Qualifying Exam
Logic
August 2006

Instructions:

If you signed up for Model Theory, do two E and two M problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $\mathcal{L} = \{<\}$.

a. Prove that $(\omega; <) \models \varphi$ iff $(\omega + \omega; <) \models \varphi$ whenever φ is a Σ_2 sentence of $\mathcal{L}$.

b. Write a Σ_3 sentence of $\mathcal{L}$ which is true in $(\omega + \omega; <)$ and false in $(\omega; <)$.

Σ_2 sentences are of the form $\exists\vec{x}\,\forall\vec{y}\,\psi(\vec{x}, \vec{y})$, and Σ_3 sentences are of the form $\exists\vec{x}\,\forall\vec{y}\,\exists\vec{z}\,\psi(\vec{x}, \vec{y}, \vec{z})$, where ψ has no quantifiers and $\vec{x}, \vec{y}, \vec{z}$ denote finite tuples of variables.

E2. Let A be any set and $f : A \to A$ any function such that $f(x) \neq x$ for all $x \in A$. Prove that one can partition A into three sets $A = B \cup C \cup D$ such that $f(B) \cap B = \emptyset$ and $f(C) \cap C = \emptyset$ and $f(D) \cap D = \emptyset$.

E3. A theory is *quasifinitely axiomatizable* if it can be axiomatized by the usual set of sentences (just using "=") stating that the model is infinite, plus a finite number of other sentences. Characterize all quasifinitely axiomatizable complete theories of a single equivalence relation.

Model Theory

M1. If $\mathcal{L}$ is countable and Σ is a complete theory in $\mathcal{L}$ with infinite models, let $n(\Sigma)$ be the number of non-isomorphic countable models of Σ; so $1 \le n(\Sigma) \le 2^{\aleph_0}$.

If $\mathcal{L}' \subseteq \mathcal{L}$, let $\Sigma \restriction \mathcal{L}'$ be the set of all sentences of Σ which only use symbols of $\mathcal{L}'$.

a. Give an example where $\aleph_0 > n(\Sigma \restriction \mathcal{L}') > n(\Sigma)$.
b. Prove that if $n(\Sigma) = 1$ then $n(\Sigma \restriction \mathcal{L}') = 1$.

M2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be infinite structures for a countable language $\mathcal{L}$. Then $\mathfrak{A} \times \mathfrak{B}$ is also a structure for $\mathcal{L}$, with all relations and functions evaluated coordinatewise. Now suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are models for the same $\aleph_1$–categorical theory Σ and $\mathfrak{C}$ is elementarily equivalent to $\mathfrak{A} \times \mathfrak{B}$. Assume that a product of two models of Σ is also a model of Σ. Must $\mathfrak{C}$ then be isomorphic to some product, $\mathfrak{A}' \times \mathfrak{B}'$? Give either a proof or a counterexample.

M3. Characterize all $\aleph_0$-categorical theories of a single unary 1–1 function.

Answers

E1. For (b), your sentence can say "there is a limit ordinal":

$$\exists w \exists z \forall x \exists y \left[z < w \ \wedge \ [x < w \to x < y < w] \right]$$

For (a), note that if θ is a Π_1 property true of some $\vec{a}$ in $\omega + \omega$, then θ is also true (in ω and in $\omega + \omega$) of some $\vec{b}$ coming from ω. In particular, if $\vec{a}$ is $(p_1, \ldots, p_m,\ \omega + q_1, \ldots, \omega + q_n)$, where $p_1, \ldots, p_m, q_1, \ldots, q_n < \omega$, then $\vec{b}$ can be $(p_1, \ldots, p_m,\ K + q_1, \ldots, K + q_n)$, for a large enough finite K.

E2. View A as the set of nodes of a graph, where there is an edge between x and y iff $f(x) = y$ or $f(y) = x$. Now, you want to prove that this graph is 3-colorable. By the Compactness Theorem, it's enough to prove that every finite subgraph is 3-colorable, and this can be done by induction on the size, n, of the subgraph. Assuming that it works for sizes $< n$, choose a note z with at most two other nodes connected to it, color the other $n - 1$ nodes, and then assign z a color different from the nodes connected to it. There is such a node z because the subgraph with n nodes can have at most n edges (since f is a function).

E3. There are two cases: Either there is an infinite equivalence class; then, by quasifinite axiomatizability, this equivalence class must be cofinite, and on the finite complement, any configuration is quasifinitely axiomatizable. Or there is no infinite equivalence class; then by quasifinite axiomatizability, there must be a fixed bound on the size of all equivalence classes, and only one of these sizes can occur infinitely often, so the model is the union of a finite set with arbitrary classes plus infinitely many classes of a fixed finite size.

M1. For (b), use the fact that $n(\Sigma) = 1$ iff there are finitely many k–types for each k. For (a), let $\mathcal{L}' = \{<\} \cup \{c_j : j \in \mathbb{Z}\}$, and let $\Sigma \restriction \mathcal{L}'$ say that $\cdots c_{-2} < c_{-1} < c_0 < c_1 < c_2 < \cdots$ and that $<$ is a dense total order without endpoints. So, this is like Ehrenfeucht's example at both ends, and $n(\Sigma \restriction \mathcal{L}') = 3 \cdot 3 = 9$. Let $\mathcal{L} = \mathcal{L}' \cup \{f\}$, where f is a unary function symbol, and let Σ say that f is an order-reversing bijection, with $f(f(x)) = x$ and $f(c_j) = c_{-j}$ for each j. This forces both ends to be the same, so $n(\Sigma) = 3$.

M2. Let Σ be the axioms for torsion-free divisible abelian groups. Here, we can view models of Σ as vector spaces over $\mathbb{Q}$, and a product of two models must have dimension at least 2, so the model of dimension 1 cannot be a product.

M3. Suppose first that there is some element x such that $f^n(x) \neq x$ for all $n > 0$. Then any 2-type containing the formulas $f^n(x) \neq f^m(x)$ for any $n, m \geq 0$ cannot be isolated (and some such 2-type is consistent with the theory), contradicting Ryll-Nardzewski. A similar argument using Ryll-Nardzewski shows that in fact there is a fixed bound n_0 such that for any element x, $f^n(x) = x$ for some n with $0 < n \leq n_0$. These are the only limitations, i.e., any theory specifying for a finite number of cycle sizes that there are such and such finite number, or infinitely many, cycles of that size is $\aleph_0$-categorical.