Qualifying Exam
Logic
January 2006

**Instructions:**

If you signed up for Computability Theory, do two E and two C problems. If you signed up for Model Theory, do two E and two M problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

**E1.** Prove that a subset $E$ of $\omega$ is first-order definable in $(\omega; <)$ iff $E$ is either finite or cofinite. Here, "definable" means that there is a formula $\varphi$ with just $x$ free such that $E = \{n \in \omega : (\omega; <) \models \varphi[n]\}$.

**E2.** Let $\xi$ be a non-zero ordinal.

a. Prove that there are ordinals $\alpha < \beta$ such that $\alpha + \beta = \beta + \alpha + \xi$.
b. Find the least possible $\beta$ such that there is an $\alpha < \beta$ satisfying $\alpha + \beta = \beta + \alpha + \xi$.

**E3.** Prove that the theory of atomless boolean algebras admits quantifier elimination. Here, $\mathcal{L} = \{\vee, \wedge, ', 0, 1\}$ (where $'$ denotes complement). You have to show that for each formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$, there is a quantifier-free formula $\psi(x_1, \ldots, x_n)$ of $\mathcal{L}$ such that $\varphi$ and $\psi$ are equivalent in all atomless boolean algebras.

**C1.** Let $Dom(A) = \{x \mid W_x \subseteq A\}$. Show that if $A$ is productive then so is $Dom(A)$. *Note.* A set $P$ is *productive* iff there is a p.c. function $\psi$ such that for all $x$: if $W_x \subseteq P$ then $\psi(x) \downarrow \in P - W_x$.

**C2.** Show that the $m$-degrees form a *distributive* upper semilattice, i.e., whenever $A \leq_m B \oplus C$ then there are sets $A_0 \leq_m B$ and $A_1 \leq_m C$ such that $A \equiv_m A_0 \oplus A_1$.

**C3.** Call an infinite set $A = \{a_0 < a_1 < \dots\}$ *retraceable* if there is a p.c. function $\psi$ such that $\psi(a_0) = a_0$ and $\psi(a_{i+1}) = a_i$ for all $i \in \omega$. Show that if $A$ and $\overline{A}$ are both retraceable then $A$ is computable. *Hint.* Start by showing that $A \oplus \overline{A}$ is retraceable.

Model Theory

**M1.** Let $\mathcal{L}$ consist of a single unary function symbol $f$, and let $T$ say that $f$ is a bijection admitting infinitely many cycles of every finite length.

Show that $T$ is complete and $\omega$-stable. Classify its 1-types over a model and calculate their Morley rank. Describe the prime model and the countable saturated model of $T$.

**M2.** Fix a complete theory $T$. Say that a formula $\phi(\bar{x}, \bar{y})$ has the *independence property* if for every $n$, every model $M \models T$ contains $\bar{b}_i$ for $i < n$ and $\bar{a}_\sigma$ for $\sigma \in \{0, 1\}^n$ such that for each $i, \sigma$, $M \models \phi(\bar{a}_\sigma, \bar{b}_i)$ iff $\sigma_i = 1$. Here, $\bar{a}_\sigma$ is a tuple of the length of $\bar{x}$ and $\bar{b}_i$ is a tuple of the length of $\bar{y}$. Then, $T$ has the *independence property* if some formula does.

Assume that the complete theory $T$ does NOT have the independence property, and let $(a_i : i < \omega)$ be an indiscernible sequence in some model $M \models T$.

a. Let $\phi(x, \bar{b})$ be a formula with parameters in $\bar{b} \in M$. Show that the sequence of truth values of $\phi(a_i, \bar{b})$ is eventually constant.

b. Let $\mathrm{Avg}(a_i : i < \omega, M)$ consist of all formulae $\phi(x, \bar{b})$ as in (a) such that $\phi(a_i, \bar{b})$ is eventually true. Show that $\mathrm{Avg}(a_i : i < \omega, M)$ is a 1–type over $M$.

**M3.** Let $T$ be a complete theory in a countable language, $M$ a model of $T$, and $a, b \in M$. Let $p(x, y) = \mathrm{tp}(a, b)$, $q(y) = \mathrm{tp}(b)$. Say that *$a$ semi-isolates $b$* if there is a formula $\phi(x, y) \in p$ such that $\phi(a, y) \vdash q(y)$.

Show that the following are equivalent for types $q, r \in S(T)$:

1. Every model of $T$ realizing $r$ also realizes $q$.
2. There is a model $M$ of $T$, and $a, b \in M$ realizing $r$ and $q$, respectively, such that $a$ semi-isolates $b$.

Answers

**E1.** Finite and cofinite sets are definable (without parameters) because every element of $\omega$ is definable in $(\omega; <)$. Now, suppose that $E$ is neither finite nor cofinite and $E = \{n \in \omega : (\omega; <) \models \varphi[n]\}$. Then $(\omega; <) \models \psi$, where $\psi$ is $\forall x \exists y > x\, [\varphi(y) \wedge \neg\varphi(S(y))]$. Note that $S(y)$, the successor of $y$, is definable in $(\omega; <)$.

