LOGIC QUALIFYING EXAM, JANUARY 1992

INSTRUCTIONS: Do any four problems, including at most two elementary problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

ELEMENTARY PROBLEMS

E1. Answer true or false for each of the following statements. If true, indicate a reason. If false, give a counter-example. $\alpha, \beta, \gamma$ range over ordinals.

(Sample). $\forall\alpha\forall\beta\forall\gamma(\alpha \cdot \beta = \beta \cdot \alpha)$.
Answer: False. $\omega \cdot 2 \neq 2 \cdot \omega$.

a. $\forall\alpha\forall\beta\forall\gamma(\alpha < \beta \Rightarrow \alpha + \gamma < \beta + \gamma)$.

b. $\forall\alpha\forall\beta\forall\gamma(\alpha < \beta \Rightarrow \gamma + \alpha < \gamma + \beta)$.

c. $\forall\alpha\forall\beta\forall\gamma((\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma)$.

d. $\forall\alpha\forall\beta\forall\gamma(\gamma \cdot (\alpha + \beta) = \gamma \cdot \alpha + \gamma \cdot \beta)$.

e. $\forall\alpha\forall\beta\forall\gamma((\alpha + \beta = \beta + \alpha) \land (\beta + \gamma = \gamma + \beta) \Rightarrow (\alpha + \gamma = \gamma + \alpha))$.

E2. Let $T$ be a recursive set of sentences in a finite language $L$. Assume that for each sentence $\phi$ of $L$, either $T \cup \{\phi\}$ is inconsistent or $T \cup \{\phi\}$ has a finite model. Prove that the set $\{\phi : T \models \phi\}$ is recursive.

E3. In the language with one unary function symbol $f$, prove that the theory $\{\forall x f(f(x)) = x\}$ has countably many complete extensions, and describe them.
M1. Let $D(\mathfrak{A})$ denote the diagram of a model $\mathfrak{A}$, that is, the set of all atomic and negated atomic sentences true in $\mathfrak{A}$. Suppose that $T$ is a complete theory, $\mathfrak{A}$ is a model of $T$, and $T \cup D(\mathfrak{A})$ is complete. Prove that for every elementary submodel $\mathfrak{B}$ of $\mathfrak{A}$, $T \cup D(\mathfrak{B})$ is complete.

M2. Let $\mathfrak{A}$ be an arbitrary model of Peano arithmetic. Prove that $\mathfrak{A}$ has an ultrapower $\mathfrak{B}$ with an element $b \in B$ such that $\{c \in B : \mathfrak{B} \models c \leq b\}$ has size $2^\omega$.

M3. Let $T$ be a complete theory in a countable language and let $\kappa$ be a cardinal. Prove that $T$ has a countable model $\mathfrak{A}$ and a model $\mathfrak{B}$ of size $\kappa$ such that every countable elementary submodel of $\mathfrak{B}$ is elementarily embeddable in $\mathfrak{A}$. Hint: Use indiscernibles.
LOGIC QUALIFYING EXAM, JANUARY 1992, RECURSION THEORY

Notation: $\phi_n$ is the recursive function with Gödel number $n$ and $W_n$ is the domain of $\phi_n$. $\leq_T$ means Turing reducible, and $\equiv_T$ means Turing equivalent. $B'$ denotes the jump of $B$.

**R1.** a. Show that there is an index $n$ such that $W_n = \{n\}$.
b. Use this and the Padding Lemma to show that

$$K = \{e \mid \phi_e(e) \text{ converges}\}$$

is not an index set.

**R2.** Given a nonrecursive r.e. set $A$, give a construction to show that there is a simple set $S$ such that $S \leq_T A$.

**R3.** Prove that for all sets $A, B \subseteq \omega$, if $A \leq_T B'$ then there is a binary relation $C \equiv_T B$ such that $\lim_s C(s, \cdot) = B$. 
S1. Prove that there is a totally ordered set \((X, <)\) of size \(\aleph_1\) such that every ordinal \(\alpha < \omega_2\) is isomorphic to a subset of \(X\). \textit{Don't} assume CH. \textit{Hint.} Consider \(\omega_1^{<\omega}\) ordered lexically, and use induction.

S2. In the following, forcing always refers to the Cohen partial order – that is, finite partial functions from \(\omega\) into 2. In the ground model, \(M\), assume that \(F\) is a \textit{closed} set of real numbers. Prove that the following are equivalent:
1. \(1 \models (\bar{F} \text{ is closed})\).
2. \(F\) is countable in \(M\).

\textit{Hint.} Uncountable closed sets contain a copy of the Cantor set.

S3. \textit{Notation:} An antichain is a pairwise incompatible family.

Let \(T = \{s \mid (\exists \alpha < \omega_1)(s : \alpha \to \omega \text{ and } s \text{ is } 1 - 1)\}\), ordered by inclusion.

Prove:

a. \(T\) has no \(\omega_1\)-branches.

b. Every uncountable subset of \(T\) contains an uncountable antichain.