Qualifying Exam
Logic
January 15, 1998

Instructions: If you signed up for Recursion Theory, do two E and two R problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $\mathcal{L}$ be the language of one unary function f, and T the $\mathcal{L}$-theory stating that f is a bijection. Call a subset of a T-model an orbit if it is of the form $\{f^n(x) \mid n \in \mathbf{Z}\}$ where $f^n(x)$ is obtained from x by applying n many times f to x (or $-n$ many times applying f^{-1} to x if n is negative). For a T-model $\mathfrak{M}$, call the spectrum of $\mathfrak{M}$ the set of all cardinals $\kappa \leq \omega$ such that some T-model $\mathcal{N}$ elementarily equivalent to $\mathfrak{M}$ has an orbit of size κ. Determine all possible spectra of T-models.

E2. Show that PA has a model M such that: For some formula $\theta(x)$

$$M \models \exists x\, \theta(x)$$

but for every $n = 0, 1, 2, \ldots$

$$M \models \neg\theta(n).$$

E3. Define the product order on pairs of ordinals by:

$$(\xi, \eta) < (\alpha, \beta) \iff \xi \leq \alpha \ \&\ \eta \leq \beta \ \&\ (\xi, \eta) \neq (\alpha, \beta) \quad .$$

This is well-founded, so we can define:

$$rank(\alpha, \beta) = sup\{rank(\xi, \eta) + 1 : (\xi, \eta) < (\alpha, \beta)\} \quad .$$

Compute $rank(\omega^3 + \omega \cdot 2 + 2,\ \omega^2 + \omega \cdot 3 + 1)$ explicitly in terms of ordinal arithmetic ($+$, $\cdot$, and exponentiation).

In the Recursion Theory problems, φ_e is the e^{th} partial recursive function of one variable, using some standard enumeration.

The words "computable" and "recursive" are assumed to have the same meaning. A fixed point of a total computable function f is an index e such that $\varphi_e = \varphi_{f(e)}$.

R1. Let f be a total computable function, and assume that the set S of its fixed points is computable. Show that S must contain an index for every partial computable function.

R2. Let K be the halting problem. Show that $K \oplus \overline{K}$ is many-one reducible to $K \times \overline{K}$, but not conversely. *Hint:* Show that the converse would imply K computable.

R3. Show: There is a Δ^0_2-set A which is not the Boolean combination of computably enumerable sets. (A Boolean combination of computably enumerable sets is obtained from the computably enumerable sets by a finite number of unions, intersections, and complementations.)

Answers to Logic Qual January 1998

E1. All finite subsets of $\omega+1$; and all infinite subsets of $\omega+1$ containing ω. Use compactness theorem and explicit construction.

E2. Let $\theta(x)$ say that x is the code of a proof of 0=1 from the axioms of PA. Take M to be a model of PA + not Con(PA).

E3. $\omega^3 + \omega^2 + \omega \cdot 5 + 3$.

If you start to write out a table of values, you will see the pattern. The general formula (not needed here) is: $rank(\alpha, \beta) = \alpha \# \beta$, where # is natural sum: In Cantor Normal Form, if $\alpha = \omega^{\mu_1} \cdot m_1 + \cdots + \omega^{\mu_k} \cdot m_k$ and $\beta = \omega^{\mu_1} \cdot n_1 + \cdots + \omega^{\mu_k} \cdot n_k$ then $\alpha \# \beta = \omega^{\mu_1} \cdot (m_1 + n_1) + \cdots + \omega^{\mu_k} \cdot (m_k + n_k)$; here, $\mu_1 > \cdots > \mu_k$ and $0 \leq m_i, n_i < \omega$ for $i = 1, \ldots, k$.

R1. Otherwise, fix e_0 such that φ_{e_0} has no index in S. Then the following total computable function, g, has no fixed point, contradicting the Recursion Theorem: Let $g(x) = f(x)$ for $x \notin S$, and $g(x) = e_0$ for $x \in S$.

R2. In one direction, simply fix $e \in K$ and $e' \notin K$, and map $2x$ to (x, e'), and $2x + 1$ to (e, x). For the other direction, assume f is such a reduction and distinguish cases: If for all $x \in \overline{K}$, there is some $y \in K$ with $f(y, x)$ even, then $\overline{K}$ is seen to be Σ_1; else there is some $x_0 \in \overline{K}$ such that for all $y \in K$, $f(y, x_0)$ is odd, so $g(y) = f(y, x_0)$ gives a reduction of K to $\emptyset \oplus \overline{K} \equiv_m \overline{K}$.

R3. We can list all Boolean combinations of c.e. sets as $\{V_e : e \in \omega\}$, where $\{(x, e) : x \in V_e\}$ is computable in $0'$, and hence Δ^0_2. Now, let $A = \{x : x \notin V_x\}$.