Instructions: Do all four problems. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

E1. Describe the set of all ordinals, $\alpha$, such that $\alpha + \omega^2 = \omega^2 + \alpha$. Justify your answer.

E2. Two elements of a partial order are compatible iff there exists an element $\leq$ to both of them. An antichain in a partial order is a set of pairwise incompatible elements. Suppose $P$ is a partially ordered set with an antichain of size greater than $n$ for each $n \in \omega$. Prove that for every infinite cardinal $\kappa$ there exists a partially ordered set elementarily equivalent to $P$ with a maximal antichain of cardinality $\kappa$.

M1. Prove that there is an uncountable model for PA which is $\omega$-homogeneous but not $\omega_1$-homogeneous.

M2. Given two models

$$A = (U, R_1, R_1, \ldots), \quad B = (V, S_1, S_2, \ldots)$$

of models for the same language such that $U$ and $V$ are disjoint, define the union to be

$$A \cup B = (U \cup V, R_1 \cup S_1, R_2 \cup S_2, \ldots).$$

Suppose that $A_1 \equiv A_2$, $B_1 \equiv B_2$, $A_1, B_1$ have disjoint universes, and $A_2, B_2$ have disjoint universes. Prove that $A_1 \cup B_1 \equiv A_2 \cup B_2$. 
Instructions: Do any four problems. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

E1. Describe the set of all ordinals, \( \alpha \), such that \( \alpha + \omega^2 = \omega^2 + \alpha \). Justify your answer.

E2. Two elements of a partial order are compatible iff there exists an element \( \leq \) to both of them. An antichain in a partial order is a set of pairwise incompatible elements. Suppose \( P \) is a partially ordered set with an antichain of size \( n \) for each \( n \in \omega \). Prove that for every infinite cardinal \( \kappa \) there exists a partially ordered set elementarily equivalent to \( P \) with a maximal antichain of cardinality \( \kappa \).

S1. Prove that adding 1 Cohen real isn’t the same as adding \( \omega_1 \) Cohen reals. More precisely, let \( Fn(\alpha, 2) \) be the partial order of all finite partial functions from \( \alpha \) to 2. Let \( M \) be a countable transitive model of ZFC, and let \( \delta \) be any ordinal in \( M \) which is not countable in \( M \). Let \( G \) be \( Fn(\omega, 2) \)-generic over \( M \) and let \( H \) be \( Fn(\delta, 2) \)-generic over \( M \). Prove that \( M[G] \neq M[H] \).

S2. A ladder is a sequence of the form \( \langle C_\alpha : \alpha \in Lim \rangle \) where \( Lim \) is the set of countable limit ordinals and each \( C_\alpha \) is a cofinal subset of \( \alpha \) of order type \( \omega \). A coloring of the ladder \( \langle C_\alpha : \alpha \in Lim \rangle \) is a sequence \( \langle x_\alpha : \alpha \in Lim \rangle \) where each \( x_\alpha \) is a function from \( C_\alpha \) to 2.

A ladder has the uniformization property iff for every coloring of it there exists \( f : \omega_1 \to 2 \) such that for every \( \alpha \in Lim \)

\[ f(\beta) = x_\alpha(\beta) \]

holds for all but finitely many \( \beta \in C_\alpha \).

(a) Show that \( \diamond \omega_1 \) implies that no ladder has the uniformization property.

(b) Show that \( MA + notCH \) implies that every ladder has the uniformization property.
S3. A formula $\phi$ in $\in,=$ is called $\Delta_0$ iff all quantifiers in $\phi$ are bounded (i.e., occur as $\forall x \in y \ldots$ or $\exists x \in y \ldots$). Define $\hat{L}$ exactly as $L$ is defined, but use just $\Delta_0$ formulas. That is, $\hat{L}(0) = \emptyset$, and $\hat{L}(\alpha + 1)$ is the set of all subsets of $\hat{L}(\alpha)$ definable over $\hat{L}(\alpha)$ using a $\Delta_0$ formula. As with $L$, the definition may mention a finite number of parameters from $\hat{L}(\alpha)$. Take unions at limit ordinals, as usual. Prove that $\hat{L} = L$. Hint. Prove $\hat{L} \subseteq L \subseteq \hat{L}$. It’s not true that each $\hat{L}(\alpha) = L(\alpha)$. 
E1. $\alpha = \omega^2 n$ for some $n \in \omega$.

E2. Use compactness and Lowenheim-Skolem to get a model of size $\kappa$ with a $\kappa$ antichain. Use choice to extend it to a maximal antichain.

M1. Start with an $\omega_1$ saturated model and build an $\omega$-chain of $\omega$-homogenous models each with a new element on the end. Then cofinal a $\omega$ sequence has same type as some bounded sequence from first model.

M2. Add extra unary relations for the universes of $A_i$ and $B_i$, and form the model pairs $(A_i, B_i)$. Then take special or saturated models $(A'_i, B'_i) \equiv (A_i, B_i), i = 1, 2$ of the same sufficiently large cardinality and prove that $(A'_1, B'_1)$ is isomorphic to $(A'_2, B'_2)$. Finish up by taking reducts to the original language.

Alternatively, you can use Ehrenfeucht games.

S1. In $M[G]$ every uncountable subset of $\omega_1$ contains an uncountable subset of the ground model $M$.

S2.(b) Let $P$ be the partial order of functions whose domain is a finite union of $C_\alpha$’s and which agree with the corresponding $x_\alpha$ with finitely many exceptions. Use delta-system and push-down arguments to show $P$ has ccc.

S3. It’s enough to prove $\hat{L}$ is a transitive model for $ZF$ and contains all the ordinals; then, by absoluteness, $L$ and $\hat{L}$ are subsets of each other.

The basic stuff about $L$, e.g.: each $L(\alpha)$ is transitive, $ON \cap L(\alpha) = \alpha$, and $L(\alpha) \in L(\alpha + 1)$; goes over unchanged to $\hat{L}$, since the formulas used are all $\Delta_0$.

This yields all the axioms except comprehension. To prove comprehension with the formula $\phi$: reflect as usual to get an $\alpha$ such that $\phi$ relativized to $\hat{L}$ is equivalent to $\phi$ relativized to $L(\alpha)$. Then, since $\hat{L}(\alpha)$ is a member of $\hat{L}(\alpha + 1)$, the set you’re trying to construct by quantifying over $\hat{L}(\alpha)$ becomes $\Delta_0$ over $\hat{L}(\alpha + 1)$, so it’s collected in $\hat{L}(\alpha + 2)$. 