Qualifying Exam
Logic
January 2002

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to
the proctor and indicate your interpretation in your solution. In such cases,
do not interpret the problem in such a way that it becomes trivial.

E1. Let \( L \) be a language containing a single binary relation symbol \( E \), and
let \( G \) be an \( L \)-structure. An element \( x \in G \) has finite out-degree if there
are only finitely many \( y \) such that \( x E y \) holds in \( G \). Prove that there is no
\( L \)-sentence \( \varphi \) such that \( G \) satisfies \( \varphi \) if and only if all elements in \( G \) have finite out-degree.

E2. Show in ZFC that there exists a subset \( A \) of \( \mathbb{R}^2 \) that intersects every
circle in \( \mathbb{R}^2 \) in exactly three points.

Hint. You may take the reals as a given and use without proof that there
are exactly continuum many closed sets of reals and any uncountable closed
set of reals has cardinality the continuum.

E3. Fix a real \( x \in (0, 1) \), and assume that the \( n^{th} \) bit (past the ‘.’) in the
binary representation of \( x \) is a computable function of \( n \). Prove that the \( n^{th} \)
digit in the decimal representation of \( x \) is a computable function of \( n \).

Hint. It may be easier to break your proof into two cases, depending on
whether or not \( x \) is rational.

E4. Show that a set of natural numbers \( A \) is finite iff every subset of \( A \) is
computably enumerable.
Computability Theory

C1. Let $A \subseteq \omega$ be simple. Prove that there exists sets $B$ and $C$ such that
(1) both $B$ and $C$ are simple,
(2) $A = B \cup C$, and
(3) both $A - B$ and $A - C$ are infinite.

C2. Define
\[
\Phi_e(x) = \begin{cases} 
\mu s \varphi_{e,s}(x) \downarrow & \text{if } \varphi_e(x) \text{ converges} \\
\infty & \text{otherwise}
\end{cases}
\]
Prove that for every computable function $g : \omega \to \omega$ there exists a computable $f : \omega \to 2$ such that for every $e$:
\[
\text{if } \varphi_e = f \text{ then } \Phi_e(x) > g(x) \text{ for all but finitely many } x.
\]

C3. A learner is a computable mapping $M : \omega^{<\omega} \to \omega$. We say that $M$ learns a total computable function $f : \omega \to \omega$ iff there is an index $e$ such that $\varphi_e = f$ and
\[
M(f(0), f(1), \ldots, f(n)) = e \text{ for almost all } n.
\]
A family $S$ of functions is learnable iff there is a learner $M$ which learns every $f \in S$.

Prove that:
(a) Every computably enumerable family \{f_0, f_1, \ldots\} of total computable functions is learnable.
(b) The class of all total computable functions is not learnable.
Model Theory

**M1.** Let $F$ be a field of characteristic zero, and let $L$ be the first-order language with a constant symbol 0, a one-place function symbol $f_\lambda$ for each $\lambda \in F$ and a two-place function symbol $\cdot$. Let also $V$ be a nontrivial vector space over $F$, and consider 

$$V = (V, +, 0, f_\lambda)_{\lambda \in F}$$

as an $L$-structure where $+$ is vector addition, 0 is the zero vector, and each $f_\lambda : V \to V$ is scalar multiplication by $\lambda$.

1. Show that the theory of $V$ admits quantifier elimination. (You may use any standard facts from Linear Algebra.)

2. Let $S \subseteq V$. Show that the algebraic closure in the model theoretic sense of $S$ in $V$ is equal to the linear subspace of $V$ generated by $S$.

The algebraic closure in the model theoretic sense of $S$ in $V$ is defined to be the smallest subset $A$ of $V$ such that $S \subseteq A$ and for every first order formula $\varphi(x)$ with parameters from $A$ if there are only finitely many $v \in V$ such that $\varphi(v)$ holds in $V$, then all of these $v$ are in $A$.