Fix $\mathfrak{A}$ such that $\mathfrak{A} \equiv (\omega; <)$ and $\mathfrak{A} \not\cong (\omega; <)$. Then $\mathfrak{A}$ consists of an $\omega$ at the beginning, followed by some blocks of order type $\mathbb{Z}$.

Since $\mathfrak{A} \models \psi$, fix $a \in A$ such that $a \notin \omega$ and $\mathfrak{A} \models \varphi(a) \wedge \neg\varphi(S(a))$. But this is a contradiction, since there is an automorphism of $\mathfrak{A}$ moving $a$ to $S(a)$.

**E2.** Fix $\mu$ with $\omega^\mu \leq \xi < \omega^{\mu+1}$. Let $\alpha = \omega^{\mu+1}$ and let $\beta = \omega^{\mu+1} + \xi$. Then $\alpha + \beta = \beta + \alpha + \xi = \omega^{\mu+1} \cdot 2 + \xi$. Now, suppose $\alpha < \beta < \omega^{\mu+1} + \xi$. We show that $\alpha, \beta$ cannot work.

First, say $\beta = \omega^{\mu+1} + \eta$, where $\eta < \xi$. If $\alpha < \omega^{\mu+1}$ then $\alpha + \beta = \beta < \beta + \alpha + \xi$. If $\alpha = \omega^{\mu+1} + \zeta$ with $\zeta < \eta$, then $\alpha + \beta = \omega^{\mu+1} \cdot 2 + \eta$ and $\beta + \alpha + \xi = \omega^{\mu+1} \cdot 2 + \zeta + \xi > \alpha + \beta$.

So, $\alpha < \beta < \omega^{\mu+1}$. If $\beta < \omega^\mu$, then $\alpha + \beta < \xi \leq \beta + \alpha + \xi$, so $\omega^\mu \leq \beta < \omega^{\mu+1}$. Then, if $\alpha < \omega^\mu$, we have $\alpha + \beta = \beta < \beta + \alpha + \xi$. Thus, $\omega^\mu \leq \alpha < \beta < \omega^{\mu+1}$.

Say $\beta = \omega^\mu \cdot n + \eta$ and $\alpha = \omega^\mu \cdot m + \zeta$, where $m, n \neq 0$ and $\eta, \zeta < \omega^\mu$. Then $\alpha + \beta = \omega^\mu \cdot (m + n) + \eta < \omega^\mu \cdot (m + n) + \xi \leq \beta + \alpha + \xi$.

**E3.** By Marker, Cor. 3.1.6, it is sufficient to show that for all quantifier-free formulas $\varphi(\vec{v}, w)$, all atomless boolean algebras $M, N$ having a common subalgebra $A$, and all $\vec{a} \in A$: If there is a $b \in M$ such that $M \models \varphi(\vec{a}, b)$, then there is a $c \in N$ such that $N \models \varphi(\vec{a}, c)$. WLOG, $A$ is the subalgebra generated by $\vec{a}$, and hence finite. By the Downward Löwenheim-Skolem-Tarski Theorem, WLOG $|M| = |N| = \aleph_0$. But then it's trivial, since there is an isomorphism from $M$ onto $N$ which is the identity on $A$.

**C1.** Let $A$ be productive via $\psi$, set $W_{\varphi(x)} = W_i \cup \{\psi(i)\}$ where $W_i = \bigcup_{e \in W_x} W_e$. Now if $W_x \subseteq Dom(A)$ then for all $e \in W_x$, we have $W_e \subseteq A$ and so $W_i \subseteq A$. Thus $\psi(i) \in A - W_i$, so $W_{\varphi(x)} \subseteq A$ and $\varphi(x) \in Dom(A)$. On the other hand, if $\varphi(x) \in W_x$ then $W_{\varphi(x)} \subseteq W_i$, a contradiction.

**C2.** Without loss of generality assume that $A$ (and so $B$ and $C$) is neither empty nor all of $\omega$. Let $A \leq_m B \oplus C$ via $f$, set $A_0 = f^{-1}(B \oplus \emptyset)$ and $A_1 = f^{-1}(\emptyset \oplus C)$.

**C3.** See Odifreddi I, page 240. Note that by changing the retrace function $\psi$ we may assume that it has the properties:

(1) for any $x$ if $\psi(x)\downarrow= y$, then $\psi(y)\downarrow$ and either $y < x$ or $y = a_0$
(2) for any $x$ if $\psi(x)\downarrow$, then $\psi^n(x) = a_0$ for some $n$.

The idea is simply to refuse for say $\psi^*(x)$ to converge until finitely many iterates of $\psi$ converge and descend down to $a_0$.

