LOGIC QUALIFYING EXAM, JANUARY 1993 – RECURSION THEORY

INSTRUCTIONS: Do two elementary problems and two recursion theory problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

ELEMENTARY PROBLEMS

E1. Find three ordinals: α, β, γ such that the 6 sums:

$\alpha+\beta$, $\alpha+\gamma$, $\beta+\alpha$, $\beta+\gamma$, $\gamma+\alpha$, $\gamma+\beta$

are all distinct.

E2. Suppose T is a decidable set of axioms in a countable language, all models of T are infinite, and T has only finitely many non-isomorphic countable models. Prove that the set of all logical consequences of T is decidable.

E3. Assume ZFC is consistent. Prove that there is a Turing machine M such that:

1. M does not halt.
2. ZFC cannot prove that M does not halt.

RECURSION THEORY PROBLEMS

R1. One of the following statements is true and the other is false. Prove the true one and disprove the false one.

1. Suppose f is a recursive function such that whenever ϕ_n is total, $\phi_{f(n)}$ is total. Then there exists n such that ϕ_n is total and $\phi_{f(n)} = \phi_n$.

2. Suppose f is a recursive function such that whenever $\phi_n(0)$ is defined, $\phi_n(0) = \phi_{f(n)}(0)$. Then there exists n such that $\phi_n(0)$ is defined and $\phi_{f(n)} = \phi_n$.

R2. Prove that the index set $\{\langle x,y\rangle \mid W_x \subseteq W_y\}$ is Π_2 – complete.

R3. A set $A \subseteq \omega$ is called <u>immune</u> if it is infinite but does not contain any infinite r.e. subset. It is called <u>retraceable</u> if there is a partial recursive function ϕ such that $\phi(x) = x$ for the least element $x \in A$, and for all other elements $y \in A$, $\phi(y)$ is the next smaller element of A. Show that every retraceable set is recursive or immune.

LOGIC QUALIFYING EXAM, JANUARY 1993 – MODEL THEORY

INSTRUCTIONS: Do two elementary problems and two model theory problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

ELEMENTARY PROBLEMS

E1. Find three ordinals: α, β, γ such that the 6 sums:

$\alpha+\beta$, $\alpha+\gamma$, $\beta+\alpha$, $\beta+\gamma$, $\gamma+\alpha$, $\gamma+\beta$

are all distinct.

E2. Suppose T is a decidable set of axioms in a countable language, all models of T are infinite, and T has only finitely many non-isomorphic countable models. Prove that the set of all logical consequences of T is decidable.

E3. Assume ZFC is consistent. Prove that there is a Turing machine M such that:

1. M does not halt.
2. ZFC cannot prove that M does not halt.

MODEL THEORY PROBLEMS

M1. Prove that the complete theory of the model $\langle \mathbb{Z}, \leq \rangle$ has exactly 2 countable ω–homogeneous models up to isomorphism.

M2. Let T be a complete theory with infinite models in a countable language. Prove that there is an elementary chain $\mathfrak{A}_\alpha$, $\alpha < \omega_1$, of countable models of T such that whenever $\alpha < \beta < \omega_1$, $\mathfrak{A}_\alpha \cong \mathfrak{A}_\beta$ but $\mathfrak{A}_\alpha \neq \mathfrak{A}_\beta$.

M3. Let $\mathfrak{A}$ be a model for a countable language and let $\Gamma(x)$ be a set of formulas which is consistent but not realized in $\mathfrak{A}$. Let $\mathfrak{B} = \Pi_U \mathfrak{A}$ be an ultrapower of $\mathfrak{A}$ with respect to a countably incomplete ultrafilter U. Prove that $\Gamma(x)$ is satisfied by infinitely many elements of $\mathfrak{B}$.