**M2.** Let $L$ be a first-order language and $T$ an $L$-theory, and assume that $T$ is model-complete and universally axiomatizable. Let $p$ be a complete 1-type (over the empty set) consistent with $T$, and let $\phi(x)$ be an $L$-formula without parameters with at most one free variable $x$. The formula $\phi(x)$ isolates $p$ with respect to $T$ if and only if $\phi(x)$ is in $p$ and 

$$T \models \phi(x) \rightarrow \psi(x)$$

for every formula $\psi(x)$ in $p$. For any $L$-structure $A$ and any $a \in A$ we denote by $\langle a \rangle$ the substructure of $A$ generated by $a$.

Show that $\phi(x)$ isolates $p$ with respect to $T$ if and only if for any $M \models T$, $N \models T$, $a \in M$ and $b \in N$ such that $M \models \phi[a]$ and $N \models \phi[b]$, there is an $L$-isomorphism $f : \langle a \rangle \to \langle b \rangle$ such that $f(a) = b$. 

3
**M3.** Let $L$ be the language with one binary relation symbol $<$ and one unary operation symbol $f$. Let $T$ be the $L$-theory stating that $<$ is a dense linear ordering without endpoints and $f$ is an order preserving bijection such that $f(x) > x$ for all $x$.

1. Prove that $T$ admits quantifier elimination.

2. Prove that every model of $T$ is o-minimal.

3. Give, with justification, two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that the structures $(\mathbb{R}, <, f)$ and $(\mathbb{R}, <, g)$ are models of $T$, but the structure

$$(\mathbb{R}, <, f, g)$$

is not o-minimal.

A structure is o-minimal iff any subset of it which is definable with parameters is a finite union of sets each of which is a point, or an open interval with end points in the structure, or a ray with end point in the structure.
Set Theory

S1. Prove that the following are equivalent:

1. There is a family $F$ consisting of $\aleph_2$ stationary subsets of $\omega_1$ such that the intersection of any two distinct elements of $F$ is nonstationary.
2. There is a family $F$ consisting of $\aleph_2$ stationary subsets of $\omega_1$ such that the intersection of any two distinct elements of $F$ is countable.

Hint: The diagonal intersection $D$ of a sequence $\{C_\alpha | \alpha < \omega_1\}$ of closed unbounded sets is defined as

$$D = \{ \beta < \omega_1 | \beta \in \bigcap_{\alpha < \beta} C_\alpha \}$$

Show that $D$ is a closed unbounded set.

S2. Call $\mathcal{H}$ a MAD family iff

a. $\mathcal{H} \subseteq \mathcal{P}(\omega_1)$.

b. Each $A \in \mathcal{H}$ is uncountable.

c. $A \cap B$ is countable whenever $A, B$ are distinct elements of $\mathcal{H}$.

d. $\mathcal{H}$ is maximal with respect to (a,b,c).

Let $M$ be a countable transitive model for ZFC, let $\mathbb{P}$ be ccc partial order of $M$, and let $G$ be $\mathbb{P}$-generic over $M$. Assume that $\mathcal{H} \in M$ and that $M \models [\mathcal{H}$ is a MAD family]. Prove that $M[G] \models [\mathcal{H}$ is a MAD family].

S3. (Do not assume that $V = L$.) Let $\kappa$ be an uncountable regular cardinal. $ZC$ denotes $ZFC$ minus the Replacement Axiom. Prove that

$$\{ \alpha < \kappa : L_\alpha \models ZC \text{ but } L_\alpha \not\models ZFC \}$$

is unbounded in $\kappa$ but not stationary.
Answers

E1. Let \( c_n \) and \( d \) be new constant symbols. Let \( \theta_n \) be the first order sentence saying \( c_i \neq c_n \) for \( i < n \) and \( E(d, c_n) \). Then by the compactness theorem it is easy to check that the set of sentences \( \{ \varphi \} \cup \{ \theta_n : n < \omega \} \) has a model.

E2. Well-order the circles \( \{ C_\alpha : \alpha < \omega \} \). Inductively construct increasing \( A_\alpha \subseteq \mathbb{R}^2 \) so that

1. \( A_\alpha \) and no four points of it lie on a circle,
2. \( A_{\alpha+1} \) contains three points of \( C_\alpha \),
3. \( A_{\alpha+1} - A_\alpha \) is finite, and
4. at limits take unions.

Since three points determine a circle and any two circles intersect in at most two points, it is possible to do (1) and (2).