To prove the Hint, let $\psi_0$ be the retrace function for $A$ and $\psi_1$ the retrace function for $\overline{A}$. Let $(x, 0) = 2x$ and $(x, 1) = 2x + 1$. Define the retrace function $\psi$ on $A \oplus \overline{A}$ as follows. Given $(x, i)$ if $\psi_i(x)\downarrow= y$ and $y < x - 1$ then put $\psi(x, i) = (x - 1, 1 - i)$, i.e., switch to the other side. If $y = x - 1$, then put $\psi(x, i) = (x - 1, i)$. Note that for any $x$ exactly one of the $(x, 0), (x, 1)$ is correct, i.e., in the set $A \oplus \overline{A}$. It is easy to see that if $(x, i)$ is correct, then $\psi(x, i)$ converges and is correct. By a finite modification we can have it end on the minimal element of the set.

Now we use $\psi$ to prove that the set $A$ is computable. Given any $x$ $\psi$ must converge on the correct one of $(x, 0)$ and $(x, 1)$ but may or may not converge on the other.

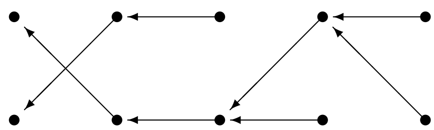
Case 1. For all but finitely many $x$, $\psi$ only converges on one of the two.

Then it must only converge on the correct one and so we can easily compute $A$ by going beyond the exceptions and waiting for the correct side to converge.

Case 2. There are infinitely many $x$ such that $\psi$ converges on both $(x, 0)$ and $(x, 1)$.

When it does there are four possible outcomes, either they crisscross or both stay on the same side or they both move to the same side. If they move to the same side (the last two squares in the figure) then the node they move to is correct since at least one of the two inputs was correct. Hence, if they are infinitely many squares that look like either of the last two in the figure we can easily compute $A$.

Lastly, we assume that for all but finitely many $x$ if $\psi$ converges on both sides then it either criss-crosses or stays on the same side, i.e, the first two squares in the diagram. In this case, with finitely many exceptions, $\psi$ always converges. We get two isolated branches one of which is always correct and the other always wrong. Since they are isolated we can compute them and hence $A$.



**M1.** Add a predicate symbol $P_n$ for each $n > 0$ meaning "$x$ belongs to a cycle of length $n$". Then $T$ eliminates quantifiers in this expanded language. Thus a 1-type over a model is either realized (rank 0) in the model, or is "$x$ belongs to an $n$-cycle not in the model" (rank 1) or "$x$ belongs to a $\mathbb{Z}$–chain not in the model" (rank 1).

Prime model: $\omega$ many cycles of each finite length. Saturated model: same plus $\omega$ many $\mathbb{Z}$–chains.

**M2.** Assume for a contradiction that there is a formula $\phi(x, \bar{b}) \in L(M)$ such that the sequence of truth values of $\phi(a_i, \bar{b})$ never stabilizes. Then for every $n$, and every $\sigma \in \{T, F\}^n$, we can find $i_0 < i_1 < ... < i_{n-1}$ such that $\phi(a_{i_j}, \bar{b}) = \sigma_j$ for all $j < n$. Embedding $M$ elementarily in a monster model for $T$, there is an automorphism $f$ sending $a_{i_0}, ..., a_{i_{n-1}}$ to $a_0, ..., a_{n-1}$, and let $\bar{b}_\sigma = f(\bar{b})$.

Then for all $\sigma \in \{T, F\}^n$ and $i < n$: $\phi(a_i, \bar{b}_\sigma) = \sigma_i$. As this can be done for all $n < \omega$, $\phi$ has the independence property.

Assume $\phi(x, \bar{b}), \psi(x, \bar{c}) \in \text{Avg}(\bar{a}, M)$. Then both are true from some point onwards, whereby their conjunction is eventually true, so $\phi(x, \bar{b}) \wedge \psi(x, \bar{c}) \in \text{Avg}(\bar{a}, M)$. Thus $\text{Avg}(\bar{a}, M)$ is closed under finite conjunction. Since every single formula in $\text{Avg}(\bar{a}, M)$ is consistent, $\text{Avg}(\bar{a}, M)$ is consistent. By the first item, every formula $\phi(x, \bar{b}) \in L(M)$ or its negation is there, so it is a complete 1-type.

**M3.** $2 \to 1$: Let $\phi(x, y)$ witness this. Then $r(x) \vdash \exists y \phi(x, y)$.

$1 \to 2$. Let $M \models T$, $a \in M$ realism $r$. Let $T(a) = Th(M, a)$. Then $q$ defines a closed subset $X \subseteq S(T(a))(= S(a))$. If (2) fails then $X$ has empty interior, i.e., is nowhere dense, and can be omitted in a model of $T(a)$, so (1) fails as well.