Instructions:
Do two E problems and two C problems.
Write your letter code on on all of your answer sheets.
If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1.
Prove that there is a computable equivalence relation on $\omega$ all of whose equivalence classes are finite such that the set of these finite sizes is a non-computable set.

E2.
Let $A$ be any set. Prove that there is a subset $E \subseteq \mathcal{P}(A)$ such that

1. Whenever $X \subseteq A$ is finite: $X \in E$ iff $|X|$ is even.

2. Whenever $X, Y \in \mathcal{P}(A)$ are disjoint: $X \cup Y \in E$ iff either $X, Y \in E$ or $X, Y \notin E$.

E3.
In this problem, a real-valued function means a partial function $F$ with $\text{dom}(F) \subseteq \mathbb{R}$ and $\text{ran}(F) \subseteq \mathbb{R}$; then, as a set, $F \subseteq \mathbb{R} \times \mathbb{R}$. Call such an $F$ monotonic iff it satisfies either $\forall x_1, x_2 \in \text{dom}(F) \ [x_1 < x_2 \rightarrow F(x_1) \leq F(x_2)]$ or $\forall x_1, x_2 \in \text{dom}(F) \ [x_1 < x_2 \rightarrow F(x_1) \geq F(x_2)]$. Assuming the Continuum Hypothesis, prove that there is a real-valued function $G$ such that $\text{dom}(G) = \mathbb{R}$ and $G \cap F$ is countable for all monotonic real-valued functions $F$. 
Computability Theory

C1.
Recall that $A \leq_{wtt} B$ (weak truth table reducible) iff $A$ is Turing reducible to $B$ by an algorithm for which the use is computably bounded. This means there exists an oracle machine $e$ such that $A = \{e\}^B$ and a computable function $f$ such that for every $n$ the computation $\{e\}^B(n)$ only asks the oracle about $k$'s bounded by $f(n)$.

Define $A \leq_{bqtt} B$ (bounded query truth table reducible) iff $A$ is Turing reducible to $B$ by an algorithm for which there is a computable bound on the number of queries to the oracle.

Prove or disprove: weak truth table reducible is the same as bounded query truth table reducible.

C2.
Recall that a set $G \in 2^\omega$ is 1-generic iff for any computably enumerable set $D \subseteq 2^{\omega}$ there exists $\tau$ an initial segment of $G$ such that either $\tau \in D$ or no extension of $\tau$ is in $D$. Show that no 1-generic computes a non-computable c.e. set.

C3.
A *computable numbering* of a family $F$ of c.e. sets is a surjective and infinite-to-one function $\nu : \omega \rightarrow F$ such that the predicate “$x \in \nu(e)$” is (uniformly) c.e. Call two computable numberings $\mu$ and $\nu$ *equivalent* if there is a computable permutation $p$ of $\omega$ such that $\mu \circ p = \nu$.

Show that a finite family $F$ of c.e. sets has only one computable numbering (up to equivalence) iff there are do not exist distinct sets $A, B \in F$ with $A \subseteq B$. 
Answers - Sketch

E1.
Let \( k_n \) for \( n \in \omega \) be a computable enumeration of \( K \). Define
\[
a_{n+1} = a_n + k_n + 1
\]
Take the equivalence relation whose equivalence classes are \([a_n, a_{n+1})\).

E2.
If \( \mathcal{F} \) is any finite subalgebra of \( \mathcal{P}(A) \) and \( F \) is the union of all finite elements of \( \mathcal{F} \), we could define \( X \in \mathcal{E}_\mathcal{F} \) iff \( |X \cap F| \) is even. Use the Compactness Theorem.

E3.
Under CH, there are \( 2^{\aleph_1} \) monotonic functions, but there are only \( \aleph_1 \) closed sets. Note that the closure of a monotonic function has the property that each vertical slice has size 0, 1, or 2. Inductively construct \( G \).

C1.
They are not the same. Let \( B_0 \) be the set of all \( e \) such that \( \psi_e \) is total and strictly increasing. Define
\[
x_n = \max\{\psi_e(n) + 1 : e < n \text{ and } e \in B_0\}
\]
Construct \( A \) and \( B_1 \) so that \( n \in A \) iff \( x_n \in B_1 \), but \( A \) is not weak truth table reducible to \( B = B_0 \oplus B_1 \).

C2.
Suppose \( G \) is 1-generic, \( X \) is c.e., and \( \{e\}^G = X \). Consider
\[
D = \{\tau \in 2^{<\omega} : \exists n \{e\}^{\tau}(n) \downarrow = 0 \text{ and } n \in X\}.
\]

C3.
Suppose \( A, B \in F \) and \( A \) is a proper subset of \( B \). Let \( \mu \) be a numbering of \( F \) for which the inverse image of each element of \( F \) is an infinite computable set. Define \( \nu \) by \( \nu(2n) = \mu(n) \) and
\[
\nu(2n + 1) = \begin{cases} B & \text{if } n \in K \\ A & \text{otherwise} \end{cases}
\]