E3. If \( x \) is rational, then the decimal expansion of \( x \) is eventually periodic and hence computable. So we may assume that \( x \) is irrational. Let

\[
x = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = \sum_{n=1}^{\infty} \frac{d_n}{10^n}
\]

where each \( b_n \) is 0 or 1 and \( d_n \) is 0, 1, \ldots, 9. Let

\[
q_n = \sum_{k=1}^{n} \frac{b_k}{2^k}
\]

and suppose we have already computed

\[
r_N = \sum_{k=1}^{N} \frac{d_k}{10^k}
\]

then we just search for the least \( n \) such that for some \( i = 0, \ldots, 9 \)

\[
r_N + \frac{i}{10^{N+1}} < q_n < q_n + \frac{1}{2^n} < r_N + \frac{i + 1}{10^{N+1}}
\]

This \( i \) must be \( d_{N+1} \). (Note that the above comparison can be made by the usual grade school algorithms for adding fractions and comparing them.)

E4. Suppose \( A \) is infinite. Then \( A \) contains uncountably many subsets. Since there are only countably many ce sets, one of these must not be ce. On the other hand if \( A \) is finite, then all of its subsets are finite and hence ce.
Alternative solution by student on exam. Let \( f : \omega \to A \) be a one-to-one, onto, computable function. Let \( B = f(\overline{K}) \). Then \( B \) is not ce, because \( f \) shows that \( \overline{K} \leq_1 B \).

C1. Let \( f : \omega \to A \) be a one-to-one and onto, computable function. Let \( B = f(K) \). Then \( B \) is not ce, because \( f \) shows that \( K \leq 1_B \).

C2. At stage \( n \) let \( s = g(n) \).

Def \( e < n \) is not canceled is \( \forall x < n \ \varphi_{e,s}(x) \downarrow \varphi_e(x) = f(x) \).

Find the least \( e < n \) such that \( \phi_e \) has not been canceled and \( \varphi_{e,s}(n) \downarrow \) and put \( f(n) = 1 - \varphi_e(x) \).

C3.
(a) Let \( h \) be computable so that \( f_e = \varphi_{h(e)} \) for all \( e \). On input
\[
f(0) f(1) \ldots f(n)
\]
the learner searches for the first \( e \) such that \( f(m) = f_e(m) \) for \( m = 0, 1, \ldots, n \) and then outputs \( h(e) \).

(b) Assume that \( M \) is a learner which learns all computable functions. Start with the empty string \( \sigma_0 \) and extend \( \sigma_n \) inductively to \( \sigma_{n+1} \) such that one obtains an infinite computable sequence on which \( M \) does not converge.

Given \( \sigma_n \), there is a computable function \( f \supseteq \sigma_n \) which does not have an index below \( n \). Since \( M \) learns \( f \), there is an extension \( \sigma_{n+1} \subseteq f \) such that \( M(\sigma_{n+1}) > n \).

As one can search for the extension \( \sigma_{n+1} \) effectively only requiring that \( M(\sigma_{n+1}) > n \), the whole process gives a computable sequence \( \sigma_0, \sigma_1, \ldots \) of strings, each one properly extending the previous one. Therefore, the union of the \( \sigma_n \) is a computable function \( f \) such that \( M \) outputs arbitrarily large indices while reading \( f \). Contradiction, \( M \) does not learn \( f \).

M1.

1. Let \( \exists x \ \phi(x, y_1, \ldots, y_n) \) be a formula such that \( \phi \) is a conjunction of atomic and negation of atomic formulas. By using elementary linear algebra we may assume each of these conjunctions is of the form
\[
x = \alpha_1 y_1 + \cdots + \alpha_n y_n \text{ or } x \neq \alpha_1 y_1 + \cdots + \alpha_n y_n
\]
If the first case ever occurs, then just substitute \( \alpha_1 y_1 + \cdots + \alpha_n y_n \) for \( x \) in all the others and hence eliminate \( x \). If all the conjunctions are \( \neq \) then the formula is equivalent to True.
(2) Suppose $\theta(x, a_1, \ldots, a_n)$ has only finitely many solutions. Then by part (1) it is clear that $\theta$ is logically equivalent to saying that $x$ is one of a finite set of linear combinations of the $a_i$.

