Qualifying Exam
Logic
January 15, 1998

Instructions: If you signed up for Recursion Theory, do two E and two R problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $L$ be the language of one unary function $f$, and $T$ the $L$-theory stating that $f$ is a bijection. Call a subset of a $T$-model an orbit if it is of the form \{ $f^n(x)$ $|$ $n \in \mathbb{Z}$ \} where $f^n(x)$ is obtained from $x$ by applying $n$ many times $f$ to $x$ (or $-n$ many times applying $f^{-1}$ to $x$ if $n$ is negative). For a $T$-model $\mathfrak{m}$, call the spectrum of $\mathfrak{m}$ the set of all cardinals $\kappa \leq \omega$ such that some $T$-model $\mathcal{N}$ elementarily equivalent to $\mathfrak{m}$ has an orbit of size $\kappa$. Determine all possible spectra of $T$-models.

E2. Show that PA has a model $M$ such that: For some formula $\theta(x)$

$$M \models \exists x \theta(x)$$

but for every $n = 0, 1, 2, \ldots$

$$M \models \neg \theta(n).$$

E3. Define the product order on pairs of ordinals by:

$$(\xi, \eta) < (\alpha, \beta) \iff \xi \leq \alpha \land \eta \leq \beta \land (\xi, \eta) \neq (\alpha, \beta).$$

This is well-founded, so we can define:

$$rank(\alpha, \beta) = \sup \{ rank(\xi, \eta) + 1 : (\xi, \eta) < (\alpha, \beta) \}.$$ 

Compute $rank(\omega^3 + \omega \cdot 2 + 2, \omega^2 + \omega \cdot 3 + 1)$ explicitly in terms of ordinal arithmetic (+, $\cdot$, and exponentiation).
In the Recursion Theory problems, $\varphi_e$ is the $e$th partial recursive function of one variable, using some standard enumeration.

The words “computable” and “recursive” are assumed to have the same meaning. A fixed point of a total computable function $f$ is an index $e$ such that $\varphi_e = \varphi_{f(e)}$.

R1. Let $f$ be a total computable function, and assume that the set $S$ of its fixed points is computable. Show that $S$ must contain an index for every partial computable function.

R2. Let $K$ be the halting problem. Show that $K \oplus K$ is many-one reducible to $K \times \overline{K}$, but not conversely. **Hint:** Show that the converse would imply $K$ computable.

R3. Show: There is a $\Delta^0_2$-set $A$ which is not the Boolean combination of computably enumerable sets. (A Boolean combination of computably enumerable sets is obtained from the computably enumerable sets by a finite number of unions, intersections, and complementations.)
E1. All finite subsets of $\omega + 1$; and all infinite subsets of $\omega + 1$ containing $\omega$. Use compactness theorem and explicit construction.

E2. Let $\theta(x)$ say that $x$ is the code of a proof of $0=1$ from the axioms of PA. Take $M$ to be a model of PA + not Con(PA).

E3. $\omega^3 + \omega^2 + \omega \cdot 5 + 3$.

If you start to write out a table of values, you will see the pattern. The general formula (not needed here) is: \( \text{rank}(\alpha, \beta) = \alpha \# \beta \), where \( \# \) is natural sum: In Cantor Normal Form, if $\alpha = \omega^{\mu_1} \cdot m_1 + \cdots + \omega^{\mu_k} \cdot m_k$ and $\beta = \omega^{\mu_1} \cdot n_1 + \cdots + \omega^{\mu_k} \cdot n_k$ then $\alpha \# \beta = \omega^{\mu_1} \cdot (m_1 + n_1) + \cdots + \omega^{\mu_k} \cdot (m_k + n_k)$; here, $\mu_1 > \cdots > \mu_k$ and $0 \leq m_i, n_i < \omega$ for $i = 1, \ldots, k$.

R1. Otherwise, fix $e_0$ such that $\varphi_{e_0}$ has no index in $S$. Then the following total computable function, $g$, has no fixed point, contradicting the Recursion Theorem: Let $g(x) = f(x)$ for $x \notin S$, and $g(x) = e_0$ for $x \in S$.

R2. In one direction, simply fix $e \in K$ and $e' \notin K$, and map $2x$ to $(x, e')$, and $2x + 1$ to $(e, x)$. For the other direction, assume $f$ is such a reduction and distinguish cases: If for all $x \in K$, there is some $y \in K$ with $f(y, x)$ even, then $\overline{K}$ is seen to be $\Sigma_1$; else there is some $x_0 \in K$ such that for all $y \in K$, $f(y, x_0)$ is odd, so $g(y) = f(y, x_0)$ gives a reduction of $K$ to $\emptyset \oplus \overline{K} \equiv_m \overline{K}$.

R3. We can list all Boolean combinations of c.e. sets as $\{V_e : e \in \omega\}$, where $\{(x, e) : x \in V_e\}$ is computable in $0'$, and hence $\Delta^0_3$. Now, let $A = \{x : x \notin V_x\}$. 
