Qualifying Exam
Logic (Set Theory)
January 21, 2000

Instructions:

Do all six problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Assume the Continuum Hypothesis. Let $\mathcal{L} = \{R\}$, where R is binary, and let T be the theory in $\mathcal{L}$ whose axioms are: $\forall x\,[xRx]$, $\forall xy\,[xRy \leftrightarrow yRx]$, and $\forall xyz\,[xRy \wedge yRz \to xRz]$. Prove that T has exactly $\aleph_1$ non-isomorphic models of size $\aleph_1$.

E2. Define $0! = 1$, $(\alpha+1)! = \alpha! \cdot (\alpha+1)$, and $\gamma! = \sup_{\alpha<\gamma} \alpha!$ for limit γ. So, $\omega! = \omega$, $(\omega+2)! = \omega \cdot (\omega+1) \cdot (\omega+2)$, and $(\omega+\omega)! = \omega^\omega$. Prove that:

1. $\alpha! \leq \alpha^\alpha$ when $\alpha \neq 0$.
2. $(\beta+\alpha)! \geq \beta^{1+\alpha}$.
3. ε_0 is the least $\alpha > \omega$ such that $\alpha! = \alpha$.

$\varepsilon_0 = \sup_{n\in\omega} \tau_n$, where $\tau_0 = \omega$ and $\tau_{n+1} = \omega^{\tau_n}$. All exponentiation, $+$, and $\cdot$ refers to *ordinal* arithmetic.

E3. Let $\mathbb{Q}$ be the rationals, with the usual $<$ and $+$. Prove that $+$ is not first-order definable in the structure $(\mathbb{Q};<)$ – that is, there is no formula $\varphi(x,y,z)$ in $<$ and $=$ such that for all $a,b,c \in \mathbb{Q}$: $(\mathbb{Q};<) \models \varphi(a,b,c)$ iff $a+b=c$.

S1. Assume $MA + \neg CH$. Let A_α, for $\alpha < \omega_1$, be infinite sets. Prove that there is a set E such that for all α, both $A_\alpha \cap E$ and $A_\alpha \backslash E$ are infinite.

S2. In M (a countable transitive model of ZFC): Let λ be an uncountable cardinal with countable cofinality and $\kappa = \lambda^+$. Let $\mathbb{P}$ be the set of all pairs (a,A) such that $a, A \subset \kappa$, a is finite, $|A| < \lambda$, and $a \cap A = \emptyset$. Order $\mathbb{P}$ by: $(a,A) \leq (b,B)$ iff $a \supseteq b$ and $A \supseteq B$.

Let G be $\mathbb{P}$-generic over M. Prove that κ is the ω_1 of $M[G]$.

Hint: Show that in $M[G]$: κ remains regular, and there is a cofinal subset of κ all of whose initial segments are countable.

S3. Assume $V = L$. Let $o(x)$ be the least α such that $x \in L(\alpha+1)$. Let $W = \{o(x) : x \in \mathcal{P}(\omega)\}$. Prove that W is unbounded and non-stationary in ω_1.

Qualifying Exam
Logic (Computability Theory)
January 21, 2000

Instructions:

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E1. Assume the Continuum Hypothesis. Let $\mathcal{L} = \{R\}$, where R is binary, and let T be the theory in $\mathcal{L}$ whose axioms are: $\forall x\,[xRx]$, $\forall xy\,[xRy \leftrightarrow yRx]$, and $\forall xyz\,[xRy \wedge yRz \rightarrow xRz]$. Prove that T has exactly $\aleph_1$ non-isomorphic models of size $\aleph_1$.

E2. Define $0! = 1$, $(\alpha+1)! = \alpha! \cdot (\alpha+1)$, and $\gamma! = \sup_{\alpha<\gamma} \alpha!$ for limit γ. So, $\omega! = \omega$, $(\omega+2)! = \omega \cdot (\omega+1) \cdot (\omega+2)$, and $(\omega+\omega)! = \omega^\omega$. Prove that:

1. $\alpha! \leq \alpha^\alpha$ when $\alpha \neq 0$.
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$\varepsilon_0 = \sup_{n\in\omega} \tau_n$, where $\tau_0 = \omega$ and $\tau_{n+1} = \omega^{\tau_n}$. All exponentiation, $+$, and $\cdot$ refers to *ordinal* arithmetic.