M2. Suppose $\phi(x)$ isolates $p$. Given $a, b$ define $f : \langle a \rangle \rightarrow \langle b \rangle$ by $f(\tau(a)) = \tau(b)$ where $\tau(x)$ is any term with one free variable. Then since $p$ is complete we have that $\tau(a) = \tau'(a)$ iff $\tau(b) = \tau'(b)$ and so $f$ is well-defined and similarly it is an isomorphism.

Suppose on the other hand that $\phi(x)$ does not isolate $p$, then there exists $M \models T, N \models T, a \in M$ and $b \in N$ such that $M \models \phi[a]$ and $N \models \phi[b]$, where $p$ is the type of $a$ in $M$ but not the type of $b$ in $N$. Since $\langle a \rangle$ and $\langle b \rangle$ are elementary substructures there can be no isomorphism $f$ taking $a$ to $b$.

M3.

(1) Let $\exists x \phi(x, y_1, \ldots, y_n)$ be a formula such that $\phi$ is a conjunction of atomic and negation of atomic formulas. Temporarily add the symbol $f^{-1}$ to the language. By using the properties of a linear order and $f$ (ie. we can replace $x < f(x)$ by True) these conjunctions can be taken to be of the form

$x = f^n(y_i), \ x < f^n(y_i)$ or $\ x > f^n(y_i)$ where $n$ is an integer (possibly negative or zero).

If the “=” case occurs, then we may substitute and eliminate $x$. If one of the other cases doesn’t occur then the formula is equivalent to True. If both of the other cases occur then just replace each pair $x < f^n(y_i), \ x > f^m(y_j)$ by $f^n(y_i) > f^m(y_j)$. To get rid of negative exponents just apply $f$ repeatedly to both sides of the equation or inequality, e.g. replace $f^{-3}(y_1) = f^2(y_2)$ by $y_1 = f^5(y_2)$, etc.

(2) Each atomic formula defines either a point or ray or empty set or the whole model. Hence by qe every definable set is a finite boolean combination of these.

(3) Let $f(x) = x + 2$ and $g(x) = f(x) + \sin(x)$. Then the set of $x$ where $f(x) = g(x)$ is the set of multiples of $\pi$.

S1. Suppose $\langle S_\alpha : \alpha < \omega_2 \rangle$ is a family. satisfying (1). By the hint: for any $\alpha$, there exists a club $C_\alpha$ such that $S_\alpha \cap S_\beta \cap C_\alpha$ is countable for all $\beta < \alpha$. Then, $\langle S_\alpha \cap C_\alpha : \alpha < \omega_2 \rangle$ satisfies (2).

S2. Suppose $\dot{X}$ is a name for a new set which is forced by $q$ to be uncountable and almost disjoint from all the members of MAD family $\mathcal{H}$. Define

$$S = \{ \alpha : \exists p \leq q \ p \vdash \alpha \in \dot{X} \}$$

Then $S$ is in the ground model and is uncountable. Hence there exists $Y \in \mathcal{H}$
which has uncountable intersection with $S$. Since the forcing is ccc we can find $\beta < \omega_1$ such that $q \Vdash X \cap Y \subseteq \beta$. Now, to get a contradiction, consider any $\alpha \in S \cap Y$ above $\beta$.

S3. To prove the set is unbounded: Let $\gamma$ be the $\omega^{th}$ cardinal of $L$ larger than $\kappa$. Then $L_\gamma$ is a model of ZC but not ZFC (because the last $\omega$-sequence of $L$-cardinals is definable). By elementary substructures and Mostowski collapse there are unboundedly many $\delta < \kappa$ such that $L_\delta$ can be elementarily embedded into $L_\gamma$.

To prove the set is nonstationary: Let $C$ be set of $\alpha < \kappa$ such that $L_\alpha$ is an elementary substructure of $L_\kappa$. Since $\kappa$ is regular, $L_\kappa$, and hence $L_\alpha$ for $\alpha \in C$, is a model for the Replacement Axiom, so none of these $L_\alpha$ can be a model of ZC without being a model of ZFC.