Qualifying Exam
Logic
August 31, 1995

Instructions: If you signed up for model theory, do two E and two M problems. If you signed up for recursion theory, do two E and two R problems. If you signed up for set theory, do two E and two S problems. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $\mathbb{R}$ be the set of real numbers. Prove that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that f maps every perfect subset of $\mathbb{R}$ *onto* $\mathbb{R}$. We say $P \subseteq \mathbb{R}$ is perfect iff it is closed, non-empty, and has no isolated points (for example, the Cantor set, or any interval).

E2. Let $\mathcal{L}$ be the language consisting of $=$, two binary functions, $+, *$, and one unary function, f. Let $\mathfrak{A}$ be the structure whose domain of discourse is the set of real numbers, where $+, *$ are interpreted as the usual addition and multiplication, and f is interpreted as the *sin* function. Prove that the theory of $\mathfrak{A}$ is undecidable.

E3. Let $\mathcal{L}$ be any language in predicate logic *without* equality, consisting of finitely many constant, function, and predicate symbols. A *clause* of $\mathcal{L}$ is a logical sentence of the form $\forall x_1 \ldots x_n(\phi_1 \vee \ldots \vee \phi_k)$, where each ϕ_i is either an atomic formula or the negation of an atomic formula. For example, $\forall xy(p(x, f(g(y))) \vee \neg p(g(x), y))$ is a clause. Prove that it is decidable whether a clause of $\mathcal{L}$ is logically valid.

M1. Let L and L' be first order languages such that $L' \subseteq L$. Let T be a theory in L. Suppose that for any two models M, N for L whose L'-reducts M' and N' are isomorphic, M is a model of T if and only if N is a model of T. Prove that T is equivalent to a theory in L'.

M2. Let T be a model complete theory in a countable language. Suppose R is a binary relation in the language of T, and let K be the class of all models M of T such that M is well ordered by R^M. We say that N is an *end extension* of M if N is a proper extension of M and $N \models R(a, b)$ for all $a \in M$ and $b \in N - M$.

Suppose that K is nonempty and each $M \in K$ has an end extension $N \in K$. Prove that for each uncountable cardinal κ there exists $M \in K$ such that R^M has order type κ.

M3. Let J be an uncountable set, let A be the set of all finite subsets of J, and let $M = \langle A, R \rangle$ where R is the subset relation on A. Let $N = \langle B, S \rangle$ be a countably indexed ultrapower of M such that the natural embedding $d : M \prec N$ is proper. Prove that:

a) For each $b \in B$ the set $E_b = \{a \in A : S(d(a), b)\}$ is at most countable.

b) For each countable subset $C \subset B$, there exists $b \in B$ such that $S(c, b)$ for all $c \in C$.

R1. Let $f : \omega \to \omega$ be a recursive function, and let $S = \{e \mid \varphi_e = \varphi_{f(e)}\}$. Show that if S is recursive then it contains an index for every partial recursive function.

R2. Let $\mathcal{A}$ be a uniformly recursively enumerable collection of recursively enumerable sets. Assume that $\mathcal{A}$ contains all finite sets. Show that there is a uniform enumeration of $\mathcal{A}$ without repetitions.

R3. Let A be a simple set. Prove that A is Turing complete iff there is a function $f \leq_T A$ such that for all e, $W_e \subseteq \bar{A}$ implies $|W_e| \leq f(e)$.

S1. Let M be a countable transitive model for $ZFC + GCH$, and let I be an infinite set in M. Let $\mathbb{P}$ be the partial order of finite partial functions from I to I, and let G be $\mathbb{P}$-generic over M. Prove that $M[G] \models GCH$. *Note*: I need not be countable in M.

S2. Let $\mathbb{Q}$ be the set of rational numbers. Call $S \subset \mathbb{Q}$ *small* iff $S \cap (-\infty, x)$ is finite for all $x \in \mathbb{Q}$. Assume Martin's Axiom, and let $\mathcal{F}$ be a family of fewer than $2^{\aleph_0}$ small sets. Prove that there is a small set $T \subset \mathbb{Q}$ such that $S \backslash T$ is finite for all $S \in \mathcal{F}$.

S3. Assume $\alpha < \beta$, $R(\alpha) \prec R(\beta)$, and α is regular. Prove that for some $\delta < \alpha$, $R(\delta) \prec R(\alpha)$. Here, $R(\alpha) = V(\alpha)$ is the set of sets of rank less than α, and $\prec$ means "elementary submodel".

