Qualifying Exam
Model Theory
August 31, 1993

Instructions: Do all four problems. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

E1. Describe the set of all ordinals, α, such that $\alpha + \omega^2 = \omega^2 + \alpha$. Justify your answer.

E2. Two elements of a partial order are compatible iff there exists an element $\leq$ to both of them. An antichain in a partial order is a set of pairwise incompatable elements. Suppose P is a partially ordered set with an antichain of size greater than n for each $n \in \omega$. Prove that for every infinite cardinal κ there exists a partially ordered set elementarily equivalent to P with a maximal antichain of cardinality κ.

M1. Prove that there is an uncountable model for PA which is ω-homogeneous but not ω_1-homogeneous.

M2. Given two models

$$A = (U, R_1, R_1, \ldots), \qquad B = (V, S_1, S_2, \ldots)$$

of models for the same language such that U and V are disjoint, define the union to be

$$A \cup B = (U \cup V, R_1 \cup S_1, R_2 \cup S_2, \ldots).$$

Suppose that $A_1 \equiv A_2$, $B_1 \equiv B_2$, A_1, B_1 have disjoint universes, and A_2, B_2 have disjoint universes. Prove that $A_1 \cup B_1 \equiv A_2 \cup B_2$.

Qualifying Exam
Set Theory
August 31, 1993

Instructions: Do any four problems. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

E1. Describe the set of all ordinals, α, such that $\alpha + \omega^2 = \omega^2 + \alpha$. Justify your answer.

E2. Two elements of a partial order are compatible iff there exists an element $\leq$ to both of them. An antichain in a partial order is a set of pairwise incompatable elements. Suppose P is a partially ordered set with an antichain of size n for each $n \in \omega$. Prove that for every infinite cardinal κ there exists a partially ordered set elementarily equivalent to P with a maximal antichain of cardinality κ.

S1. Prove that adding 1 Cohen real isn't the same as adding ω_1 Cohen reals. More precisely, let $Fn(\alpha, 2)$ be the partial order of all finite partial functions from α to 2. Let M be a countable transitive model of ZFC, and let δ be any ordinal in M which is not countable in M. Let G be $Fn(\omega, 2)$-generic over M and let H be $Fn(\delta, 2)$-generic over M. Prove that $M[G] \neq M[H]$.

S2. A ladder is a sequence of the form $\langle C_\alpha : \alpha \in Lim \rangle$ where Lim is the set of countable limit ordinals and each C_α is a cofinal subset of α of order type ω. A coloring of the ladder $\langle C_\alpha : \alpha \in Lim \rangle$ is a sequence $\langle x_\alpha : \alpha \in Lim \rangle$ where each x_α is a function from C_α to 2.

A ladder has the uniformization property iff for every coloring of it there exists $f : \omega_1 \to 2$ such that for every $\alpha \in Lim$

$$f(\beta) = x_\alpha(\beta)$$

holds for all but finitely many $\beta \in C_\alpha$.

(a) Show that $\diamondsuit_{\omega_1}$ implies that no ladder has the uniformization property.

(b) Show that MA+notCH implies that every ladder has the uniformization property.

S3. A formula ϕ in $\in, =$ is called Δ_0 iff all quantifiers in ϕ are bounded (i.e., occur as $\forall x \in y \ldots$ or $\exists x \in y \ldots$). Define $\hat{L}$ exactly as L is defined, but use just Δ_0 formulas. That is, $\hat{L}(0) = \emptyset$, and $\hat{L}(\alpha+1)$ is the set of all subsets of $\hat{L}(\alpha)$ definable over $\hat{L}(\alpha)$ using a Δ_0 formula. As with L, the definition may mention a finite number of parameters from $\hat{L}(\alpha)$. Take unions at limit ordinals, as usual. Prove that $\hat{L} = L$. *Hint.* Prove $\hat{L} \subseteq L \subseteq \hat{L}$. It's *not* true that each $\hat{L}(\alpha) = L(\alpha)$.

Answers to Logic Qual Aug 93

E1. $\alpha = \omega^2 n$ for some $n \in \omega$.

E2. Use compactness and Lowenheim-Skolem to get a model of size κ with a κ antichain. Use choice to extend it to a maximal antichain.

M1. Start with an ω_1 saturated model and build an ω-chain of ω-homogenous models each with a new element on the end. Then cofinal a ω sequence has same type as some bounded sequence from first model.

M2. Add extra unary relations for the universes of A_i and B_i, and form the model pairs (A_i, B_i). Then take special or saturated models $(A'_i, B'_i) \equiv (A_i, B_i), i = 1, 2$ of the same sufficiently large cardinality and prove that (A'_1, B'_1) is isomorphic to (A'_2, B'_2). Finish up by taking reducts to the original language.

Alternatively, you can use Ehrenfeucht games.

S1. In $M[G]$ every uncountable subset of ω_1 contains an uncountable subset of the ground model M.

S2.(b) Let P be the partial order of functions whose domain is a finite union of C_α's and which agree with the corresponding x_α with finitely many exceptions. Use delta-system and push-down arguments to show P has ccc.

S3. It's enough to prove $\hat{L}$ is a transitive model for ZF and contains all the ordinals; then, by absoluteness, L and $\hat{L}$ are subsets of each other.

The basic stuff about L, e.g.: each $L(\alpha)$ is transitive, $ON \cap L(\alpha) = \alpha$, and $L(\alpha) \in L(\alpha + 1)$; goes over unchanged to $\hat{L}$, since the formulas used are all Δ_0.

This yields all the axioms except comprehension. To prove comprehension with the formula ϕ: reflect as usual to get an α such that ϕ relativized to $\hat{L}$ is equivalent to ϕ relativized to $\hat{L}(\alpha)$. Then, since $\hat{L}(\alpha)$ is a member of $\hat{L}(\alpha+1)$, the set you're trying to construct by quantifying over $\hat{L}(\alpha)$ becomes Δ_0 over $\hat{L}(\alpha + 1)$, so it's collected in $\hat{L}(\alpha + 2)$.