E3. Let $\mathbb{Q}$ be the rationals, with the usual $<$ and $+$. Prove that $+$ is not first-order definable in the structure $(\mathbb{Q}; <)$ – that is, there is no formula $\varphi(x,y,z)$ in $<$ and $=$ such that for all $a,b,c \in \mathbb{Q}$: $(\mathbb{Q}; <) \models \varphi(a,b,c)$ iff $a+b=c$.

c.e. = r.e. = recursively enumerable = computably enumerable = Σ^0_1.
$\langle W_e : e \in \omega\rangle$ is a uniformly c.e. enumeration of c.e. sets; *uniformly* c.e. means that $\{(e,n) : n \in W_e\}$ is c.e.

C1. Show that there is an effective enumeration of all c.e. sets without repetition; that is, there is a computable function f such that the sequence $\langle W_{f(e)} : e \in \omega\rangle$ is uniformly c.e. and contains each c.e. set exactly once.

C2. Show that no 1-generic set is of minimal Turing degree. (A set A is *1-generic* if for all c.e. sets $S \subseteq 2^{<\omega}$ of binary strings, there is an initial segment σ of A such that either $\sigma \in S$ or no τ extending σ is in S. Here we identify sets and infinite binary strings.) *Hint*: Consider the set $B = \{x : 2x \in A\}$.

C3. Show that the collection of hyperimmune Turing degrees is closed upward. (A Turing degree **a** is *hyperimmune* if it contains a *hyperimmune set*, i.e., an infinite set A such that for any computable function f, there is some n such that the interval $[0, f(n)]$ contains fewer than n elements of A.)

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THE ANSWERS

E1. T says that R is an equivalence relation, so the models are determined up to isomorphism by the number of equivalence classes of each cardinality. For models of size $\aleph_1$, there are exactly $2^{\aleph_0} = \aleph_1$ possibilities.

E2. (1) and (2) are by induction on α; for (2), first note that $\beta! \geq \beta$. Using (1), show that $\alpha < \varepsilon_0 \to \alpha! < \varepsilon_0$, so that $\varepsilon_0! = \sup_{\alpha<\varepsilon_0} \alpha! \leq \varepsilon_0$; hence $\varepsilon_0! = \varepsilon_0$. Using (2) with $\beta = \omega$, show that $\tau_n! \geq \tau_{n+1}$ for $n > 0$, so that $\alpha! > \alpha$ whenever $\omega < \alpha < \varepsilon_0$.

E3. Consider a permutation of $\mathbb{Q}$ which is an automorphism of $(\mathbb{Q}; <)$ but not of $(\mathbb{Q}; +)$.

C1. See, e.g., Odifreddi, *Classical Recursion Theory* (Vol. 1, p. 230).

C2. Show that B is 1-generic, and that A is 1-generic relative to B, so $\emptyset <_T B <_T A$.

C3. Suppose H is hyperimmune and $S >_T H$. Find a set T such that $H \oplus T$ is hyperimmune and Turing equivalent to S by "spreading out" S.

S1. Choose a countably infinite $a_\alpha \subseteq A_\alpha$, let $E = \bigcup_\alpha a_\alpha$, and apply MA to finite partial functions from E to 2.

S2. $\mathbb{P}$ has the κ-cc in M, so κ remains a regular cardinal in $M[G]$. In $M[G]$, form $X = \bigcup\{a : \exists A[(a, A) \in G]\}$. Then X is unbounded in κ. For $E \in \mathcal{P}(\kappa) \cap M$, if $(|E| < \lambda)^M$, then $E \cap X$ is finite. Hence, for each $\xi < \kappa$, $(|\xi \cap X| \leq \omega)^{M[G]}$ (since, in M, one can cover ξ by countably many sets of size less than λ). Thus, the order type of X is $\leq (\omega_1)^{M[G]}$, so that $\kappa = (\omega_1)^{M[G]}$.

S3. W is unbounded because each $L(\alpha)$ is countable for $\alpha < \omega_1$. To prove that W is non-stationary, let $\{M_\alpha : \alpha < \omega_1\}$ be a continuous increasing chain of countable elementary submodels of $L(\omega_2)$. Let $\gamma_\alpha = M_\alpha \cap \omega_1$. Make sure that $\alpha < \beta \to \gamma_\alpha < \gamma_\beta$. Then $\{\gamma_\alpha : \alpha < \omega_1\}$ is a club and is disjoint from W.