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E1. List all the perfect sets as $\{P_\alpha : \alpha < \mathfrak{c}\}$, and list all reals as $\{r_\alpha : \alpha < \mathfrak{c}\}$. Let $\phi, \psi : \mathfrak{c} \to \mathfrak{c}$ so that the map $\xi \mapsto (\phi(\xi), \psi(\xi))$ maps $\mathfrak{c}$ onto $\mathfrak{c} \times \mathfrak{c}$. Choose $x_\xi \in P_{\phi(\xi)} \setminus \{x_\eta : \eta < \xi\}$, and let $f(x_\xi) = r_{\psi(\xi)}$.

E2. In $\mathfrak{A}$, one may define π as the first positive x such that $sin(x) = 0$, and then define y to be an integer iff $sin(\pi y) = 0$. Then, just use the fact that the theory of the integers with $+, *$ is undecidable.

E3. The only way a clause can be logically valid is if one of the ϕ_i is the negation of some other ϕ_j.

M1. Let T' be the set of all consequences of T in L'. Let M be a model of T'. Let S' be the set of all sentences in L' true in M. Then $S' \bigcup T$ is finitely satisfiable. By the compactness theorem $S' \bigcup T$ has a model N. The L'-reducts of M and N are elementarily equivalent. Then M and N have elementary extensions M_1 and N_1 whose L'-reducts are isomorphic. N_1 is a model of T, so by hypothesis M_1 and hence M is a model of T. Therefore T' is a theory in L' which is equivalent to T.

M2. Since T is model complete, whenever $M \subseteq N$ and $M, N \in K$, N is an elementary extension of M. By the elementary chain theorem, the union of a chain of end elementary extensions of M is again an elementary extension and thus a model of T. Since each extension is an end extension, this union is also well ordered by R and hence belongs to K. Given a cardinal κ, by transfinite recursion we may form a sequence of models $M_\alpha \in K, \alpha \leq \kappa$ such that $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for limit ordinals α, and M_α is an end extension of M_β. By the downward Lowenheim-Skolem theorem we may also take M_α to be of cardinality $\omega \cup |\alpha|$. Then $M_\kappa \in K$ has order type κ.

M3. a) $N = \Pi_U M$ for some nonprincipal ultrafilter U over ω. Each $b \in B$ has the form $b = \langle b_n : n \in \omega \rangle_U$, and each b_n belongs to A and hence is finite. If $a \in E_b$, then $\{n \in \omega : a \subseteq b_n\} \in U$. Therefore E_b is contained in the countable set $\bigcup_n P(b_n)$. b) For each $n \in \omega$ the model M satisfies the sentence ϕ_n which says that for each $C \subset A$ of size $\leq n$ there exists $a \in A$ such that $S(c, a)$ for all $c \in C$. By Łos' theorem, $N \models \phi_n$ for each n. Then b) follows because N is ω_1-saturated.

R1. Else find a recursive function g without fixed points by setting $g(e) = f(e)$ if e is not a fixed point and equal to an index for a function not represented in S otherwise.

R2. A variation on Friedberg's theorem that this holds for the class of all r.e. sets (see, e.g., Odifreddi's book, page 230).

R3. Left-to-right: Since A is complete, A can compute whether $W_e \subseteq \bar{A}$. If so then, by simplicity of A, W_e must be finite, and A can compute the size of W_e by the completeness of A. Right-to-left: Mimic Martin's proof that every effectively simple set is Turing complete.

S1. In M, let $\kappa = |I|$. Note that κ becomes countable in $M[G]$. In $M[G]$, let λ be any infinite cardinal; so $\lambda = \omega$ or $\lambda > \kappa$. Then $2^\lambda = \lambda^+$ holds in $M[G]$ because in M, there are only λ^+ nice $\mathbb{P}$-names for subsets of λ.

S2. Let $\mathcal{F} = \{S_\alpha : \alpha < \kappa\}$. Let $\mathbb{P}$ be the set of all pairs $p = (f_p, b_p)$ such that f_p is a finite partial function from κ to ω and $b_p \in \mathbb{Q}$. Let $T(p)$ be $\bigcup\{S_\alpha \cap (f_p(\alpha), \infty) : \alpha \in dom(f_p)\}$. Think of $T(p)$ as an approximation to T and b_p as a promise that no further rationals below b_p will be added to T. So, define $p \leq q$ iff f_p extends f_q, $b_p \geq b_q$, and $(-\infty, b_q) \cap T(p) = (-\infty, b_q) \cap T(q)$. Let $T = \bigcup\{T(p) : p \in G\}$, where G meets enough dense sets.

S3. Using $R(\alpha) \prec R(\beta)$, α is strongly inaccessible. Hence, by a Löwenheim-Skolem-Tarski argument, there is a $\delta < \alpha$ such that $R(\delta) \prec R(\alpha)$.