Computability
Logic Qualifying Exam
January 2009

Instructions:

Do two E problems and two C problems.

Write your letter code on on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1.

Prove that there is a computable equivalence relation on ω all of whose equivalence classes are finite such that the set of these finite sizes is a non-computable set.

E2.

Let A be any set. Prove that there is a subset $\mathcal{E} \subseteq \mathcal{P}(A)$ such that

1. Whenever $X \subseteq A$ is finite: $X \in \mathcal{E}$ iff $|X|$ is even.

2. Whenever $X, Y \in \mathcal{P}(A)$ are disjoint: $X \cup Y \in \mathcal{E}$ iff either $X, Y \in \mathcal{E}$ or $X, Y \notin \mathcal{E}$.

E3.

In this problem, a *real-valued function* means a partial function F with $\mathrm{dom}(F) \subseteq \mathbb{R}$ and $\mathrm{ran}(F) \subseteq \mathbb{R}$; then, as a set, $F \subseteq \mathbb{R} \times \mathbb{R}$. Call such an F *monotonic* iff it satisfies either $\forall x_1, x_2 \in \mathrm{dom}(F)\,[x_1 < x_2 \rightarrow F(x_1) \leq F(x_2)]$ or $\forall x_1, x_2 \in \mathrm{dom}(F)\,[x_1 < x_2 \rightarrow F(x_1) \geq F(x_2)]$. Assuming the Continuum Hypothesis, prove that there is a real-valued function G such that $\mathrm{dom}(G) = \mathbb{R}$ and $G \cap F$ is countable for all monotonic real-valued functions F.

Computability Theory

C1.

Recall that $A \leq_{wtt} B$ (weak truth table reducible) iff A is Turing reducible to B by an algorithm for which the use is computably bounded. This means there exists an oracle machine e such that $A = \{e\}^B$ and a computable function f such that for every n the computation $\{e\}^B(n)$ only asks the oracle about k's bounded by $f(n)$.

Define $A \leq_{bqtt} B$ (bounded query truth table reducible) iff A is Turing reducible to B by an algorithm for which there is a computable bound on the number of queries to the oracle.

Prove or disprove: weak truth table reducible is the same as bounded query truth table reducible.

C2.

Recall that a set $G \in 2^\omega$ is 1-generic iff for any computably enumerable set $D \subseteq 2^{<\omega}$ there exists τ an initial segment of G such that either $\tau \in D$ or no extension of τ is in D. Show that no 1-generic computes a non-computable c.e. set.

C3.

A *computable numbering* of a family F of c.e. sets is a surjective and infinite-to-one function $\nu : \omega \to F$ such that the predicate "$x \in \nu(e)$" is (uniformly) c.e. Call two computable numberings μ and ν *equivalent* if there is a computable permutation p of ω such that $\mu \circ p = \nu$.

Show that a *finite* family F of c.e. sets has only one computable numbering (up to equivalence) iff there are do not exist distinct sets $A, B \in F$ with $A \subseteq B$.

Answers - Sketch

E1.

Let k_n for $n \in \omega$ be a computable enumeration of K. Define

$$a_{n+1} = a_n + k_n + 1$$

Take the equivalence relation whose equivalence classes are $[a_n, a_{n+1})$.

E2.

If $\mathcal{F}$ is any finite subalgebra of $\mathcal{P}(A)$ and F is the union of all finite elements of $\mathcal{F}$, we could define $X \in \mathcal{E}_{\mathcal{F}}$ iff $|X \cap F|$ is even. Use the Compactness Theorem.

E3.

Under CH, there are $2^{\aleph_1}$ monotonic functions, but there are only $\aleph_1$ closed sets. Note that the closure of a monotonic function has the property that each vertical slice has size 0, 1, or 2. Inductively construct G.

C1.

They are not the same. Let B_0 be the set of all e such that ψ_e is total and strictly increasing. Define

$$x_n = \max\{\psi_e(n) + 1 : e < n \text{ and } e \in B_0\}$$

Construct A and B_1 so that $n \in A$ iff $x_n \in B_1$, but A is not weak truth table reducible to $B = B_0 \oplus B_1$.

C2.

Suppose G is 1-generic, X is c.e., and $\{e\}^G = X$. Consider

$$D = \{\tau \in 2^{<\omega} : \exists n \ \{e\}^\tau(n) \downarrow = 0 \text{ and } n \in X\}.$$

C3.

Suppose $A, B \in F$ and A is a proper subset of B. Let μ be a numbering of F for which the inverse image of each element of F is an infinite computable set. Define ν by $\nu(2n) = \mu(n)$ and

$$\nu(2n+1) = \begin{cases} B & \text{if } n \in K \\ A & \text{otherwise} \end{cases}$$