(\exists k) R(m, n, k)\}$. If $m \in A$, then $W_f(m) = \omega$. If $n \notin A$, then $W_f(m) =^* W$. Thus $A \leq_m I$.

C3. Let $A_s$ be a uniformly computable enumeration of $A$. Construct sequence $(\tau_n : n < \omega)$ of elements of $2^{<\omega}$ computable in $A$ as follows. At
stage $n$ let $s$ be the least such that $A_s \cap n = A \cap n$. Look for an $e < n$ such that $W_{e,s} \cap \{ \rho : \rho \subseteq \tau_n \}$ is empty but there exists $\rho \in W_{e,s}$ such that $\tau_n \subseteq \rho$. For the least such $e$ (if there is one) put $\tau_{n+1} = \rho$ for the least such $\rho$.

**S1.** If $\lambda = \aleph_0$, this is easy by the Generic Filter Existence Theorem, so assume that $\lambda > \aleph_0$. Let $Q$ be the set of all $\vec{q} \in P_\omega$ such that $q_n = 1$ for all but finitely many $n$. Order $Q$ coordinate-wise; then $Q$ is ccc by MA($\aleph_1$). Let $H$ be a filter on $Q$ meeting the dense sets $\{ \vec{q} : q_n \in D_\alpha \}$ for each $n, \alpha$, and also meeting $\{ \vec{q} : \exists n [q_n \leq e] \}$ for each $e \in E$.

**S2.** Let $T = \bigcup_{n \in \omega} A_n$, where each $A_n$ is an antichain. WLOG, the $A_n$ are disjoint. Also, since each $L_\alpha$ is countable, WLOG each $|A_n \cap L_\alpha| \leq 1$. Now, define $\varphi|A_n$ in the standard way by induction on $n$; let $\varphi(A_0) = \{0\}$, $\varphi(A_1) = \{-1, 1\}$, $\varphi(A_2) = \{-3/2, -1/2, 1/2, 3/2\}$, etc.

**S3.** If this fails, then we have a $p \in P$ and a name $\dot{f}$ such that $p$ forces $\dot{f}$ to be a new function from $\omega_2$ into 2. Working always in the ground model, define a subtree $T \subset 2^{<\omega_2}$, where the nodes at level $\alpha$ of $T$ are those $s : \alpha \rightarrow 2$ such that some $q \leq p$ forces $\dot{f}|\alpha = s$. Then $T$ is a tree of height $\omega_2$ such that all levels of $T$ are countable and non-empty, but $T$ has no cofinal path, which is a contradiction.